

Article

Local Softening of Information Geometric Indicators of Chaos in Statistical Modeling in the Presence of Quantum-Like Considerations

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Abstract: In a previous paper (C. Cafaro *et al.*, 2012), we compared an uncorrelated 3D Gaussian statistical model to an uncorrelated 2D Gaussian statistical model obtained from the former model by introducing a constraint that resembles the quantum mechanical canonical minimum uncertainty relation. Analysis was completed by way of the information geometry and the entropic dynamics of each system. This analysis revealed that the chaoticity of the 2D Gaussian statistical model, quantified by means of the Information Geometric Entropy (IGE), is *softened* or weakened with respect to the chaoticity of the 3D Gaussian statistical model, due to the accessibility of more information. In this companion work, we further constrain the system in the context of a correlation constraint among the system's micro-variables and show that the chaoticity is further weakened, but only *locally*. Finally, the physicality of the constraints is briefly discussed, particularly in the context of quantum entanglement.

Keywords: probability theory; Riemannian geometry; chaos; complexity; entropy

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1. Introduction

In the paper by L. A. Caron *et al.* [1], classical chaos is compared with quantum chaos, and the authors discuss why the former is *weaker* than the latter. It was suggested that the weakness of quantum chaos may arise from quantum fluctuations that give rise to Heisenberg's uncertainty relation. It is also known that a quantum description of chaos is qualitatively different from a classical description and that the latter cannot simply be considered an approximation of the former. In fact, the only aspect of quantum theory that may be retained by a corresponding classical description is the canonical Heisenberg's uncertainty relation, specifically, a minimum spread of order \hbar^n in the $2n$ -dimensional phase space [2] (where $\hbar \stackrel{\text{def}}{=} \frac{h}{2\pi}$ and h is Planck's constant).

In a previous paper [3], we studied the information geometry and the information-constrained dynamics of a $3D$ uncorrelated Gaussian statistical model and compared it with that of a $2D$ uncorrelated Gaussian statistical model, which was obtained from the higher-dimensional model via introduction of an additional information constraint that resembled the Heisenberg uncertainty relation. We showed that the chaoticity (temporal complexity) of the $2D$ uncorrelated Gaussian statistical model (quantum-like model), quantified by means of the Information Geometric Entropy (IGE) [4] and the Jacobi vector field intensity, was *softened* relative to the chaoticity of the $3D$ uncorrelated Gaussian statistical model (classical-like model). By *softened*, we mean any attenuation in the asymptotic temporal growth of the indicators of chaoticity. It is worth noting that the statistical models in question were limited to the extent that we assumed that the correlation between the micro-variables of the system was unknown.

In this paper, we will again discuss the manner in which the degree of complexity changes for a statistical model (the probabilistic description of a physical system) in the presence of incomplete knowledge when the information-constrained dynamics, the so-called entropic dynamics [5], on the underlying curved statistical manifolds becomes even more constrained. Furthermore, we will reduce the probabilistic description of the dynamical systems in the presence of partial knowledge to information geometry (Riemannian geometry applied to probability theory, see [6]) and inductive inference [7–10]. We employ the same theoretical framework developed for this, termed the *Information Geometric Approach to Chaos* (IGAC) [11,12], where information geometric techniques are combined with maximum relative entropy methods [7–10] to study the complexity of informational geodesic flows on curved statistical manifolds (statistical models) underlying the probabilistic description of physical systems in the presence of incomplete information. We expand our previous findings by further constraining the quantum-like $2D$ uncorrelated model, herein denoted as $2Du$, with knowledge of the correlation between the microscopic degrees of freedom of the system by way of a covariance term, σ_{xy} . Our analysis not only provides evidence that the degree of chaoticity of statistical models is related to the existence of uncertainty relation-like information constraints, as was seen before, it also demonstrates that the chaoticity is also dependent upon the covariance term parameterized in terms of a correlation coefficient.

This constraint, specifically the correlation coefficient, may well have a physical interpretation. It is known, for example, that a realistic approach to generate entangled quantum systems is via dynamical interaction, of which local scattering events (collisions) are a natural, ubiquitous type [13]. Indeed, we have shown in a recent work [14] how the IGAC can be used to examine the quantum entanglement of two spinless, structureless, non-relativistic particles, where the entanglement is produced by two Gaussian wave-packets interacting via a scattering process. In that work, it was shown how the correlation coefficient can be related to two-particle squeezing parameters [15] for the case of continuous variable quantum systems with Gaussian continuous degrees of freedom [16].

