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Article

Thermodynamics in Curved Space-Time and Its Application to Holography

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Abstract: The thermodynamic behaviors of a system living in a curved space-time are different from those of a system in a flat space-time. We have investigated the thermodynamics for a system consisting of relativistic massless bosons. We show that a strongly curved metric will produce a large enhancement of the degrees of freedom in the formulae of energy and entropy of the system, as a comparison to the case in a flat space-time. We are mainly concerned with its implications to holography, including the derivations of holographic entropy and holographic screen.

Keywords: black hole; holographic principle; entropy bounds

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1. Introduction

Inspired by the area scaling of the black hole entropy and the generalized second law of thermodynamics, it is conjectured that the maximum entropy contained in a system can not exceed its boundary area in Planck units [1–3]. The conjecture afterwards is called holographic principle. The entropy with an area scaling is usually called holographic entropy.

The microscopic origin of holographic entropy has always been a question to be answered. In the original paper of holographic pripnle, t' Hooft [1] has studied the maximum entropy that can be provided by an ordinary quantum field theory. Assuming the system is at some temperature T, the corresponding energy and entropy of the system can be evaluated as

$$\begin{aligned} E =& c_1 z V T^4, \\ S =& c_2 z V T^3, \end{aligned} \tag{1}$$

where z is explained as the number of different particle species, c_1, c_2 are numerical constants of order one which depend on the concrete properties of the particles. Such thermodynamic behaviors can easily be found by dimensional analysis. For the system to be gravitational stable, we must require $E = c_1 z V T^4 \leq E_{bh} \sim L$ which means the energy of the system should be less than the energy of a black hole of the same size. This leads to the critical temperature $T \leq c_3 z^{-1/4} L^{-1/2}$. Substituting it into the formula of entropy, one gets an entropy bound for conventional field theory

$$S \leq c_4 z^{1/4} A^{3/4}, \tag{2}$$

where A is the boundary area of the system. We have adopted the Planck units where $G = \hbar = c = k_B = 1$ in order to simplify the expressions. If the number of particle species z is an unimportant constant, the maximum entropy will be $\left(A/l_p^2\right)^{3/4}$ by recovering units, where l_p is the Planck length. Thus there is an entropy gap between the $A^{3/4}$ entropy bound for conventional filed theory and the holographic entropy [4–8]. The result is also verified by a direct examination of gravitational stable Hilbert space for bosonic and fermionic fields [6].

To fill the entropy gap from $A^{3/4}$ to A , one must resort to other mechanisms for extra degrees of freedom (DOFs). A fast observation is that, if one assumes a large number of particle species z, the entropy gap can be filled continuously. Especially, assuming $z \sim L^2/l_p^2$, the setting of $E = E_{bh} \sim L$ leads to the temperature $T = c_3 z^{-1/4} L^{-1/2} \sim 1/L$ and the entropy

$$S = c_4 z^{1/4} A^{3/4} \sim A. \tag{3}$$

The scaling behaviors exactly coincide with those of black hole thermodynamics. Obviously, the large z plays an essential role in the derivation. This scheme of introducing a large number of particle species for overcoming the entropy gap is first suggested in [9] by Horvat. It is amazing that the entire holographic thermodynamics can be derived in such a straightforward way. Whatever, in this context, explaining z as the number of particle species may lead to some problems. First, it is hard to explain the origin of such a large number of particle species which are absent in standard model of particle physics. Second, it is hard to answer the question why the number of particle species z is related to the size of the system.

On the other hand, as suggested by Hsu [8], one should consider the influence of space-time metric in order to fill the entropy gap. The $A^{3/4}$ entropy bound may be only applicable to weak-gravitational systems, because the original derivation only takes $E \leq E_{bh}$ as a global limitation to the system. When the system has such a large energy, the dominant self-gravitational interaction will lead to a strongly curved space-time metric. So a complete derivation should use the thermodynamics in a curved space-time rather than directly using the results in the Minkowski space-time.

