*Entropy* **2015**, *17*, 1123-1134; doi:10.3390/e17031123

OPEN ACCESS

*entropy*

**ISSN 1099-4300**

www.mdpi.com/journal/entropy

*Article*

# Projective Synchronization for a Class of Fractional-Order Chaotic Systems with Fractional-Order in the (1, 2) Interval

**Ping Zhou [1,*], Rongji Bai [2] and Jiming Zheng [1]**

[1] Center of System Theory and Its Applications, Chongqing University of Posts and Telecommunications, Chongqing 400065, China; E-Mail: zhengjm@cqupt.edu.cn

[2] Key Laboratory of Network Control and Intelligent Instrument of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing 400065, China; E-Mail: bairongji1990@163.com

* Author to whom correspondence should be addressed; E-Mail: zhouping@cqupt.edu.cn; Tel./Fax: +86-23-62471795.

Academic Editor: J. A. Tenreiro Machado

*Received: 10 February 2015 / Accepted: 4 March 2015/ Published: 10 March 2015*

**Abstract:** In this paper, a projective synchronization approach for a class of fractional-order chaotic systems with fractional-order $1 < q < 2$ is demonstrated. The projective synchronization approach is established through precise theorization. To illustrate the effectiveness of the proposed scheme, we discuss two examples: (1) the fractional-order Lorenz chaotic system with fractional-order $q = 1.1$; (2) the fractional-order modified Chua's chaotic system with fractional-order $q = 1.02$. The numerical simulations show the validity and feasibility of the proposed scheme.

**Keywords:** fractional-order in interval (1, 2); chaotic systems; projective synchronization

## 1. Introduction

Many real-world physical systems can be well and more accurately described by fractional-order differential equations [1–4]. In recent years, chaotic phenomena has been found in many fractional-order nonlinear systems, such as the fractional-order Lorenz chaotic system [5,6], Chua's fractional-order chaotic circuit system [6], the fractional-order modified Duffing chaotic system [7], the fractional-order

Rössler chaotic system [8,9], the fractional-order Chen chaotic system [6–8], the fractional-order memristor chaotic system [10], and so on.

Over the last two decades, due to its potential applications in the field of science and engineering [11,12], more and more attention has been focused on synchronization of chaotic systems, and many synchronization schemes have been proposed. Among all, one particular synchronization scheme named projective synchronization has been proposed by Mainieri and Rehacek [13]. A master and slave system could be synchronized up to a scaling factor in PS, which can be used to extend binary digital to *M*-nary digital communication for getting faster communications [13,14].

However, many previous synchronization methods [6–9,13–16] for fractional-order chaotic systems only focused on the fractional-order $0 < q < 1$, when in fact, there are many fractional-order systems with fractional-order $1 < q < 2$ in the real world. For example, the time fractional heat conduction equation [17], the fractional telegraph equation [18], the time fractional reaction-diffusion systems [19], the fractional diffusion-wave equation [20], the space-time fractional diffusion equation [21], the super-diffusion systems [22], *etc.*, but the chaos phenomenon was not considered in [17–22]. Meanwhile, based on numerical simulation, Ge and Jhuang [23] reported some results on synchronization of the fractional order rotational mechanical system with fractional-order $q = 1.1$. Up to now, to the best of our knowledge, there seem to be no results on chaotic synchronization for fractional-order chaotic systems with $1 < q < 2$ through precise theorization. So, how to achieve the chaotic synchronization for fractional-order nonlinear systems with $1 < q < 2$ through precise theorization is an interesting and opening question of academic significance as well as practical importance.

Motivated by the abovementioned discussion, in this paper we propose a projective synchronization approach for a class of fractional-order chaotic systems with fractional-order $1 < q < 2$ through precise theorization. To show the effectiveness of the proposed scheme, the projective synchronization for a fractional-order Lorenz chaotic system with fractional-order $q = 1.1$ and Chua's fractional-order modified chaotic system with fractional-order $q = 1.02$ are discussed, respectively. The numerical simulations have indicated the validity and feasibility of our scheme.

## 2. Problem Statement and Main Result

In this paper, the Caputo derivative of fractional order $q$ for function $f(t)$, is defined as $D^q f(t) = \frac{1}{\Gamma(m-q)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}}d\tau$ where $m-1 < q < m$, $D^q$ denotes the Caputo derivative of fractional order $q$ for function $f(t)$, $m$ is the smallest integer larger than $q$, $f^{(m)}(t)$ is the $m$-th derivative in the usual sense, and $\Gamma(n-q) = \int_0^{+\infty} t^{(n-q-1)}e^{-t}dt$ denote the gamma function.

Now, the following fractional-order chaotic system is considered:

$$D^q x = Ax(t) + h(x(t)) \tag{1}$$

where $1 < q < 2$ is the fractional order, and $x(t) \in R^{n\times 1}$ is state vector. $A \in R^{n\times n}$ is one constant real matrix. $Ax(t) \in R^{n\times 1}$ and $h(x(t)) \in R^{n\times 1}$ are the linear part and nonlinear part in system (1), respectively.

In this paper, we only discuss a class of fractional-order chaotic systems which satisfy:

$$h(x(t)) - h(x'(t)) = H_l(x'(t))(x(t) - x'(t)) + H_n(x(t) - x'(t), x'(t)) \tag{2}$$

Here $x'(t) \in R^{n\times 1}$ is a real variable. $H_l(x'(t)) \in R^{n\times n}$ and $H_n(x(t)-x'(t), x'(t)) \in R^{n\times 1}$ are real matrices. $H_l(x'(t))(x(t)-x'(t))$ and $H_n(x(t)-x'(t), x'(t))$ are the linear part and nonlinear part with respect to $(x(t)-x'(t))$, respectively. In fact, the nonlinear part $h(x(t))$ in many fractional-order chaotic systems such as the fractional-order Lorenz chaotic system [5,6], Chua's fractional-order modified chaotic system [24], the fractional-order Duffing chaotic system [7], the fractional-order Rossler chaotic system [8,9], the fractional-order Chen chaotic system [6–8], *etc.*, all satisfy Equation (2).