The layout of this article is as follows. In Section 2, we present the basic differential geometric properties of the quantum-like two-dimensional uncorrelated statistical model, $2Du$, and the further constrained version, namely, the two-dimensional correlated statistical model, herein denoted as $2Dc$. In Section 3, we describe the geodesic paths on the curved statistical manifolds underlying the entropic dynamics of the two statistical models. In Sections 4 and 5, we study the chaotic properties of the information-constrained dynamics on the underlying curved statistical manifolds by means of the IGE. Our final remarks appear in Section 6.

2. The Information Geometry of Statistical Models

The statistical models studied in [3] were a $3D$ uncorrelated Gaussian statistical model and a $2D$ uncorrelated Gaussian statistical model obtained from the higher-dimensional model via the introduction of an additional information constraint that resembles the canonical minimum uncertainty relation in quantum theory. For a brief and recent overview on the IGAC, we refer to [4]. Note that the dimensionality ($2D$ and $3D$) pertains to the *macroscopic* variables. Specifically, the dimensionality of a curved statistical manifold equals the cardinality of the set of time-varying statistical macro-variables necessary to parametrize points on the manifold itself. Below, we examine the geometry of the two-dimensional uncorrelated case, $2Du$, and then examine a new model, namely, the $2Dc$ model, where the microscopic variables are further constrained by a *covariance* term.

2.1. The $2D$ Uncorrelated Model

This section follows our previous work [3], where we studied the information geometry and the entropic dynamics of a $3D$ Gaussian statistical model. We compared our analysis to that of a $2D$ Gaussian statistical model obtained from the higher-dimensional model via the introduction of an additional information constraint that resembled the quantum mechanical canonical minimum uncertainty relation. We showed that the chaoticity of the $2D$ Gaussian statistical model, quantified by means of the Information Geometric Entropy and the Jacobi vector field intensity, is softened with respect to the chaoticity of the $3D$ Gaussian statistical model. In view of the similarity between the information constraint on the variances and the phase-space coarse-graining imposed by the Heisenberg uncertainty relations, we suggested that this provides a possible way of explaining the phenomenon of the

suppression of classical chaos operated by quantization. The constraints on the *microscopic* variables, x and y , are:

$$\begin{aligned} \int p(x, y) y dx dy &= 0 \\ \int p(x, y) x dx dy &= \mu_x \\ \int p(x, y) (x - \mu_x)^2 dx dy &= \sigma_x^2 \\ \int p(x, y) y^2 dx dy &= \sigma_y^2 \end{aligned} \tag{1}$$

When these constraints are applied to the system, we use the method of maximum (relative) entropy [7] to obtain a family of probability distributions that characterize the 3D uncorrelated Gaussian statistical model:

$$p(x, y | \mu_x, \sigma_x, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp \left[-\frac{1}{2\sigma_x^2} (x - \mu_x)^2 - \frac{1}{2\sigma_y^2} y^2 \right] \tag{2}$$

with σ_x and σ_y in \mathbb{R}_0^+ and μ_x in \mathbb{R} . The Gaussian here is two-dimensional in its microscopic space (x, y) , but three-dimensional in its macroscopic (contextual or conditionally, given parameters) space $(\mu_x, \sigma_x, \sigma_y)$. For the 2D uncorrelated case, the probability distributions, $p(x, y | \mu_x, \sigma)$, that characterize the model are given by:

$$p(x, y | \mu_x, \sigma) \stackrel{\text{def}}{=} \frac{1}{2\pi\Sigma^2} \exp \left[-\frac{1}{2\sigma^2} (x - \mu_x)^2 - \frac{\sigma^2}{2\Sigma^4} y^2 \right] \tag{3}$$

with σ in \mathbb{R}_0^+ and μ_x in \mathbb{R} . The probability distributions Equation (3) may be obtained from Equation (2) with the addition of the following *macroscopic constraint*:

$$\sigma_x \sigma_y = \Sigma^2 \tag{4}$$

where Σ^2 is a constant belonging to \mathbb{R}_0^+ and $\sigma_x \equiv \sigma$. The macroscopic constraint Equation (4) was chosen originally, because it resembles the quantum mechanical canonical minimum uncertainty relation where x denotes the position of a particle and y , its conjugate momentum. Indeed, in view of the similarity between the constraint on the variances Equation (4) and the phase-space coarse-graining imposed by the Heisenberg uncertainty relations [2], we seek a possible way of explaining the phenomenon of the suppression of classical chaos when operated by quantization within an information geometric framework.

