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Article

Exponential Outer Synchronization between Two Uncertain Time-Varying Complex Networks with Nonlinear Coupling

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Abstract: This paper studies the problem of exponential outer synchronization between two uncertain nonlinearly coupled complex networks with time delays. In order to synchronize uncertain complex networks, an adaptive control scheme is designed based on the Lyapunov stability theorem. Simultaneously, the unknown system parameters of uncertain complex networks are identified when exponential outer synchronization occurs. Finally, numerical examples are provided to demonstrate the feasibility and effectiveness of the theoretical results.

Keywords: exponential synchronization; uncertain complex networks; nonlinear coupling; time delays

1. Introduction

Over the last decade, complex networks have received a great deal of interests from various research communities due to its wide applications in many fields including physics, mathematics, biology, engineering, social science and so on [1,2]. A complex network can be regarded as a large set of interconnected nodes and used to describe and characterize diverse complex systems. Typical examples include the World Wide Web, electrical power grids, scientific cooperation networks, biological networks, social networks and deterministic networks [3–5].

As a typical collective behavior on networks, synchronization inside a network has been widely and extensively studied [6–13]. Especially, synchronization of small-world networks, scale-free networks and tree-like networks were investigated in [14–16]. In [17], an adaptive feedback scheme was proposed to stabilize the synchronous solution of uncertain complex dynamical network with delayed coupling. Furthermore, the synchronization of networks with partial-information coupling were studied in [18,19]. In 2007, "outer synchronization", as another kind of network synchronization, was firstly put forward by Li *et al.* [20]. In that paper, the authors theoretically and numerically demonstrated the possibility of synchronization between two coupled networks. An important example of outer synchronization is the spread of infectious diseases across different communities, for instance, avian influenza spreads among domestic and wild birds, afterward infects human beings unexpectedly [21]. Improved and expanded work in this respect—*i.e.*, introducing the nonidentical topological structures, time-varying delays, circumstance noise, *etc.*—can be found in the literature [22–24], which outer synchronization is realized through the adaptive control schemes. On the other hand, generalized synchronization [25–27] and anti-synchronization [28] of complex networks have been investigated.

The above mentioned work focussed on the network problems with known dynamical properties beforehand. However, in many real-world situations, various kinds of uncertain information exist in the modern industrial systems. Therefore, data-driven methods have been introduced due to their simplicities and excellent ability to handle large amounts of data for different industrial issues under various operating conditions [29–33]. Here, this paper mainly focuses on the outer synchronization between two dynamical networks with unknown parameters and the system dynamic model can be involved in the designing of the controller. As an emerging issue, synchronization in uncertain complex networks has attracted increasing attention. Recently, the identification of network topological structure and system parameters for general uncertain complex dynamical networks was explored [34]. Xu *et al.* proposed an approach to identify the topological structure and unknown parameters for uncertain general complex networks with non-derivative and derivative couplings [35]. Exponentially adaptive synchronization, adaptive lag synchronization and generalized synchronization of uncertain general complex networks were studied in [36–38]. Che *et al.* studied the anticipatory synchronization of uncertain nonlinearly coupled complex networks with time delays [39]. However, in the most existing works on synchronization of uncertain complex networks, the topological structures of networks are always assumed to be a constant. More recently, Wu and Lu investigated the outer synchronization between two uncertain time-varying dynamical networks with same topological structures [40]. It should be pointed out that there exist different situations between drive and response networks with nonidentical topological structures. Therefore, it is necessary to study the synchronization between two nonidentical networks with uncertain information.

Inspired by the above discussions, we investigate exponential outer synchronization and parameter identification between two uncertain time-varying complex dynamical networks with nonlinear coupling, time delays and different topological structures. Based on the Lyapunov stability theorem, an adaptive control scheme is proposed to achieve exponential outer synchronization between drive and response networks. Compared with the existing results, the proposed method is efficient to control the convergence rates of outer synchronization. The unknown parameters are automatically identified when exponential outer synchronization occurs.

The outline of the rest paper is organized as follows. Section 2 gives the problem formulation of two uncertain drive-response networks and some preliminaries. Section 3 presents the main theory for identification of unknown system parameters. In Section 4, numerical example is provided to show the effectiveness of the proposed controller. The paper is concluded by Section 5.

