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Article

Solutions of Some Nonlinear Diffusion Equations and Generalized Entropy Framework

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Abstract: We investigate solutions of a generalized diffusion equation that contains nonlinear terms in the presence of external forces and reaction terms. The solutions found here can have a compact or long tail behavior and can be expressed in terms of the q-exponential functions present in the Tsallis framework. In the case of the long-tailed behavior, in the asymptotic limit, these solutions can also be connected with the Lévy distributions. In addition, from the results presented here, a rich class of diffusive processes, including normal and anomalous ones, can be obtained.

Keywords: diffusion; Tsallis entropy; Lévy distribution

Classification: PACS 66.10.Cb; 82.20.Db; 05.60.Cd; 05.40.Jc

1. Introduction

The fast growth of the phenomena-connected anomalous diffusion [1] in different fields of science has motivated the researchers to analyze several formalisms to investigate these phenomena, which are essentially characterized by non-Markovian processes. One characteristic of these situations concerns the spreading of a system that is not usual and, in several cases, given by $\langle (r - \langle r \rangle)^2 \rangle \sim t^{\alpha}$ (where

$\alpha < 1$ and $\alpha > 1$ represent the sub- and super- diffusive cases, respectively). Typical situations are the relaxation to equilibrium in systems (such as polymer chains and membranes) with long temporal memory, anomalous transport in disordered systems [2], non-Markovian dynamical processes in protein folding [3], percolation of gases through porous media [4], thin saturated regions in porous media [5], the standard solid-on-solid model for surface growth, thin liquid films spreading under gravity [6], transport of fluid in porous media and viscous fingering [7] and the overdamped motion of interacting particles [8]. One of these approaches is based on a nonlinear diffusion equation, the porous media equation [9–23], by taking the presence of external forces and reaction terms into account. For these equations, the solutions can be expressed in terms of the q-exponentials present in the Tsallis framework [24] suggesting that the thermodynamic formalism connected with the scenarios has to be extended to incorporate the effects that are not conveniently described by the usual one. Other approaches, such as Langevin equations [25,26], master equations, random walk [27] and fractional linear [28–31] or nonlinear [32–34] diffusion equations have also been used to investigate anomalous diffusion processes.

Here, we investigate the following d-dimensional nonlinear diffusion equation with radial symmetry:

$$\begin{aligned}\frac{\partial}{\partial t}\rho(r,t) &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left\{r^{d-1}\left[\widetilde{\mathcal{D}}(r,t)\left|\frac{\partial}{\partial r}\rho(r,t)\right|^{\eta}\frac{\partial}{\partial r}\rho^{\nu}(r,t)\right]\right\} \\ &- \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}F(r,t)\rho(r,t)\right] - \overline{\alpha}(\rho,r,t) \qquad (1)\end{aligned}$$

where $\widetilde{\mathcal{D}}(r,t)$ is a spatial and time-dependent diffusion coefficient, $F(r,t)$ is an external force applied to the system and $\overline{\alpha}(\rho,r,t)$ is a reaction term. In the absence of a reaction term, *i.e.*, $\overline{\alpha}(\rho,r,t) = 0$, it can be verified that $\int_0^\infty dr\ r^{d-1}\rho(r,t)$ is time-independent (hence, if ρ is normalized at $t = 0$, it will remain so forever). Indeed, if we write the equation in the $\partial_t\rho = -r^{1-d}\partial_r\left(r^{d-1}\mathcal{J}\right)$ form with:

$$\mathcal{J}(r,t) = -\widetilde{\mathcal{D}}(r,t)\left|\frac{\partial}{\partial r}\rho(r,t)\right|^{\eta}\frac{\partial}{\partial r}\rho^{\nu}(r,t) + F(r,t)\rho(r,t) \qquad (2)$$

and assume the boundary condition $\mathcal{J}(\infty,t) = 0$, it can be shown that $\int_0^\infty dr\ r^{d-1}\rho(r,t)$ is a constant of motion. Note that the solutions that emerge from Equation (1), by considering a suitable boundary condition, have, as particular cases, several situations, such as the nonlinear model of heat conduction worked out in [35], nonlinear diffusion problems related to hydraulics [36], the Chezy's law for very large cross sections and viscous flows with different rheologies [37]. Furthermore, Equation (1), in connection with nonlinear diffusion processes coupled with reaction dynamics [19], may be used to investigate scenarios characterized, for example, by population dynamics [38], recombination processes in plasma physics and the kinetics of phase transitions. These features imply that depending on the choice of the diffusive term, *i.e.*, η, ν and $\widetilde{\mathcal{D}}(r,t)$, Equation (1) may interpolate several situations, since nonlinear diffusion to Richardson diffusion leads us to a flexible approach to be compared with experimental data.