Now, we study how to realize the projective synchronization for fractional-order chaotic system (1). Select the fractional-order chaotic system (1) as master system, and choose the following controlled fractional-order system as slave system:

$$D^q y(t) = Ay(t) + \alpha^{-1} h(\alpha y(t)) + u(x(t), y(t)) \tag{3}$$

where $\alpha \neq 0$ is a constant named scaling factor, and $u(x(t), y(t)) \in R^{n\times 1}$ is the real feedback controller which will be determined later.

**Definition.** *For the fractional-order chaotic master systems (1) and slave system (3), it is said to be projective synchronization if there exists a non-zero constant* α *such that* $\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|\alpha y(t) - x(t)\| = 0$ *where* $e(t) = (\alpha y(t) - x(t)) \in R^{n\times 1}$ *is the synchronization error. The symbol* $\|\bullet\|$ *represents the matrix norm.*

**Theorem.** *The real feedback controller in slave system (3) is chosen as* $u(x(t), y(t)) = [K - \alpha^{-1} H_l(x(t))]e(t)$. *It is said to be a projective synchronization between master system (1) and slave system (3) if there exists a real matrix K such that:*

(i) $H_n[e(t), x(t)]\big|_{e(t)=0} = 0$, $\lim_{e(t)\to 0}\dfrac{\|H_n[e(t), x(t)]\|}{\|e(t)\|} = 0$ for any $x(t)$,

(ii) $\mathrm{Re}[\lambda(A+\alpha K)] < 0$, $\omega = -\max[\mathrm{Re}\,\lambda(A+\alpha K)] > [\Gamma(q)]^{1/q}$.

*where* $h(\alpha y(t)) - h(x(t)) = H_l(x(t))e(t) + H_n[e(t), x(t)]$. ω *is the minimum absolute value of the real part of the eigenvalue of matrix* $(A+\alpha K)$.

**Proof.** According to $e(t) = \alpha y(t) - x(t)$, the error system between fractional-order system (1) and system (3) is described as:

$$D^q e(t) = D^q(\alpha y(t) - x(t)) = Ae(t) + h(\alpha y(t)) - h(x(t)) + \alpha u(x(t), y(t)) \tag{4}$$

Since $u(x(t), y(t)) = [K - \alpha^{-1} H_l(x(t))]e(t)$, the system (4) can be rewritten as:

$$D^q e(t) = Ae(t) + h(\alpha y(t)) - h(x(t)) + [\alpha K - H_l(x(t))]e(t) \tag{5}$$

By $h(\alpha y(t)) - h(x(t)) = H_l(x(t))e(t) + H_n[e(t), x(t)]$, the system (5) can be changed to:

$$D^q e(t) = (A+\alpha K)e(t) + H_n[e(t), x(t)] \tag{6}$$

Since $H_n[e(t), x(t)]\big|_{e(t)=0} = 0$ for any $x(t)$, hence $e(t) = 0$ is the zero solution in error system (6). Now, let $e_{10}$ and $e_{20}$ be the initial conditions for system (6), so the solution $e(t)$ for system (6) can be shown as:

$$e(t)=E_{q,1}[(A+\alpha K)t^q]e_{10}+tE_{q,2}[(A+\alpha K)t^q]e_{20}+\int_0^t (t-s)^{q-1}E_{q,q}[(A+\alpha K)(t-s)^q]H_n[e(s),x(s)]ds \tag{7}$$

where $E_{q,1}$, $E_{q,2}$ and $E_{q,q}$ are the two-parameter function of Mittag-Leffler type, *i.e.*, $E_{q,p}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(qn+p)}$ ($q>0$, $p>0$), and $z$ is a variable.

For the Mittag-Leffler function $E_{q,p}(z)$, the inequality (8) has been obtained in Reference [25]:

$$\left\|E_{q,p}[(A+\alpha K)t^q]\right\|\le\left\|e^{(A+\alpha K)t^q}\right\| \tag{8}$$

According to Equation (7) and inequality (8), one has:

$$\begin{aligned}\|e(t)\|&\le\left\|E_{q,1}[(A+\alpha K)t^q]e_{10}\right\|+\left\|tE_{q,2}[(A+\alpha K)t^q]e_{20}\right\|+\left\|\int_0^t (t-s)^{q-1}E_{q,q}[(A+\alpha K)(t-s)^q]H_n[e(s),x(s)]ds\right\|\\&\le\left\|e^{(A+\alpha K)t^q}e_{10}\right\|+\left\|e^{(A+\alpha K)t^q}e_{20}\right\|t+\int_0^t (t-s)^{q-1}\left\|e^{(A+\alpha K)(t-s)^q}H_n[e(s),x(s)]\right\|ds\end{aligned} \tag{9}$$

Since $\mathrm{Re}[\lambda(A+\alpha K)]<0$, therefore $(A+\alpha K)$ is a stability matrix. So, $\left\|e^{(A+\alpha K)t}\right\|\le N_0e^{-\omega t}$, and $\left\|e^{(A+\alpha K)t^q}\right\|\le N_0e^{-\omega t^q}\le N_0e^{-\omega t}$, here $N_0>0$ is a suitable constant.

