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Article

Generalized Stochastic Fokker-Planck Equations

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Abstract: We consider a system of Brownian particles with long-range interactions. We go beyond the mean field approximation and take fluctuations into account. We introduce a new class of stochastic Fokker-Planck equations associated with a generalized thermodynamical formalism. Generalized thermodynamics arises in the case of complex systems experiencing small-scale constraints. In the limit of short-range interactions, we obtain a generalized class of stochastic Cahn-Hilliard equations. Our formalism has application for several systems of physical interest including self-gravitating Brownian particles, colloid particles at a fluid interface, superconductors of type II, nucleation, the chemotaxis of bacterial populations, and two-dimensional turbulence. We also introduce a new type of generalized entropy taking into account anomalous diffusion and exclusion or inclusion constraints.

Keywords: nonlinear Fokker-Planck equations; generalized entropies; long-range systems

1. Introduction

The theory of Brownian motion initiated by Einstein [1,2], Langevin [3] and Smoluchowski [4] is one of the greatest scientific achievement of the twentieth century. It culminated with the derivation of the Fokker-Planck (FP) equation [5,6] which is one of the most important, and most useful, equations of statistical mechanics and kinetic theory [7]. Indeed, this equation has applications in many area of physics, biology, chemistry, and economy. In its simplest form, the FP equation reduces to the Smoluchowski equation for overdamped particles [4] and to the Klein-Kramers equation for inertial particles [8,9]. These equations describe the diffusive motion of a Brownian particle (or an ensemble of non-interacting Brownian particles) in an external potential. Since the particles are coupled to a thermal bath fixing the temperature, the relevant statistical ensemble is the canonical ensemble. The diffusion coefficient, the mobility (or the friction coefficient) and the temperature are related to each other by the celebrated Einstein relation [1,2]. The FP equation is fully consistent with Boltzmann's thermodynamics [7]. It satisfies an H-theorem for the Boltzmann free energy and relaxes towards an equilibrium state that minimizes the Boltzmann free energy under the normalization condition constraint. This equilibrium state corresponds to the Boltzmann distribution.

In the recent years, two new topics have emerged in statistical mechanics and kinetic theory.

The first one concerns the physics of systems with long-range interactions [10]. We can distinguish two types of systems with long-range interactions depending whether they are isolated or dissipative. Hamiltonian systems with long-range interactions are isolated and evolve at fixed energy E. This corresponds to the microcanonical ensemble. In the mean field limit (collisionless regime), valid when the number of particles $N \to +\infty$, these systems are described by the Vlasov equation. When finite N effects are taken into account (collisional regime), their evolution is governed by the Landau or by the Lenard-Balescu equation [11,12]. These equations satisfy an H-theorem for the Boltzmann entropy and relax towards the mean field Boltzmann distribution with a temperature $T(E)$. Examples of Hamiltonian systems with long-range interactions include stellar systems, Coulombian plasmas, two-dimensional vortices, and the Hamiltonian mean field (HMF) model [13]. Brownian systems with long-range interactions are dissipative and evolve at fixed temperature (they are coupled to a thermal bath). This corresponds to the canonical ensemble. When $N \to +\infty$, these systems are described by mean field FP equations [14]. These equations satisfy an H-theorem for the Boltzmann free energy and relax towards the mean field Boltzmann distribution with a temperature T. Contrary to the case of ordinary Brownian particles that evolve in a fixed external potential, Brownian particles with long-range interactions move in a mean field potential that they create themselves. When finite N effects are taken into account, the deterministic mean field FP equations are replaced by stochastic FP equations that take fluctuations into account [15]. Examples of Brownian systems with long-range interactions include self-gravitating Brownian particles [16], the Brownian mean field (BMF) model [17], and bacterial populations undergoing chemotaxis [18].

Another emerging topic in statistical mechanics and kinetic theory concerns the notion of generalized thermodynamics pioneered by Tsallis [19]. It has been observed in many occasions that the Boltzmann entropy does not provide a correct description of the system under consideration and that other forms of entropies may be more relevant. This is the case for non-ergodic systems having a complex dynamics. The derivation of the Boltzmann equation assumes that the probability of transition from one site to the other is proportional to the density of the starting site and independent on the density of the arrival site. However, one may easily imagine situations in which the transition from one site to the other is inhibited or, on the contrary, stimulated so that the expression of the probability of transition depends in a complicated manner on the density of the starting and arrival sites. This leads to a generalized class of Boltzmann equations [20–25] associated with generalized entropies and having equilibrium states different from the Boltzmann distribution. Similarly, in the context of Brownian motion, the Boltzmann distribution emerges naturally from the FP equation when the particles have a constant diffusion coefficient and a constant mobility. However, one may imagine situations in which the particles are hampered in their motion by some small-scale constraints so that the diffusion coefficient and the mobility depend on the local density. In that case, the motion of the particles is biased, resulting in anomalous diffusion or anomalous mobility. The corresponding generalized FP equations are associated with generalized free energies and have equilibrium states different from the Boltzmann distribution [24,26–44] (see [45,46] for reviews). In the situations described above, generalized entropies arise because the system experiences small-scale (hidden) constraints so that the *a priori* accessible microstates are not equiprobable.

If we combine these two emerging topics, long-range interactions and generalized thermodynamics, we obtain a rich class of generalized kinetic equations [25,39,46] having a source of nonlinearity arising from the long-range interaction and a source of nonlinearity arising from the fact that the coefficients in these equations depend on the density itself (generalized thermodynamics). In the present paper, we restrict ourselves to Brownian particles in interaction (canonical description) and consider the generalized mean field FP equations introduced in [46]. These generalized mean field FP equations may have several stable steady states. In the absence of fluctuations, the system generically relaxes towards one of these states and stays there permanently. In the presence of fluctuations due, for example, to finite N effects, these equilibrium states become metastable and the system undergoes random transitions from one state to the other. Such switches can be described in terms of stochastic FP equations including a random forcing due to fluctuations (finite N effects) [15]. This type of stochastic equations has been studied numerically in [47] for a model of self-gravitating Brownian particles and bacterial populations presenting two symmetric metastable states. The system experiences random transitions from one state to the other. The lifetime of the metastable states is related to the Kramers escape rate formula which can be derived from the theory of instantons. This model is associated with Boltzmann's thermodynamics and the stochastic FP equation can be derived from first principles [15]. In the present paper, we develop a more general theoretical framework to include situations associated with non-Boltzmannian distributions. In particular, we introduce a new class of stochastic FP equations associated with a generalized thermodynamical formalism.

