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Article

Bayesian Reliability Estimation for Deteriorating Systems with Limited Samples Using the Maximum Entropy Approach

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Abstract: In this paper the combinations of maximum entropy method and Bayesian inference for reliability assessment of deteriorating system is proposed. Due to various uncertainties, less data and incomplete information, system parameters usually cannot be determined precisely. These uncertainty parameters can be modeled by fuzzy sets theory and the Bayesian inference which have been proved to be useful for deteriorating systems under small sample sizes. The maximum entropy approach can be used to calculate the maximum entropy density function of uncertainty parameters more accurately for it does not need any additional information and assumptions. Finally, two optimization models are presented which can be used to determine the lower and upper bounds of systems probability of failure under vague environment conditions. Two numerical examples are investigated to demonstrate the proposed method.

Keywords: deteriorating system; maximum entropy method; small sample sizes; Bayesian inference; fuzzy numbers

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Acronyms and Abbreviations

RV	Random variable
PDF	Probability density function
CDF	Cumulative distribution function
MM	Method of moments
MLE	Maximum likelihood estimation
MEE	Maximum entropy estimation
S.	Integral domain
λ	Lagrange multipliers
$\tilde{a}$	A fuzzy number
$\xi_{\tilde{a}}(x)$	Membership function
$(\tilde{a})_\alpha$	The α-level set of $\tilde{a}$
$(\bullet,\bullet,\bullet)$	Triangular fuzzy number
$\pi(\bullet)$	Prior distribution
$p(\bullet)$	Maximum entropy density function
$p(\bullet \mid \mathbf{y}.)$	Posterior PDF
$D_i(t)$	Degradation at time t of ith unit
D_f	Critical threshold
y.	Given samples

1. Introduction

Degradation is a common cause of failure of many products. In structural reliability analysis, traditionally, the uncertain deterioration and degradation are usually modeled using lifetime distributions [1], and the information of systems can be analyzed using degradation data. Generally, system reliability assessment using degradation data has many advantages because the analysts can use less collected degradation data to assess system reliability, which is useful for applications [2–5].

In engineering practice, time-dependent reliability analysis for modeling deterioration systems is necessary because the performance of many products is a deterioration process [6]. Traditionally, the deteriorations of structures usually are modeled by using random variables models (such as the deterioration rate), cumulative damage models and gamma processes [6–9], respectively. The random variable (RV) models (also called degradation curve models) are used widely in engineering applications for assessing the reliability of deteriorating components [10]. Examples of the RV models for reliability estimation can be found in references [11–15]. Another widely used method for modeling deteriorating systems is the gamma process, which belongs to a general class of stochastic processes [6]. The gamma process can be used to model gradual damage monotonically accumulating over time, such as wear, fatigue, corrosion, erosion, crack growth, consumption, creep, swell, *etc*., and these failure models are common causes of failure of many engineering components [8,16].

In engineering practice, it is well known that data sometimes cannot be recorded or collected precisely due to various uncertainties [17,18] such as human errors, machine errors, incomplete information or some other unexpected situations [19]. In this case, the Bayesian inference has shown to be useful for modeling uncertainty. A lot of researchers have paid more attention to Bayesian reliability analysis under vague information conditions. For example, Bayesian reliability estimations under fuzzy environments constraints were proposed by Wu [19–21], Taheri and Zarei [22], respectively, in recent years.

Despite many efforts, reliability assessment for deteriorating systems using traditional RV models also has some limitations. In the traditional RV models, the distributions of random variables (such as the deterioration rate) usually are assumed known and precisely determined, such as normal distribution and Weibull distribution. However, determining lifetime distributions requires sufficient data which is often impossible to acquire in engineering practice, especially for some reliable and long lifetime products such as satellites, spaceships and so on. Furthermore, these assumptions are not reasonable in the case of less data and incomplete information. For example, we may face the problem that the distribution of one random variable can be viewed as a normal or Weibull distribution under small sample sizes. In order to solve the problem and consider the advantages of both the maximum entropy method and the Bayesian inference for systems under small sample sizes and incomplete information constraints, combinations of the two methods are considered for the deteriorating systems, and the fuzzy Bayesian reliability assessment for deteriorating components is proposed in this paper.

The remainder of the paper is organized as follows: in Section 2, Bayesian inference and fuzzy theory are introduced. The maximum entropy method is given in Section 3. The proposed fuzzy Bayesian reliability assessment for deteriorating components under small sample sizes is given in Section 4. Two numerical examples are presented in Section 5. Finally, Section 6 presents brief discussions and our conclusions.

2. Review of Bayesian Inference and Fuzzy Theory

2.1. Bayesian Inference

Bayesian inference is based on the subjective view of probability. Let Y be a random variable with a probability density function (PDF) $p\left(y|\mathbf{\Theta}\right)$, which is indexed by a parameter vector Θ. Bayesian inference with observations $\mathbf{y}=\left(y_1,y_2,\cdots,y_n\right)$ can be given by [23–25]:

$$p\left(\mathbf{\Theta}|\mathbf{y}\right)=\frac{p\left(\mathbf{y}|\mathbf{\Theta}\right)\pi\left(\mathbf{\Theta}\right)}{\int_{\mathbf{\Theta}}p\left(\mathbf{y}|\mathbf{\Theta}\right)\pi\left(\mathbf{\Theta}\right)d\mathbf{\Theta}} \tag{1}$$

where $\pi\left(\mathbf{\Theta}\right)$ is the prior PDF, $p\left(\mathbf{\Theta}|\mathbf{y}\right)$ is the posterior PDF of Θ, and $p\left(\mathbf{y}|\Theta\right)$ is the sampling PDF of the observations, respectively. To apply Equation (1), the prior PDF is needed. Generally, the prior PDF plays an important role in reliability analysis. The more accurate of the prior PDF is, the more accurate of the posterior PDF will be. There are five methods to formulate a prior PDF based on the past data [24]: (I) the method of moments (MM); (II) maximum likelihood estimation (MLE); (III) maximum entropy estimation (MEE); (IV) two-state updating of a non-informative "pre-prior"; and (V) credible interval-based method. Furthermore, non-informative prior PDF is used widely in engineering under the case of no prior available information. More information about the non-informative prior PDF can be found in references [25,26].

