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Article

Violation of the Third Law of Black Hole Thermodynamics in Higher Curvature Gravity

Takashi Torii

Department of General Education, Osaka Institute of Technology, 5-16-1, Omiya, Asahi-ku, Osaka, 535-8585, Japan; E-Mail: torii@ge.oit.ac.jp

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Abstract: We examine the weak version of the third law of black hole thermodynamics in the n-dimensional Einstein–Gauss–Bonnet system with a negative cosmological constant. To see whether the extreme black hole solution with zero temperature is formed, we investigate the motion of the thin shell that has the equal mass to the extreme black hole in the background described by the non-GR branch solutions. The interior of the shell is empty. Our analysis using the generalized Israel's junction condition shows that the shell can contract beyond a certain radius and the degenerate horizon is formed for same range of parameters. Hence, this model can be a counterexample of the third law.

Keywords: black holes; thermodynamics; modified gravity

1. Introduction

Black holes are characteristic objects to general relativity (GR). From the observational point of view, recent data show the existence of one or more huge black holes in the central region of a number of galaxies, and it becomes recognized commonly that the galaxies evolve together with such black holes.

From the theoretical point of view, there are a large number of interesting topics. Among them, black hole thermodynamics is one of the most important themes (for a review see, e.g., [1]). It was pointed out that there are remarkable relationship between the ordinary laws of thermodynamics and the certain properties of black hole [2,3]. After that, the concept of a black hole temperature was established by the discovery of the Hawking radiation [4,5], and evidences for the "generalized second law" were proposed [6–10].

The third law of thermodynamics is formulated in a several ways. The statement given by Nernst, which is sometimes called the strong version, asserts that the entropy of a system tends to a universal constant as its temperature approaches absolute zero. However, it can be easily checked that the analogue of the strong version does not hold for Kerr–Newman black hole at least theoretically. There are still some discussions on this violation [11]. In black hole thermodynamics the weak version of the third law is more often discussed [2], which states that the temperature of a system cannot be reduced to zero in a finite number of operations, *i.e.*, the extreme black hole of which temperature is zero cannot be produced by any physical processes. Several evidences for this unattainability principle were shown in GR [12–15] except for one interesting counterexample in a black hole system with a non-Abelian hair [16].

On the other hand, black holes and their thermodynamics are one of the main topics also in superstring-M theories, in particular, following the discoveries of the microscopic origin of black hole entropy [17,18] and the AdS/CFT correspondence [19,20]. Since no much is known about the non-perturbative aspects of the theory, taking string effects perturbatively into classical gravity is one approach to the study of the quantum effects of gravity. Among the correction terms with respect to the curvature in the low-energy supergravity action coming from superstrings, the simplest one can be described by the Gauss–Bonnet (GB) term [21,22], which is ghost-free combinations [23,24] and gives the second order field equations. The gravitational theory including the GB term is considered as one of the modified gravity theories, which attracts much attention recently. For instance, the problem of the dark energy is intensively studied in the context of modified gravity (See [25] for recent review and references therein).

The black hole solutions in GB gravity were discovered by Boulware and Deser [26]. Since then, many types of black hole solutions have been intensively studied (See [27,28], and references therein). The black hole solutions are classified into the GR and the non-GR branches according to whether there is GR limit or not, respectively. By the recent work, it was found that the black hole solutions in the non-GR branch have peculiar properties [29]. In this paper, we show that the peculiar black hole solutions also become the counterexample for the third law of black hole thermodynamics in the Einstein–GB-Λ system. The only one counterexample of the weak version is given in [16] to the best of our knowledge. There the discussion relies on a quasi-static treatment and the result can be considered as a prediction. However, we consider an infalling matter and show that the extreme black hole is formed through such dynamical process. This can be the direct evidence for the violation of the weak version of the third law.

It should be commented here that when we consider these objects, we need thermodynamics of the black holes in generalized theories of gravity. Iyer and Wald formulated the black hole thermodynamics in arbitrary diffeomorphism invariant theories of gravity [30–32]. In their formulation, the black hole entropy is defined as a Noether charge and satisfies the first law of black hole thermodynamics.

In Section 2, we introduce our model and briefly review the black hole solutions in the Einstein–GB-Λ system in higher dimensions. In Section 3, we derive the equation of motion of a thin shell around the black hole by using the generalized Israel's junction condition [33–35]. In Section 4, we investigate the motion of the thin shell and show the violation of the third law for certain parameter range. In Section 5, we give conclusions and discuss related issues and future work. Throughout this paper we use units such that $c = \hbar = k_B = 1$. The Greek indices run $\mu, \nu = 0, 1, \cdots, n-1$.