In this paper, we study the thermodynamics for a system living in a curved space-time using a formulism originated and developed in [10,11]. From our result, the large value of z in Equation (3) can be seen as an effect of space-time metric. Notice that the parameter z will not be explained as the number of particle species from now on, because we shall start from a system consisting of only one kind of particles. Strong gravitational field naturally induces a large enhancement of the DOFs in the formulae of energy and entropy for the thermodynamics of the system, which is reflected by the parameter z. Hence we can understand the successful filling-up of the entropy gap in Equation (3) because it has considered gravitational effect in an effective way. We shall also analyze the important role of a large z in searching for the holographic screen for a black hole.

2. Thermodynamics in Static Curved Space-Time

In this section, we shall study the thermodynamics of a system living in a static space-time with the aim of seeing how a large z can be produced in a strong gravitational region.

First, we count the number of quantum states for a particle to occupy when it is put in a curved space-time. In a static space-time, the energy of a particle can be defined as $E = \xi_\mu k^\mu$, where $\xi^\mu = (1, \vec{0})$ is the killing vector of the space-time and k^μ is the four momentum of the particle. Using the fact that $p_\mu p^\mu = m^2$, the energy for a single particle is calculated as $E = \sqrt{g_{00}}(\gamma^{\alpha\beta}p_\alpha p_\beta + m^2)^{\frac{1}{2}}$, where $\gamma_{\alpha\beta}$ is the spatial part of the metric. We also verify the energy formula from the lagrangian $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ combined with a Legendre transformation. According to the uncertainty principle, the number of possible single-particle states with energy less than w is

$$P(w) = \int \Theta\left(w - \sqrt{g_{00}}(\gamma^{\alpha\beta}p_\alpha p_\beta + m^2)^{\frac{1}{2}}\right) \frac{dx^\alpha dp_\alpha}{(2\pi)^3}, \tag{4}$$

where $dx^\alpha dp_\alpha$ is the invariant phase space element and the Heaviside step function $\Theta(x)$ is used. Accordingly, the momentum integral requires to count all the momenta satisfying $\gamma^{\alpha\beta}p_\alpha p_\beta \leq w^2/g_{00} - m^2$ and gives a volume $\frac{4\pi}{3}\sqrt{\gamma}(w^2/g_{00} - m^2)^{3/2}$ in the momentum space [10]. It leads to

$$P(w) = \frac{1}{(2\pi)^3}\int \sqrt{\gamma}d^3x\frac{4\pi}{3}(\frac{w^2}{g_{00}} - m^2)^{3/2}. \tag{5}$$

With Equation (5) in hand, we can study the thermodynamic behaviors for a system living in a curved space-time. For simplicity, we consider an idea gas consisting of one kind of relativistic massless bosons with $m = 0$. The number of possible polarizations is also set to be 1. Then the number of quantum states with energy between w to $w + dw$ is

$$g(w)dw = \frac{\partial P(w)}{\partial w}dw = \frac{1}{(2\pi)^3}\int \sqrt{\gamma}d^3x\frac{4\pi w^2 dw}{(\sqrt{g_{00}})^3}. \tag{6}$$

At the temperature $T = 1/\beta$, the particles are assigned to the quantum states according to the Bose-Einstein distribution. Then we get the energy of the system as

$$\begin{aligned} U &= \int \frac{w}{e^{\beta w} - 1}g(w)dw \\ &= \frac{1}{(2\pi)^3}\int \sqrt{\gamma}d^3x \int \frac{w}{e^{\beta w} - 1}\frac{4\pi w^2 dw}{(\sqrt{g_{00}})^3} \\ &= c_1 T^4 \int \frac{\sqrt{\gamma}}{(\sqrt{g_{00}})^3}d^3x, \end{aligned} \tag{7}$$

where $c_1 = \pi^2/30$. The temperature and energy of the system are measured by the Killing observer of the space-time, which can be seen from the definition of the single-particle energy. The entropy of the system is