The paper is organized as follows. In Section 2, we consider a system of overdamped Brownian particles with long-range interactions and show from very general arguments how a notion of generalized thermodynamics emerges in the case of complex systems experiencing small-scale constraints. In Section 3, we derive a generalized mean field FP equation describing the deterministic dynamics of these systems. In Section 4, we take fluctuations into account (arising, for example, from finite N effects) and derive a generalized stochastic FP equation. In Section 5, we present a new form of generalized entropy that takes into account anomalous diffusion and exclusion or inclusion constraints in physical space. In Sections 6 and 7, we consider a limit of short-range interactions and derive a generalized stochastic Cahn-Hilliard equation. In Section 8, we give physical applications of our formalism for self-gravitating Brownian particles, colloid particles at a fluid interface, superconductors of type II, nucleation, the chemotaxis of bacterial populations, and two-dimensional turbulence. In Appendix A, we apply the Landau-Lifshitz theory of fluctuations. In Appendix B, we recall the stochastic Ginzburg-Landau and Cahn-Hilliard equations. In Appendix C, we derive the generalized Smoluchowski equation from the dynamic density functional theory (DDFT) for systems with long and short-range interactions.

2. Overdamped Brownian Particles with Long-Range Interactions

2.1. The N-body Smoluchowski Equation

We consider a system of N Brownian particles in interaction in the strong friction limit $\xi \to +\infty$, where ξ is the friction coefficient, for which the inertia of the particles can be neglected. This corresponds to the overdamped Brownian model (we refer to [14] for the description of the inertial Brownian model from which the overdamped Brownian model can be derived). For the sake of generality, we work in a space of dimension d. The dynamics of the particles is described by the stochastic Langevin equations

$$\frac{d\mathbf{r}_i}{dt} = -\frac{\chi}{m}\nabla_i U(\mathbf{r}_1, ..., \mathbf{r}_N) + \sqrt{2D}\,\mathbf{R}_i(t), \tag{1}$$

where $\chi = 1/\xi$ is the mobility (by unit of mass), D is the diffusion coefficient, $U(\mathbf{r}_1, ..., \mathbf{r}_N)$ is the potential of interaction, and $\mathbf{R}_i(t)$ is a Gaussian white noise satisfying $\langle \mathbf{R}_i(t)\rangle = \mathbf{0}$ and $\langle R_i^\alpha(t) R_j^\beta(t')\rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$, where $i = 1, ..., N$ labels the particles and $\alpha = 1, ..., d$ labels the coordinates of space. The coefficients of diffusion and mobility are related to each other by the Einstein relation

$$D = \frac{\chi k_B T}{m}, \tag{2}$$

where T is the temperature of the bath to which the Brownian particles are coupled. Since $D \propto T$, the temperature measures the strength of the stochastic force acting on the particles. Since the system is in contact with a thermal bath fixing the temperature T, the proper statistical ensemble is the canonical ensemble.

The evolution of the N-body distribution function $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is governed by the N-body FP equation [14]:

$$\frac{\partial P_N}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i}\cdot\left[D\frac{\partial P_N}{\partial \mathbf{r}_i} + \frac{\chi}{m}P_N\frac{\partial}{\partial \mathbf{r}_i}U(\mathbf{r}_1, ..., \mathbf{r}_N)\right]. \tag{3}$$

This is the so-called N-body Smoluchowski equation. It can be derived directly from the stochastic Langevin equations (1).

The stationary solution of the N-body Smoluchowski equation is the canonical distribution [14]:

$$P_N(\mathbf{r}_1, ..., \mathbf{r}_N) = \frac{1}{Z(\beta)}\left(\frac{2\pi}{\beta m}\right)^{dN/2} e^{-\beta U(\mathbf{r}_1, ..., \mathbf{r}_N)}, \tag{4}$$

where

$$Z(\beta) = \left(\frac{2\pi}{\beta m}\right)^{dN/2}\int e^{-\beta U(\mathbf{r}_1, ..., \mathbf{r}_N)}\, d\mathbf{r}_1...d\mathbf{r}_N \tag{5}$$

is the partition function determined by the normalization condition $I \equiv \int P_N\, d\mathbf{r}_1...d\mathbf{r}_N = 1$. In order to obtain Equation (4), we have used the Einstein relation (2) and we have defined $\beta = 1/(k_B T)$. The N-body distribution (4) corresponds to the statistical equilibrium state of the Brownian particles in the canonical ensemble. It gives the probability density of the microstate $\{\mathbf{r}_1, ..., \mathbf{r}_N\}$.

The proper thermodynamical potential in the canonical ensemble is the free energy

$$F[P_N] = E[P_N] - TS[P_N], \tag{6}$$

where the energy and the entropy of the N-body system are given by [14]:

$$E[P_N] = \frac{d}{2}Nk_BT + \int P_N U \, d\mathbf{r}_1...d\mathbf{r}_N, \tag{7}$$

$$S[P_N] = -k_B \int P_N \ln P_N \, d\mathbf{r}_1...d\mathbf{r}_N + \frac{d}{2}Nk_B \ln\left(\frac{2\pi k_B T}{m}\right) + \frac{d}{2}Nk_B. \tag{8}$$

The free energy is explicitly given by

$$F[P_N] = \int P_N U \, d\mathbf{r}_1...d\mathbf{r}_N + k_BT \int P_N \ln P_N \, d\mathbf{r}_1...d\mathbf{r}_N - \frac{d}{2}Nk_BT \ln\left(\frac{2\pi k_B T}{m}\right). \tag{9}$$

The N-body Smoluchowski equation satisfies an H-theorem for the free energy (9). Indeed, a simple calculation gives

$$\dot{F} = -\sum_{i=1}^{N} \int \frac{m}{\xi P_N} \left(\frac{k_BT}{m}\frac{\partial P_N}{\partial \mathbf{r}_i} + \frac{1}{m}P_N \frac{\partial U}{\partial \mathbf{r}_i}\right)^2 d\mathbf{r}_1...d\mathbf{r}_N. \tag{10}$$

Therefore, $\dot{F} \le 0$ and $\dot{F} = 0$ if, and only if, the term in parenthesis vanishes. This leads to the canonical distribution (4). Because of the H-theorem, the system converges towards the canonical distribution (4) for $t \to +\infty$ (provided that $Z < +\infty$). The canonical distribution (4) is the (unique) minimum of $F[P_N]$ satisfying the normalization condition constraint. Indeed, the cancelation of the first variations $\delta F - \mu \delta I = 0$, where μ is a Lagrange multiplier, returns Equation (4), and the second variations $\delta^2 F = \frac{1}{2}\int \frac{(\delta P_N)^2}{P_N} \, d\mathbf{r}_1...d\mathbf{r}_N > 0$ are positive. At equilibrium, substituting Equation (4) in Equation (9), we get $F[P_N] = F(T)$ where $F(T) = -k_BT \ln Z(T)$.