The plan of this work is to investigate Equation (1), as mentioned above, by taking several situations into account. We start by considering the stationary case in the presence of an arbitrary external force, $F(r,t)$, in the absence of reaction terms. In this case, it is assumed that $\widetilde{\mathcal{D}}(r,t) \to \mathcal{D}(r)$ and $F(r,t) \to f(r)$ in the limit, $t \to \infty$, to obtain $\rho(r,t) \to \rho_s(r)$. After, we analyze the time-dependent cases, which emerge when we consider $\widetilde{\mathcal{D}}(r,t) = \overline{\mathcal{D}}(t)r^{-\theta}$, the external force $F(r,t) = -k(t)r$ with

$\overline{\alpha}(\rho,r,t) = \alpha(t)\rho(r,t) - \alpha_\gamma(t)r^\lambda\rho^\gamma(r,t)$. Note that this choice for the source or absorbent term is connected to the Verhulst growth dynamics. In all cases, we express our results in terms of the q-exponentials that appear in the Tsallis framework, and in the asymptotic limit, the solutions are related to the Lévy distributions. These developments are performed in Section II, and in Section III, we present our discussions and conclusions.

2. Nonlinear Diffusion Equation

Let us start our discussion about the stationary solutions, by applying in Equation (1) the conditions previously discussed in the introduction. For this case, it can be written as:

$$\mathcal{D}(r)\left|\frac{\partial}{\partial r}\rho_s(r)\right|^\eta \frac{\partial}{\partial r}\rho_s^\nu(r) - f(r)\rho_s(r) = 0 \tag{3}$$

with the solution subjected to the boundary condition, $\lim_{r\to\infty}\rho_s(r) = 0$, in order to satisfy the normalization condition, where $f(r) = -\partial_r V(r)$ and $V(r)$ represent a potential consistent with the presence of stationary states. In order to obtain the solution of Equation (3), it is interesting to note that it should recover, in the appropriate limits, the solutions of the porous media and usual diffusion equations. For these reason, we propose the *ansatz*:

$$\rho_s(r) = \exp_q\left[-\beta\mathcal{G}(r)\right]/\mathcal{Z} \tag{4}$$

as a solution, where:

$$\exp_q[x] = \begin{cases} (1+(1-q)x)^{\frac{1}{1-q}}, & \text{if } 1/(1-q) \leq x \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

is the q-exponential function present in the Tsallis framework [24], $\mathcal{G}(r)$ is a function to be determined and the constants, β and $\mathcal{Z}$, are related to the normalization condition. In particular, it is important to note that in Equation (5), there is a cut-off in order to preserve the probabilistic interpretation of the solutions. Here, it is not out of place to mention that Equation (4) could be obtained from the maximum entropy principle when $S_q = (1-\int_0^\infty r^{d-1}\rho_s(r))/(q-1)$ is considered with suitable constraints. In addition, the entropy plays an important role in the Htheorem and its generalizations associated with nonlinear Fokker-Planck equations, as discussed in [13,14]. By substituting Equation (4) in Equation (3) and solving the differential equation obtained for $\mathcal{G}(r)$, we obtain that:

$$\mathcal{G}(r) = \int^r \left(\frac{1}{\nu\mathcal{D}(\xi)}\frac{\partial}{\partial\xi}V(\xi)\right)^{\frac{1}{1+\eta}} d\xi \tag{6}$$