$$
\begin{aligned}
S &= ln\Xi + \beta U = -\int \ln(1-e^{-\beta w})g(w)dw + \beta U \\
&= c_2 T^3 \int \frac{\sqrt{\gamma}}{(\sqrt{g_{00}})^3} d^3x,
\end{aligned}
\tag{8}
$$

where $c_2 = 2\pi^2/45$, and Ξ is the partition function of the thermodynamic system [12]. Obviously, the influence of the metric is reflected by the non-trivial factor $\sqrt{\gamma}/(\sqrt{g_{00}})^3$.

In order to see clearly how the corresponding thermodynamic behaviors can be changed in a strongly curved space-time, an example will be useful. Consider a thermodynamic system which is subject to an external space-time metric, e.g., the Schwarzschild metric

$$
ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1-\frac{2M}{r}} dr^2 - r^2 d\Omega. \tag{9}
$$

For convenience of comparison, after substituting the metric into Equations (7) and (8), the formulae of energy and entropy can be recast into the form

$$
\begin{aligned}
U &= c_1 z V T^4, \\
S &= c_2 z V T^3,
\end{aligned}
\tag{10}
$$

where $V = \int \sqrt{\gamma} d^3x$ is the spatial volume of the system, and z is defined as

$$
z = \frac{\int \sqrt{\gamma}/(\sqrt{g_{00}})^3 d^3x}{V} = \frac{\int \frac{1}{(1-\frac{2M}{r})^2} r^2 dr d\Omega}{\int \frac{1}{\sqrt{1-\frac{2M}{r}}} r^2 dr d\Omega}. \tag{11}
$$

The integral above is over the region occupied by the thermodynamic system. When the system is far from the black hole, $z \simeq 1$, so we recover the familiar thermodynamics for a bosonic system consisting of relativistic massless particles like photon gas in a flat space-time, which is $U = c_1 V T^4$ and $S = c_2 V T^3$. When the system lives in a strong gravitational region close to the black hole, a large number z emerges and changes the thermodynamic behaviors of the system. It is extremely noticeable that z diverges when one edge of the system touches the event horizon, so one must choose some cutoff near the horizon.

We have obtained our main result that the large z in Equation (10) includes certain effects of the curved space-time. It explains the successful derivation of the holographic entropy in Equation (3) by introducing a large number of particle species [9], since the relevant influence of space-time metric is included effectively. As we argued in the introduction part, directly inserting a very large number z in Equation (3) and explaining it as the number of particle species are questionable. By comparison, due to our formulism, the emergence of z in Equation (10) is a deduced result rather than an assumption. In fact, to strictly derive the holographic thermodynamics, one should consider a large self-gravitational system, use a metric generated by itself and construct the thermodynamics based on the metric. The details could be very complex. Interestingly, the thermodynamics with a large z may be taken as an effective theory for such a complex system, and the derivation of the holographic thermodynamics becomes possible due to the large z.

For clarity, here we provide a detailed discussion on the thermodynamics of a system in a black hole background and show how the energy and entropy of the system change when it is lowering to the horizon of the black hole. We shall discuss two distinct processes, which are respectively an isothermal process and an adiabatic process. First is the isothermal process. Our result (10) shows a continuous increase of the energy and entropy when the system approaches the black hole. To be precise, consider a small box of photon gas and lower the system slowly to the horizon by means of a string. To maintain the temperature observed by a distinct observer, the system must be contact with a thermal reservoir and absorbs heat to compensate the gravitational redshift of the temperature in the lowering process. One can easily expect a continuous increase of the entropy of the system because of the absorption of heat. In contrast, the thermodynamic behaviors of the system will be very different when the system is lowering adiabatically to the horizon along with an extraction of work to the outside through the string attached to it. The increase of the value of z will not cause any increase of the entropy in the lowering process. Because $z = \frac{\int \sqrt{\gamma}/(\sqrt{g_{00}})^3 d^3x}{\int \sqrt{\gamma} d^3x} \sim 1/(\sqrt{g_{00}})^3$, its effect is offset by the redshift of T^3 and consequently the entropy is left unchanged. Actually according to Equation (10) there is always $S \sim U/T$. The entropy is unchanged because of the simultaneous redshift of the energy and the temperature.