Remark 1. *We have derived the canonical distribution (4) from the N-body stochastic dynamics (1). For systems with short-range interactions, one usually proceeds the other way round. We first derive the canonical distribution from the microcanonical distribution by considering a subsystem of a large system. Then, we introduce a Langevin dynamics that reproduces the canonical distribution at equilibrium. However, for systems with long-range interactions, we cannot proceed in this manner because the canonical distribution cannot be derived from the microcanonical distribution since the energy is non-additive* [48]. *Still, the canonical distribution is perfectly well-defined for systems with long-range interactions described at the start by a Langevin dynamics instead of a Hamiltonian dynamics.*

2.2. Long-Range Interactions

The preceding results are completely general. From now on, we assume that the particles interact via a binary potential so that $U(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{i<j} m^2 u_{ij}$ where $u_{ij} = u(|\mathbf{r}_i - \mathbf{r}_j|)$. We assume furthermore that the potential of interaction is long-ranged which means that it decays at large distances as $r^{-\gamma}$ with $\gamma < d$ [48]. For systems with long-range interactions, it can be shown that the correlations between the particles can be neglected in a proper thermodynamic limit $N \to +\infty$ [48]. Therefore, the mean field approximation becomes exact in this limit and the N-body distribution function can be factorized in a product of N one-body distribution functions

$$P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \prod_{i=1}^{N} P_1(\mathbf{r}_i, t). \tag{11}$$

Substituting Equation (11) in Equation (7), we find that the mean field energy is given by

$$E = \frac{d}{2} N k_B T + \frac{1}{2} \int \rho \Phi \, d\mathbf{r}, \tag{12}$$

where $\rho(\mathbf{r}, t) = N m P_1(\mathbf{r}, t)$ is the mean density,

$$\Phi(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}', t) \, d\mathbf{r}' \tag{13}$$

is the mean potential, and

$$W = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} \tag{14}$$

is the mean field energy of interaction. The mass can be expressed in terms of the density as

$$M = N m = \int \rho \, d\mathbf{r}. \tag{15}$$

Remark 2. *In Equation (13) the mean field potential appears as a convolution product ($\Phi = u * \rho$) and the factor $1/2$ in the energy of interaction (14) is introduced in order to avoid double counting. The particles may also experience an external potential $\Phi_{\rm ext}(\mathbf{r})$. The energy of the particles in the external potential is $W_{\rm ext} = \int \rho \Phi_{\rm ext} \, d\mathbf{r}$.*

2.3. Complex Systems: Generalized Thermodynamics

If the particles only interact via a long-range potential, we shall say that the system is "simple" (which does not mean of course that it is trivial!). In that case, it is possible to develop the thermodynamics and the kinetic theory of these systems rigorously in the limit $N \to +\infty$ and show that the entropy $S[\rho]$ of the macrostate $\rho(\mathbf{r})$ is the Boltzmann entropy. In this paper, we consider the case of "complex" systems for which small-scale constraints act on the particles (in addition to the long-range interaction) and modify their simple dynamics. We shall not try to model these constraints because they would lead to very complicated equations of motion that, most of the time, cannot be written explicitly. Therefore, these small-scale constraints appear as "hidden constraints". We shall take them into account indirectly by using a form of "generalized entropy". The fact that complex systems exhibit non-Boltzmannian distributions and non-Boltzmannian entropies has been observed in a wide variety of situations [19].

2.4. The Free Energy $F[\rho]$

For a simple Brownian system at statistical equilibrium, the N-body distribution is given by the canonical distribution (4) and (5). This distribution says that the probability density of a microstate $\{\mathbf{r}_1, ..., \mathbf{r}_N\}$ (any) with energy $U(\mathbf{r}_1, ..., \mathbf{r}_N)$ is proportional to $e^{-\beta U(\mathbf{r}_1, ..., \mathbf{r}_N)}$. For complex systems, we assume that some microstates are forbidden in physical space because of microscopic constraints that we shall not try to describe (in a future contribution [49], we shall consider the case where some microstates are forbidden in phase space). We denote by Ω the set of physically accessible microstates in physical space. We assume that the probability of an accessible microstate $\{\mathbf{r}_1, ..., \mathbf{r}_N\} \in \Omega$ is given by Equation (4). On the other hand, $P_N(\mathbf{r}_1, ..., \mathbf{r}_N) = 0$ for a forbidden microstate $\{\mathbf{r}_1, ..., \mathbf{r}_N\} \notin \Omega$. Therefore, we replace the partition function (5) by

$$Z(\beta) = \left(\frac{2\pi}{\beta m}\right)^{dN/2} \int_\Omega e^{-\beta U(\mathbf{r}_1,...,\mathbf{r}_N)}\, d\mathbf{r}_1...d\mathbf{r}_N. \tag{16}$$

We now introduce the smooth density $\rho(\mathbf{r})$ in position space and denote by $\Omega[\rho]$ the number of microstates $\{\mathbf{r}_i\}$ corresponding to the macrostate $\rho(\mathbf{r})$. The smooth density $\rho(\mathbf{r})$ corresponds to the coarse-grained density of Section 4.1 denoted $\overline{\rho}(\mathbf{r})$ but, to simplify the notations, we do not write the bar on ρ. The unconditional entropy of the macrostate $\rho(\mathbf{r})$ is defined by $S_0[\rho] = k_B \ln \Omega[\rho]$. In principle, the number of complexions $\Omega[\rho]$ can be obtained by a combinatorial analysis. If there is no microscopic constraint (simple systems) then, for $N \gg 1$, we obtain the Boltzmann entropy $S_0[\rho] = -k_B \int (\rho/m) \ln(\rho/Nm)\, d\mathbf{r}$. However, if some microstates are forbidden (complex systems), we will have instead $S_0[\rho] = -k_B \int C(\rho)\, d\mathbf{r}$ where $S_0[\rho]$ is a generalized entropy that can be different from the Boltzmann entropy. The function $C(\rho)$ can take various forms. For future purposes, we assume that $C(\rho)$ is convex ($C'' > 0$).