2.2. Fuzzy Theory

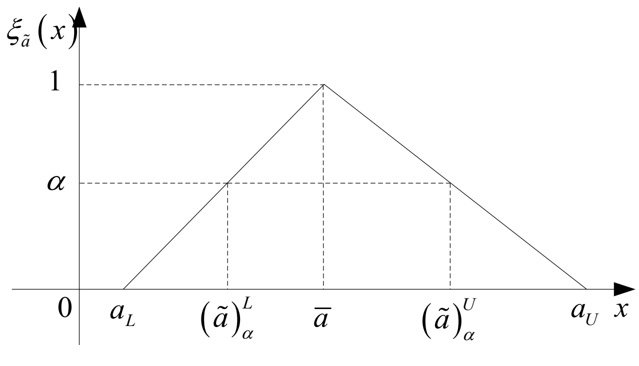
In engineering practice, the value or distribution of a parameter usually cannot be determined precisely under small sample sizes or vague information conditions. For example, we often state that the critical threshold is "about 10" or "about 9–11", and the statements "about 10" and "about 9-11" can be viewed as fuzzy numbers. Therefore, the fuzzy sets theory provides an appropriate tool for modeling the situations where some parameters are fuzzy numbers. Let $\tilde{a}$ be a fuzzy real number with membership function $\xi_{\tilde{a}}$. Its α-level set is defined by $(\tilde{a})_\alpha = \{x : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in [0,1]$. Generally, $(\tilde{a})_\alpha$ is convex, closed and bounded in real number [20], and $(\tilde{a})_\alpha$ is a closed interval which can be expressed as $(\tilde{a})_\alpha = \left[(\tilde{a})_\alpha^L, (\tilde{a})_\alpha^U\right]$.

There are many kinds of membership functions that can be used in engineering applications. For illustration purposes, a special kind of fuzzy real number is introduced in the paper. We say that $\tilde{a}$ is a triangular fuzzy real number if its membership function is given by [22]:

$$\xi_{\tilde{a}}(x) = \begin{cases} (x - a_L)/(\bar{a} - a_L) & \text{if } a_L \leq x \leq \bar{a} \\ (a_U - x)/(a_U - \bar{a}) & \text{if } \bar{a} \leq x \leq a_U \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Triangular fuzzy number $\tilde{a}$ can be denoted by $\tilde{a} = (a_L, \bar{a}, a_U)$. The α-level set of triangular fuzzy real number $\tilde{a}$ is calculated by $(\tilde{a})_\alpha = \left[(\tilde{a})_\alpha^L, (\tilde{a})_\alpha^U\right]$ = $\left[(\bar{a} - a_L)\alpha + a_L, (\bar{a} - a_U)\alpha + a_U\right]$ for all $\alpha \in [0,1]$. In this paper, $(\tilde{a})_\alpha$ is also denoted by $(\tilde{a})_\alpha = \left\{(a)_\alpha : (\tilde{a})_\alpha^L \leq (a)_\alpha \leq (\tilde{a})_\alpha^U\right\}$. The α-level set of $\tilde{a}$, shown in Figure 1, is used widely in system reliability analysis.

Figure 1. The α-level set of the fuzzy real number $\tilde{a}$.



3. Maximum Entropy Methods

In order to improve the accuracy in systems reliability assessment, additional assumptions should be avoided. The maximum entropy method [27–29] is a flexible and powerful tool for density approximation as it does not need any additional information or assumptions. The algorithms for calculating maximum entropy density can be found in [29–31]. The maximum entropy density function (also called FDF of sample in order to keep the naming consistent with PDF) is calculated by maximizing Shannon's entropy [29]:

$$M(x) = \int_{S_x} -p(x) \ln p(x) dx \tag{3}$$

with the constraints:

$$\int_{S_x} p(x) dx = 1 \tag{4}$$

$$\int_{S_x} x^i p(x) dx = m_i, \quad i = 1, 2, \cdots k \tag{5}$$

where $p(x)$ is the maximum entropy density function, S_x is the integral domain, and m_i is the known *ith* moment.

From the Lagrange's method, the $p(x)$ can be expressed as [30]:

$$p(x) = \exp\left(-\lambda_0 - \sum_{i=1}^{k} \lambda_i x^i\right) \tag{6}$$

where λ_i is the Lagrangian multiplier for the *ith* moment constraint, and:

$$\lambda_0 = \ln\left[\int_{S_x} \exp\left(-\sum_{i=1}^{k} \lambda_i x^i\right) dx\right] \tag{7}$$

To solve for the Lagrange multipliers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \cdots, \lambda_k)$, a feasible method, named the Newton's iterative algorithm, can be adopted [29,31]. Some studies, such as references [32,33], have been shown that the first three or four moments are sufficient to describe a wide range of distribution types.

4. Fuzzy Bayesian Reliability Assessment for Deteriorating Components under Small Sample Size Conditions

4.1. The Degradation Curve of Deteriorating Components

In order to assess the reliability of deteriorating components using RV models, the degradation curves of components should be determined firstly. Suppose that a linear degradation curve exists and its initial value is D_0, the degradation at time t for the *ith* unit is given by

$$D_i(t) = D_0 + \theta_i t \tag{8}$$

where θ is a random variable and θ_i is a unit specific parameter. When the unit's degradation reaches a critical threshold D_f, the unit is considered to fail. From Equation (8), the pseudo lifetime of the *ith* unit is $(D_f - D_0)/\theta_i$. Suppose the measurement error ε_{ij} is a normally distributed random variable with distribution $N(0, \sigma_\varepsilon^2)$. The observed degradation at t_j is denoted by y_{ij}, and the degradation of the *ith* unit at t_j is denoted by $D_i(t_j)$. The relationships between y_{ij} and $D_i(t_j)$ under considering measurement errors can be expressed as:

$$Y_{ij} = D_i(t_j) + \varepsilon_{ij} \tag{9}$$

In reality, the degradation curves are more complicated than linear degradation curves such as power laws and exponential curves. In these situations, these complicated curves can be transformed by logarithms. For example, we can transform an exponential curve $D_i\left(t\right)=\theta_1 e^{\theta_2 t}$ to a linear curve by logarithmic arithmetic, which is given by:

$$\ln D_i\left(t\right)=\ln\theta_1+\theta_2 t \tag{10}$$

In the situation of considering measurement errors, Equation (10) can be rewritten as:

$$\ln D_i\left(t\right)=\ln\theta_1+\theta_2 t+\ln\varepsilon \tag{11}$$

Where $\ln\varepsilon\sim N\left(0,\sigma_\varepsilon^2\right)$. Let $\bar{D}_i(t)=\ln D_i\left(t\right)$, $B_1=\ln\theta_1$, and $\varepsilon^{'}=\ln\varepsilon$, Equation (11) can be rewritten as:

$$\bar{D}_i(t)=B_1+\theta_2 t+\varepsilon^{'} \tag{12}$$

For simplicity, we let $\ln\varepsilon\sim N\left(0,\sigma_\varepsilon^2\right)$. Consider the situation of $D_i\left(t\right)=\theta_1 e^{\theta_2 t}+\varepsilon$, where $\varepsilon\sim N\left(0,\sigma_\varepsilon^2\right)$, which is a very complicated regression model and is called non-linear regression model [34]. The information of non-linear regression model is omitted here, but can be found in some related books. The following degradation curves are used widely for deterioration system reliability assessment [35], which can be expressed as:

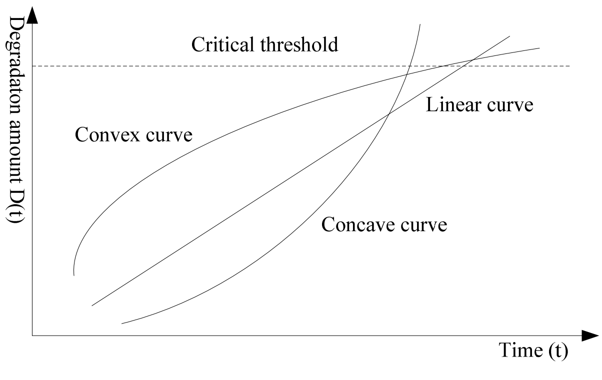
$$D_i\left(t\right)=D_0+\theta_i t \tag{13}$$

$$\ln\left[D_i^1\left(t\right)\right]=\theta_1^i+\theta_2^i t \tag{14}$$

$$\ln\left[D_i^2\left(t\right)\right]=\theta_3^i+\theta_4^i\ln t \tag{15}$$

The different kinds of degradation curves are shown in Figure 2.

Figure 2. Different degradation curves.



4.2. Bayesian Reliability Assessment for Deteriorating Components under Small Sample Sizes

From the aforementioned discussions, many commonly used degradation curves can be transformed into a linear curve by using some corresponding transformations. For simplicity and illustration purposes, the linear degradation curve $D\left(t\right)=D_0+\theta t$ is considered in this paper.

Suppose that n ($n<30$) units' degradation data of a component is shown in Table 1 [9], the degradation curve is given by $D\left(t\right)=D_0+\theta t$, and the critical threshold is D_f. θ is a random variable, and specific parameters $\theta_i\ (i=1,2,\cdots,n)$ can be determined by the least squares method for the *ith* unit degradation data. Three cases are considered in order to assess the component reliability, respectively.

Table 1. Degradation data.

Time t	t_0	t_1	t_2	t_3	**…**	t_j
1(unit)	$D_1(t_0)$	$D_1(t_1)$	$D_1(t_2)$	$D_1(t_3)$		$D_1(t_j)$
2(unit)	$D_2(t_0)$	$D_2(t_1)$	$D_2(t_2)$	$D_2(t_3)$		$D_2(t_j)$
…	…	…	…	…	…	…
n(unit)	$D_n(t_0)$	$D_n(t_1)$	$D_n(t_2)$	$D_n(t_3)$		$D_n(t_j)$

Case 1: There is no prior information for random variables θ and T.

According to Equation (13), the pseudo lifetime T_i of the corresponding *ith* unit can be calculated as $\left(D_f-D_0\right)/\theta_i$. Then we can obtain n pseudo lifetimes $T_i\left(i=1,2,\cdots,n\right)$. The maximum entropy density (PDF of sample) *p(t)* of the random variable T can be given by:

$$p\left(t\right)=\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right) \tag{16}$$

From Equation (16), the cumulative distribution function (CDF) of T is given by:

$$F_T\left(t\right)=P\left(T\le t\right)=\int_{S_t}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)dt \tag{17}$$

From Equation (17), the probability of failure $P_f^{t_0}$ under $t=t_0$ becomes:

$$P_f^{t_0}=P\left(T\le t_0\right)=F_T\left(t_0\right) \tag{18}$$

Generally, Equation (17) has no analytical and close form solutions. We can solve it by using numerical integration methods such as the Simpson and Romberg algorithms [36].

Case 2: There is prior information for random variable T.

Suppose that the prior PDF of T is $\pi\left(t\right)$. From Equations (1) and (16), the posterior PDF $p\left(t\middle|\mathbf{y}_t\right)$ is given by:

$$p\left(t\middle|\mathbf{y}_t\right)=\frac{\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)\pi\left(t\right)}{\int_{S_t}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)\pi\left(t\right)dt} \tag{19}$$

The integration $\int_{S_t}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)\pi\left(t\right)dt$ is a constant which can be denoted by $C\left(t,\boldsymbol{\lambda},\pi\right)$. Therefore, Equation (19) can be rewritten as:

$$p\left(t\middle|\mathbf{y}_t\right)=\frac{\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)\pi\left(t\right)}{C\left(t,\boldsymbol{\lambda},\pi\right)} \tag{20}$$

From Equation (20), the CDF $F_T\left(t\middle|\mathbf{y}_t\right)$ can be expressed as:

$$F_T\left(t\middle|\mathbf{y}_t\right)=p\left(T\le t\middle|\mathbf{y}_t\right)=\int_{S_t}\frac{\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i t^i\right)\pi\left(t\right)}{C\left(t,\boldsymbol{\lambda},\pi\right)}dt \tag{21}$$

From Equation (21), the probability of failure $P_f^{t_0}$ under $t=t_0$ becomes:

$$P_f^{t_0}=p\left(T\le t_0\middle|\mathbf{y}_t\right)=F_T\left(t_0\middle|\mathbf{y}_t\right) \tag{22}$$

Case 3: There is prior information for the random variable θ.

Suppose that the prior PDF of θ is $\pi\left(\theta\right)$. According to Equations (1) and (16), the posterior PDF $p\left(\theta\middle|\mathbf{y}_\theta\right)$ is given by:

$$p\left(\theta\middle|\mathbf{y}_\theta\right)=\frac{\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i\right)\pi\left(\theta\right)}{C\left(\theta,\boldsymbol{\lambda},\pi\right)} \tag{23}$$

where $C\left(\theta,\boldsymbol{\lambda},\pi\right)$ = $\int_{S_\theta}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i\right)\pi\left(\theta\right)d\theta$ is a constant.