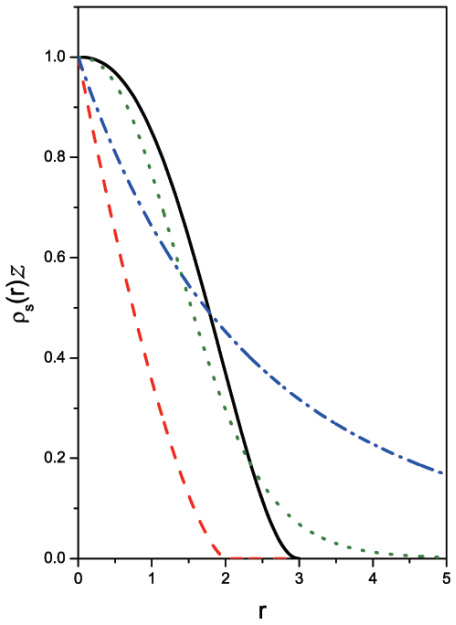
with $\beta = \mathcal{Z}^{1-q}$ and $q = (2-\nu)/(1+\eta)$, and consequently:

$$\rho_s(r) = \exp_q\left[-\beta\int^r \left(\frac{1}{\nu\mathcal{D}(\xi)}\frac{\partial}{\partial\xi}V(\xi)\right)^{\frac{1}{1+\eta}} d\xi\right]\Big/\mathcal{Z} \tag{7}$$

For $\eta = 0$, Equation (7) recovers the stationary solution of the porous media equation, and the usual one is obtained with $\eta = 0$ and $\nu = 1$. Figure 1 illustrates the behavior of the stationary solution for the

harmonic potential $V(r) = kr^2/2$ and $\mathcal{D}(r) = \overline{\mathcal{D}} r^{-\theta}$ for different values of the parameters, q, θ and η. Note that depending on the values of q, the solution may present a compact (see the red dashed and black solid lines for q less than one) or a long-tailed behavior (see the green dotted and blue dashed-dotted lines for q greater than one). For the case that q equals one, we obtain from the stationary solution a stretched exponential, which recovers the usual case for $\eta = \theta = 0$.

Figure 1. The behavior of Equation (7) *versus* r is illustrated for different values of q, θ and η in the absence of an absorbent (source) term by considering, for simplicity, $\overline{\mathcal{D}} = 1$ and $V(r) = kr^2/2$ with $k = 1$. The red dashed and the black solid lines were obtained for $q = 1/2$, $\theta = 1$, $\eta = 1/2$ and $q = 1/3$, $\theta = -1$ and $\eta = 1$. The green dotted and red dashed-dotted lines were obtained for $q = 6/5$, $\theta = 1$, $\eta = 1/2$ and $q = 6/5$, $\theta = -1$ and $\eta = 1/3$.



Now, we focus our attention on the time-dependent solutions of Equation (1). We first consider the case characterized by the diffusion coefficient $\widetilde{\mathcal{D}}(r,t) = \overline{\mathcal{D}}(t) r^{-\theta}$ and the external force $F(r,t) = -k(t)r$ in the absence of the reaction term. Thus, we have that:

$$\begin{aligned} \frac{\partial}{\partial t}\rho(r,t) &= \frac{\overline{\mathcal{D}}(t)}{r^{d-1}}\frac{\partial}{\partial r}\left\{ r^{d-1}\left[r^{-\theta}\left|\frac{\partial}{\partial r}\rho(r,t)\right|^{\eta}\frac{\partial}{\partial r}\rho^{\nu}(r,t)\right]\right\} \\ &- \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}\left(-k(t)r\right)\rho(r,t)\right] \end{aligned} \tag{8}$$

Note that the external force may exhibit a stationary solution given by Equation (7), which satisfies the required boundary conditions, if $k(t) \to const.$ for $t \to \infty$. This feature lead us to consider that a time-dependent solution for Equation (8) connected to Equation (7) is given by:

$$\rho(r,t) = \exp_q\left[-\beta(t) r^{\lambda}\right] / \mathcal{Z}(t) \tag{9}$$

with $\lambda = 1+(1+\theta)/(1+\eta)$ and $q = (2-\nu)/(1+\eta)$. The time-dependent functions, $\beta(t)$ and $\mathcal{Z}(t)$, can be obtained by substituting Equation (9) in Equation (8). In this sense, after performing some calculations, it is possible to show that the functions, $\beta(t)$ and $\mathcal{Z}(t)$, satisfy the following set of coupled equations:

$$\frac{1}{\mathcal{Z}}\frac{d}{dt}\mathcal{Z} = \nu d\mathcal{D}(t)\mathcal{Z}^{(q-1)(1+\eta)}\lambda\beta|\lambda\beta|^{\eta} - dk(t) \quad (10)$$

$$\frac{d}{dt}\beta = \lambda k(t)\beta - \nu\mathcal{D}(t)\mathcal{Z}^{(q-1)(1+\eta)}(\lambda\beta)^2|\lambda\beta|^{\eta} \quad (11)$$