Instead of integrating on the microstates $\{\mathbf{r}_1, ..., \mathbf{r}_N\}$ in Equation (16), we can integrate on the macrostates $\{\rho(\mathbf{r})\}$. If we consider a system with long-range interactions, so that a mean field approximation applies at the thermodynamic limit $N \to +\infty$, we obtain for $N \gg 1$:

$$\begin{aligned} Z(\beta) &\simeq e^{\frac{dN}{2}\ln\left(\frac{2\pi}{\beta m}\right)} \int e^{-\beta W[\rho]} \Omega[\rho]\, \delta(M[\rho] - M)\, \mathcal{D}\rho \\ &\simeq e^{\frac{dN}{2}\ln\left(\frac{2\pi}{\beta m}\right)} \int e^{S_0[\rho]/k_B - \beta W[\rho]}\, \delta(M[\rho] - M)\, \mathcal{D}\rho \\ &\simeq \int e^{-\beta F[\rho]}\, \delta(M[\rho] - M)\, \mathcal{D}\rho, \end{aligned} \tag{17}$$

where the δ-function accounts for the mass constraint. The free energy $F[\rho]$ is given by

$$F[\rho] = E[\rho] - TS[\rho], \tag{18}$$

where the mean field energy E is given by Equation (12) and the generalized entropy S is given by

$$S[\rho] = -k_B \int C(\rho)\, d\mathbf{r} + \frac{d}{2} N k_B \ln\left(\frac{2\pi k_B T}{m}\right) + \frac{d}{2} N k_B. \tag{19}$$

The free energy explicitly writes

$$F[\rho] = \frac{1}{2}\int \rho\Phi\, d\mathbf{r} + k_B T \int C(\rho)\, d\mathbf{r} - \frac{dN}{2} k_B T \ln\left(\frac{2\pi k_B T}{m}\right). \tag{20}$$

The canonical probability density of the distribution ρ is

$$P[\rho] = \frac{1}{Z(\beta)} e^{-\beta F[\rho]} \delta(M[\rho] - M). \tag{21}$$

2.5. Variational Principle

For systems with long-range interactions, for which the mean field approximation is exact in the proper thermodynamic limit $N \to +\infty$, we can make the saddle point approximation

$$Z(\beta) \equiv e^{-\beta F(\beta)} \simeq e^{-\beta F[\rho_*]}, \tag{22}$$

i.e.,

$$F(\beta) \simeq F[\rho_*], \tag{23}$$

where ρ_* is the global minimum of free energy at fixed mass. This corresponds to the most probable macrostate in the canonical ensemble. We therefore have to solve the minimization problem

$$F(T) = \min_{\rho}\{F[\rho] \,|\, M[\rho] = M\}. \tag{24}$$

Remark 3. *The variational principle (24) where ρ_* is the global minimum of free energy at fixed mass determines the strict statistical equilibrium state of the system. For systems with long-range interactions, it is very relevant to consider also local minima of free energy at fixed mass. They correspond to metastable states. For systems with long-range interactions, these metastable states have tremendously long lifetimes, scaling as e^N, so they can be considered as fully stable states in practice* [50].

2.6. Equilibrium States

According to the results of Section 2.5, the equilibrium state of the system in the canonical ensemble, corresponding to the most probable macrostate, is obtained by solving the minimization problem (24) where $F[\rho]$ is given by Equation (20).

The critical points of free energy at fixed mass are determined by the condition

$$\delta F - \mu \delta M = 0, \tag{25}$$

where μ is a Lagrange multiplier (chemical potential) associated with the conservation of mass. This yields

$$\frac{\delta F}{\delta \rho} = \mu. \tag{26}$$

Using the expression (20) of the free energy, we obtain

$$C'(\rho) = -\beta\Phi - \alpha, \tag{27}$$

where we have defined $\alpha = -\mu/k_B T$. Substituting Equation (13) in Equation (27), we find that the density is determined by the integro-differential equation

$$C'(\rho) = -\beta \int u(|\mathbf{r} - \mathbf{r}'|)\rho(\mathbf{r}')\, d\mathbf{r} - \alpha. \tag{28}$$

Since C is convex, Equation (27) can be reversed to give

$$\rho(\mathbf{r}) = F[\beta\Phi(\mathbf{r}) + \alpha], \tag{29}$$

where $F(x) = (C')^{-1}(-x)$. We note that, at equilibrium, the density is a function of the potential: $\rho = \rho(\Phi)$. Taking the derivative of Equation (27) with respect to ρ, we get

$$\rho'(\Phi) = -\frac{\beta}{C''(\rho)}. \tag{30}$$

Since $C'' > 0$, the relation $\rho = \rho(\Phi)$ is monotonic. It is decreasing at positive temperatures (which is the normal case) and increasing at negative temperatures.

A critical point of free energy at fixed mass is a (local) minimum if, and only if,

$$\delta^2 F = k_B T \int C''(\rho)\frac{(\delta\rho)^2}{2}\, d\mathbf{r} + \frac{1}{2}\int \delta\rho\delta\Phi\, d\mathbf{r} > 0, \tag{31}$$

or equivalently if, and only if,

$$\delta^2 F = -\frac{1}{2}\left\{\int \frac{(\delta\rho)^2}{\rho'(\Phi)}\, d\mathbf{r} - \int \delta\rho\delta\Phi\, d\mathbf{r}\right\} > 0 \tag{32}$$

for all perturbations $\delta\rho$ that conserve mass: $\delta M = \int \delta\rho\, d\mathbf{r} = 0$.

Remark 4. *if the system is only subjected to an external potential $\Phi_{\rm ext}(\mathbf{r})$, the equilibrium state is given by $\rho(\mathbf{r}) = F[\beta\Phi_{\rm ext}(\mathbf{r}) + \alpha]$. The second variations of free energy are always positive*

$$\delta^2 F = k_B T \int C''(\rho)\frac{(\delta\rho)^2}{2}\, d\mathbf{r} > 0, \tag{33}$$

so that a critical point of F at fixed mass is necessarily a minimum. Therefore, the system has only one (global) minimum of free energy at fixed mass.

Remark 5. *Equation (29) determines the equilibrium state from the knowledge of the generalized entropy. Inversely, in certain cases, we know the equilibrium state and we want to determine the corresponding entropy. If we prescribe the equilibrium state in the form $\rho = F(\beta\Phi + \alpha)$, we have $C'(\rho) = -F^{-1}(\rho)$ so the generalized entropy is given by Equation (19) with* [46]*:*

$$C(\rho) = -\int^{\rho} F^{-1}(x)\, dx. \tag{34}$$

Equation (34) determines the generalized entropy from the knowledge of the equilibrium state.