For example, if the prior PDF of θ is normally distributed such as $\theta\sim N\left(\mu_\theta,\sigma_\theta^2\right)$, according to Equations (1) and (23), the posterior PDF of θ becomes:

$$p\left(\theta\middle|\mathbf{y}_\theta\right)=\frac{\frac{1}{\sqrt{2\pi}\sigma_\theta}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i-\left(\theta-\mu_\theta\right)^2/2\sigma_\theta^2\right)}{C\left(\theta,\mu_\theta,\sigma_\theta\right)} \tag{24}$$

where $C\left(\theta,\mu_\theta,\sigma_\theta\right)$ = $\int_{S_\theta}\frac{1}{\sqrt{2\pi}\sigma_\theta}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i-\left(\theta-\mu_\theta\right)^2/2\sigma_\theta^2\right)d\theta$.

Similarly, if the prior PDF of θ is a Weibull distribution with $\pi\left(\theta\right)$ = $\frac{\beta}{\eta}\theta^{\beta-1}\exp\left[-\left(\theta\right)^\beta/\eta\right]$, from Equations (1) and (23), the posterior PDF of θ is given by:

$$p\left(\theta\middle|\mathbf{y}_\theta\right)=\frac{\frac{\beta}{\eta}\theta^{\beta-1}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i-\left(\theta\right)^\beta/\eta\right)}{C\left(\theta,\beta,\eta\right)} \tag{25}$$

Where β is the shape parameter, η is the scale parameter and $C\left(\theta,\beta,\eta\right)$ = $\int_{S_\theta}\frac{\beta}{\eta}\theta^{\beta-1}\exp\left(-\lambda_0-\sum_{i=1}^{k}\lambda_i\theta^i-\left(\theta\right)^\beta/\eta\right)d\theta$, respectively.

Considering a random variable X with PDF $p_X(x)$, $-\infty < x < \infty$, $g(x)$ is a monotonically differentiable function of x with a unique inverse function. We have $Y = g(X)$ is a continuous random variable, and its PDF $p_Y(y)$ is given by [34]:

$$p_Y(y) = \begin{cases} p_X\left[h(y)\right]\left|h(y)'\right| & if\ \gamma_1 < y < \gamma_2 \\ 0 & otherwise \end{cases} \tag{26}$$

where $\gamma_1 = \min\{g(-\infty), g(\infty)\}$, $\gamma_2 = \max\{g(-\infty), g(\infty)\}$, and $h(y)$ is unique inverse function of $g(x)$, respectively.

According to Equation (26), the posterior PDF $p_T(t|\mathbf{y}_\theta)$ of T is calculated by:

$$p_T(t|\mathbf{y}_\theta) = \left|\frac{D_0 - D_f}{t^2}\right| p\left[\left.\frac{(D_f - D_0)}{t}\right|\mathbf{y}_\theta\right] \tag{27}$$

From Equation (27), the CDF $F_T(t|\mathbf{y}_\theta)$ is given by:

$$F_T(t|\mathbf{y}_\theta) = \int_{S_t} p_T(t|\mathbf{y}_\theta)dt = \int_{S_t}\left|\frac{D_0 - D_f}{t^2}\right| p\left[\left.\frac{(D_f - D_0)}{t}\right|\mathbf{y}_\theta\right]dt \tag{28}$$

From Equation (28), the probability of failure $P_f^{t_0}$ under $t = t_0$ becomes:

$$P_f^{t_0} = P(T \le t_0|\mathbf{y}_\theta) = F_T(t_0|\mathbf{y}_\theta) \tag{29}$$

For example, let $D_f - D_0 = D_r$, when the prior PDF of θ is normally distributed such as $\theta \sim N(\mu_\theta, \sigma_\theta^2)$, from Equations (24) and (27), the PDF $p_T(t|\mathbf{y}_\theta)$ is given by:

$$p_T(t|\mathbf{y}_\theta) = \frac{\frac{1}{\sqrt{2\pi}\sigma_\theta}\left|\frac{-D_r}{t^2}\right|\exp\left[-\lambda_0 - \sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i - \left(\frac{D_r}{t} - \mu_\theta\right)^2 \Big/ 2\sigma_\theta^2\right]}{C(\theta, \mu_\theta, \sigma_\theta)} \tag{30}$$

From Equation (30), the CDF $F_T(t|\mathbf{y}_\theta)$ of T becomes:

$$F_T(t|\mathbf{y}_\theta) = \int_{S_t}\frac{\frac{1}{\sqrt{2\pi}\sigma_\theta}\left|\frac{-D_r}{t^2}\right|\exp\left[-\lambda_0 - \sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i - \left(\frac{D_r}{t} - \mu_\theta\right)^2 \Big/ 2\sigma_\theta^2\right]}{C(\theta, \mu_\theta, \sigma_\theta)}dt \tag{31}$$

Similarly, when the prior PDF of θ is a Weibull distribution with the shape parameter β and the scale parameter η, from Equations (24) and (27), the posterior PDF $p_T(t|\mathbf{y}_\theta)$ and CDF $F_T(t|\mathbf{y}_\theta)$ are respectively given by:

$$p_T(t|\mathbf{y}_\theta) = \frac{\left|\frac{-D_r}{t^2}\right|\frac{\beta}{\eta}\left(\frac{D_r}{t}\right)^{\beta-1}\exp\left[-\lambda_0 - \sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i - \left(\frac{D_r}{t}\right)^\beta \Big/ \eta\right]}{C(\theta, \beta, \eta)} \tag{32}$$

and:

$$F_T\left(t\middle|\mathbf{y}_\theta\right)=\int_{S_t}\frac{\left|\frac{-D_r}{t^2}\right|\frac{\beta}{\eta}\left(\frac{D_r}{t}\right)^{\beta-1}\exp\left[-\lambda_0-\sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i-\left(\frac{D_r}{t}\right)^\beta\Big/\eta\right]}{C\left(\theta,\beta,\eta\right)}dt \tag{33}$$

4.3. Fuzzy Bayesian Reliability Assessment for Deteriorating Components

In Section 4.2, we described how to combine Bayesian inference and the maximum entropy method to assess component reliability in details. However, due to various uncertainties and lacking of data, it is impossible to determine all parameters precisely. For example, the critical threshold cannot be determined precisely but can be modeled by the fuzzy number $\tilde{D}_f$. The prior distribution of the random variable θ is $N\left(\tilde{\mu}_\theta,\sigma_\theta^2\right)$, that is, the mean value is a fuzzy number $\tilde{\mu}_\theta$. In this section, we will discuss how to assess reliability when fuzzy parameters exist in the systems.