2. Black Hole Solution in the Einstein–GB-Λ System

2.1. Model

We start with the n-dimensional action:

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{2\kappa_n^2} (R - 2\Lambda + \alpha \mathcal{L}_{GB}) \right] \tag{1}$$

where R and Λ are the n-dimensional Ricci scalar and the cosmological constant, respectively. $\kappa_n := \sqrt{8\pi G_n}$, where G_n is the n-dimensional gravitational constant. The GB Lagrangian $\mathcal{L}_{GB}$ is,

$$\mathcal{L}_{GB} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \tag{2}$$

α is the coupling constant of the GB term. This type of action is derived from the superstring theory in the low-energy limit [21,22]. Since α is related to inverse string tension in string theory, we assume that α is non-negative.

The gravitational equation of the system Equation (1) is,

$$G_{\mu\nu} + \alpha H_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3}$$

where,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4}$$

$$H_{\mu\nu} = 2\Big[RR_{\mu\nu} - 2R_{\mu\alpha}R^{\alpha}_{\ \nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_{\mu}^{\ \alpha\beta\gamma}R_{\nu\alpha\beta\gamma} \Big] - \frac{1}{2} g_{\mu\nu}\mathcal{L}_{GB} \tag{5}$$

We assume the static spacetime and adopt the following line element:

$$ds^2 = -f(r)e^{-2\delta(r)}dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{n-2}^2 \tag{6}$$

where $d\Omega_{n-2}^2 = \gamma_{ij}dx^i dx^j$ is the metric of the unit $(n-2)$-dimensional constant curvature space with curvature $k = 0, \pm 1$.

2.2. Solutions

The solution of the gravitational equation is obtained as [26,36]:

$$f = k + \frac{r^2}{2\tilde{\alpha}}(1 + \epsilon x) \tag{7}$$

$$\delta \equiv 0 \tag{8}$$

where we have defined $\tilde{\alpha} := (n-3)(n-4)\alpha$, $\Lambda = -(n-1)(n-2)/2\ell^2$,

$$x := \sqrt{1 + 4\tilde{\alpha}\left(\frac{\tilde{M}}{r^{n-1}} - \frac{1}{\ell^2} \right)} \tag{9}$$

$$\tilde{M} := \frac{2\kappa_n^2 M}{(n-2)\Sigma_{n-2}^k} \tag{10}$$

The integration constant M is the mass of the black hole. Although we will also consider not only the black hole solutions but also the AdS solution and singular solutions, we call M the mass of the solution. Σ_{n-2}^k is the volume of the unit $(n-2)$-dimensional constant curvature space. There are two families of solutions, which correspond to $\epsilon = \pm 1$.

In the asymptotic region with large r, the solution is expanded as:

$$f = k + \frac{r^2}{2\tilde{\alpha}}(1 + \epsilon x_0) + \frac{\epsilon \tilde{M}}{x_0 r^{n-3}} \tag{11}$$

where,

$$x_0 := \sqrt{1 - \frac{4\tilde{\alpha}}{\ell^2}} \tag{12}$$

It is noted that the conserved mass of the solution is M while the gravitational mass of the solution is,

$$M_g := -\frac{\epsilon M}{x_0} \quad \left(\tilde{M}_g := -\frac{\epsilon \tilde{M}}{x_0}\right) \tag{13}$$

In the $\alpha \to 0$ limit,

$$f = k - \frac{\tilde{M}}{r^{n-3}} + \frac{r^2}{\ell^2} \tag{14}$$

for $\epsilon = -1$ of Equation (7). This is the higher dimensional black hole solution in GR of which spherical version ($k = 1$) was found by Tangherlini [37]. On the other hand, there is no GR limit for $\epsilon = +1$. We call solutions that have a minus (plus) sign the GR (non-GR) branch solution.