From these equations, we obtain the following relation between $\beta(t)$ and $\mathcal{Z}(t)$:

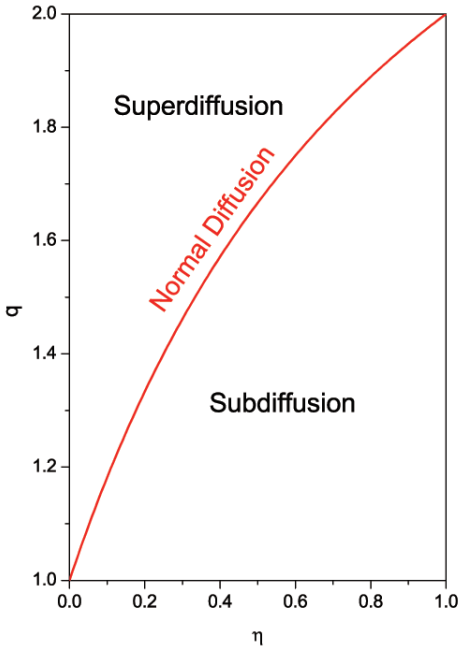
$$\mathcal{Z}(t)\beta^{\frac{d}{\lambda}}(t) = \mathcal{N} \quad (12)$$

connected with the normalization condition of the $\rho(r,t)$, *i.e.*, $\int_0^\infty drr^{d-1}\rho(r,t) = 1$, from which the constant, $\mathcal{N}$, can be obtained. By using Equation (12), we obtain that:

$$\beta(t) = \beta(0)e^{-\lambda\int_0^t dt'k(t')}\left[1 + \mathcal{C}\int_0^t d\bar{t}\mathcal{D}(\bar{t})e^{\lambda(\sigma-1)\int_0^{\bar{t}}dt'k(t')}\right]^{\frac{1}{1-\sigma}} \quad (13)$$

where $\mathcal{C} = \nu(\sigma-1)\lambda^{2+\eta}/\left(\mathcal{N}^{(1-q)(1+\eta)}\beta^{1-\sigma}(0)\right)$, $\sigma = 2+\eta+(1-q)(1+\eta)d/\lambda$ and $\beta(0)$ is a constant defined by the initial condition. The time-dependent function, $\mathcal{Z}(t)$, is obtained by using Equation (12) and Equation (13). By using Equations (9), (12) and (13), one can find that the spreading of the system is governed by $\langle(r-\langle r\rangle)^2\rangle \propto t^{\zeta}$ with $\zeta = 2/\left[(1+\eta)(\lambda+(1-q)d)\right]$ when $\mathcal{D}(t) = const.$ and $k(t) = 0$. In particular, in Figure 2 is illustrated, for simplicity, for $d = 1$ and $\lambda = 2$, the regions for which the behavior of the mean square displacement is usual or anomalous. In this sense, it is also interesting to point out that the case $\zeta = 1$ leads us to a linear time dependence on time, and $\zeta \neq 1$ produces a nonlinear time dependence on the mean square displacement. The first case is considered as usual diffusion, and the other case, which can be characterized by $\zeta < 1$ (subdiffusive) or $\zeta > 1$ (superdiffusive), is an anomalous diffusion.

Figure 2. This figure illustrates the regions where the system may present an usual or anomalous behavior for the mean square displacement depending on the values of q and η; for simplicity, for $d = 1$ and $\lambda = 2$.



It is important to mention that, depending on the q values, *i.e.*, $q > 1$ with $1 + \lambda/d > q > 1 + \lambda/(d+2)$, the second moment obtained from the solution is not defined and, consequently, the distribution may asymptotically be connected with the Lévy distributions, as performed for the Tsallis distribution [39–41].