Now, the inequality (9) can be transformed as:

$$\|e(t)\|\le N_0e^{-\omega t}\|e_{10}\|+N_0e^{-\omega t}\|e_{20}\|t+N_0\int_0^t (t-s)^{q-1}e^{-\omega(t-s)}\|H_n[e(s),x(s)\|ds \tag{10}$$

Due to $H_n[e(t),x(t)]\big|_{e(t)=0}=0$, $\lim_{e(t)\to 0}\frac{\|H_n[e(t),x(t)]\|}{\|e(t)\|}=0$ for any $x(t)$, so there exists a constant $\varepsilon>0$ such that $\|H_n[e(t),x(t)]\|\le\|e(t)\|/N_0$ as $\|e(t)\|<\varepsilon$.

So, the inequality (10) can be rewritten as:

$$\|e(t)\|\le N_0e^{-\omega t}\|e_{10}\|+N_0e^{-\omega t}\|e_{20}\|t+\int_0^t (t-s)^{q-1}e^{-\omega(t-s)}\|e(s)\|ds \tag{11}$$

That is:

$$\|e(t)\|e^{\omega t}\le N_0\|e_{10}\|+N_0\|e_{20}\|t+\int_0^t (t-s)^{q-1}e^{-\omega s}\|e(s)\|ds \tag{12}$$

According to the result in Reference [26], the inequality (12) can be changed as:

$$\|e(t)\|e^{\omega t}\le(N_0\|e_{10}\|+N_0\|e_{20}\|t)E_{q,1}[\Gamma(q)t^q] \tag{13}$$

Using the result in Reference [4]: $\left|E_{q,p}(z)\right|\le N_1(1+|z|)^{(1-p)/q}e^{\mathrm{Re}(z^{1/q})}+N_2(1+|z|)^{-1}$, where $N_i>0(i=1,2)$, $|z|\ge 0$, $\arg(z)\le\rho$ ,and $0.5\pi q<\rho<\min(\pi,\pi q)$, therefore, the inequality (13) can be changed to:

$$\|e(t)\|e^{\omega t}\le(N_0\|e_{10}\|+N_0\|e_{20}\|t)E_{q,1}[\Gamma(q)t^q]\le(N_0\|e_{10}\|+N_0\|e_{20}\|t)[N_1e^{t(\Gamma(q))^{1/q}}+N_2/(1+\Gamma(q)t^q)] \tag{14}$$

That is:

$$\begin{aligned}\|e(t)\| &\le (N_0\|e_{10}\| + N_0\|e_{20}\|t)[N_1 e^{t(\Gamma(q))^{1/q}} + N_2/(1+\Gamma(q)t^q)]e^{-\omega t} \\ &= (N_0\|e_{10}\| + N_0\|e_{20}\|t)N_1 e^{t[(\Gamma(q))^{1/q}-\omega]} + (N_0\|e_{10}\| + N_0\|e_{20}\|t)N_2/\{[1+\Gamma(q)t^q]e^{\omega t}\}\end{aligned} \tag{15}$$

Since $\omega = -\max[\mathrm{Re}\,\lambda(A+\alpha K)] > [\Gamma(q)]^{1/q}$, one has $\omega > 0$ and $[\Gamma(q)]^{1/q} - \omega < 0$. Therefore, $\lim_{t\to+\infty}(N_0\|e_{10}\| + N_0\|e_{20}\|t)N_1 e^{t[(\Gamma(q))^{1/q}-\omega]} = 0, \lim_{t\to+\infty}(N_0\|e_{10}\| + N_0\|e_{20}\|t)N_2/\{[1+\Gamma(q)t^q]e^{\omega t}\} = 0$

So:

$$\lim_{t\to+\infty}\|e(t)\| = 0 \tag{16}$$

Equation (16) indicates that the zero solution in error system (6) is asymptotically stable, so $\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|\alpha y(t) - x(t)\| = 0$. Hence, the projective synchronization between fractional-order chaotic system (1) and system (3) will be obtained. The proof is completed. □

We notice that some academics [27–29] have discussed complex dynamical networks with non-delayed and delayed coupling, the application of chaos synchronization to secure communication, and Takagi-Sugeno fuzzy systems with multiple state delays. Applying our results to these issues is our ongoing work.

## 3. Illustrative Example

To demonstrate the effectiveness of the projective synchronization method proposed in Section 2, we apply the synchronization scheme to the fractional-order Lorenz chaotic system with fractional-order $q = 1.1$ and the fractional-order modified Chua chaotic system with fractional-order $q = 1.02$, respectively.

### *3.1. Projective Synchronization of Fractional-Order Lorenz Chaotic System with* $1 < q < 2$.

The fractional-order Lorenz system [30] is given by:

$$\begin{pmatrix} D^{qx} x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} \sigma(x_2 - x_1) \\ \gamma x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \beta x_3 \end{pmatrix} \tag{17}$$

where $\sigma = 10$, $\gamma = 28$, and $\beta = 8/3$. The fractional-order Lorenz system (17) displays a chaotic attractor with $q = 1.1$. The chaotic attractor is shown in Figure 1.

In order to realize the projective synchronization for the fractional-order Lorenz chaotic system (17), the system (17) is selected as master system, so the slave system can be constructed as follows:

$$\begin{pmatrix} D^q y_1 \\ D^q y_2 \\ D^q y_3 \end{pmatrix} = \begin{pmatrix} \sigma(y_2 - y_1) \\ \gamma y_1 - y_2 \\ -\beta y_3 \end{pmatrix} + \alpha^{-1} h(\alpha y) + [K - \alpha^{-1} H_l(x)]e \tag{18}$$

where $e = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}}$, and $e_i = \alpha y_i - x_i (i = 1,2,3)$. $\mathbf{T}$ denotes the transposition for matrix:

Obviously, $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$, $h(y) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{pmatrix}$.