2. Problem formulation and preliminaries

In this paper, we consider the following uncertain drive-response networks:

$$\begin{aligned} \dot{x}_i(t) &= F_i(t, x_i(t), x_i(t-\tau_1), \theta_i) + \sum_{j=1}^{N} a_{ij}(t) h(x_j(t-\tau_2)), \quad (1) \\ \dot{y}_i(t) &= F_i(t, y_i(t), y_i(t-\tau_1), \tilde{\theta}_i) + \sum_{j=1}^{N} b_{ij}(t) h(y_j(t-\tau_2)) + u_i, \quad (2) \end{aligned}$$

where $x_i, y_i \in R^n$, $i = 1, 2, \ldots, N$. $F_i(\cdot) \in R^n$ is a continuously differentiable nonlinear vector field, $\theta_i = (\theta_i^{(1)}, \theta_i^{(2)}, \ldots, \theta_i^{(m)})^T \in R^m$ are unknown parameters, $\tilde{\theta}_i = (\tilde{\theta}_i^{(1)}, \tilde{\theta}_i^{(2)}, \ldots, \tilde{\theta}_i^{(m)})^T \in R^m$ are the estimation of the unknown θ_i, $h(\cdot) \in R^n$ is the nonlinear inner-coupling function, τ_1 is node delay and τ_2 is coupling delay. $A(t) = (a_{ij}(t))_{N\times N}, B(t) = (b_{ij}(t))_{N\times N}$ are respective time-varying coupling matrices representing the topological structures of networks X and Y. The entries $a_{ij}(t)$ $(b_{ij}(t))$ are defined as follows: $a_{ij}(t)\,(b_{ij}(t)) > 0$ if there is a connection between node i and node j $(i \neq j)$; otherwise $a_{ij}(t)\,(b_{ij}(t)) = 0$ $(i \neq j)$, and the diagonal entries $a_{ii}(t) = -\sum_{j=1, j\neq i}^{N} a_{ij}(t), b_{ii}(t) = -\sum_{j=1, j\neq i}^{N} b_{ij}(t)$, $i = 1, 2, \ldots, N$. u_i is a controller to be designed.

Suppose that the unknown parameter θ_i is linearly dependent of the nonlinear functions of the ith node, and we can rewrite networks Equations (1) and (2) in the following forms:

$$\begin{aligned} \dot{x}_i(t) &= f_{i1}(t, x_i(t), x_i(t-\tau_1)) + f_{i2}(t, x_i(t), x_i(t-\tau_1))\theta_i + \sum_{j=1}^{N} a_{ij}(t) h(x_j(t-\tau_2)), \quad (3) \\ \dot{y}_i(t) &= f_{i1}(t, y_i(t), y_i(t-\tau_1)) + f_{i2}(t, y_i(t), y_i(t-\tau_1))\tilde{\theta}_i + \sum_{j=1}^{N} b_{ij}(t) h(y_j(t-\tau_2)) + u_i, \quad (4) \end{aligned}$$

where $f_{i1}(\cdot) \in R^n$ is a continuous vector function and $f_{i2}(\cdot) \in R^{n\times m}$ is a continuous matrix function, $i = 1, 2, \ldots, N$.

Denote $e_i(t) = y_i(t) - x_i(t)$, $\bar{\theta}_i = \tilde{\theta}_i - \theta_i$. According to the drive network Equation (3) and the response network Equation (4), the error dynamical network is then described by

$$\begin{aligned} \dot{e}_i(t) &= f_{i1}(t, y_i(t), y_i(t-\tau_1)) + f_{i2}(t, y_i(t), y_i(t-\tau_1))(\bar{\theta}_i + \theta_i) \\ &\quad - f_{i1}(t, x_i(t), x_i(t-\tau_1)) - f_{i2}(t, x_i(t), x_i(t-\tau_1))\theta_i \\ &\quad + \sum_{j=1}^{N} b_{ij}(t) h(y_j(t-\tau_2)) - \sum_{j=1}^{N} a_{ij}(t) h(x_j(t-\tau_2)) + u_i \\ &= f_i(t, y_i(t), y_i(t-\tau_1), \theta_i) - f_i(t, x_i(t), x_i(t-\tau_1), \theta_i) \\ &\quad + f_{i2}(t, y_i(t), y_i(t-\tau_1))\bar{\theta}_i + \sum_{j=1}^{N} b_{ij}(t) h(y_j(t-\tau_2)) - \sum_{j=1}^{N} a_{ij}(t) h(x_j(t-\tau_2)) + u_i, \quad (5) \end{aligned}$$

where $f_i(\cdot, \theta_i) = f_{i1}(\cdot) + f_{i2}(\cdot)\theta_i$, $i = 1, 2, \ldots, N$.

Throughout the rest of this paper, some useful assumptions, lemmas and definitions are presented.