Before starting our analysis about the time-dependent solutions of Equation (1) with:

$$\overline{\alpha}(\rho, r, t) = \alpha(t)\rho(r,t) - \alpha_\gamma(t) r^{\overline{\lambda}} \rho^\gamma(r,t) \,, \tag{14}$$

it is interesting to note that this term incorporated in Equation (1) extends the Fisher equation usually used to model population biology [42], providing a simple generalization of the Verhulst logistic equation [43–46] for population dynamics. In this context, the first term on the right side of Equation (14) can be related to the growing of the system, and the second one gives a regulation for the size of the system. In this manner, Equation (14) incorporated in Equation (1) may be used to describe the local changes in the population dynamics and the diffusive term governs the diffusion of this population in space, taking into account the usual and the porous media equation as particular cases. Particularly, performing an integration in Equation (1) with the $\overline{\alpha}(\rho, r, t)$ given by Equation (14), we obtain:

$$\frac{d}{dt}\mathcal{P}(t) = \alpha(t)\mathcal{P}(t) - \alpha_\gamma(t) \int_0^\infty dr r^{d-1} \left[r^{\overline{\lambda}} \rho^\gamma(r,t) \right] \tag{15}$$

where $\mathcal{P}(t) = \int_0^\infty dr r^{d-1} \rho(r,t)$ can be considered as a total population. In this context, we may consider two mechanisms for the global regulation of the population: (i) one leading to a Verhulst-like equation of motion for the population; and the other (ii) with a constant total population. For case (i), a global regulation mechanism is given by the function:

$$\alpha_\gamma(t) = \Phi(t) \left(\int_0^\infty dr r^{d-1} \rho(r,t) \right)^q \Big/ \int_0^\infty dr r^{d-1} \left[r^{\overline{\lambda}} \rho(r,t) \right] \tag{16}$$

By substituting Equation (16) in Equation (15), we obtain the following equation:

$$\frac{d}{dt}\mathcal{P}(t) = \alpha(t)\mathcal{P}(t) - \Phi(t)\mathcal{P}^q(t) \tag{17}$$

which is an extension of the Verhulst equation and has as the solution:

$$\mathcal{P}(t) = \mathcal{P}(0) e^{\int_0^t dt' \alpha(t')} \exp_q \left[-\frac{1}{\mathcal{P}^{1-q}(0)} \int_0^t d\bar{t} \Phi(\bar{t}) e^{-(1-q)\int_0^{\bar{t}} dt' \alpha(t')} \right] \tag{18}$$

For $q > 1$ with $\Phi(t)$ and $\alpha(t)$ constants, in the limit of $t \to \infty$, we obtain from Equation (18) that $\mathcal{P}(t) \to (\alpha/\Phi)^{1/(q-1)}$. The alternative form of obtaining a global regulation mechanism corresponds to relating $\alpha(t)$ and $\alpha_\gamma(t)$ as follows:

$$\alpha(t) = \alpha_\gamma(t) \int_0^\infty dr r^{d-1} \left[r^{\overline{\lambda}} \rho^\gamma(r,t) \right] \Big/ \int_0^\infty dr r^{d-1} \rho(r,t) \tag{19}$$

Now, let us focus on the time-dependent solutions of Equation (1) with the reaction term given by Equation (14). To obtain the time-dependent solutions for this case is a hard task when $\overline{\lambda}$ and γ are arbitrary parameters for the required boundary condition. However, for the case $\overline{\lambda} = \lambda$ with $\gamma = q$, it

is possible to verify that Equation (9) is the solution with the time-dependent functions governed by the following equations:

$$\frac{1}{\mathcal{Z}}\frac{d}{dt}\mathcal{Z} = \nu d\mathcal{D}(t)\mathcal{Z}^{(q-1)(1+\eta)}\lambda\beta|\lambda\beta|^{\eta} - dk(t) - \alpha(t) \tag{20}$$

$$\frac{d}{dt}\beta = \lambda\beta k(t) - \nu\mathcal{D}(t)\mathcal{Z}^{(q-1)(1+\eta)}|\lambda\beta|^{\eta}(\lambda\beta)^2 + \mathcal{Z}^{1-q}\alpha_{\gamma}(t) \tag{21}$$

Note that Equation (12) is satisfied if $\alpha(t) = d\alpha_{\gamma}(t)\mathcal{Z}^{1-q}/(\lambda\beta)$ and implies:

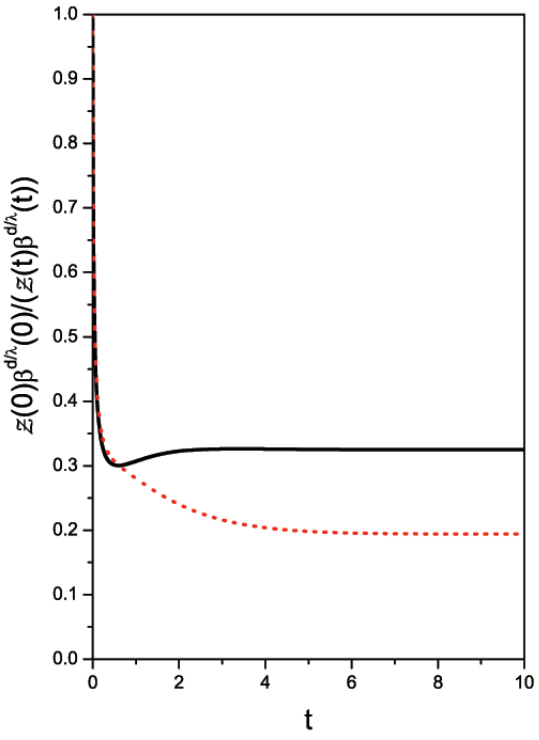
$$\beta(t) = \beta(0)e^{-\lambda\int_0^t dt'(k(t')+\alpha(t')/d)}\left[1 + \mathcal{C}\int_0^t d\bar{t}\mathcal{D}(\bar{t})e^{\lambda(\sigma-1)\int_0^{\bar{t}}dt'(k(t')+\alpha(t')/d)}\right]^{\frac{1}{1-\sigma}} \tag{22}$$

otherwise, this point may not be verified by the solution when $\alpha(t)$ and $\alpha_{\gamma}(t)$ are arbitraries. This feature is illustrated in Figure 3 for a particular choice of $\alpha(t)$ and $\alpha_{\gamma}(t)$. Another interesting aspect is obtained from Equation (20) for $\alpha(t)$ an arbitrary with $\alpha_{\gamma}(t) = 0$, *i.e.*, a diffusion equation with a reaction term of the first order, yielding:

$$\beta(t) = \beta(0)e^{-\lambda\int_0^t dt'k(t')}\left[1 + \mathcal{C}\int_0^t d\bar{t}\mathcal{D}(\bar{t})e^{\lambda(\sigma-1)\int_0^{\bar{t}}dt'\left(k(t')+\bar{d}\alpha(t')\right)}\right]^{\frac{1}{1-\sigma}} \tag{23}$$

with $\bar{d} = (1-q)(1+\eta)d/\lambda$ and $\mathcal{Z}\beta^{\frac{d}{\lambda}} = \mathcal{N}e^{-d\int_0^t\alpha(t')dt'}$ implying a nonconservation of the probability.

Figure 3. The behavior of $\mathcal{Z}(0)\beta^{\frac{d}{\lambda}}(0)/\left(\mathcal{Z}(t)\beta^{\frac{d}{\lambda}}(t)\right)$ *versus* t is illustrated for typical values of q, η and θ by considering the presence of the reaction term with $\alpha(t) = \alpha e^{-t}$ and $\alpha_{\gamma}(t) = \alpha_{\gamma}e^{-t}$ with, for simplicity, $\alpha = 1$ and $\alpha_{\gamma} = 1$. The red dotted and solid black lines were obtained for $q = 1/2$ and $q = 6/5$ with $\theta = 1$ and $\eta = 1/2$.