Use $h(\alpha y)-h(x)=H_l(x)e+H_n(e,x)$ , the matrix $H_l(x)$ and matrix $H_n(e,x)$ can be derived as follows:

$$H_l(x)=\begin{pmatrix} 0 & 0 & 0 \\ -x_3 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{pmatrix},\ H_n(e,x)=\begin{pmatrix} 0 \\ e_1e_3 \\ -e_1e_2 \end{pmatrix}$$

According to $u(x,y)=[K-\alpha^{-1}H_l(x)]e$, the controller $u(x,y)$ in slave system (18) is chosen as:

$$u(x,y)=\left(K-\alpha^{-1}\begin{pmatrix} 0 & 0 & 0 \\ -x_3 & 0 & -x_1 \\ x_2 & x & 0 \end{pmatrix}\right)e$$

Now, it is easy to verify the following:

$$H_n(e,x)\big|_{e=0}=\begin{pmatrix} 0 \\ e_1e_3 \\ -e_1e_2 \end{pmatrix}\Bigg|_{e=0}=0$$

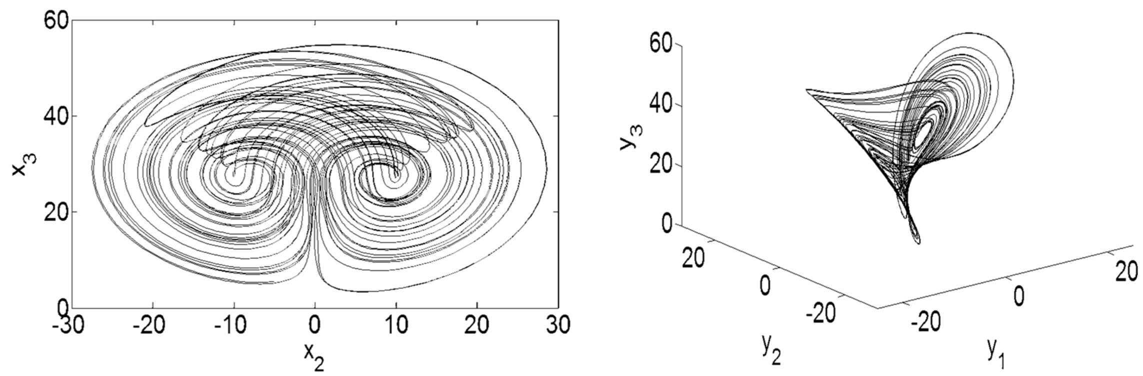
and:

$$\frac{\|H_n(e,x)\|}{\|e\|}=\frac{\sqrt{(e_1e_3)^2+(e_1e_2)^2}}{\sqrt{e_1{}^2+e_2{}^2+e_3{}^2}}\le\frac{\sqrt{(e_1e_3)^2+(e_1e_2)^2+(e_1^2)^2}}{\sqrt{e_1{}^2+e_2{}^2+e_3{}^2}}=|e_1|$$

Therefore:

$$H_n(e,x)\big|_{e=0}=0,\ \lim_{e\to 0}\frac{\|H_n(e,x)\|}{\|e\|}\le\lim_{e\to 0}|e_1|=0$$

The above results imply that the Condition (i) in the Theorem are satisfied.




(**a**) (**b**)

**Figure 1.** Chaotic attractor of system (17) with $q=1.1$. (**a**) Chaotic attractor in x2-x3 plane. (**b**) Chaotic attractor in x1-x2-x3 space.

Based on the Theorem in Section 2, a suitable non-zero constant $\alpha$ and constant real matrix $K$ can be selected such that $\mathrm{Re}[\lambda(A+\alpha K)]<0$ and $-\max[\mathrm{Re}\,\lambda(A+\alpha K)]>[\Gamma(q)]^{1/q}$ are held, so the projective synchronization between the fractional-order Lorenz chaotic system (17) and the controlled fractional-order Lorenz chaotic system (18) can be achieved.

For example, let $K = \begin{pmatrix} 0 & -10/3 & 0 \\ -28/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\alpha$ =3, respectively, so $\lambda_1(A+\alpha K) = -10$, $\lambda_2(A+\alpha K) = -1$, $\lambda_3(A+\alpha K) = -8/3$, and $-\max[\mathrm{Re}\,\lambda(A+\alpha K)] = 1 > [\Gamma(1.1)]^{1/1.1}$ =0.9557, respectively. Simulation results are shown in Figure 2, in which the initial conditions are $(x_{10}, x_{20}, x_{30})$ = (10, 20, 30), and $(y_{10}, y_{20}, y_{30})$ = (10, 20, 30), respectively. In Figures 2a–c, the solid lines refer to attractors of system (17), the dashed ones refer to attractors of system (18). Projective synchronization errors between system (17) and system (18) are shown in Figure 2d.