3. Thermodynamic Spheres and Holography

In this section, we still use the Schwarzschild metric (9) but turn to analyze the thermodynamic behaviors of 2-dimensional systems living on the spheres surrounding the black hole. There have been some works on this kind of topic in the literature. In [13] the thermodynamic properties and the Bose-Einstein condensations on stretched horizons are extensively studied. In [14] the thermodynamics for quantum photons living on astrophysical spheres is constructed. We will still focus on the effect of the curved space-time metric on the thermodynamics of the system and point out that the induced large z always plays an important role in the derivation of a holographic screen.

The formulism above can be easily generalized to a 2-dimensional system living in a curved space-time. Consider an idea gas of relativistic massless bosons confined in the system. The number of quantum states with energy between w and $w + dw$ will be

$$g(w)dw = \frac{1}{(2\pi)^2} \int \sqrt{\gamma_2} d^2x \frac{2\pi w dw}{(\sqrt{g_{00}})^2}. \tag{12}$$

At the temperature $T = 1/\beta$, using the Bose-Einstein distribution, the energy of the system is

$$\begin{aligned} U &= \int \frac{w}{e^{\beta w} - 1} g(w)dw \\ &= \frac{1}{(2\pi)^2} \int \sqrt{\gamma_2} d^2x \frac{w}{e^{\beta w} - 1} \frac{2\pi w dw}{(\sqrt{g_{00}})^2}, \\ &= c_1 T^3 \int \frac{\sqrt{\gamma_2}}{g_{00}} d^2x, \end{aligned} \tag{13}$$

where $c_1 = \zeta(3)/\pi \simeq 1.202/\pi$. Notice the energy distribution $\frac{1}{g_{00}}\frac{w^2}{e^{\beta w}-1}dw$ was obtained in [14] by a standard quantization procedure for the photons on spheres, which verifies the present formulism we adopt. The corresponding entropy is

$$\begin{aligned} S &= ln\Xi + \beta U = -\int \ln(1-e^{-\beta w})g(w)dw + \beta U \\ &= c_2 T^2 \int \frac{\sqrt{\gamma_2}}{g_{00}} d^2x, \end{aligned} \tag{14}$$

where $c_2 = 3\zeta(3)/(2\pi) \simeq 1.803/\pi$.

On the sphere of some radius R which surrounds the Schwarzschild black hole, the term $g_{00}(R)$ is just a constant, so the formulae (13) and (14) can be directly calculated as

$$U = c_1 z A_R T^3, \quad S = c_2 z A_R T^2, \tag{15}$$

with

$$A_R = \int \sqrt{\gamma_2} d^2x = \int R^2 \sin\theta d\theta d\varphi = 4\pi R^2, \tag{16}$$

$$z = 1/g_{00}(R) = 1/(1-\frac{2M}{R}). \tag{17}$$

Now A_R is the volume of the 2-dimensional system and z reflects the effect of the curved space-time metric. As in the 3-dimensional case, the emergence of z is a deduced result rather than an assumption. When the sphere is far from the black hole, $z \to 1$, we recover the conventional thermodynamics $U = c_1 A_R T^3, S = c_2 A_R T^2$ for a 2-dimensional bosonic gas in a flat space-time. In contrast, when the sphere is located in a strong gravitational region close to the horizon of the black hole, we get a very large z in Equation (15).