We then relaxed the conditionality on the microscopic space to explore the space of Gaussians described by μ_x and σ . The infinitesimal Fisher-Rao line element, ds_{2Du}^2 , for this model reads:

$$ds_{2Du}^2 = g_{lm}^{(2Du)}(\theta) d\theta^l d\theta^m = \frac{1}{\sigma^2} d\mu_x^2 + \frac{4}{\sigma^2} d\sigma^2 \tag{5}$$

where the Fisher-Rao information metric, $g_{lm}^{(2Du)}(\theta)$, is defined as [6]:

$$g_{lm}^{(2Du)}(\theta) \stackrel{\text{def}}{=} \int dx dy p(x, y | \mu_x, \sigma) \frac{\partial \log p(x, y | \mu_x, \sigma)}{\partial \theta^l} \frac{\partial \log p(x, y | \mu_x, \sigma)}{\partial \theta^m} \tag{6}$$

with $\theta \equiv (\theta^1, \theta^2) \stackrel{\text{def}}{=} (\mu_x, \sigma)$. Using Equation (5), it follows that the non-vanishing connection coefficients, Γ_{ij}^k , are given by:

$$\Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{\sigma}, \Gamma_{11}^2 = \frac{1}{4\sigma}, \Gamma_{22}^2 = -\frac{1}{\sigma} \tag{7}$$

The scalar curvature, $\mathcal{R}^{(2Du)}$, of the probability distributions in Equation (3) is given by:

$$\mathcal{R}^{(2Du)} = g^{11}(\theta) R_{11} + g^{22}(\theta) R_{22} = -\frac{1}{2} \tag{8}$$

with $g^{lm}g_{mk} = \delta_k^l$ and where the only non-vanishing Ricci curvature tensor components, R_{ij} , are:

$$R_{11} = -\frac{1}{4\sigma^2}, R_{22} = -\frac{1}{\sigma^2} \tag{9}$$

The sectional curvature [17] is independent of the tangent plane chosen on any point of the manifold and is therefore constant, with value:

$$\mathcal{K}^{(2Du)} = -\frac{1}{4} \tag{10}$$

As shown in [3], the scalar curvature for the uncorrelated 3D case was $\mathcal{R}^{(3Du)} = -1$. This implies that the 3D uncorrelated statistical model is *globally* more negatively curved than the 2D uncorrelated statistical model. This suggests that the 3D model might exhibit more chaotic features than the 2D model.

2.2. The 2D Correlated Model

In addition to the original 3D uncorrelated Gaussian statistical model constraints in Equation (2), we now add the following *covariance* constraint to the model:

$$\int p(x, y) xy dx dy = \sigma_{xy} \tag{11}$$

The probability distributions that characterize the 3D *correlated* Gaussian statistical model Equation (2) now read:

$$p(x, y | \mu_x, \sigma_x, \sigma_y, \sigma_{xy}) = \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}} \exp\left\{ \frac{-1}{2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)} [(x - \mu_x)^2\sigma_y^2 + y^2\sigma_x^2 - 2(x - \mu_x)y\sigma_{xy}] \right\} \tag{12}$$

Making a change of variable with regard to the covariance, $\sigma_{xy} = r\sigma_x\sigma_y$, we obtain the standard bivariate normal distribution, where the parameter, r , is the correlation coefficient between x and y and assumes values within the ranges $-1 \leq r \leq 1$. Applying the macroscopic constraint Equation (4) to the covariance constraint yields $\sigma_{xy} = r\Sigma^2$, where $\Sigma^2 = \sigma_x\sigma_y$ is a constant belonging to \mathbb{R}_0^+ and $\sigma_x \equiv \sigma$. The 3D correlated model Equation (12) now becomes:

$$p(x, y | \mu_x, \sigma, r) \stackrel{\text{def}}{=} \frac{1}{2\pi\Sigma^2\sqrt{1-r^2}} \exp\left\{ \frac{-1}{2(1-r^2)} \left[\frac{(x - \mu_x)^2}{\sigma^2} + \frac{y^2\sigma^2}{\Sigma^4} - \frac{2r(x - \mu_x)y}{\Sigma^2} \right] \right\} \tag{13}$$