3. Summary and Conclusions

We have investigated solutions of a nonlinear diffusion equation in the presence of a reaction term. First, we have considered the solutions of the stationary case in the absence of a reaction term for an arbitrary external force. For this case, we have shown that the solutions obtained can be expressed in terms of the q-exponentials present in the Tsallis framework, which suggests that Equation (1), similar to the porous media equation, may find in this framework a thermostatics base. After, we have considered the time-dependent case in the presence of a linear external force connected to a harmonic potential. The solution can present a compact or long-tailed behavior depending on the choice of q, and in particular, the last case can be connected to the Lévy distributions. In this scenario, a reaction term was incorporated, and solutions were obtained.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Pekalski, A., Sznajd-Weron, K., Eds. *Anomalous Diffusion: From Basics to Applications*; Lecture Notes in Physics; Springer: Berlin, Germany, 1999.
2. Metzler, R.; Barkai, E.; Klafter, J. Anomalous transport in disordered systems under the influence of external fields. *Physica A* **1999**, *266*, 343–350.
3. Plotkin, S.S.; Wolynes, P.G. Non-Markovian configurational diffusion and reaction coordinates for protein folding. *Phys. Rev. Lett.* **1998**, *80*, 5015–5018.
4. Muskat, M. *The Flow of Homogeneous Fluid Through Porous Media*; McGraw-Hill: New York, NY, USA, 1937.
5. Polubarinova-Kochina, P.Y. *Theory of Ground Water Movement*; Princeton University Press: Princeton, NJ, USA, 1962.
6. Buckmaster, J. Viscous sheets advancing over dry bed. *J. Fluid Mech.* **1977**, *81*, 735–756.
7. Grosfils, P.; Boon, J.B. Nonextensive statistics in viscous fingering. *Physica A* **2006**, *362*, 168–173.
8. Andrade, J.S., Jr.; da Silva, G.F.T.; Moreira, A.A.; Nobre, F.D.; Curado, E.M.F. Thermostatistics of overdamped motion of interacting particles. *Phys. Rev. Lett.* **2010**, *105*, 260601.
9. Compte, A.; Jou, D.; Katayama, Y. Anomalous diffusion in linear shear flows. *J. Phys. A* **1997**, *30*, 1023–1030.
10. Giordano, C.; Plastino, A.R.; Casas, M.; Plastino, A. Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and entropy. *Eur. Phys. J. B* **2001**, *22*, 361–368.
11. Tsallis, C.; Bukman, D.J. Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and entropy. *Phys. Rev. E* **1996**, *54*, R2197–R2200.

12. Drazer, G.; Wio, H.S.; Tsallis, C. Anomalous diffusion with absorption: Exact time-dependent solutions. *Phys. Rev. E* **2000**, *61*, 1417–1422.
13. Curado, E.M.F.; Nobre, F.D. Derivation of nonlinear Fokker-Planck equations by means of approximations to the master equation. *Phys. Rev. E* **2003**, *67*, 021107.
14. Schwämmle, V.; Curado, E.M.F.; Nobre, F.D. Dynamics of normal and anomalous diffusion in nonlinear Fokker-Planck equations. *Eur. Phys. J. B* **2009**, *70*, 107–116.
15. Chavanis, P.H. Nonlinear mean-field Fokker-Planck equations and their applications in physics, astrophysics and biology. *Phys. Rev. E* **2003**, *68*, 036108.
16. Frank, T.D. On a general link between anomalous diffusion and nonextensivity. *J. Math. Phys.* **2002**, *43*, 344–350.
17. Plastino, A.R.; Plastino, A. Non-Extensive statistical mechanics and generalized Fokker-Planck equation. *Physica A* **1995**, *222*, 347–354.
18. Silva, P.C.; da Silva, L.R.; Lenzi, E.K.; Mendes, R.S.; Malacarne, L.C. Nonlinear diffusion equation, Tsallis formalism and exact solutions. *Physica A* **2004**, *342*, 16–19.
19. Plastino, A.R.; Casas, M.; Plastino, A. A nonextensive maximum entropy approach to a family of nonlinear reaction-diffusion equations. *Physica A* **2000**, *280*, 289–303.
20. Wada, T.; Scarfone, A.M. On the non-linear Fokker-Planck equation associated with κ-entropy. *AIP Conf. Proc.* **2007**, *965*, 177–180.
21. Wada, T.; Scarfone, A.M. Asymptotic solutions of a nonlinear diffusive equation in the framework of κ-generalized statistical mechanics. *Eur. Phys. J. B* **2009**, *70*, 65–71.
22. Frank, T.D. *Nonlinear Fokker-Planck Equations*; Springer: Heidelberg, Germany, 2005.
23. Reisner, J.; Serencsa, J.; Shkoller, S. A spacetime smooth artificial viscosity method for nonlinear conservation laws. *J. Comput. Phys.* **2013**, *235*, 912–933.
24. Tsallis, C.; Plastino, A.R.; Mendes, R.S. The role of constraints within generalized nonextensive statistics. *Physica A* **1998**, *261*, 534–554.
25. Coffey, W.T.; Kalmykov, Y.P.; Waldron, J.T. *The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry and Electrical Engineering*; World Scientific Publishing Company: Singapore, Singapore, 2004.
26. Sandev, T.; Metzler, R.; Tomovski, Z. Velocity and displacement correlation functions for fractional generalized Langevin equations. *Fract. Calc. Appl. Anal.* **2012**, *15*, 426–450.
27. Weiss, G.W. *Aspects and Applications of the Random Walk*; North Holland: Amesterdan, The Netherlands, 1994.
28. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77.
29. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, Singapore, 2000.
30. Priya, G.S.; Prakash, P.; Nieto, J.J.; Kayar, Z. Higher-order numerical scheme for the fractional heat equation with dirichlet and neumann boundary conditions. *Numer. Heat Transf. Part B* **2013**, *63*, 540–559.
31. Tomovski, Z.; Sandev, T.; Metzler, R.; Dubbeldam, J. Generalized space-time fractional diffusion equation with composite fractional time derivative. *Physica A* **2012**, *391*, 2527–2542.