Assumption 1. *For any $x_i(t)$, $y_i(t) \in R^n$, there exists a positive constant L satisfying*

$$\|f_i(t, y_i(t), y_i(t-\tau_1), \theta_i) - f_i(t, x_i(t), x_i(t-\tau_1), \theta_i)\| \leq L(\|e_i(t)\| + \|e_i(t-\tau_1)\|).$$

Hereafter, the norm $\|x\|$ of vector x is defined as $\|x\| = \sqrt{x^T x}$.

Assumption 2. *There exists a positive constant α such that*

$$\|h(x) - h(y)\| \leq \alpha\|x - y\|$$

hold for any $x, y \in R^n$.

Assumption 3. *Denote $f_{i2}(t, y_i(t), y_i(t-\tau_1)) = (f_{i2}^{(1)}(t, y_i(t), y_i(t-\tau_1)), f_{i2}^{(2)}(t, y_i(t), y_i(t-\tau_1)), \ldots, f_{i2}^{(m)}(t, y_i(t), y_i(t-\tau_1)))$. Assume that $f_{i2}^{(k)}(t, y_i(t), y_i(t-\tau_1))$, $h(y_j(t-\tau_2))$ ($k = 1, 2, \ldots, m$, $j = 1, 2, \ldots, N$) are linearly independent on the synchronized orbit $x_i(t) = y_i(t)$ of synchronization manifold for any given $i \in \{1, 2, \ldots, N\}$.*

Lemma 1. *For any vectors $x, y \in R^n$, the following inequality holds:*

$$2x^T y \leq x^T x + y^T y.$$

Definition 1. *We say that networks X and Y achieve exponential outer synchronization if there exist positive constants M and μ such that*

$$\|e_i(t)\| \leq M \exp(-\mu t), \quad i = 1, 2, \ldots, N.$$

Moreover, the constant μ is defined as the exponential synchronization rate [41].

3. Theoretical Results

In this section, we consider exponential outer synchronization between the drive network Equation (3) and the response network Equation (4). The main results are summarized in the following theorem.

Theorem 1. *Suppose that Assumptions 1–3 hold. Then exponential outer synchronization between the drive network Equation (3) and the response network Equation (4) can be achieved and the unknown parameters θ_i can be identified by using the estimation $\tilde{\theta}_i$ with the following adaptive controllers and the corresponding updating laws:*

$$\begin{cases} u_i(t) = -d_i(t)e_i(t), \\ \dot{d}_i(t) = q_i e_i^T(t)e_i(t)\exp(\mu t), \\ \dot{\tilde{\theta}}_i = -f_{i2}^T(t, y_i(t), y_i(t-\tau_1))e_i(t)\exp(\mu t), \\ \dot{b}_{ij}(t) = -l_i e_i^T(t)h(y_j(t-\tau_2))\exp(\mu t), \\ \dot{a}_{ij}(t) = k_i e_i^T(t)h(x_j(t-\tau_2))\exp(\mu t), \end{cases} \tag{6}$$

where q_i, l_i and k_i are positive constants, $i = 1, 2, \ldots, N$.

Proof. Construct the following Lyapunov functional:

$$\begin{aligned} V(t) &= \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t)\exp(\mu t) + \frac{1}{2}\sum_{i=1}^{N}\bar{\theta}_i^T\bar{\theta}_i + \sum_{i=1}^{N}\frac{1}{2q_i}(d_i(t)-d_i^*)^2 \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2k_i}(a_{ij}(t)-H_{ij}^*)^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2l_i}(b_{ij}(t)-H_{ij}^*)^2 \\ &+ \frac{L}{2}\sum_{i=1}^{N}\int_{t-\tau_1}^{t} e_i^T(s)e_i(s)\exp[\mu(s+\tau_1)]\mathrm{d}s + \frac{\alpha N H^*}{2}\sum_{i=1}^{N}\int_{t-\tau_2}^{t} e_i^T(s)e_i(s)\exp[\mu(s+\tau_2)]\mathrm{d}s, \end{aligned} \tag{7}$$

where d_i^* and H_{ij}^* $(i,j=1,2,\dots,N)$ are sufficiently large positive constants, $H^* = \max_{1\le i,j\le N}\{H_{ij}^*\}$.