Assuming that the correlation between x and y is constant, we then relax the conditionality on the microscopic space to explore the space of Gaussians described by μ_x and σ only. Thus, the infinitesimal Fisher-Rao line element, ds_{2Dc}^2 , reads:

$$ds_{2Dc}^2 = g_{lm}^{(2Dc)}(\theta) d\theta^l d\theta^m = \frac{1}{\sigma^2(1-r^2)} d\mu_x^2 + \frac{4}{\sigma^2(1-r^2)} d\sigma^2 \tag{14}$$

where $\theta \equiv (\theta^1, \theta^2) \stackrel{\text{def}}{=} (\mu_x, \sigma)$. Observe that line element Equation (14) is only valid provided $(1-r^2) > 0$. Using Equation (14), it follows that the non-vanishing connection coefficients, Γ_{ij}^k , are given by:

$$\Gamma_{12}^1 = -\frac{1}{\sigma}, \Gamma_{11}^2 = \frac{1}{4\sigma}, \Gamma_{21}^1 = -\frac{1}{\sigma}, \Gamma_{22}^2 = -\frac{1}{\sigma} \tag{15}$$

The Ricci scalar curvature, $\mathcal{R}^{(2Dc)}$, of the probability distributions in Equation (13) is given by:

$$\mathcal{R}^{(2Dc)} = g^{11}(\theta) R_{11} + g^{22}(\theta) R_{22} = -\frac{1}{2} + \frac{r^2}{2} \tag{16}$$

with $g^{lm} g_{mk} = \delta_k^l$. The only non-vanishing Ricci curvature tensor components, R_{ij} , are:

$$R_{11} = -\frac{1}{4\sigma^2} \text{ and } R_{22} = -\frac{1}{\sigma^2} \tag{17}$$

As in the previous case, the sectional curvature is independent of the tangent plane chosen on any point of the manifold and is therefore constant, with value:

$$\mathcal{K}^{(2Dc)} = -\frac{1}{4} (1-r^2) \tag{18}$$

Notice that the Ricci scalar of the correlated 2D model, $\mathcal{R}^{(2Dc)}$ in Equation (16), is r -dependent, with $\mathcal{R}^{(2Dc)}$ approaching zero as r^2 tends to unity, while in the limit, $r \rightarrow 0$, we recover the scalar curvature of the 2D uncorrelated Gaussian model Equation (8). By including the macroscopic constraint Equation (4) with the correlation constraint in Equation (11), we limit what Σ^2 can be; $\sigma_{xy}/r = \Sigma^2$ and since $r^2 < 1$,

$$\sigma_{xy}^2/\Sigma^4 < 1, \text{ and, therefore, } \sigma_{xy} < \Sigma^2 \tag{19}$$

Moreover, observe that the sectional curvature is correlation-dependent, while the covariant Ricci tensor components are identical in both 2D cases.

3. Geodesic Motion on Curved Statistical Manifolds

In this section, we present the geodesic paths on the curved statistical manifolds underlying the entropic dynamics of both the two-dimensional correlated and uncorrelated Gaussian statistical models. Such paths are obtained by integrating the geodesic equations given by [18]:

$$\frac{d^2\theta^k}{d\tau^2} + \Gamma_{lm}^k(\theta) \frac{d\theta^l}{d\tau} \frac{d\theta^m}{d\tau} = 0 \tag{20}$$

where $\Gamma_{lm}^k(\theta)$ are the connection coefficients.

Substituting Equation (15) into Equation (20), the set of nonlinear and coupled ordinary differential equations in Equation (20) reads:

$$\begin{aligned}
 0 &= \frac{d^2\mu_x}{d\tau^2} - \frac{2}{\sigma} \frac{d\mu_x}{d\tau} \frac{d\sigma}{d\tau} \\
 0 &= \frac{d^2\sigma}{d\tau^2} + \frac{\sigma^2 + 1}{4\sigma} \left(\frac{d\mu_x}{d\tau}\right)^2 - \frac{1}{\sigma} \left(\frac{d\sigma}{d\tau}\right)^2
 \end{aligned}
 \tag{21}$$