Here the purpose is to study the idea of holographic screen and see which sphere can be seen as the holographic screen for the black hole. "Holographic screen" means all the information contained in the bulk can be encoded on such a screen. Usually, one may prefer the event horizon at $R_h = 2M$ as the holographic screen for the black hole. However, on the event horizon $g_{00} = 0$ leads to an infinite state density $g(w)dw$ and further causes an embarrassing divergent thermodynamic behaviors according to Equations (13) and (14). So we should use the stretched horizon, which is located at a Planck proper distance away from the event horizon, as suggested in [11,13,15]. The concept of stretched horizon is a useful idea in the discussion of black hole thermodynamics [16]. Because of quantum uncertainty, one can not distinguish between a physical event on the stretched horizon and the one on the event horizon. The radius R of the stretched horizon can be calculated from the requirement

$$l_p = \int_{R_h}^{R} \frac{1}{\sqrt{1-\frac{2M}{r}}} dr \simeq 2\sqrt{R_h(R-R_h)}. \tag{18}$$

It gives $R = R_h + l_p^2/(4R_h)$. From Equation (17), there is $z = 1/g_{00}(R) = 4R_h^2/l_p^2$ on the stretched horizon. If we require the corresponding 2-dimensional system has energy $E = E_{bh} \sim R$, the thermodynamics (15) immediately leads to

$$T \sim 1/R, \quad S \sim R^2, \tag{19}$$

which captures the familiar scaling behaviors of the black hole thermodynamics. Thus the stretched horizon can surely be taken as the holographic screen for the black hole and the DOFs of the black hole can be viewed as living on the stretched horizon. Notice that the large z still plays an important role in the derivation of the holographic screen. Besides, the locally measured temperature at the stretched horizon is calculated as $T_{loc} = T/\sqrt{g_{00}(R)} \sim T_p$, which is of the order of the Planck temperature. It justifies the choice of the location of the stretched horizon, because if we put the stretched horizon closer to the black hole, we will get a higher locally measured temperature which is physically unacceptable.

4. Conclusions

We have analyzed the thermodynamics of a system consisting of relativistic massless bosons which lives in a static space-time. We find that a parameter z enters the formulae of energy and entropy of the system, which includes certain effects of the curved space-time. When the system is in a flat space-time, there is $z = 1$, then we retrieve the conventional thermodynamics. When the system is in a strong gravitational region, z becomes a very large number. The non-trivial z greatly changes the thermodynamic behaviors of the system and plays an essential role in the derivation of the holographic entropy and holographic screen. Actually, our result indicates an important fact that, for two systems with the same temperature T (from the point of view of a Killing observer), the system located in a more curved space-time contains a larger entropy density.

Finally, we want to make two further comments. First, the thermodynamics (10) in a curved space-time has been recast into the same form as the thermodynamics (1) with a large number of particle species. It is tempting to give this a physical explanation, even though it is a vague one. As we have shown in Equation (6), gravitational field expands the single-particle state space by a factor $\sqrt{\gamma}/(\sqrt{g_{00}})^3$. Roughly speaking, it implies that the microstates provide $\sqrt{\gamma}/(\sqrt{g_{00}})^3$ distinct "copies" or "species" for the particles to occupy, compared to the situation in a flat space-time. Second, in order to reach the holographic entropy in Equation (3), the number of particle species must be taken as L^2/l_p^2. At the same time, the total number of particles in the system is calculated as $N = E_{bh}/T \sim L^2/l_p^2$. The two quantities are of the same order of magnitude. This implies that all the particles are distinguishable from each other by their species and this kind of model is equivalent to a model based on distinguishable particles. In fact, it is aware very early that holographic thermodynamics can be derived from distinguishable particles. Banks et al. have noticed that bosonic and fermionic systems do not have enough DOFs for the holographic entropy and turn to construct a black hole model from Matrix theory which consists of a Bolztmann gas of $D0$ branes [17,18]. Moreover, in loop quantum gravity, the holographic entropy can be derived by counting the DOFs on the isolated horizon of a black hole, which consists of horizon punctures pierced by the edges of spin network. In [19] these punctures are confirmed to be distinguishable and the corresponding Gibbs paradox is analyzed. Moreover, it is proved in [20,21] that infinite statistics (quantum Boltzmann statistics) saturates the holographic entropy bound. Our results may shed light on the understanding of those works. It seems that gravitational field plays a role in causing the particle distinguishability at least effectively and it is worthy of further study.