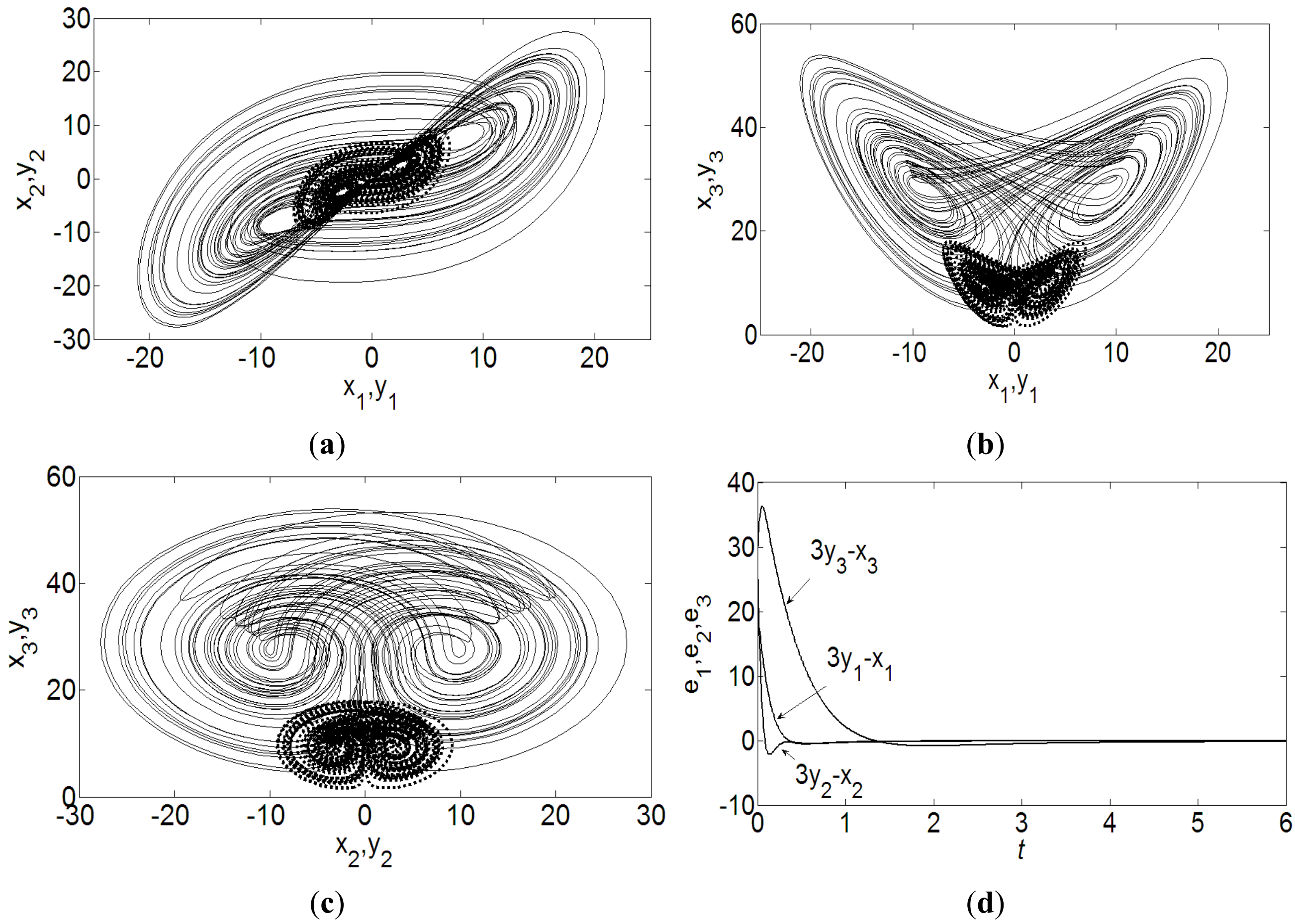



**Figure 2.** The PS result for fractional-order Lorenz chaotic system with $q = 1.1$ and $\alpha$ = 3. (**a**) The PS result in $x_1$-$x_2$ and $y_1$-$y_2$ plane; (**b**) The PS result in $x_1$-$x_3$ and $y_1$-$y_3$ plane; (**c**) The PS result in $x_2$-$x_3$ and $y_2$-$y_3$ plane; (**d**) The projective synchronization errors.

### *3.2. Projective Synchronization of Fractional-Order Modified Chua's Chaotic System with* $1 < q < 2$.

In 1971, Chua's chaotic circuit was discovered by Chua [30]. In 2010, Muthuswamy and Chua [24] reported the simplest modified Chua chaotic circuit, which consists of a linear passive inductor, a linear passive capacitor, and a non-linear active memristor. The simplest modified Chua chaotic circuit system [24] can be shown as:

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \begin{pmatrix} x_2 \\ -[x_1 + 1.7(x_3^2 - 1)x_2]/3.3 \\ -x_2 - 0.2x_3 + x_2 x_3 \end{pmatrix}$$

Its fractional-order system is named fractional-order modified Chua chaotic system, and it can be described as follows:

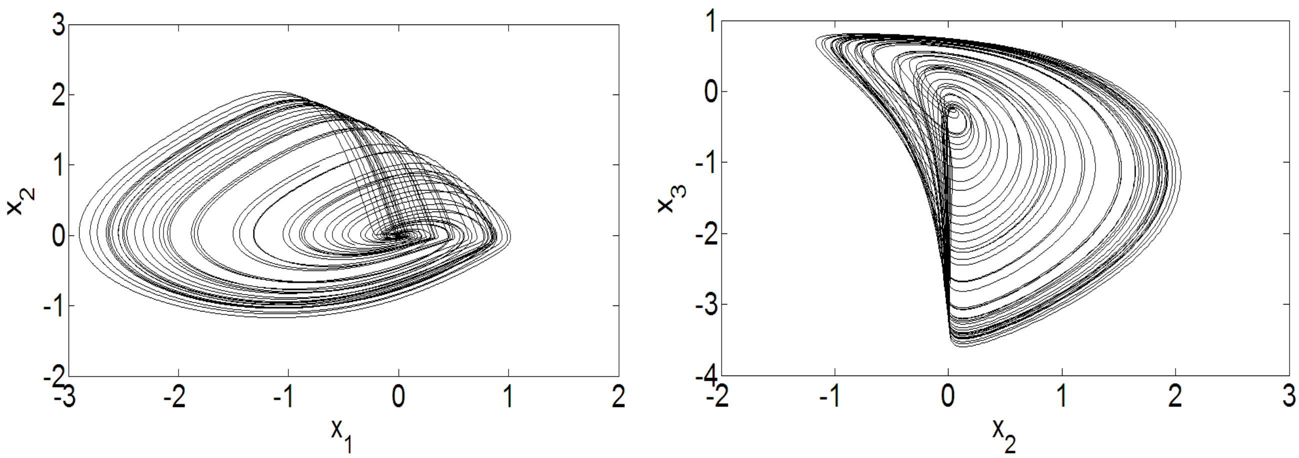
$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -[x_1 + 1.7(x_3^2 - 1)x_2]/3.3 \\ -x_2 - 0.2x_3 + x_2 x_3 \end{pmatrix} \tag{19}$$