3. Generalized Mean Field Fokker-Planck Equations

3.1. Simple Systems: The Mean Field Smoluchowski Equation

For simple systems with long-range interactions, the equation governing the dynamical evolution of the density of Brownian particles $\rho(\mathbf{r}, t) = NmP_1(\mathbf{r}, t)$ can be obtained as follows [14]. One starts from the N-body Smoluchowski Equation (3) and writes down the equivalent of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy for the reduced probability distributions $P_j(\mathbf{r}_1, ..., \mathbf{r}_j, t)$. The equation for the one-body distribution function is

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[D\nabla\rho + \chi\int \rho_2(\mathbf{r}, \mathbf{r}', t)\nabla u(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}'\right], \tag{35}$$

where $\rho_2(\mathbf{r},\mathbf{r}',t) = N(N-1)m^2 P_2(\mathbf{r},\mathbf{r}',t)$ is the two-body distribution function. One then closes the hierarchy of equations in the limit $N \to +\infty$ by using the mean field approximation (11). In that limit, $\rho_2(\mathbf{r},\mathbf{r}',t) = \rho(\mathbf{r},t)\rho(\mathbf{r}',t)$. This leads to the mean field Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D\nabla\rho + \chi\rho\nabla\Phi)\,, \tag{36}$$

where $\Phi(\mathbf{r},t)$ is given by Equation (13). Using the Einstein relation (2), the mean field Smoluchowski equation can be rewritten as

$$\xi\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{k_B T}{m}\nabla\rho + \rho\nabla\Phi\right). \tag{37}$$

Introducing the Boltzmann free energy

$$F_B[\rho] = \frac{1}{2}\int \rho\Phi\, d\mathbf{r} + k_B T\int \frac{\rho}{m}\ln\frac{\rho}{m}\, d\mathbf{r} - \frac{dN}{2}k_B T\ln\left(\frac{2\pi k_B T}{m}\right), \tag{38}$$

we can write Equation (37) in the form

$$\xi\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho\nabla\frac{\delta F_B}{\delta\rho}\right). \tag{39}$$

This equation decreases the Boltzmann free energy (38) at fixed mass (H-theorem) and relaxes towards the mean field Boltzmann distribution

$$\rho(\mathbf{r}) = Ae^{-\beta m\Phi(\mathbf{r})} \tag{40}$$

which is the minimum of $F_B[\rho]$ at fixed mass. Therefore, the mean field Smoluchowski equation is fully consistent with Boltzmann's thermodynamics.

3.2. Complex Systems: The Generalized Mean Field Smoluchowski Equation

For complex systems, the diffusion and the mobility coefficients depend on the density [45,46]. This is because the transition probabilities from one site to the other depend on the occupation number of the initial and arrival sites [24]. This is due to the effect of small-scale constraints acting on the system. As a result, the mean field Smoluchowski equation (36) is replaced by the generalized mean field Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\nabla(D(\rho)\rho) + \chi(\rho)\rho\nabla\Phi\right]. \tag{41}$$

We can rewrite this equation in the form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (Dh(\rho)\nabla\rho + \chi g(\rho)\nabla\Phi)\,, \tag{42}$$

where D and $\chi = 1/\xi$ are constants. The generalized mean field Smoluchowski equation (42) is a generalized mean field Fokker-Planck (GFP) equation [45,46]. It can be derived from a kinetic theory (see [24] and Section 2.11 of [46]) in which the transition probabilities from one site to the other can be inhibited or, inversely, stimulated. In that case, the functions $h(\rho)$ and $g(\rho)$ can be related to the functions $a(\rho)$ and $b(\rho)$ appearing in the transition probabilities (see [24] and Section 2.11 of [46]). The case of simple systems is recovered for $a(\rho) = \rho$ and $b(\rho) = 1$, leading to $h(\rho) = 1$ and $g(\rho) = \rho$.

We assume that the coefficients of diffusion D and mobility χ are related to each other by the generalized Einstein relation

$$D = T\chi, \tag{43}$$

where T plays the role of a generalized temperature (it has not, in general, the dimension of a temperature). Actually, this relation can be regarded as a definition of the generalized temperature T for given D and χ. We note that the form of the Einstein relation (2) is preserved in the generalized thermodynamics framework.

Using Equation (43), and setting $\beta = 1/T$, we can write the GFP equation (42) in the equivalent forms

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(T h(\rho) \nabla \rho + g(\rho) \nabla \Phi \right), \tag{44}$$

and

$$\frac{\partial \rho}{\partial t} = D \nabla \cdot \left(h(\rho) \nabla \rho + \beta g(\rho) \nabla \Phi \right). \tag{45}$$

3.3. Gradient Flow

The GFP equation (44) can be rewritten as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ g(\rho) \left[T \frac{h(\rho)}{g(\rho)} \nabla \rho + \nabla \Phi \right] \right\}. \tag{46}$$

For the generalized free energy

$$F[\rho] = E[\rho] - TS[\rho] = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + T \int C(\rho) \, d\mathbf{r}, \tag{47}$$

we have the identities

$$\frac{\delta F}{\delta \rho} = TC'(\rho) + \Phi, \tag{48}$$

$$\nabla \frac{\delta F}{\delta \rho} = TC''(\rho) \nabla \rho + \nabla \Phi. \tag{49}$$

Comparing Equations (46) and (49), we see that the GFP equation (44) can be written as a gradient flow of the form

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[g(\rho) \nabla \frac{\delta F}{\delta \rho} \right] \tag{50}$$

provided that we make the identification

$$C''(\rho) = \frac{h(\rho)}{g(\rho)}. \tag{51}$$

This is a very important relation that connects the generalized entropy $S = -\int C(\rho) \, d\mathbf{r}$ obtained by general arguments in Section 2.4 to the functions $h(\rho)$ and $g(\rho)$ occurring in the GFP equation (44) and, correspondingly, to the functions $a(\rho)$ and $b(\rho)$ determining the microscopic process underlying the dynamics (see [24] and Section 2.11 of [46]). Since h and g are positive in general, the function C is convex. We note that there exists an infinity of FP equations with the same entropy $C(\rho)$ but different functions $g(\rho)$ and $h(\rho)$, namely all the equations with the same ratio $h(\rho)/g(\rho)$ [46].

The GFP equation (50) can be put in the conservative form

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}, \tag{52}$$

with the current

$$\mathbf{J} = -\frac{1}{\xi} g(\rho) \nabla \frac{\delta F}{\delta \rho}. \tag{53}$$

Since the right hand side of Equation (52) is the divergence of a current $\mathbf{J}$, the mass $M = \int \rho \, d\mathbf{r}$ is conserved provided that the current vanishes at infinity or that the normal component of the current vanishes on the boundary of the system.