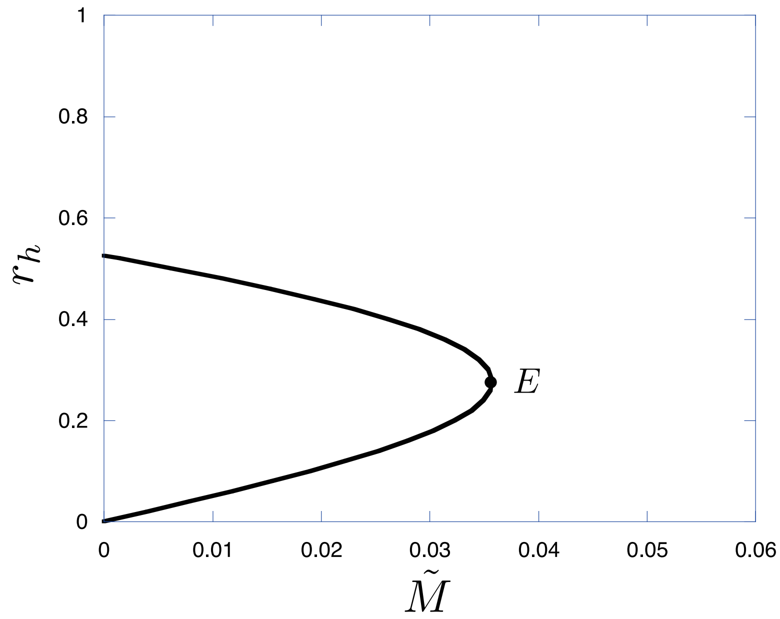
2.3. Properties of the Solutions

When the mass parameter vanishes $\tilde{M} = 0$, the spacetime is pure vacuum expressed by Equation (7) with $x = x_0$. For a well-defined theory, the condition $4\tilde{\alpha} \leq \ell^2$ should be satisfied. When $4\tilde{\alpha} = \ell^2$, the pure vacuum solutions of both branches coincide. In this paper, we assume $4\tilde{\alpha} < \ell^2$.

We briefly show the properties of the solutions with the parameters $n \geq 6$, $k = -1$ and $\epsilon = +1$. With this parameter, there is an extreme black hole solution without an electromagnetic charge and the solutions have interesting properties from the viewpoint of the third law, as we will see below. The properties of other solutions are summarized in [27,28].

The horizon radius r_h is obtained by the condition $f(r_h) = 0$. Figure 1 shows the $\tilde{M}$-r_h diagram. This diagram shows that the horizon radius (or the area) decreases as the mass of the black hole increases. This implies that the area theorem does not hold in this system. (It is confirmed that the second law of black hole thermodynamics holds [27].) For the Reissner–Nordström black hole in GR, the charged shell must be fallen into the non-extreme black hole against the Coulomb force, or the evaporation through Hawking radiation must be completed to zero temperature within a finite time to obtain the extreme black hole. However, these processes are forbidden [14]. For our black hole, we do not need such process but only have to add some matter to the black hole. The black hole solutions with this type exist only for the parameter $n \geq 6$, $k = -1$ and $\epsilon = +1$. Although there is a possibility that the third law would be violated for other parameters and dimensions, we explore this case in this paper as the first step.

Figure 1. The $\tilde{M}$-r_h diagrams of the static solutions in the six-dimensional Einstein–GB-Λ system. We set $1/\ell^2 = 1$ (negative cosmological constant), $k = -1$, $\epsilon = +1$ (the non-GR branch) and $\tilde{\alpha} = 0.2$. For $0 < \tilde{M} < \tilde{M}_{ex}$, the solution has a black hole horizon (the upper one) and an inner horizon (the lower one). The dot with character "E" implies the degenerate horizon.



The pure vacuum solution has a black hole event horizon, of which radius r_0 is given by:

$$r_0^2 = \frac{\ell^2}{2}\left(1 - \sqrt{1 - \frac{4\tilde{\alpha}}{\ell^2}}\right) \tag{15}$$

Although the solution has the black hole event horizon, the center is not singular but regular and spacelike.

For $\tilde{M} = \tilde{M}_{ex}$, the solution has a degenerate horizon r_{ex} and represents the extreme black hole spacetime. The mass of the extreme solution is,

$$\tilde{M} = \tilde{M}_{ex} = \frac{2}{n-1}\left(\frac{2\tilde{\alpha}}{r_{ex}^2} - 1\right) r_{ex}^{n-3} \tag{16}$$

where the radius of the degenerate horizon is,

$$r_{ex}^2 = \frac{(n-3)\ell^2}{2(n-1)}\left[1 - \sqrt{1 - \frac{4\tilde{\alpha}}{\ell^2}\frac{(n-1)(n-5)}{(n-3)^2}}\right] \tag{17}$$

For $0 < \tilde{M} < \tilde{M}_{ex}$, the solution has a black hole horizon and an inner horizon. The center of the spacetime is timelike singularity. For $\tilde{M} > \tilde{M}_{ex}$, the solution has no horizon and represents the spacetime with a globally naked singularity.