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Author Contributions

Yong Xiao proposed the main idea of the paper. The authors together proceeded and finished the research. Li-Hua Feng wrote the paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. 't Hooft, G. Dimensional reduction in quantum gravity. **1993**, arXiv:gr-qc/9310026.
2. Susskind, L. The world as a hologram. *J. Math. Phys.* **1995**, *36*, 6377–6396.
3. Bousso, R. The holographic principle. *Rev. Mod. Phys.* **2002**, *74*, 825–874.
4. Cohen, A.G.; Kaplan, D.B.; Nelson, A.E. Effective field theory, black holes, and the cosmological constant. *Phys. Rev. Lett.* **82**, 4971–4974.
5. Buniy, R.V.; Hsu, S.D.H. Entanglement entropy, black holes and holography. *Phys. Lett. B* **2007**, *644*, 72–76.
6. Chen, Y.X.; Xiao, Y. The entropy bound for local quantum field theory. *Phys. Lett. B* **2008**, *662*, 71–74.
7. Barrow, J.D. Entropic Principles. *New Astron.* **1999**, *4*, 333–338.
8. Hsu, S.D.H.; Reeb, D. Black hole entropy, curved space and monsters. *Phys. Lett. B* **2008**, *658*, 244–248.
9. Horvat, R. Effective field theory, large number of particle species, and holography. *Phys. Lett. B* **2009**, *674*, 1–3.
10. Padmanabhan, T. Phase volume occupied by a test particle around an incipient black hole. *Phys. Lett. A* **1989**, *136*, 203–205.
11. Kolekar, S.; Padmanabhan, T. Ideal Gas in a strong Gravitational field: Area dependence of Entropy. *Phys. Rev. D* **2011**, *83*, 064034.
12. Kadanoff, L.P. *Statistical Physics: Statics, Dynamics, and Renormalization*; World Scientific: Singapore, Singapore, 2000.
13. Mirza, B.; Mohammadzadeh, H.; Raissi, Z. Condensation of an ideal gas with intermediate statistics on the horizon. *Eur. Phys. J. C* **2012**, *72*, 2152.
14. Baldiotti, M.C.; Elias, W.S.; Molina, C.; Pereira, T.S. Thermodynamics of quantum photon spheres. *Phys. Rev. D* **2014**, *90*, 104025.

15. Iizuka, N.; Kabat, D.N.; Lifschytz, G.; Lowe, D.A. Stretched horizons, quasiparticles and quasinormal modes. *Phys. Rev. D* **2003**, *68*, 084021.
16. Susskind, L.; Thorlacius, L.; Uglum, J. The Stretched horizon and black hole complementarity. *Phys. Rev. D* **1993**, *48*, 3743–3761.
17. Banks, T.; Fischler, W.; Klebanov, I.R.; Susskind, L. Schwarzschild black holes in matrix theory. II. *J. High Energy Phys.* **1998**, *9801*, 008.
18. Minic, D. Infinite statistics and black holes in matrix theory. **1997**, arXiv:hep-th/9712202.
19. Pithis, A.G.A. The Gibbs paradox, Black hole entropy and the thermodynamics of isolated horizons. *Phys. Rev. D* **2013**, *87*, 084061.
20. Xiao, Y.; Chen, Y.X. Three types of statistics and the entropy bounds. *Phys. Lett. B* **2011**, *700*, 385–388.
21. Xiao, Y.; Guo, G.S.; Wang, P. Counting states for the entropy bound of infinite statistics. *EPL* **2012**, *97*, 20014.

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