According to Equations (5) and (6), we get the time derivative of $V(t)$ as

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N}\left[e_i^T(t)\dot{e}_i(t)\exp(\mu t) + \frac{\mu}{2}e_i^T(t)e_i(t)\exp(\mu t)\right] + \sum_{i=1}^{N}\bar{\theta}_i^T\dot{\bar{\theta}}_i + \sum_{i=1}^{N}\frac{1}{q_i}(d_i(t)-d_i^*)\dot{d}_i(t) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{k_i}(a_{ij}(t)-H_{ij}^*)\dot{a}_{ij}(t) + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{l_i}(b_{ij}(t)-H_{ij}^*)\dot{b}_{ij}(t) \\ &+ \frac{L}{2}\sum_{i=1}^{N}\left\{e_i^T(t)e_i(t)\exp[\mu(t+\tau_1)] - e_i^T(t-\tau_1)e_i(t-\tau_1)\exp(\mu t)\right\} \\ &+ \frac{\alpha N H^*}{2}\sum_{i=1}^{N}\left\{e_i^T(t)e_i(t)\exp[\mu(t+\tau_2)] - e_i^T(t-\tau_2)e_i(t-\tau_2)\exp(\mu t)\right\} \\ &\le \sum_{i=1}^{N} e_i^T(t)\left[f_i(t,y_i(t),y_i(t-\tau_1),\theta_i) - f_i(t,x_i(t),x_i(t-\tau_1),\theta_i)\right]\exp(\mu t) \\ &+ \left(\frac{\mu}{2} + \frac{L}{2}\exp(\mu\tau_1) + \frac{\alpha N H^*}{2}\exp(\mu\tau_2) - d^*\right)\sum_{i=1}^{N} e_i^T(t)e_i(t)\exp(\mu t) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij}^* e_i^T(t)\left[h(y_j(t-\tau_2)) - h(x_j(t-\tau_2))\right]\exp(\mu t) \\ &- \frac{L}{2}\sum_{i=1}^{N} e_i^T(t-\tau_1)e_i(t-\tau_1)\exp(\mu t) - \frac{\alpha N H^*}{2}\sum_{i=1}^{N} e_i^T(t-\tau_2)e_i(t-\tau_2)\exp(\mu t), \end{aligned}$$

where $d^* = \min_{1\le i\le N}\{d_i^*\} > 0$.

From Lemma 1, Assumptions 1 and 2, one gets

$$\begin{aligned} & e_i^T(t)\left[f_i(t,y_i(t),y_i(t-\tau_1),\theta_i) - f_i(t,x_i(t),x_i(t-\tau_1),\theta_i)\right]\exp(\mu t) \\ \le\ & \frac{3L}{2}e_i^T(t)e_i(t)\exp(\mu t) + \frac{L}{2}e_i^T(t-\tau_1)e_i(t-\tau_1)\exp(\mu t), \end{aligned}$$

$$\begin{aligned} & e_i^T(t)\left[h(y_j(t-\tau_2) - h(x_j(t-\tau_2))\right] \\ \le\ & \|e_i(t)\|\|h(y_j(t-\tau_2) - h(x_j(t-\tau_2))\| \le \alpha\|e_i(t)\|\|e_j(t-\tau_2)\| \\ \le\ & \frac{\alpha}{2}\left(e_i^T(t)e_i(t) + e_j^T(t-\tau_2)e_j(t-\tau_2)\right). \end{aligned}$$

Then, one has

$$\begin{aligned}\dot{V}(t) &\le \left(\frac{\mu+3L}{2}+\frac{L}{2}\exp(\mu\tau_1)+\frac{\alpha NH^*}{2}\exp(\mu\tau_2)-d^*\right)\sum_{i=1}^{N}e_i^T(t)e_i(t)\exp(\mu t)\\ &\quad+\frac{\alpha H^*}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[e_i^T(t)e_i(t)+e_j^T(t-\tau_2)e_j(t-\tau_2)\right]\exp(\mu t)\\ &\quad-\frac{\alpha NH^*}{2}\sum_{i=1}^{N}e_i^T(t-\tau_2)e_i(t-\tau_2)\exp(\mu t)\\ &= \left(\frac{\mu+3L}{2}+\frac{L}{2}\exp(\mu\tau_1)+\frac{\alpha NH^*}{2}(1+\exp(\mu\tau_2))-d^*\right)\sum_{i=1}^{N}e_i^T(t)e_i(t)\exp(\mu t)\end{aligned}$$

If H^* is fixed, then we can choose $d^* \ge \frac{\mu+3L}{2}+\frac{L}{2}\exp(\mu\tau_1)+\frac{\alpha NH^*}{2}(1+\exp(\mu\tau_2))+1$. Thus, one has $\dot{V}(t) \le -\sum_{i=1}^{N}e_i^T(t)e_i(t)\exp(\mu t) \le 0$. It follows that $V(t) \le V(0)$ for any $t \ge 0$. From the Lyapunov function Equation (7), one gets

$$\frac{1}{2}\|e_i(t)\|^2\exp(\mu t) = \frac{1}{2}e_i^T(t)e_i(t)\exp(\mu t) \le V(t) \le V(0).$$

Therefore, one obtains $\|e_i(t)\| \le M\exp(-\frac{\mu}{2}t)$, where $M = \sqrt{2V(0)} > 0$. Hence, every trajectory $y_i(t)$ of network Equation (4) must synchronize exponentially toward the $x_i(t)$ with a convergence rate of $\frac{\mu}{2}$. It implies that exponential outer synchronization between the drive network Equation (3) and the response network Equation (4) has been achieved. Thus, $\lim_{t\to\infty}\|e_i(t)\| = 0$ $(i = 1, 2, \ldots, N)$.