3. Equation of Motion of the Thin Dust Shell

We define the trajectory of the $(n-1)$-dimensional thin shell moves as $t = t(\tau)$ and $r = R(\tau)$ in the n-dimensional spacetime with the metric Equations (6)–(8), where τ is the proper time on the shell. The induced metric on the shell defined by $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is written as:

$$dl^2 = -d\tau^2 + R(\tau)^2 d\Omega_{n-2}^2 \tag{18}$$

The tangent vector of the shell is $u^\mu = (\dot{t}, 0, \cdots, 0, \dot{R})$. By the normalization condition $u_\alpha u^\alpha = -1$, we find:

$$f^2 \dot{t}^2 - \dot{R}^2 = f \tag{19}$$

The unit normal vector defined by $u^\alpha n_\alpha = 0$ becomes $n_\mu = (-\dot{R}, 0, \cdots, 0, \dot{t})$, where the normal vector points to the outward direction $r > R(\tau)$.

The non-zero components of the extrinsic curvature $K_{\mu\nu} := h_\mu^{\ \alpha} h_\nu^{\ \beta} \nabla_\alpha n_\beta$ are:

$$K_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{f\dot{t}} \left(\ddot{R} + \frac{f'}{2} \right) \tag{20}$$

$$K_{ij} = \frac{f\dot{t}}{a} g_{ij} \tag{21}$$

where a prime denotes the differentiation with respect to r.

The junction condition at the shell is [34,35],

$$[K_{ij}]_\pm - h_{ij}[K]_\pm + 2\alpha \Big(3[J_{ij}]_\pm - h_{ij}[J]_\pm - 2P_{iajb}[K^{ab}]_\pm \Big) = -\kappa_n^2 \tau_{ij} \tag{22}$$

where,

$$J_{ij} = \frac{1}{3} \big(2KK_{ia}K^a_{\ j} + K_{ab}K^{ab}K_{ij}, -2K_{ia}K^{ab}K_{bj} - K^2 K_{ij} \big) \tag{23}$$

$$P_{ijkl} = R_{ijkl} + 2h_{i[l}R_{k]j} + 2h_{j[k}R_{l]i} + Rh_{i[k}h_{l]j} \tag{24}$$

and τ_{ij} is the energy-momentum tensor on the shell. We have introduced the bracket:

$$[X]_\pm := X^+ - X^- \tag{25}$$

where $X^\pm$ are X's evaluated either on the outer (for the plus sign) or inner (for the minus sign) side of the shell, and $P_{\mu\nu\rho\sigma}$ is the divergence free part of the Riemann tensor, *i.e.*,

$$D_\alpha P^\alpha_{\ ijk} = 0 \tag{26}$$

The shell is assumed to be dust with the surface energy density ρ. Since the surface energy of the dust shell is conserved [38], it behaves as:

$$\frac{d}{d\tau}(\rho R^{n-2}) = 0 \tag{27}$$

The proper mass of the dust shell defined by $M_s = \Sigma_{n-2}^k R^{n-2} \rho$ remains constant. The difference between the gravitational mass and the proper mass $M_s - M_g$ can be interpreted as a kind of binding energy of the shell.

The (τ, τ) component of the junction condition (22) is,

$$\left[1 - \frac{2\alpha(n-1)(n-6)\dot{R}^2}{3R^2} + \frac{4\alpha k}{R^2}\right]\frac{[f\dot{t}]_\pm}{R} - \frac{2\tilde{\alpha}}{3}\frac{[f^2\dot{t}]_\pm}{R^3} = -\frac{\kappa_n^2}{n-2}\rho = -\frac{\tilde{M}_s}{2R^{n-2}} \tag{28}$$

where,

$$\tilde{M}_s := \frac{2\kappa_n^2 M_s}{(n-2)\Sigma_{n-2}^k} \tag{29}$$

By using the normalization condition (19), the equation of the shell becomes:

$$\left[1 - \frac{2\alpha(n-1)(n-6)\dot{R}^2}{3R^2} + \frac{4\alpha k}{R^2}\right]\left[\sqrt{f + \dot{R}^2}\right]_\pm - \frac{2\tilde{\alpha}}{3R^2}\left[f\sqrt{f + \dot{R}^2}\right]_\pm = -\frac{\tilde{M}_s}{2R^{n-3}} \tag{30}$$