Explicating the expression of the current (53) using Equation (49), we get

$$\mathbf{J} = -\frac{1}{\xi} g(\rho) \left(T C''(\rho) \nabla \rho + \nabla \Phi \right). \tag{54}$$

Therefore, the GFP equation (50) can be rewritten as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[g(\rho) \left(T C''(\rho) \nabla \rho + \nabla \Phi \right) \right]. \tag{55}$$

At $T = 0$, we get

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(g(\rho) \nabla \Phi \right). \tag{56}$$

3.4. Generalized H-Theorem

The form of Equation (50) ensures that the generalized free energy (47) decreases monotonically. Indeed, the time derivative of the generalized free energy is given by

$$\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} \, d\mathbf{r} = -\int \frac{\delta F}{\delta \rho} \nabla \cdot \mathbf{J} \, d\mathbf{r} = \int \mathbf{J} \cdot \nabla \frac{\delta F}{\delta \rho} \, d\mathbf{r} = -\frac{1}{\xi} \int g(\rho) \left(\nabla \frac{\delta F}{\delta \rho} \right)^2 d\mathbf{r}. \tag{57}$$

We note that

$$\dot{F} = -\xi \int \frac{\mathbf{J}^2}{g(\rho)} \, d\mathbf{r}. \tag{58}$$

Using Equation (54), we explicitly obtain

$$\dot{F} = -\frac{1}{\xi} \int g(\rho) \left(T C''(\rho) \nabla \rho + \nabla \Phi \right)^2 d\mathbf{r}. \tag{59}$$

From these expressions, we conclude that $\dot{F} \leq 0$. This is the generalized version of the H-theorem in the canonical ensemble.

A steady state of Equation (50) satisfies $\dot{F} = 0$, implying $\mathbf{J} = \mathbf{0}$, *i.e.*, $\nabla(\delta F/\delta \rho) = \mathbf{0}$. Therefore, a steady state is determined by the condition (26), where μ is a constant of integration which plays the role of a chemical potential. This condition is equivalent to Equations (27)–(29). Therefore, a steady state of the GFP equation (50) is a critical point of free energy F at fixed mass. Using Lyapunov's direct method [45], one can show that a steady state of the GFP equation (50) is dynamically stable if, and only if, it is a (local) minimum of F at fixed mass (maxima or saddle points of F are dynamically unstable). In this sense, dynamical and generalized thermodynamical stability in the canonical ensemble coincide. In general, the GFP equation relaxes towards a stable steady state for $t \to +\infty$ and stays there permanently. If several stable steady states exist, the choice of equilibrium is determined by a complicated notion of *basin of attraction*. Of course, there exist situations where the system has an even more complex dynamics. This is the case, for example, of self-gravitating Brownian particles (or bacterial populations) that can experience gravitational collapse (or chemotactic collapse) and never reach a steady state [16].

3.5. Onsager's Linear Thermodynamics

If we define an out-of-equilibrium chemical potential by

$$\mu(\mathbf{r}, t) \equiv \frac{\delta F}{\delta \rho} = TC'(\rho) + \Phi, \tag{60}$$

we can write the current (53) as

$$\mathbf{J} = -\frac{1}{\xi} g(\rho) \nabla \mu. \tag{61}$$

Accordingly, the GFP equation (50) becomes

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (g(\rho) \nabla \mu) \,. \tag{62}$$

This corresponds to Onsager's linear thermodynamics where the current is proportional to the gradient of the chemical potential that is uniform at equilibrium [see Equation (26)]. In the present case, the coefficient of proportionality depends on the density through the function $g(\rho)$. Even for simple systems we have a density dependence since $g(\rho) = \rho$.

Remark 6. *The GFP equation (50) can also be obtained from a variational principle called the principle of maximum dissipation of free energy (see Section 2.10.3 of* [46]*). Indeed, the current* $\mathbf{J}_* = -(1/\xi) g(\rho) \nabla(\delta F/\delta \rho)$ *minimizes the functional* $\dot{F} + E_d$ *where* $\dot{F}[\mathbf{J}] = \int \mathbf{J} \cdot \nabla(\delta F/\delta \rho)\, d\mathbf{r}$ *and* $E_d[\mathbf{J}] = \xi \int \mathbf{J}^2/(2g(\rho))\, d\mathbf{r}$. *We have* $\delta(\dot{F} + E_d) = 0$ *and* $\delta^2(\dot{F} + E_d) = \xi \int (\delta \mathbf{J})^2/(2g(\rho))\, d\mathbf{r} > 0$ *(there is a misprint in* [46]*)). Furthermore,* $E_d[\mathbf{J}_*] = -(1/2)\dot{F}[\mathbf{J}_*]$.

3.6. Particular Form: Normal Mobility and Generalized Diffusion

If we assume that $g(\rho) = \rho$ and $h(\rho) = \rho C''(\rho)$, the GFP equation (50) becomes

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \frac{\delta F}{\delta \rho} \right). \tag{63}$$

It can be explicitly written in the equivalent forms

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (T \rho C''(\rho) \nabla \rho + \rho \nabla \Phi) \,, \tag{64}$$

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \rho C''(\rho) \nabla \rho + \chi \rho \nabla \Phi) \,, \tag{65}$$

$$\frac{\partial \rho}{\partial t} = D \nabla \cdot (\rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi) \,. \tag{66}$$

In that case, we have a constant mobility $\chi(\rho) = \chi$ and a density-dependent diffusion coefficient $D(\rho) = D\rho[C(\rho)/\rho]'$ (see Equation (41)). We note that the condition $D(\rho) \geq 0$ requires that $[C(\rho)/\rho]' \geq 0$. This gives a constraint on the possible forms of $C(\rho)$.

3.7. Particular Form: Normal Diffusion and Generalized Mobility

If we assume that $h(\rho) = 1$ and $g(\rho) = 1/C''(\rho)$, the GFP equation (50) becomes

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{C''(\rho)} \nabla \frac{\delta F}{\delta \rho} \right]. \tag{67}$$

It can be explicitly written in the equivalent forms

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T \nabla \rho + \frac{1}{C''(\rho)} \nabla \Phi \right], \tag{68}$$

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[D \nabla \rho + \frac{\chi}{C''(\rho)} \nabla \Phi \right], \tag{69}$$

$$\frac{\partial \rho}{\partial t} = D \nabla \cdot \left[\nabla \rho + \frac{\beta}{C''(\rho)} \nabla \Phi \right]. \tag{70}$$

In that case, we have a constant diffusion coefficient $D(\rho) = D$ and a density-dependent mobility $\chi(\rho) = \chi/(\rho C''(\rho))$.