Case 1: The critical threshold is a fuzzy number $\tilde{D}_f$

The pseudo lifetime $\left(\tilde{T}_i\right)_\alpha$ of the corresponding *ith* unit under the α-level set can be obtained by $\left[\left(\tilde{D}_f\right)_\alpha-D_0\right]\Big/\theta_i$. Then n lower and upper bounds pseudo lifetimes $\left(\tilde{T}_1\right)_\alpha^L,\left(\tilde{T}_2\right)_\alpha^L,\cdots,\left(\tilde{T}_n\right)_\alpha^L$ and $\left(\tilde{T}_1\right)_\alpha^U,\left(\tilde{T}_2\right)_\alpha^U,\cdots,\left(\tilde{T}_n\right)_\alpha^U$ under the α-level set can be obtained, respectively. According to Equation (16), the maximum entropy density (PDF of sample) under $\left(\tilde{T}_1\right)_\alpha^L,\left(\tilde{T}_2\right)_\alpha^L,\cdots,\left(\tilde{T}_n\right)_\alpha^L$ and $\left(\tilde{T}_1\right)_\alpha^U,\left(\tilde{T}_2\right)_\alpha^U,\cdots,\left(\tilde{T}_n\right)_\alpha^U$ can be expressed as:

$$\dot{p}\left(t\right)=\exp\left(-\dot{\lambda}_0-\sum_{i=1}^{k}\dot{\lambda}_i t^i\right)_\alpha \tag{34}$$

and:

$$\ddot{p}\left(t\right)=\exp\left(-\ddot{\lambda}_0-\sum_{i=1}^{k}\ddot{\lambda}_i t^i\right)_\alpha \tag{35}$$

respectively.

From Equations (34) and (35), the lower and upper bounds of probability of failure under the α-level sets can be calculated as:

$$\left(P_f^{\min}\right)_\alpha=\int_{S_t}\exp\left(-\ddot{\lambda}_0-\sum_{i=1}^{k}\ddot{\lambda}_i t^i\right)_\alpha dt \tag{36}$$

and:

$$\left(P_f^{\max}\right)_\alpha=\int_{S_t}\exp\left(-\dot{\lambda}_0-\sum_{i=1}^{k}\dot{\lambda}_i t^i\right)_\alpha dt \tag{37}$$

respectively.

Case 2: Some parameters in the prior distribution are fuzzy numbers

Suppose that the prior PDF of the random variable θ is $\pi\left(\theta\right)$, and the parameters $\tilde{\varphi}_1,\tilde{\varphi}_2,\cdots,\tilde{\varphi}_l$ in $\pi\left(\theta\right)$ are fuzzy numbers. The lower and upper bounds of probability of failure under the α-level set can be expressed as the following optimization model:

$$\begin{aligned}
&\left(P_f^{\min}\right)_\alpha\left(P_f^{\max}\right)_\alpha = \min(\max)\int_{S_t}\left|\frac{-D_r}{t^2}\right| p\left[\left(\frac{D_r}{t}\right)\middle|\left(\mathbf{y}_\theta,\tilde{\varphi}_1,\tilde{\varphi}_2,\cdots,\tilde{\varphi}_l\right)\right]dt \\
&s.t.\ \left(\tilde{\varphi}_1\right)_\alpha^L \le \left(\varphi_1\right)_\alpha \le \left(\tilde{\varphi}_1\right)_\alpha^U \\
&\qquad\ \vdots \qquad\qquad \vdots \\
&\left(\tilde{\varphi}_l\right)_\alpha^L \le \left(\varphi_l\right)_\alpha \le \left(\tilde{\varphi}_1\right)_\alpha^U
\end{aligned} \tag{38}$$

With different α-level, the membership function of $\tilde{P}_f$ can be determined. The minimum and maximum probability of failure can be calculated by:

$$\left(P_f^{\min}\right)\left(P_f^{\max}\right) = \min(\max)\left(\tilde{P}_f\right)_{\alpha=0} \tag{39}$$

For example, suppose that the prior PDF of θ is $N\left(\tilde{\mu}_\theta,\sigma_\theta^2\right)$, where the mean value $\tilde{\mu}_\theta$ is a fuzzy number. The lower and upper bounds of probability of failure under the α-level sets can be determined as:

$$\begin{aligned}
&\left(p_f^{\min}\right)_\alpha = \min\int_{S_t}\left\{\frac{\frac{1}{\sqrt{2\pi}\sigma_\theta}\left|\frac{-D_r}{t^2}\right|\exp\left[-\lambda_0-\sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i-\left(\frac{D_r}{t}-\left(\tilde{\mu}_\theta\right)_\alpha\right)^2\Big/2\sigma_\theta^2\right]}{C\left[\theta,\left(\tilde{\mu}_\theta\right)_\alpha,\sigma_\theta\right]}\right\}dt \\
&s.t.\ \left(\tilde{\mu}_\theta\right)_\alpha^L \le \left(\mu_\theta\right)_\alpha \le \left(\tilde{\mu}_\theta\right)_\alpha^U
\end{aligned} \tag{40}$$

and:

$$\begin{aligned}
&\left(p_f^{\max}\right)_\alpha = \max\int_{S_t}\left\{\frac{\frac{1}{\sqrt{2\pi}\sigma_\theta}\left|\frac{-D_r}{t^2}\right|\exp\left[-\lambda_0-\sum_{i=1}^{k}\lambda_i\left(\frac{D_r}{t}\right)^i-\left(\frac{D_r}{t}-\left(\tilde{\mu}_\theta\right)_\alpha\right)^2\Big/2\sigma_\theta^2\right]}{C\left[\theta,\left(\tilde{\mu}_\theta\right)_\alpha,\sigma_\theta\right]}\right\}dt \\
&s.t.\ \left(\tilde{\mu}_\theta\right)_\alpha^L \le \left(\mu_\theta\right)_\alpha \le \left(\tilde{\mu}_\theta\right)_\alpha^U
\end{aligned} \tag{41}$$

respectively.

5. Illustrative Examples and Discussion

In this section, two examples are analyzed to demonstrate the proposed method. The first example is associated with only one linear degradation cure, and its critical threshold is a fuzzy number. The degradation curve of the second example is a non-linear curve, and some fuzzy numbers exist in the prior distribution.