32. Silva, A.T.; Lenzi, E.K.; Evangelista, L.R.; Lenzi, M.K.; da Silva, L.R. Fractional nonlinear diffusion equation, solutions and anomalous diffusion. *Physica A* **2007**, *375*, 65–71.
33. Alibaud, N.; Cifani, S.; Jakobsen, E.R. Continuous dependence estimates for nonlinear fractional convection-diffusion equations. *SIAM J. Math. Anal.* **2012**, *44*, 603–632.
34. Cifani, S.; Jakobsen, E.R.; Karlsen, K.H. The discontinuous Galerkin method for fractional degenerate convection-diffusion equations. *BIT Numer. Math.* **2011**, *51*, 809–844.
35. Pascal, H. A nonlinear model of heat conduction. *J. Phys. A* **1992**, *25*, 939–948.
36. Daly, E.; Porporato, A. Similarity solutions of nonlinear diffusion problems related to mathematical hydraulics and the Fokker-Planck equation. *Phys. Rev. E* **2004**, *70*, 056303.
37. Gratton, J.; Minotti, F.; Mahajan, S.M. Theory of creeping gravity currents of a non-Newtonian liquid. *Phys. Rev. E* **1999**, *60*, 6960–6967.
38. Troncoso, P.; Fierro, O.; Curilef, S.; Plastino, A.R. A family of evolution equations with nonlinear diffusion, Verhulst growth, and global regulation: Exact time-dependent solutions. *Physica A* **2007**, *375*, 457–466.
39. Tsallis, C.; Levy, S.V.F.; Souza, A.M.C.; Maynard, R. Statistical mechanical foundation of the ubiquity of Lvy distributions in nature. *Phys. Rev. Lett.* **1995**, *75*, 3589–3782;
40. Tsallis, C.; Levy, S.V.F.; Souza, A.M.C.; Maynard, R. Errata: Statistical mechanical foundation of the ubiquity of Lvy distributions in nature [Phys. Rev. Lett. 75, 3589 (1995)]. *Phys. Rev. Lett.* **1996**, *77*, 5442.
41. Prato, D.; Tsallis, C. Nonextensive foundation of Levy distributions. *Phys. Rev. E* **2000**, *60*, 2398–2400.
42. Murray, J. *Mathematical Biology*; Springer: Berlin, Germany, 1989.
43. Verhulst, P.F. Recherches mathematiques sur la loi daccroissement de la population. *Mem. Acad. R. Belg.* **1845**, *18*, 1–38, (in French).
44. Mazumdar, J. *An Introduction to Mathematical Physiology and Biology*; Cambridge University Press: Cambridge, UK, 1999.
45. Hutchinson, G.E. *An Introduction to Population Ecology*; Yale University Press: New Haven, CT, USA, 1978.
46. Leach, P.G.L.; Andriopoulos, K. An oscillatory population model. *Chaos Soliton. Fract.* **2004**, *22*, 1183–1188.