The fractional-order modified Chua's system (19) displays a chaotic attractor with $q = 1.02$. The chaotic attractor is shown in Figure 3.

In order to realize the projective synchronization for the fractional-order chaotic system (19), the system (19) is chosen as master system, so the slave system can be constructed as follows:

$$\begin{pmatrix} D^q y_1 \\ D^q y_2 \\ D^q y_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ -(y_1 - 1.7y_2)/3.3 \\ -y_2 - 0.2y_3 \end{pmatrix} + \alpha^{-1} h(\alpha y) + [K - \alpha^{-1} H_l(x)]e \tag{20}$$

where $e = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}}$, and $e_i = \alpha y_i - x_i (i = 1,2,3)$.




(**a**) (**b**)

**Figure 3.** Chaotic attractor of the fractional-order modified Chua system (19) with $q = 1.02$. (**a**) Chaotic attractor in $x_1$-$x_2$ plane; (**b**) Chaotic attractor in $x_2$-$x_3$ plane.

Obviously, $A = \begin{pmatrix} 0 & 1 & 0 \\ -10/33 & 17/33 & 0 \\ 0 & -1 & -0.2 \end{pmatrix}, h(y) = \begin{pmatrix} 0 \\ -17/33 y_2 y_3^2 \\ y_2 y_3 \end{pmatrix}$.

Use $h(\alpha y) - h(x) = H_l(x)e + H_n(e, x)$, the matrix $H_l(x)$ and matrix $H_n(e, x)$ can be derived as follows:

$$H_l(x)=\begin{pmatrix}0 & 0 & 0\\ 0 & -17x_3^2/33 & -34x_2x_3/33\\ 0 & -x_3 & -x_2\end{pmatrix},\ H_n(e,x)=\begin{pmatrix}0\\ 17[2e_2e_3x_3+(x_2-e_2)e_3^2]/33\\ e_2e_3\end{pmatrix}$$

According to $u(x,y)=[K-\alpha^{-1}H_l(x)]e$, the controller $u(x,y)$ in slave system (20) is chosen as:

$$u(x,y)=\left(K-\alpha^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & -17x_3^2/33 & -34x_2x_3/33\\ 0 & -x_3 & -x_2\end{pmatrix}\right)e$$

Now, it is easy to verify the following:

$$H_n(e,x)\big|_{e=0}=\left.\begin{pmatrix}0\\ 17[2e_2e_3x_3+(x_2-e_2)e_3^2]/33\\ e_2e_3\end{pmatrix}\right|_{e=0}=0$$

and:

$$\frac{\|H_n(e,x)\|}{\|e\|}=\frac{\sqrt{\{1.7[2e_2e_3x_3+(x_2-e_2)e_3^2]/3.3\}^2+(e_2e_3)^2}}{\sqrt{e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}}}$$
$$\le\frac{\sqrt{[2e_2e_3x_3+(x_2-e_2)e_3^2]^2+(e_2e_3)^2}}{\sqrt{e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}}}\le\frac{\sqrt{(|2e_2e_3x_3|+|(x_2-e_2)|e_3^2)^2+(e_2e_3)^2}}{\sqrt{e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}}}$$

According to the boundedness of modified Chua's chaotic system, there exist a real positive constant $M$ such that $M\ge\max\left(2|x_3|,|x_2-e_2|=|\alpha y_2|\right)$. Symbol max is the maximum value.

So:

$$\frac{\|H_n(e,x)\|}{\|e\|}\le\frac{\sqrt{(|e_2e_3|+e_3^2)^2M^2+(e_2e_3)^2}}{\sqrt{e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}}}\le\sqrt{\frac{(|e_2e_3|+e_3^2)^2M^2+(e_2e_3)^2}{e_3^{\ 2}}}=\sqrt{(|e_2|+e_3)^2M^2+e_2^{\ 2}}$$

Therefore:

$$H_n(e,x)\big|_{e=0}=0,\ \lim_{e\to 0}\frac{\|H_n(e,x)\|}{\|e\|}\le\lim_{e\to 0}\sqrt{(|e_2|+e_3)^2M^2+e_2^{\ 2}}=0$$

The above results imply that the Conditions (i) in the Theorem are satisfied.

Based on the above Theorem in Section 2, a suitable non-zero constant $\alpha=-0.5$ and constant real matrix $K$ can be selected such that $\mathrm{Re}[\lambda(A+\alpha K)]<0$ and $-\max[\mathrm{Re}\,\lambda(A+K)]>[\Gamma(q)]^{1/q}\alpha$ hold, so the projective synchronization between the fractional-order modified Chua chaotic system (19) and the controlled fractional-order modified Chua chaotic system (20) can be achieved.