A suitable family of geodesic paths fulfilling the geodesic equations above is given by:

$$\mu_x(\tau) = \frac{(\mu_0 + 2\sigma_0) [1 + \exp(2\sigma_0\lambda_+\tau)] - 4\sigma_0}{1 + \exp(2\sigma_0\lambda_+\tau)}
 \tag{22}$$

and:

$$\sigma(\tau) = \frac{2\sigma_0 \exp(\sigma_0\lambda_+\tau)}{1 + \exp(2\sigma_0\lambda_+\tau)}
 \tag{23}$$

where $\mu_0 \stackrel{\text{def}}{=} \mu_x(0)$, $\sigma_0 \stackrel{\text{def}}{=} \sigma(0)$ and λ_+ belongs to \mathbb{R}^+ [3].

4. Information Geometric Entropy

In this section, the chaotic properties of the information-constrained (entropic) dynamics on the underlying curved statistical manifolds are quantified by means of the IGE. We point out that a suitable indicator of temporal complexity (chaoticity) within the IGAC framework is provided by the IGE, which, in the general case, reads [19]:

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \stackrel{\text{def}}{=} \log \left[\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \int_{\mathcal{D}_\theta^{(\text{geodesic})}(\tau')} \rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n) d^n\theta d\tau' \right]
 \tag{24}$$

where $\rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n)$ is the so-called Fisher density and equals the square root of the determinant of the metric tensor, $g_{im}(\theta)$:

$$\rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n) \stackrel{\text{def}}{=} \sqrt{g((\theta^1, \dots, \theta^n))}
 \tag{25}$$

The subscript, \mathcal{M}_s , in Equation (24) denotes the curved statistical manifold underlying the entropic dynamics. The integration space, $\mathcal{D}_\theta^{(\text{geodesic})}(\tau')$, in Equation (24) is defined as follows:

$$\mathcal{D}_\theta^{(\text{geodesic})}(\tau') \stackrel{\text{def}}{=} \{ \theta \equiv (\theta^1, \dots, \theta^n) : \theta^k(0) \leq \theta^k \leq \theta^k(\tau') \}
 \tag{26}$$

where $k = 1, \dots, n$ and $\theta^k \equiv \theta^k(s)$, with $0 \leq s \leq \tau'$, such that $\theta^k(s)$ satisfies Equation (20). The integration space, $\mathcal{D}_\theta^{(\text{geodesic})}(\tau')$, in Equation (26) is an n -dimensional subspace of the whole (permitted) parameter space, $\mathcal{D}_\theta^{(\text{tot})}$. The elements of $\mathcal{D}_\theta^{(\text{geodesic})}(\tau')$ are the n -dimensional macro-variables, $\{\theta\}$, whose components, θ^k , are bounded by the specified limits of integration $\theta^k(0)$ and $\theta^k(\tau')$ with $k = 1, \dots, n$. The limits of integration are obtained via integration of the n -dimensional set of coupled nonlinear second order ordinary differential equations characterizing the geodesic equations. Formally, the IGE is defined in terms of an averaged parametric $(n + 1)$ -fold integral (τ is the parameter) over the multi-dimensional geodesic paths connecting $\theta(0)$ to $\theta(\tau)$.

In the cases being investigated, using Equations (22)–(24), it follows that the asymptotic expressions of the IGE for the uncorrelated model, $\mathcal{S}_{\mathcal{M}_s}^{(2Du)}$, and for the correlated model, $\mathcal{S}_{\mathcal{M}_s}^{(2Dc)}$, become:

$$\mathcal{S}_{\mathcal{M}_s}^{(2Du)}(\tau) = \log \mathcal{V}_{\mathcal{M}_s}^{(2Du)}(\tau) \stackrel{\tau \gg 1}{\approx} \sigma_0 \lambda_+ \tau \text{ and } \mathcal{S}_{\mathcal{M}_s}^{(2Dc)} = \log \mathcal{V}_{\mathcal{M}_s}^{(2Dc)}(\tau) \stackrel{\tau \gg 1}{\approx} \sigma_0 \lambda_+ \tau \tag{27}$$

since:

$$\begin{aligned} \mathcal{V}_{\mathcal{M}_s}^{(2Du)}(\tau) &\stackrel{\tau \gg 1}{\approx} \left[\left(\frac{\mu_0 + 2\sigma_0}{\sigma_0^2 \lambda_+} \right) \frac{\exp(\sigma_0 \lambda_+ \tau)}{\tau} \right] \\ \mathcal{V}_{\mathcal{M}_s}^{(2Dc)}(\tau) &\stackrel{\tau \gg 1}{\approx} \frac{1}{(1-r^2)} \mathcal{V}_{\mathcal{M}_s}^{(2Du)}(\tau) \end{aligned} \tag{28}$$