3.8. Generalized Smoluchowski Equation

The GFP equation (64) corresponding to a normal mobility and a generalized diffusion can be written in the form of a generalized mean field Smoluchowski equation [39]:

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P + \rho \nabla \Phi), \tag{71}$$

where P is a pressure given by a barotropic equation of state $P(\rho)$ determined by

$$P'(\rho) = T \rho C''(\rho). \tag{72}$$

Since $C(\rho)$ is convex, we find that $P'(\rho) \geq 0$. The steady states of Equation (71) satisfy the condition of hydrostatic equilibrium

$$\nabla P + \rho \nabla \Phi = \mathbf{0}. \tag{73}$$

Integrating Equation (72) once, we get

$$P(\rho) = T \rho^2 \left[\frac{C(\rho)}{\rho} \right]' = T[C'(\rho)\rho - C(\rho)]. \tag{74}$$

A second integration leads to the identity

$$TC(\rho) = \rho \int^{\rho} \frac{P(\rho')}{\rho'^2} d\rho'. \tag{75}$$

Using Equation (75), the free energy (47) can be rewritten as

$$F[\rho] = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + \int \rho \int^{\rho} \frac{P(\rho')}{\rho'^2} d\rho' d\mathbf{r}. \tag{76}$$

3.9. Expression in Terms of the Enthalpy

Introducing the pressure defined by Equation (72), the GFP equation (55) can be rewritten as

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[\frac{g(\rho)}{\rho}\left(\nabla P+\rho\nabla\Phi\right)\right]. \tag{77}$$

We define the enthalpy through the relation $dh=dP/\rho$ implying

$$h'(\rho)=\frac{P'(\rho)}{\rho}=TC''(\rho). \tag{78}$$

Integrating Equation (78) once, we get

$$h(\rho)=TC'(\rho). \tag{79}$$

A second integration leads to

$$TC(\rho)=H(\rho), \tag{80}$$

where H is a primitive of h. Using Equation (80), the free energy (47) can be rewritten as

$$F[\rho]=\frac{1}{2}\int\rho\Phi\,d\mathbf{r}+\int H(\rho)\,d\mathbf{r}. \tag{81}$$

On the other hand, since $\nabla P/\rho=\nabla h$, Equation (77) can be rewritten as

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[g(\rho)\left(\nabla h+\nabla\Phi\right)\right]. \tag{82}$$

Comparing Equation (82) with Equation (62), we see that the chemical potential is related to the enthalpy by

$$\mu(\mathbf{r},t)=h(\mathbf{r},t)+\Phi(\mathbf{r},t). \tag{83}$$

At equilibrium, we have

$$h(\mathbf{r})+\Phi(\mathbf{r})=\mu, \tag{84}$$

where μ is a constant.

4. Theory of Fluctuations

The GFP equations of the previous section ignore fluctuations. They are valid in the $N\rightarrow+\infty$ limit where the mean field approximation is exact. Fluctuations become important when the number of particles is small and/or when the system is close to a critical point. In order to take fluctuations into account, we need to replace the deterministic GFP equations by stochastic GFP equations. This can be achieved by using the theory of fluctuating hydrodynamics developed by Landau and Lifshitz [51]. In [15], we have applied their procedure to the case of simple systems described by the Boltzmann entropy, and we have derived a stochastic Smoluchowski equation. Here, we generalize their procedure to complex systems described by generalized entropic functionals, and derive a generalized stochastic Smoluchowski equation.

4.1. Simple Systems: The Stochastic Smoluchowski Equation

In the case of simple systems described by the N-body stochastic equations (1), Dean [52] has shown that the discrete density $\rho_d(\mathbf{r},t)=\sum_i \delta(\mathbf{r}-\mathbf{r}_i(t))$ of particles satisfies a stochastic partial differential equation

$$\xi\frac{\partial\rho_d}{\partial t}=\nabla\cdot\left(\frac{k_B T}{m}\nabla\rho_d+\rho_d\nabla\Phi_d\right)+\nabla\cdot\left[\sqrt{2\xi k_B T\rho_d}\,\mathbf{R}(\mathbf{r},t)\right], \tag{85}$$

where $\Phi_d(\mathbf{r},t)$ is given by Equation (13) in which $\rho(\mathbf{r},t)$ is replaced by $\rho_d(\mathbf{r},t)$. On the other hand, $\mathbf{R}(\mathbf{r},t)$ is a Gaussian white noise satisfying $\langle\mathbf{R}(\mathbf{r},t)\rangle=\mathbf{0}$ and $\langle R^\alpha(\mathbf{r},t)R^\beta(\mathbf{r},t')\rangle=\delta_{\alpha\beta}\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$, where $\alpha=1,...,d$ labels the coordinates of space. This equation is exact and bears the same information as the N-body Langevin equations (1), or as the N-body Smoluchowski equation (3). In this sense, it contains too much information to be of practical use. Furthermore, $\rho_d(\mathbf{r},t)$ is a sum of Dirac δ-functions, which is not easy to handle. If we take the ensemble, or noise, average of Equation (85), and make a mean field approximation, we recover the mean field Smoluchowski equation (37). However, in that case, we have lost the effect of fluctuations.

In [15], we have argued that, for simple systems with long-range interactions, the evolution of the smooth (coarse-grained) density $\overline{\rho}(\mathbf{r},t)$ is governed by the stochastic Smoluchowski equation

$$\xi\frac{\partial\overline{\rho}}{\partial t}=\nabla\cdot\left(\frac{k_B T}{m}\nabla\overline{\rho}+\overline{\rho}\nabla\overline{\Phi}\right)+\nabla\cdot\left[\sqrt{2\xi k_B T\overline{\rho}}\,\mathbf{R}(\mathbf{r},t)\right], \tag{86}$$

where $\overline{\Phi}(\mathbf{r},t)$ is given by Equation (13) in which $\rho(\mathbf{r},t)$ is replaced by $\overline{\rho}(\mathbf{r},t)$. This equation can be obtained from the theory of fluctuating hydrodynamics (see Appendix B of [15]). Although it has a mathematical form similar to Equation (85), this equation is fundamentally different from Equation (85) since it applies to a *smooth* density $\overline{\rho}(\mathbf{r},t)$, not to a sum of δ-functions. In a sense, it describes the evolution of the system at a mesoscopic level, intermediate between Equations (85) and (37).

Introducing the Boltzmann free energy (38), we can rewrite Equation (86) in the form

$$\xi\frac{\partial\overline{\rho}}{\partial t}=\nabla\cdot\left(\overline{\rho}\nabla\frac{\delta F_B}{\delta\overline{\rho}}\right)+\nabla\cdot\left[\sqrt{2\xi k_B T\overline{\rho}}\,\mathbf{R}(\mathbf{r},t)\right]. \tag{87}$$

The stochastic Smoluchowski equation can be interpreted as a stochastic Langevin equation for the density field $\overline{\rho}(\mathbf{r},t)$. The corresponding FP equation for the probability density $P[\overline{\rho},t]$ of the density field $\overline{\rho}(\mathbf{r},t)$ at time t is

$$\xi\frac{\partial P}{\partial t}[\overline{\rho},t]=-\int\frac{\delta}{\delta\overline{\rho}(\mathbf{r}',t)}\left\{\nabla\cdot\overline{\rho}\nabla\left[k_B T\frac{\delta}{\delta\overline{\rho}}+\frac{\delta F_B}{\delta\overline{\rho}}\right]P[\overline{\rho},t]\right\}\,d\mathbf{r}'. \tag{88}$$

Its stationary solution returns the canonical distribution (21) with the Boltzmann free energy (38).