Example 1

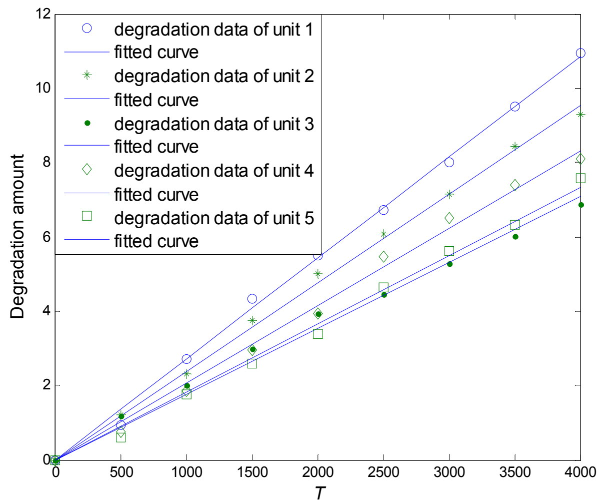
GaAs is a typical reliable and long lifetime laser which is widely used in military. Since the GaAs laser is highly reliable and very costly, there is little failure information because carrying out many experiments is impossible. Suppose the degradation data of five GaAs laser units is as shown in Table 2. The critical threshold D_f is 10, that is, the laser is failure if its current increases more than the original value by 10% [9]. The degradation curve of the GaAs lasers is a linear function.

Table 2. Degradation data of GaAs [9].

***t* (h)**	0	500	1000	1500	2000	2500	3000	3500	4000
1(unit)	0	0.93	2.72	4.34	5.48	6.72	8.00	9.49	10.94
2(unit)	0	1.22	2.30	3.75	4.99	6.07	7.16	8.42	9.28
3(unit)	0	1.17	1.99	2.97	3.94	4.45	5.27	6.02	6.88
4(unit)	0	0.74	1.85	2.95	3.92	5.47	6.50	7.39	8.09
5(unit)	0	0.61	1.77	2.58	3.38	4.63	5.62	6.32	7.59

Since the degradation curve is a linear function, it can be expressed as $D(i,t) = D_0 + \theta_i t$, where θ_i can be determined by the least squares method from the degradation data of the *ith* unit. In order to assess system reliability, θ and T usually are assumed as known distribution such as normal distribution or Weibull distribution in the traditional methods, and the values of distribution parameters are determined by using the maximum likelihood estimation. In reality, the distributions information of θ and T cannot be known clearly under the small sample size conditions. In order to avoid personal assumptions, the maximum entropy method can be used since it does not need any additional assumptions and information. The degradation data of each unit and its fitted curve are shown in Figure 3.

Figure 3. Degradation data of each unit and its fitted curve.



From Table 2 and Equation (16), the PDFs of the random variable T are shown in Figure 4, where k = 3 and k = 4 denote the results calculated using the first three and four moments, respectively. Generally, the first four moments are sufficient to describe a wide range of distribution types. From Equation (17), the CDFs of T are shown in Figure 5.

In reality, critical thresholds sometimes cannot be determined precisely due to various uncertainties. The more appropriate way to describe the thresholds is that D_f is a fuzzy number around 10. The phrase "around 10" should be regarded as a fuzzy number $\tilde{10}$. Suppose that the critical threshold D_f in this example is a triangular fuzzy number (9, 10, 11). System reliability under T equal to 4000 hours is considered. From Equations (36) and (37), the probability of failure under different α-levels are shown in Figure 6, which shows that the probability of failure is a fuzzy number rather than precise value. From Figure 6 we also know that a small variation of the critical threshold may lead to a big change of the probability of failure.

Figure 4. PDFs of T calculated by using the first three and four moments.

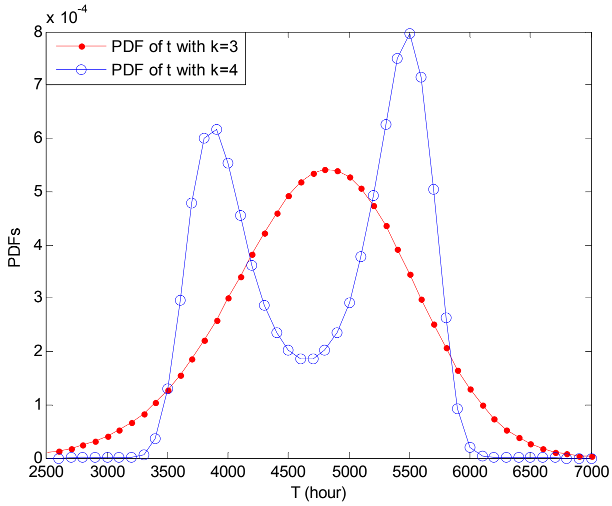


Figure 5. CDFs calculated by using the first three and four moments.

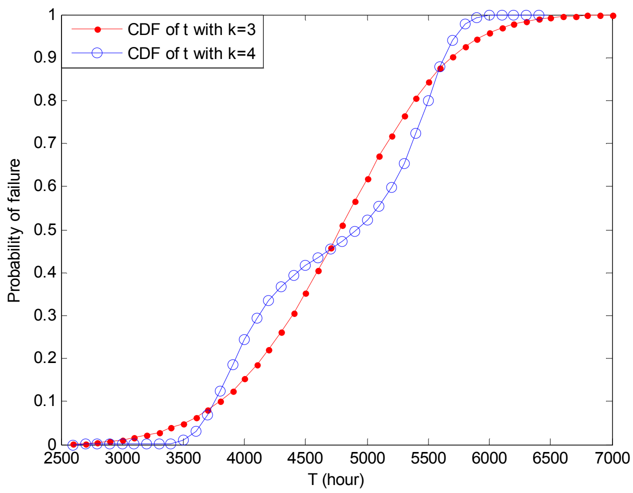
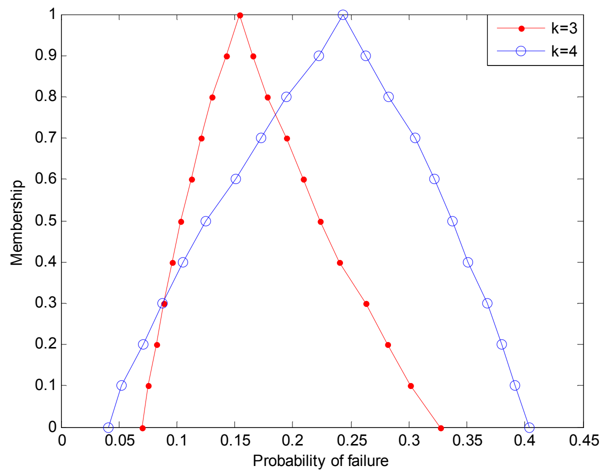


Figure 6. Probability of failure under different α-levels.