4. Motion of the Shell in 6-dimensional Spacetime

In the 5-dimensional spacetime, the solution (7)–(9) is not an extreme black hole and $T \neq 0$ for any M. For $n \geq 6$, the black hole solutions have qualitatively same properties independent of the dimensions. Among these dimensions, Equation (30) becomes simple for $n = 6$. Hence, we focus on the 6-dimensional spacetime. In 6-dimensional spacetime, the Equation of motion (30) can be written in a simple form as:

$$\left[D\sqrt{f + \dot{R}^2}\right]_\pm = -C \tag{31}$$

where,

$$C := \frac{\tilde{M}_s}{2R^3} \tag{32}$$

$$D := 1 + \frac{4\alpha k}{R^2} - \frac{2\tilde{\alpha}}{3R^2}f \tag{33}$$

This equation can be written as:

$$\frac{1}{2}\dot{R}^2 + V(R) = 0 \tag{34}$$

where,

$$\begin{aligned} V(R) \;\; = \;\; & -\frac{1}{2(D_+^2 - D_-^2)^2}\Big\{C^2(D_+^2 + D_-^2) - (D_+^2 - D_-^2)(f_+ D_+^2 - f_- D_-^2) \\ & \pm 2C\sqrt{D_+^2 D_-^2\left[C^2 - (f_+ - f_-)(D_+^2 - D_-^2)\right]}\Big\} \end{aligned} \tag{35}$$

The signs in the potential V should be chosen to satisfy the original Equation (31). Since the "energy" of the shell is zero in Equation (34), the shell can move the region where $V(R) \leq 0$.

When $\alpha = 0$, *i.e.*, in GR without the GB term, the potential of the shell is,

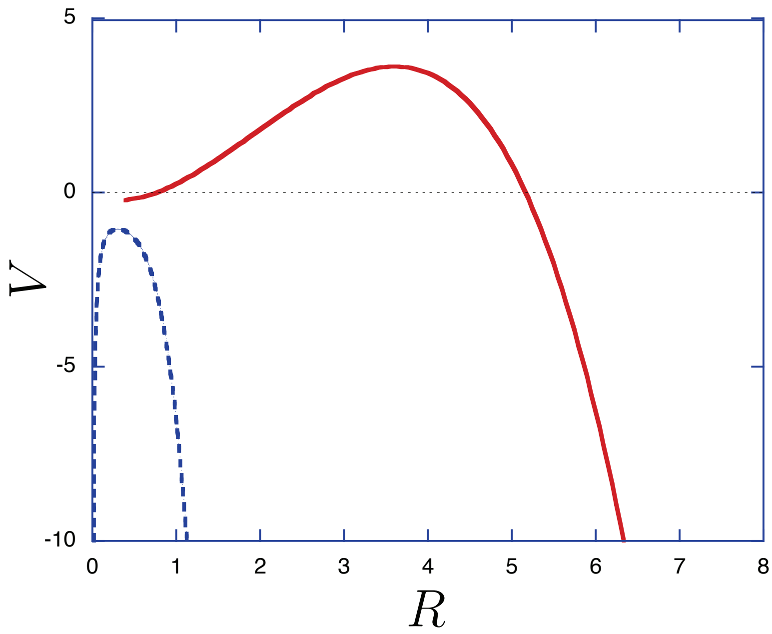
$$V(R) = \frac{1}{2C^2}\left[f_+ f_- - \frac{1}{4}(f_+ + f_- - C^2)^2\right] \tag{36}$$

We assume the inside of the dust shell is empty and its metric function f is described by Equation (7) with $x = x_0$ and that the exterior solution is described by Equations (7), (9) and (16), *i.e.*, the extreme black hole spacetime.

Here we set the parameter as $\tilde{\alpha} = 0.2$, $\ell^2 = 1$, $k = -1$ and $\epsilon = 1$. (When the cosmological constant is nonzero, the curvature radius ℓ is scales out by introducing new variables as $r := r/\ell$, $t := t/\ell$, $\tilde{M} := \tilde{M}/\ell^{n-3}$, $\alpha := \alpha/\ell^2$, $R := R/\ell$, $\tau := \tau/\ell$, and $\tilde{M}_s := \tilde{M}_s/\ell^{n-3}$.) For the exterior solution, the mass parameter is $\tilde{M}_{ex} = 0.035771$. When the shell shrinks inside the radius $r_{ex} = 0.276393$, the degenerate horizon is formed at $r = r_{ex}$.