Suppose that $\lim_{t\to\infty}\dot{e}_i(t)$ exists. Since $e_i(t)$ converges to a constant as $t \to \infty$, one has $\lim_{t\to\infty}\dot{e}_i(t) = 0$. According to the error system Equation (5), we have

$$\begin{aligned}&f_{i2}(t, y_i(t), y_i(t-\tau_1))(\tilde{\theta}_i-\theta_i)+\sum_{j=1}^{N}(b_{ij}(t)-a_{ij}(t))h(y_j(t-\tau_2))\\ =&\sum_{k=1}^{m}(\tilde{\theta}_i^{(k)}-\theta_i^{(k)})f_{i2}^{(k)}(t, y_i(t), y_i(t-\tau_1))+\sum_{j=1}^{N}(b_{ij}(t)-a_{ij}(t))h(y_j(t-\tau_2))=0,\end{aligned}$$

as $t \to \infty$. From Assumptions 3, one has $\tilde{\theta}_i^{(k)}-\theta_i^{(k)} \to 0$ and $b_{ij}(t)-a_{ij}(t) \to 0$ as $t \to \infty$ [42]. That is to say, the unknown parameters θ_i can be identified by using the control scheme Equation (6). All this completes the proof. □

4. Numerical Example

To show the effectiveness of the above adaptive control scheme, numerical example is presented in this section. Here, the dynamics at every node of both the networks X and Y follows the Hindmarsh-Rose (HR) neuronal system [43] with time delayed nonlinearity.

$$\begin{cases}\dot{x}_{i1} = x_{i2}(t-\tau_1)-\alpha_{i1}x_{i1}^3(t-\tau_1)+\alpha_{i2}x_{i1}^2(t-\tau_1)-x_{i3}(t-\tau_1)+I,\\ \dot{x}_{i2} = \alpha_{i3}-\alpha_{i4}x_{i1}^2(t-\tau_1)-x_{i2}(t-\tau_1),\\ \dot{x}_{i3} = p[s(x_{i1}(t-\tau_1)-\chi)-x_{i3}(t-\tau_1)],\end{cases} \tag{8}$$

where x_{i1} is the membrane action potential, x_{i2} is a recovery variable and x_{i3} is a slow adaptation current, I is the external direct current, α_{i1}, α_{i2}, α_{i3}, α_{i4}, s, p and χ are constants, $i = 1, 2, \ldots, N$. In the following simulations, let $\alpha_{i1} = 1.0$, $\alpha_{i2} = 3.0$, $\alpha_{i3} = 1.0$, $\alpha_{i4} = 5.0$, $s = 4.0$, $p = 0.006$, $\chi = -1.60$ and $I = 3.0$, HR system shows the chaotic firing pattern [44]. For simplicity, we only assume that α_{i4} are not known in advance, then system Equation (8) can be rewritten as

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} &= \begin{pmatrix} x_{i2}(t-\tau_1) - \alpha_{i1}x_{i1}^3(t-\tau_1) + \alpha_{i2}x_{i1}^2(t-\tau_1) - x_{i3}(t-\tau_1) + I \\ \alpha_{i3} - x_{i2}(t-\tau_1) \\ p[s(x_{i1}(t-\tau_1) - \chi) - x_{i3}(t-\tau_1)] \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -x_{i1}^2(t-\tau_1) \\ 0 \end{pmatrix} \theta_i, \\
&= f_{i1}(t, x_i(t), x_i(t-\tau_1)) + f_{i2}(t, x_i(t), x_i(t-\tau_1))\theta_i,
\end{aligned} \tag{9}$$

where $\theta_i = \alpha_{i4}$, $i = 1, 2, \ldots, N$. For convenience, set the node delay $\tau_1 = 0.1$ and the coupling delay $\tau_2 = 0.2$. Let the nonlinear inner-coupling function $h(x_i) = (0, 1 + \sin(x_{i2}), 0)^T$, so it satisfies Assumptions 2 and 3.