For example, choose $K=\begin{pmatrix}4 & 0 & 0\\ 46/33 & 166/33 & 0\\ 0 & -2 & 4\end{pmatrix}$, and $\alpha=-0.5$, respectively. So, $\lambda_\pm(A+\alpha K)=-2\pm i$, $\lambda_3(A+\alpha K)=-2.2$, and $-\max[\mathrm{Re}\,\lambda(A+\alpha K)]=2>[\Gamma(1.02)]^{1/1.02}$ = 0.9891, respectively. The numerical result is shown in Figure 4, in which the initial conditions are $(x_{10},x_{20},x_{30})$ = (0.1, 0, 0.1), and $(y_{10},y_{20},y_{30})$ = (−1, 2, 1), respectively.

In Figures 4a–c, the solid lines refer to attractors of system (19), and the dashed lines refer to attractors of system (20), respectively. Projective synchronization errors between system (19) and system (20) are shown in Figure 4d.

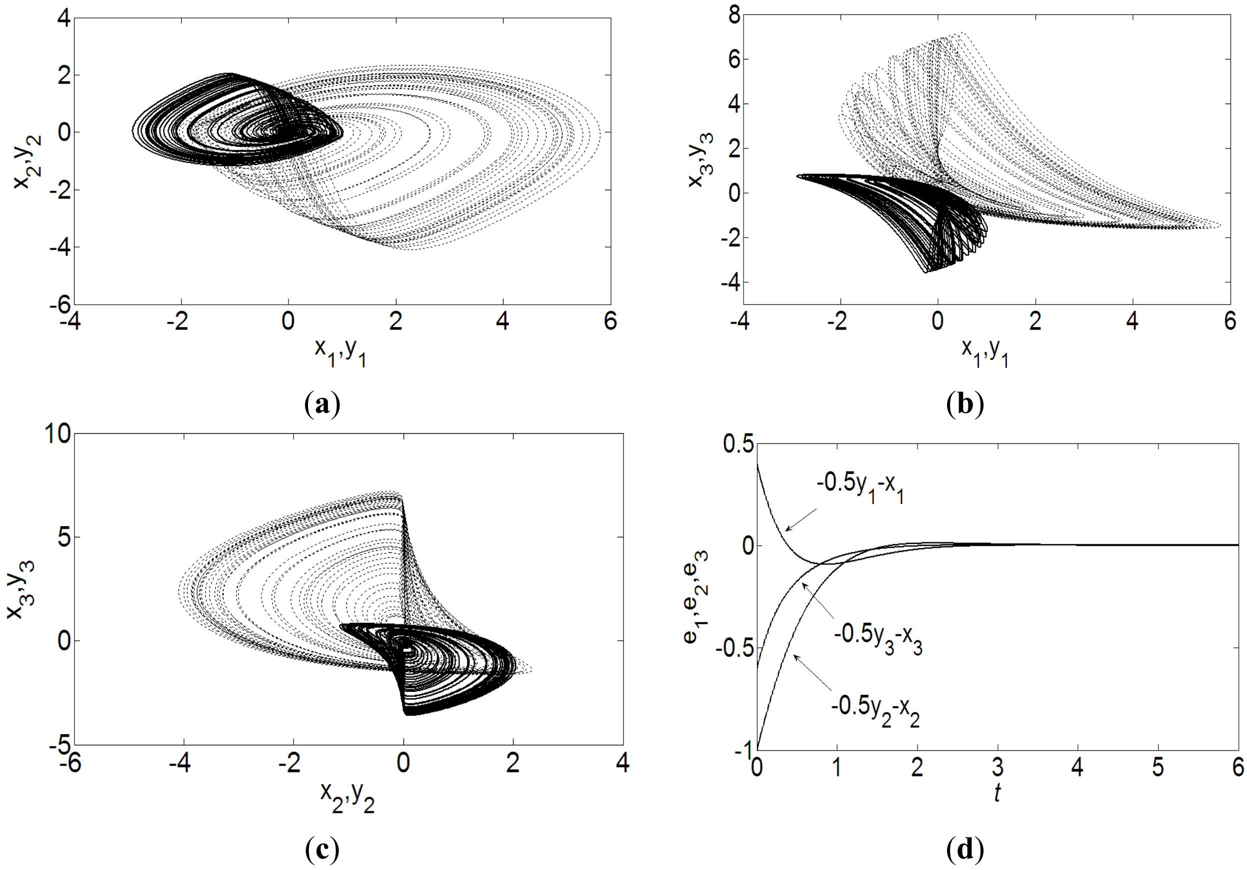



**Figure 4.** The PS result for fractional-order modified Chua's chaotic system with $q = 1.02$ and $\alpha = -0.5$. (**a**) The PS result in $x_1$-$x_2$ and $y_1$-$y_2$ plane; (**b**) The PS result in $x_1$-$x_3$ and $y_1$-$y_3$ plane; (**c**) The PS result in $x_2$-$x_3$ and $y_2$-$y_3$ plane; (**d**) The projective synchronization errors.

## 4. Conclusions

In this paper, a projective synchronization approach is proposed for a class of fractional-order chaotic system with $1 < q < 2$. Our approach can be applied to a class of nonlinear fractional-order chaotic systems, in which the nonlinear terms in the chaotic system satisfy Equation (2). To demonstrate the effectiveness of proposed projective synchronization scheme, we apply the synchronization scheme to the fractional-order Lorenz chaotic system with $q = 1.1$ and Chua's fractional-order modified chaotic system with $q = 1.02$, respectively. The numerical simulations show the validity and feasibility of the proposed scheme.