From Equation (27), we observe that:

$$\mathcal{S}_{\mathcal{M}_s}^{(2Dc)} \stackrel{\tau \gg 1}{\approx} \mathcal{S}_{\mathcal{M}_s}^{(2Du)} \tag{29}$$

The IGE does not change asymptotically for either of the 2D models being considered. Equation (28) is quite interesting, since it quantitatively shows that the information geometric complexity (IGC), $\mathcal{V}_{\mathcal{M}_s}^{(2Dc)}$, of the correlated 2Dc model diverges as the correlation coefficient (introduced via the constraint Equation (11)) approaches unity. As expected, the two cases are identical for $r = 0$. In [3], the IGE of the quantum-like model 2Du was less than the IGE of the classical model (3D). That result indicated a weaker (softer) chaoticity for the 2Du model. Here, the comparison of the uncorrelated and correlated models shows that further constraining the quantum-like model 2Du with a covariance term, σ_{xy} , does not lead to any additional global softening.

5. Jacobi Vector Field Intensity

There seems to be no change to the chaoticity when we further constrain the old, quantum-like model (2Du) with a covariance term, σ_{xy} , since the IGE for each 2D model is identical in the asymptotic limit. However, when considering chaoticity characterizations (geodesic spread), it is the *local* curvature of the manifold that must be examined. This information is encoded in the sectional curvatures of the manifold when it is isotropic (maximally symmetric). When the manifold is anisotropic (non-maximally symmetric), the Riemann curvature tensor components come into play.

From above, we see that, indeed, the sectional curvatures, \mathcal{K} , of both 2D models are constant (and, consequently, maximally symmetric) and exhibit the relationship:

$$\mathcal{K}^{(2Dc)} = -\frac{1}{4} (1 - r^2) \geq -\frac{1}{4} = \mathcal{K}^{(2Du)} \tag{30}$$

with $r^2 < 1$. However, since the sectional curvatures are different, to the extent that the 2Dc model depends on the correlation coefficient, the *local* curvature is different. We can then follow [3] in integrating the Jacobi-Levi-Civita (JLC) equation describing the geodesic spread. Omitting technical details, we find that the asymptotic temporal behavior of the Jacobi vector field intensities, \mathcal{J} , on such maximally symmetric statistical manifolds satisfy the following inequality relation, which is closely related to Equation (30):

$$\mathcal{J}^{(2Du)}(\tau) \stackrel{\tau \rightarrow \infty}{\approx} \exp\left(+\sqrt{\mathcal{K}^{(2Du)}}\tau\right) \geq \exp\left(+\sqrt{\mathcal{K}^{(2Dc)}}\tau\right) \stackrel{\tau \rightarrow \infty}{\approx} \mathcal{J}^{(2Dc)}(\tau) \tag{31}$$

This would imply that there is indeed a *local softening* of the geodesic spreads on the quantum-like model, the $2Du$ model, when it is further constrained by a covariance constraint, σ_{xy} , manifested in the correlation term, r .