Here, we investigate exponential outer synchronization of uncertain networks by using the adaptive control scheme Equation (6). We choose $q_i = k_i = l_i = 20$ and $\mu = 0.1$. The network size is taken as $N = 5$, the initial values $a_{ij}(0)$ and $b_{ij}(0)$ $(i, j = 1, \ldots, 5)$ are randomly chosen in the interval (0, 0.5) and the interval (0.5, 1) respectively. The other initial values are randomly chosen in the interval (–1, 1). The time evolution of outer synchronization error $E(t) = \max\{\|y_i(t) - x_i(t)\| : i = 1, 2, \ldots, 5\}$ is shown in Figure 1. It is clear that outer synchronization error is rapidly converging to zero. Figure 2 shows the identification of unknown parameters θ_i $(i = 1, \ldots, 5)$. Figure 3 displays the evolution of adaptive feedback gains d_i $(i = 1, \ldots, 5)$. From Figures 4 and 5, we can find that $a_{ij}(t)$ and $b_{ij}(t)$ are converged to the same constants when outer synchronization appears.

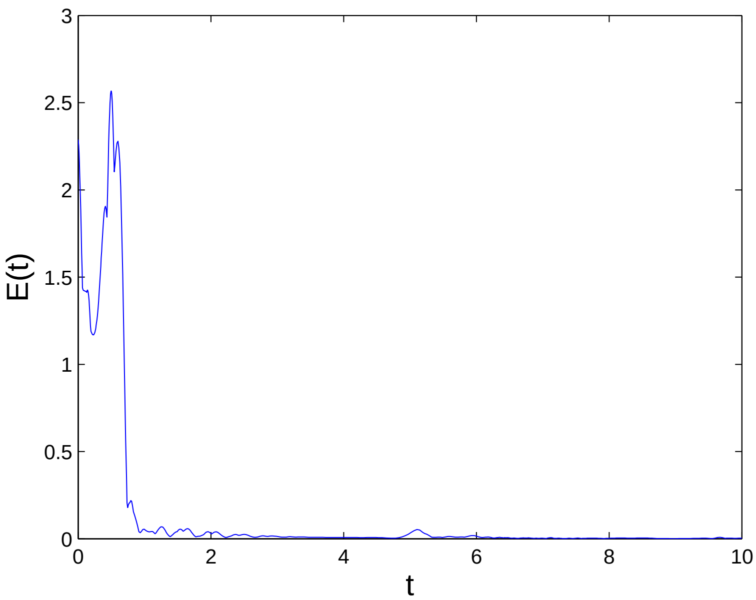


Figure 1. Evolution of outer synchronization error $E(t)$ of networks Equations (3) and (4).

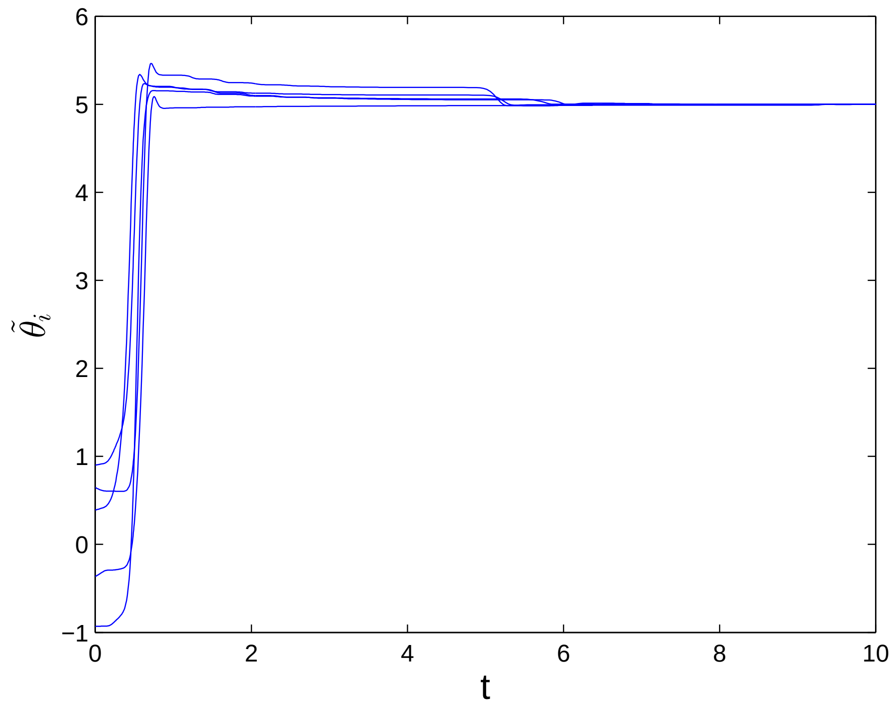


Figure 2. Identification of unknown parameters θ_i $(i = 1, \ldots, 5)$ of networks Equations (3) and (4).