Example 2

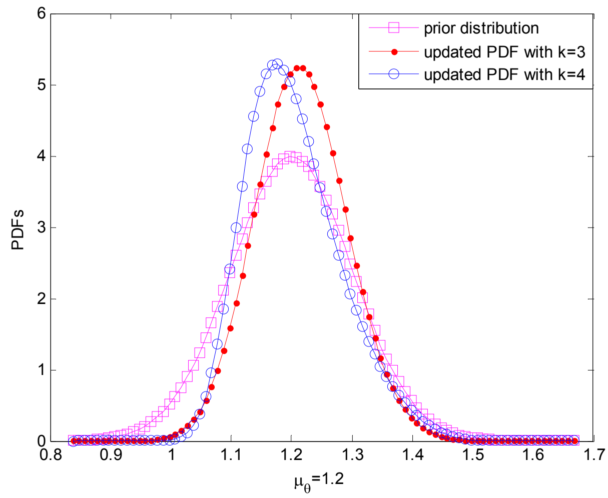
Suppose that the degradation curve of a time-depending deteriorating system is $D(i,t) = 0.02\exp(\theta_i t)$, and the critical threshold D_f = 10. θ is a random variable with fuzzy normally distributed prior distribution $N\left(\tilde{1.2}, 0.1\right)$, that is, the value of μ_θ is a triangular fuzzy number (1.1, 1.2, 1.3). The values of θ for the six units are given in Table 3.

Table 3. The values of θ from experiments.

θ (unit)	1	2	3	4	5	6
value	1.28	1.47	1.15	1.33	1.08	1.21

In engineering practice, the available information, such as the past data and expert opinions, is valuable information for us. In this example, we consider the situation where there is prior information for the random variable θ. However, the mean value of the prior distribution is a triangular fuzzy number. Suppose that μ_θ is precisely determined, that is, μ_θ = 1.2. From Equation (23) and the Bayesian inferences, the posterior distributions (also called updated distributions) of θ are shown in Figure 7. However, if μ_θ is a fuzzy number, the posterior distribution are family distributions rather single ones.

Figure 7. Posterior distribution for θ when μ_θ = 1.2.



From Equations (27), (28), (38) and (39), the probability of failure under T = 4.5 year are given in Tables 4, 5 and Figure 8, respectively. From Figure 8, we know that the probability of failure is a fuzzy number. When $\alpha = 1$, it denotes that μ_θ is precisely determined, and the probability of failure is a precise value. Furthermore, the probability of failure under different μ_θ values is shown in Figure 9.

Table 4. Probability of failure bounds under different α-levels when k = 3.

α	0	0.2	0.4	0.6	0.8	1.0
$(P_f^{min})_\alpha$	0.0019	0.0030	0.0049	0.0076	0.0116	0.0172
$(P_f^{max})_\alpha$	0.0942	0.0698	0.0507	0.0361	0.0252	0.0172

Table 5. Probability of failure bounds under different α-levels when $k = 4$.

α	0	0.2	0.4	0.6	0.8	1.0
$(P_f^{min})_\alpha$	0.0023	0.0039	0.0063	0.0101	0.0159	0.0245
$(P_f^{max})_\alpha$	0.1476	0.1088	0.0779	0.0543	0.0449	0.0245

Figure 8. Probability of failure under μ_θ is a fuzzy number.

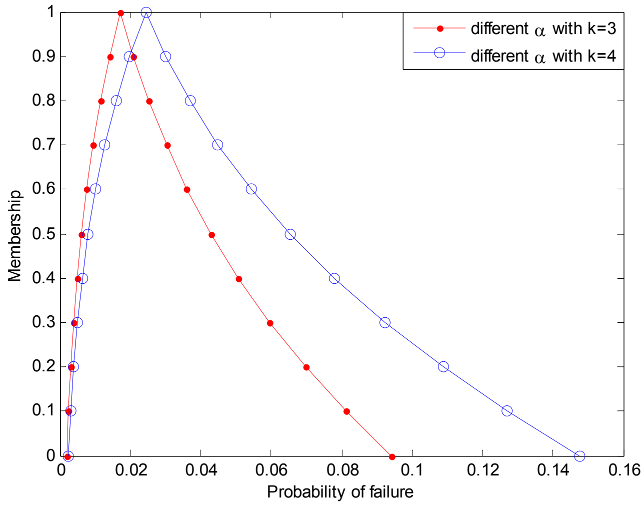
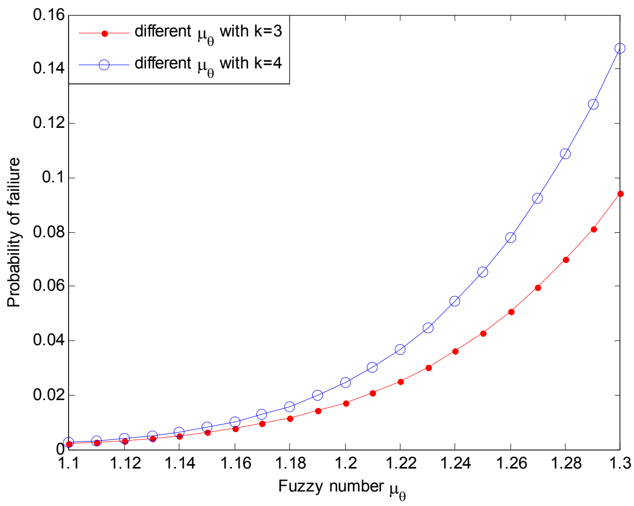