6. Other Considerations

- While $\sigma_{xy} = r\sigma_x\sigma_y$ is physically sensible for x and y representing either position-position, momentum-momentum or position-momentum pairs, the macro-quantum-like constraint $\Sigma^2 = \sigma_x\sigma_y$ with Σ^2 being constant and x and y representing either position-position or momentum-momentum pairs is physically ambiguous in view of the fact that all physical observables commute with the permutation operator. This seems to suggest that the constraint $\Sigma^2 = \sigma_x\sigma_y$ is only active when x and y refer to micro-variables that are not self-similar. Might one be able to use empirical data to trace the information-theoretic conditions that either relax the constraint $\Sigma^2 = \sigma_x\sigma_y$ (so that $\Sigma^2 \neq \text{constant}$) or renders it active? It should be further noted that even if one considers micro-variables with dissimilar dimensions, then while the covariance constraint and the macro-quantum-like constraint would be compatible, one would then have to consider a more general form of the macro-quantum-like constraint Equation (4) due to the presence of entanglement. This will be considered for a future work.
- We also stress that our information geometric analysis could accommodate non-minimum uncertainty-like relations. However, such an extension would require a more delicate analysis where maximum relative entropy methods are used to process information in the presence of inequality constraints [20]. However, for this, as well as the more general uncertainty constraint, a deeper analysis is needed, and we leave that for future investigations. Our work is especially relevant for the quantification of soft chaos effects in entropic dynamical models used to describe actual physical systems when only incomplete knowledge about them is available [21].
- Statistical complexity is a quantity that measures the amount of memory needed, on average, to statistically reproduce a given configuration [22]. In the same vein of our works in [3], a recent investigation claims that quantum mechanics can reduce the statistical complexity of classical models [23]. Specifically, it was shown that mathematical models featuring quantum effects can be as predictive as classical models, although implemented by simulators that require less memory, that is, less statistical complexity. Of course, these two works use different definitions of complexity, and their ultimate goal is definitively not the same. However, it is remarkable that both of them exploit some quantum feature, Heisenberg's uncertainty principle in [3] and the quantum state discrimination (information storage) method in [23], to exhibit the complexity softening effects. Is there any link between Heisenberg's uncertainty principle and quantum state discrimination? Recently, it was shown that any violation of uncertainty relations in quantum mechanics also leads to a violation of the second law of thermodynamics [24]. In addition, it was reported in [25] that a violation of Heisenberg's uncertainty principle allows perfect state discrimination of non-orthogonal states, which, in turn, violates the second law of thermodynamics [2]. The possibility of distinguishing non-orthogonal states is directly related

to the question of how much information we can store in a quantum state. Information storage and memory are key quantities for the characterization of statistical complexity. In view of these considerations, it would be worthwhile to explore the possible thermodynamic link underlying these two different complexity measures [26].

All these considerations will be the subject of forthcoming efforts.

7. Conclusions

In a previous paper [3], we studied the information geometry of an uncorrelated 3D Gaussian statistical model with an additional information constraint resembling the canonical minimum uncertainty relation, which, here, we called $2Du$. We showed that the chaoticity of such a modified Gaussian statistical model (quantum-like model, $2Du$), quantified by means of the Information Geometric Entropy [4] and the Jacobi vector field intensity, was indeed softened with respect to the chaoticity of the standard Gaussian statistical model (classical-like model, $3D$). However, the statistical model was limited in that we assumed there was no correlation between the constituents of the phase space.

In this paper, we expanded our previous findings by further constraining the quantum-like $2Du$ model with a covariance term, σ_{xy} . It was shown that the Ricci scalars of the two $2D$ models, Equations (8) and (16) varied by a constant related to this covariance term, Equation (11). It seems that the IGE is insensitive to the presence of correlation terms, since the asymptotic behavior of the IGE of the two $2D$ models, $S_{\mathcal{M}_s}^{(2Du)}(\tau)$ and $S_{\mathcal{M}_s}^{(2Dc)}(\tau)$, are identical. Although the IGE analysis seemed to indicate that there was no further global softening, examination of the Jacobi vector field intensity seems to indicate that the softening only appears locally (geodesic spread-deviation equations). Therefore, when the quantum-like $2Du$ model is further constrained by the knowledge of a covariance term, σ_{xy} , no softening appears at the *global* (geodesic equations) scale, but only appears *locally*, where this softness is dependent on the correlation term, r . In a forthcoming investigation, we will consider τ -dependent r quantities and study whether or not this formal identical behavior between the IGEs is preserved for such cases, as well. The stronger condition of τ -dependent r may affect the chaoticity features of the correlation constrained $2Dc$ model at a global scale.

Finally, we would like to point out a very intriguing analogy inspired by one of the referees (which we suspect to be very profound) between Einstein's Equivalence Principle [27] and the local softening effect considered in this work. Einstein's Equivalence Principle states that gravitation, like space-time curvature, works only globally, while locally, there is no gravitational field: physics is simply connected only locally. It may be worthwhile deepening this point also in future investigations, taking into proper consideration the fact that while Einstein was discussing space-time regions, our considerations concern regions on curved statistical manifolds.

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Conflicts of Interest

The authors declare no conflict of interest.

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