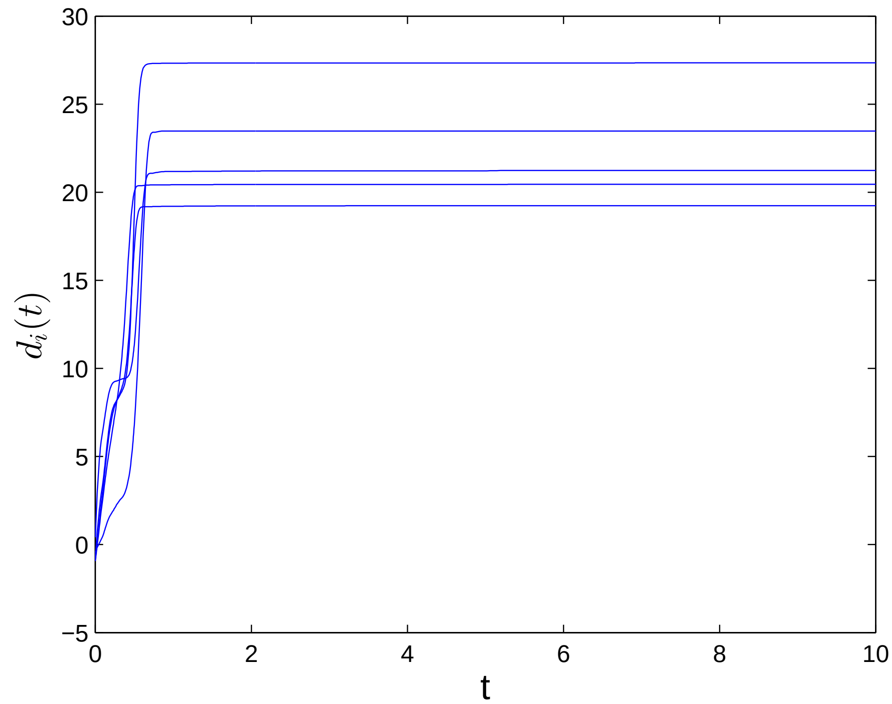
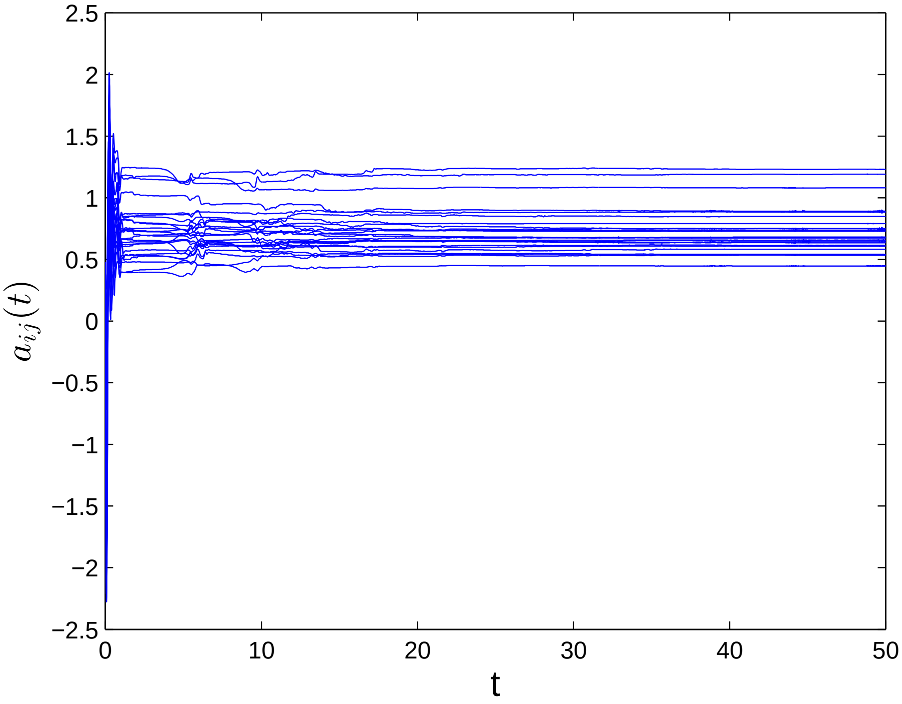
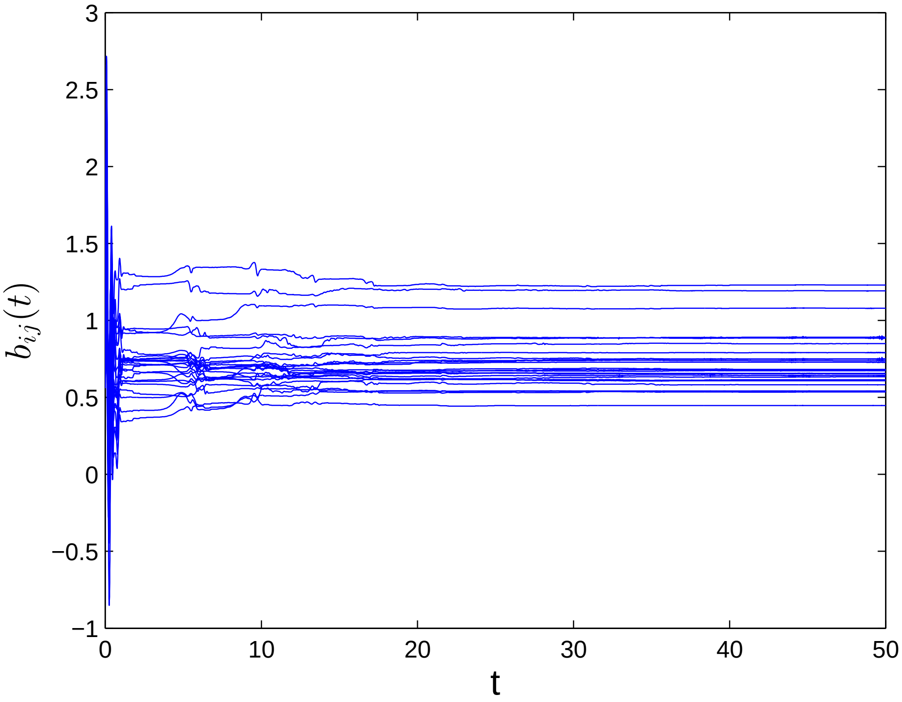