Figure 2 shows the potential $V(R)$ of the shell. $V(R)$ behaves as $-R^4$ for large R, which means that $\dot{R} \propto R^2$. (The asymptotic behavior of the shell is discussed in Appendix A.) For the case with $\tilde{M}_s = 0.1\tilde{M}_{ex}$, the contracting shell from the infinity bounces at $r = 5.1648$ and expands to infinity. This means that the degenerate horizon is not formed and the spacetime is everywhere regular.

Figure 2. "Potential" of the thin shell around the black hole. We set $\tilde{\alpha} = 0.2$, $\ell^2 = 1$, $k = -1$, $\epsilon = 1$, and $\tilde{M}_s = 0.1\tilde{M}_{ex}$ (red solid curve), $\tilde{M}_s = 2.0\tilde{M}_{ex}$ (blue dashed curve). The shell can move the region where $V(R) \leq 0$.



On the other hand, for the case with $\tilde{M}_s = 2.0\tilde{M}_{ex}$, the contracting shell from the infinity continues to collapse to the center since $V(R) < 0$ everywhere and would form the central singularity. In this case, the degenerate horizon is formed without any irrelevant phenomena. Hence, this can be the counterexample to the third law of the black hole thermodynamics.

5. Conclusions

In this paper we examined the third law of black hole thermodynamics and found a counterexample in the n-dimensional Einstein–GB system with the negative cosmological constant. The spacetime is assumed to have symmetry corresponding to the isometries of an $(n-2)$-dimensional maximally symmetric space with negative curvature. To obtain the extreme black hole solution with zero temperature, we investigate the motion of the thin shell that has the equal mass to the extreme black hole. The interior of the shell is empty. Our analysis using the generalized Israel's junction condition shows that the shell can contract beyond a certain radius and the degenerate horizon is formed for same range of parameters. This model can be a counterexample of the third law.

We only consider the six-dimensional spacetime concretely. However, that violation of the third law is expected also in higher-dimensions. In this case, it seems difficult to derive the potential function of the shell, and Equation (30) should be solved directly. It should be noted here that our counterexample presumably comes about the peculiar properties of the black hole solutions in the non-GR branch. It is not sure whether the third law is violated for more physically reasonable black holes or not. Furthermore, the shell in our analysis is infinitely thin and produces a kind of discontinuous behavior. It gives more strict result to consider a thick shell or more physically reasonable matter fields. As mentioned above, our system is highly different from the physical situation, and the third law in ordinary thermodynamics is an empirical law that is not needed to construct the thermodynamics. In this sense, it is difficult to point out direct effects of our counterexample on the physics at this stage. However, even so the discovery of the counterexample of the third law in GB gravity gives potential applicability to unsolved problem in the context of modified gravity. These prospects are left for possible future investigations.

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Appendix

A. Asymptotic Motion of the Shell

In this appendix, we briefly examine the motion of a shell in the asymptotic region $(r \to \infty)$ for the non-GR branch solution. For the large radius,

$$f_+ = f_- - \frac{\tilde{M}_g}{R^3} \tag{37}$$

Note that $\tilde{M}_g$ defined by Equation (13) is negative for the non-GR branch solution. Substituting this equation into Equation (31), we obtain,

$$x_0 + \frac{8\alpha(k + \dot{R}^2)}{R^2} = -\frac{\tilde{M}_s}{\tilde{M}_g}\sqrt{k + \dot{R}^2 + \frac{R^2}{2\tilde{\alpha}}(1 + x_0)} \tag{38}$$

The leading order gives,

$$\dot{R} = \mp \frac{\tilde{M}_s}{8\alpha x_0 \tilde{M}_g} R^2 \tag{39}$$

for the non-GR branch. The plus and minus signs mean the expanding and contracting shells, respectively. This is consistent with Figure 2. This means $R \propto \mp(\tau - \tau_0)^{-1}$, where τ_0 is the time when the shell is at infinity. Hence, the shell reaches AdS infinity within a finite time.

On the other hand, in GR, when the radius of the shell becomes large, it inevitably bounces at some finite radius in the AdS spacetime because the potential Equation (36) behaves as:

$$V(R) \sim +\frac{R^2}{2\ell^2} \tag{40}$$

The difference comes from the gravitational mass of the shell. In the Einstein–GB-Λ system, the gravitational mass of the shell defined at infinity becomes negative for the non-GR branch solution. Hence, the shell seems to have negative surface energy, and the expansion does not stop for the expanding shell. As a result, the motion of the shell shows the qualitatively different behavior from GR.