## Author Contributions

Ping Zhou proposed and designed the research; Rongji Bai performed the simulations; Ping Zhou, Rongji Bai analyzed the simulation results; Ping Zhou, Rongji Bai and Jiming Zheng wrote the paper. All authors have read and approved the final manuscript.

## Conflicts of Interest

The authors declare no conflict of interest.

## References


1. Chen, L.P.; He, Y.G.; Chai, Y.; Wu, R.C. New results on stability and stabilization of a class of nonlinear fractional-order systems. *Nonlinear Dyn*. **2014**, *75*, 633–641.
2. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, Singapore, 2000.
3. Naber, M. Time fractional Schrodinger equation. *J. Math. Phys*. **2004**, *45*, 3339–3352.
4. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
5. Grigorenko, I.; Grigorenko, E. Chaotic dynamics of the fractional Lorenz system. *Phys. Rev. Lett*. **2003**, *91*, 034101–034104.
6. Zhou, P.; Ding, R.; Cao, Y.X. Multi drive-one response synchronization of fractional-order chaotic systems. *Nonlinear Dyn*. **2012**, *67*, 1263–1271.
7. Xiao, M.; Zheng, W.X.; Cao, J.D. Approximate expressions of a fractional order Van der Pol oscillator by the residue harmonic balance method. *Math. Comput. Simul*. **2013**, *89*, 1–12.
8. Zhou, P.; Yang, F.Y. Hyperchaos, chaos, and horseshoe in a 4D nonlinear system with an infinite number of equilibrium points. *Nonlinear Dyn*. **2014**, *76*, 473–480.
9. Liu, J. Complex modified hybrid projective synchronization of different dimensional fractional-order complex chaos and real hyper-chaos. *Entropy* **2014**, *16*, 6195–6211.
10. Xi, H.L.; Li, Y.X.; Huang, X. Generation and nonlinear dynamical analyses of fractional-order memristor-based Lorenz Systems. *Entropy* **2014**, *16*, 6240–6253.
11. Chen, G.R.; Dong, X. *From Chaos to Order Perspectives, Methodologies and Applications*; World Scientific: Singapore, Singapore, 1998.
12. Sprott, J.C. *Chaos and Time-Series Analysis*; Oxford University Press: Oxford, UK, 2003.
13. Mainieri, R.; Rehacek, J. Projective synchronization in three-dimensional chaotic systems. *Phys. Rev. Lett*. **1999**, *82*, 3042–3045.
14. Zhou, P.; Zhu, W. Function projective synchronization for fractional-order chaotic systems. *Nonlinear Anal. RWA* **2011**, *12*, 811–816.
15. Wang, X.Y.; Song, J.M. Synchronization of the fractional order hyperchaos Lorenz systems with activation feedback control. *Commun. Nonlinear Sci. Numer. Simul*. **2009**, *14*, 3351–3357.
16. Zhang, R.X.; Yang, S.P. Adaptive synchronization of fractional-order chaotic systems via a single driving variable. *Nonlinear Dyn*. **2011**, *66*, 831–837.
17. Zhang, W.; Cai, X.; Holm, S. Time-fractional heat equations and negative absolute temperatures. *Comput. Math. Appl*. **2014**, *67*, 164–171.
18. Beghin, L.; Orsingher, E. The telegraph process stopped at stable-distributed times and its connection with the fractional telegraph equation. *Fract. Calc. Appl. Anal*. **2003**, *6*, 187–204.
19. Gafiychuk, V.; Datsko, B. Mathematical modeling of different types of instabilities in time fractional reaction-diffusion systems. *Comput. Math. Appl*. **2010**, *59*, 1101–1107.
20. Mainardi, F. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett*. **1996**, *9*, 23–28.

21. Gorenflo, R.; Mainardi, F.; Moretti, D.; Pagnini, G.; Paradisi, P. Fractional diffusion: Probability distributions and random walk models. *Physica A* **2002**, *305*, 106–112.
22. Li, C.P.; Zhao, Z.G.; Chen, Y.Q. Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion. *Comput. Math. Appl*. **2011**, *62*, 855–875.
23. Ge, Z.M.; Jhuang, W.R. Chaos, control and synchronization of fractional order rotational mechanical system with a centrifugal governor. *Chaos Solitons Fractals* **2007**, *33*, 270–289.
24. Muthuswamy, B.; Chua, L.O. One simplest chaotic circuit. *Int. J. Bifurc. Chaos* **2010**, *20*, 1567–8150.
25. De la Sen, M. About robust stability of Caputo linear fractional dynamics systems with time delays through fixed theory. *Fixed Point Theory A*. **2011**, doi:10.1155/2011/867932.
26. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional equation. *J. Math. Anal. Appl*. **2007**, *328*, 1075–1081.
27. Tang, J.N.; Huang, C.T. Impulsive control and synchronization analysis of complex dynamical networks with non-delayed and delayed coupling. *Int. J. Innov. Comput. Inf. Control* **2013**, *9*, 4555–4564.
28. Smaoui, N.; Karouma, A.; Zribi, M. Adaptive synchronization of hyperchaotic Chen systems with spplication to secure sommunication. *Int. J. Innov. Comput. Inf. Control* **2013**, *9*, 1127–1144.
29. Li, F.; Shi, P.; Wu, L.; Zhang, X. Fuzzy-model-based D-stability and non-fragile control for discrete-time descriptor systems with multiple delays. *IEEE Trans. Fuzzy Syst*. **2014**, *22*, 1019–1025.
30. Chua, L.O. Memristor—the missing circuit element. *IEEE Trans. Circuit Theory* **1971**, *18*, 507–519.