Figure 3. Evolution of adaptive feedback gains d_i $(i = 1, \ldots, 5)$ of networks Equations (3) and (4).



(**a**)



(**b**)

Figure 4. Evolution of $a_{ij}(t)$ and $b_{ij}(t)$ $(i, j = 1, \ldots, 5)$ of networks Equations (3) and (4).

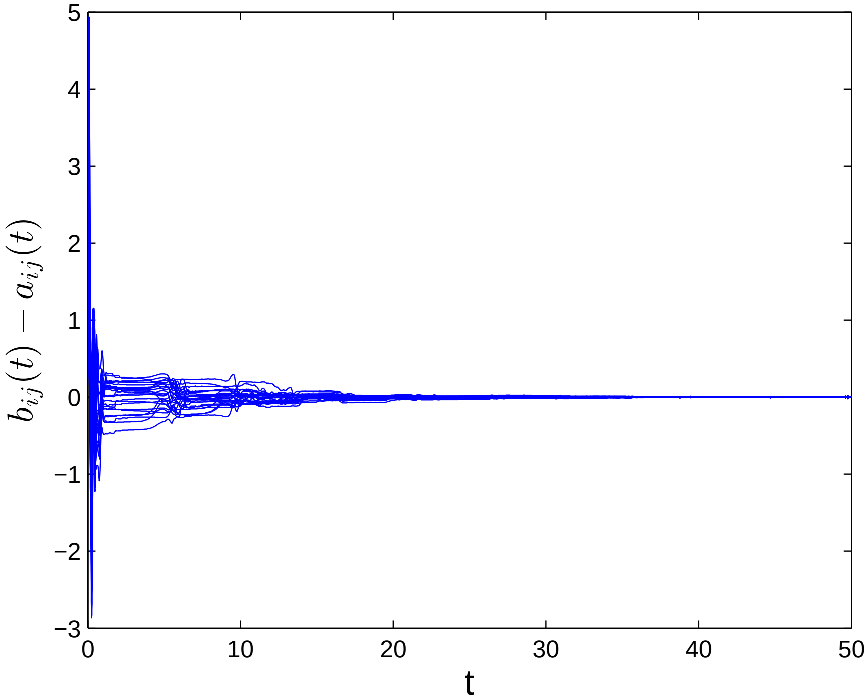


Figure 5. Evolution of $b_{ij}(t) - a_{ij}(t)$ $(i, j = 1, \ldots, 5)$ of networks Equations (3) and (4).

5. Conclusions

In this paper, exponential outer synchronization between two uncertain time-varying complex dynamical networks with nonlinear coupling and time delays has been studied both theoretically and numerically. According to the Lyapunov stability theorem, an adaptive control scheme is proposed to synchronize uncertain delayed complex networks. Meanwhile, unknown parameters are identified in the process of outer synchronization.

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Author Contributions

Yongqing Wu designed the research and wrote the paper; Li Liu performed the experiment and analyzed the data. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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