Figure 9. Probability of failure for different μ_θ values



6. Conclusions

In this paper the combination of Bayesian inference and the maximum entropy method have been applied to the problem of deteriorating system reliability assessment. Due to various uncertainties and incomplete information, precise determination of all parameters is impossible in engineering applications. In this case, fuzzy sets theory and Bayesian inference as well as the maximum entropy method have shown to be useful for the case of vague environments and less data. The maximum entropy method is robust under limited data constraints because it does not need any additional assumptions. The numerical examples have shown that the probability of failure is a fuzzy number in the case that fuzzy parameters exist in systems. Furthermore, the memberships can be determined by the optimization models proposed in the paper. From the discussions and the illustrated examples, we know that the proposed method requires neither the additional assumptions nor the large sample sizes, which are needed in other commonly used methods. It should be noted that there are some limitations to the proposed method. The RV deterioration model cannot capture temporal variability associated with evolution of degradation. The extension of the method for handling multiple failure modes will be the subject of future work in our research.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Nicolai, R.P.; Dekker, R.; van Noortwijk, J.M. A comparison of models for measurable deterioration: An application to coatings on steel structures. *Reliab. Eng. Syst. Saf.* **2007**, *92*, 1635–1650.
2. Hamada, M.S.; Martz, H.F.; Reese, C.S.; Wilson, A.G. *Bayesian Reliability*; Springer: New York, NY, USA, 2008.
3. Pandey, M.D. Probabilistic models for condition assessment of oil and gas pipelines. *NDT&E Int.* **1998**, *31*, 349–358.
4. Tsai, C.C.; Tseng, S.T.; Balakrishnan, N. Optimal burn-in policy for highly reliable products using gamma degradation process. *IEEE Trans. Reliab*. **2011**, *60*, 234–245.
5. Freitas, M.A.; de Toledo, M.L.G.; Colosimo, E.A.; Pires, M.C. Using degradation data to assess reliability: A case study on train wheel degradation. *Qual. Reliab. Eng. Int*. **2009**, *25*, 607–629.
6. Pandey, M.D.; van Noortwijk, J.M. Gamma Process Model for Time-Dependent Structural Reliability Analysis. In Proceedings of the Second International Conference on Bridge Maintenance, Safety and Management (IABMAS), Kyoto, Japan, 18–24 October 2004.
7. Huang, W.; Dietrich, D.L. An alternative degradation reliability modeling approach using maximum likelihood estimation. *IEEE Trans. Reliab*. **2005**, *54*, 310–317.
8. Van Noortwijk, J.M.; van der Weide, J.A.M.; Kallen, M.J.; Pandey, M.D. Gamma processes and peaks-over-threshold distributions for time-dependent reliability. *Reliab. Eng. Syst. Saf.* **2007**, *92*, 1651–1658.
9. Deng, A.M.; Chen, X.; Zhang, C.H.; Wang, Y.-S. Reliability assessment based on performance degradation data. *J. Astronaut*. **2006**, *27*, 546–552, (in Chinese).
10. Savage, G.J.; Son, Y.K. The set-theory method for systems reliability of structure with degrading components. *Reliab. Eng. Syst. Saf.* **2011**, *96*, 108–116.

11. Stewart, M.G.; Rosowsky, D.V. Time-dependent reliability of deteriorating reinforced concrete bridge decks. *Struct. Saf.* **1998**, *20*, 91–109.
12. Pandey, M.D.; Yuan, X.X.; van Noortwijk, J.M. The influence of temporal uncertainty of deterioration on life-cycle management of structures. *Struct. Infrastruct. Eng*. **2009**, *5*, 145–156.
13. Akgül, F.; Frangopol, D.M. Time-dependent interaction between load rating and reliability of deteriorating bridges. *Eng. Struct*. **2004**, *26*, 1751–1765.
14. Van Noortwijk, J.M.; Klatter, H.E. Optimal inspection decision for the block mats of the eastern-scheldt barrier. *Reliab. Eng. Syst. Saf.* **1999**, *65*, 203–211.
15. Frangopol, D.M.; Kallen, M.J.; van Noortwijk, J.M. Probabilistic models for life-cycle performance of deteriorating structures: Review and future directions. *Prog. Struct. Eng. Mater*. **2004**, *6*, 197–212.
16. Abdel-Hameed, M. A gamma wear process. *IEEE Trans. Reliab.* **1975**, *24*, 152–153.
17. Xiao, N.-C.; Huang, H.-Z.; Wang, Z.-L.; Liu, Y.; Zhang, X.-L. Unified uncertainty analysis by the mean value first order saddlepoint approximation. *Struct. Multidiscip. Optim*. **2012**, *46*, 803–812.
18. Kiureqhian, A.D.; Ditlevsen, O. Aleatory or epistemic? Does it matter? *Struct. Saf.* **2009**, *31*, 105–112.
19. Wu, H.C. Bayesian system reliability assessment under fuzzy environments. *Reliab. Eng. Syst. Saf.* **2004**, *83*, 277–286.
20. Wu, H.C. Fuzzy reliability estimation using Bayesian approach. *Comput. Ind. Eng*. **2004**, *46*, 467–493.
21. Wu, H.C. The fuzzy estimators of fuzzy parameters based on fuzzy random variables. *Eur. J. Oper. Res*. **2003**, *46*, 101–114.
22. Taheri, S.M.; Zarei R. Bayesian system reliability assessment under the vague environment. *Appl. Soft Comput. J.* **2011**, *11*, 1614–1622.
23. Guida, M.; Penta, F. A Bayesian analysis of fatigue data. *Struct. Saf.* **2010**, *32*, 64–76.
24. Guikema, S.D. Formulating informative, data-based priors for failure probability estimation in reliability analysis. *Reliab. Eng. Syst. Saf.* **2007**, *92*, 490–502.
25. Bernardo, J.M.; Smith, A.F.M. *Bayesian Theory*; Wiley: New York, NY, USA, 2000.
26. Koch, K.R. *Introduction to Bayesian Statistics*, 7th ed.; Springer: Berlin, Germany, 2007.
27. Jaynes, E.T. Information theory and statistical mechanics. *Phys. Rev*. **1957**, *106*, 620–630.
28. Zelluer, A.; Tobias, J. Further results on Bayesian method of moments analysis of the multiple regression model. *Int. Econ. Rev*. **2001**, *42*, 40–121.
29. Wu, X.M. Calculation of maximum entropy densities with application to income distribution. *J. Econ*. **2003**, *115*, 347–354.
30. Ormoneit, D.; White, H. An efficient algorithm to compute maximum entropy densities. *Econ. Rev*. **1999**, *18*, 127–140.
31. Zellner, A.; Highfield, R.A. Calculation of maximum entropy distributions and approximation of marginal posterior distributions. *J. Econ*. **1988**, *37*, 195–209.
32. Most, T. Efficient structural reliability methods considering incomplete knowledge of random variable distribution. *Probabilist. Eng. Mech*. **2011**, *26*, 380–386.
33. Cheng, L.; Tong, L. Measurement data processing based on maximum entropy method. *J. Electron. Meas. Instrum.* **2009**, *23*, 47–51, (in Chinese).

34. Sheng, Z.; Xie, S.Q.; Pan, C.Y. *Probability and Mathematical Statistics* (in Chinese), 4th ed.; Higher Education Press: Beijing, China, 2008,
35. Meeker, W.Q.; Escobar, L.A. *Statistical Methods for Reliability Data*; Wiley: New York, NY, USA, 1998.
36. Burden, R.L.; Faires, J.D. *Numerical Analysis*, 7th ed.; Higher Education Press: Bejing, China, 2001.