4.2. Complex Systems: The Generalized Stochastic Smoluchowski Equation

If we apply the theory of fluctuations to the generalized Smoluchowski equation (44), we obtain the stochastic partial differential equation (see Appendix A):

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot(Th(\rho)\nabla\rho+g(\rho)\nabla\Phi)+\nabla\cdot\left[\sqrt{2\xi T g(\rho)}\mathbf{R}(\mathbf{r},t)\right]. \tag{89}$$

This equation applies to the coarse-grained density $\overline{\rho}(\mathbf{r}, t)$ as explained above but, to simplify the notations, we have not written the bar on ρ. Introducing the generalized free energy functional (47), the generalized stochastic Smoluchowski equation can be rewritten as

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[g(\rho)\nabla\frac{\delta F}{\delta\rho}\right] + \nabla\cdot\left[\sqrt{2\xi T g(\rho)}\mathbf{R}(\mathbf{r}, t)\right]. \tag{90}$$

We note that, in general, the noise term is multiplicative since it depends on the density through the function $g(\rho)$. That the noise term depends on the density is also true for simple systems for which $g(\rho) = \rho$. The noise term is independent on ρ when $g(\rho) = 1$. For $h(\rho) = \rho$, corresponding to a normal diffusion, we get $C(\rho) = \rho^2/2$, *i.e.*, $S = -\frac{1}{2}\int \rho^2\, d\mathbf{r}$. This is a Tsallis entropy of index $q = 2$ (see Section 5). We note that the function $h(\rho)$ does not appear in the noise term.

Equation (89) is the main equation of this paper. It will be called the generalized stochastic Smoluchowski equation, or the generalized stochastic Fokker-Planck equation. It can be interpreted as a stochastic Langevin equation for the density field $\rho(\mathbf{r}, t)$. The corresponding FP equation for the probability density $P[\rho, t]$ of the density field $\rho(\mathbf{r}, t)$ at time t is

$$\xi\frac{\partial P}{\partial t}[\rho, t] = -\int \frac{\delta}{\delta\rho(\mathbf{r}', t)}\left\{\nabla\cdot g(\rho)\nabla\left[T\frac{\delta}{\delta\rho} + \frac{\delta F}{\delta\rho}\right]P[\rho, t]\right\} d\mathbf{r}'. \tag{91}$$

Its stationary solution returns the canonical distribution (21) with the generalized free energy (47), which shows the consistency of our approach. Actually, the form of the noise can be determined precisely in order to recover this distribution at equilibrium. Considering the form of the deterministic term in Equation (90), we see that the noise term must be multiplicative with a factor $\sqrt{g(\rho)}$.

For $N \to +\infty$, we can neglect the noise in Equation (90). In that case, we recover the deterministic GFP equation (50). In the absence of noise, the deterministic GFP equation (50) relaxes towards a stable steady state which is a (local) minimum of generalized free energy at fixed mass. If the free energy $F[\rho]$ has several (local) minima, the choice of equilibrium depends on a notion of basin of attraction. Once a stable steady state is reached, the system stays there permanently. This is because Equation (50) is valid in the $N \to \infty$ limit. In that limit, the lifetime of a metastable state is infinite [50]. For finite N systems, there are fluctuations. Because of the fluctuations, the system switches from one minimum of free energy to another and explores the whole free energy landscape [47]. The equilibrium distribution of the smooth density is given by the canonical distribution (21). The lifetime of the metastable states is given by the Kramers formula [47].

4.3. Particular Form: Normal Mobility and Generalized Diffusion

If we assume that $g(\rho) = \rho$ and $h(\rho) = \rho C''(\rho)$, the stochastic GFP equation (90) becomes

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\rho\nabla\frac{\delta F}{\delta\rho}\right) + \nabla\cdot\left(\sqrt{2\xi T\rho}\,\mathbf{R}\right). \tag{92}$$

It can be written explicitly in the equivalent forms

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left(T\rho C''(\rho)\nabla\rho+\rho\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2\xi T\rho}\,\mathbf{R}\right), \tag{93}$$

$$\frac{\partial\rho}{\partial t}=\nabla\cdot\left(D\rho C''(\rho)\nabla\rho+\chi\rho\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2D\rho}\,\mathbf{R}\right), \tag{94}$$

$$\frac{\partial\rho}{\partial t}=D\nabla\cdot\left(\rho C''(\rho)\nabla\rho+\beta\rho\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2D\rho}\,\mathbf{R}\right). \tag{95}$$

Introducing the pressure defined by Equation (72), we obtain the generalized stochastic Smoluchowski equation

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left(\nabla P+\rho\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2\xi T\rho}\,\mathbf{R}\right). \tag{96}$$

4.4. Particular Form: Normal Diffusion and Generalized Mobility

If we assume that $h(\rho)=1$ and $g(\rho)=1/C''(\rho)$, the stochastic GFP equation (90) becomes

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[\frac{1}{C''(\rho)}\nabla\frac{\delta F}{\delta\rho}\right]+\nabla\cdot\left(\sqrt{\frac{2\xi T}{C''(\rho)}}\mathbf{R}\right). \tag{97}$$

It can be written explicitly in the equivalent forms

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[T\nabla\rho+\frac{1}{C''(\rho)}\nabla\Phi\right]+\nabla\cdot\left(\sqrt{\frac{2\xi T}{C''(\rho)}}\mathbf{R}\right), \tag{98}$$

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[D\nabla\rho+\frac{\chi}{C''(\rho)}\nabla\Phi\right]+\nabla\cdot\left(\sqrt{\frac{2D}{C''(\rho)}}\mathbf{R}\right), \tag{99}$$

$$\frac{\partial\rho}{\partial t} = D\nabla\cdot\left[\nabla\rho+\frac{\beta}{C''(\rho)}\nabla\Phi\right]+\nabla\cdot\left(\sqrt{\frac{2D}{C''(\rho)}}\mathbf{R}\right). \tag{100}$$

4.5. Equivalent Forms of the Generalized Stochastic Fokker-Planck Equation

We list below different explicit expressions of the stochastic GFP equation:

$$\frac{\partial\rho}{\partial t}=\nabla\cdot\left(Dh(\rho)\nabla\rho+\chi g(\rho)\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2Dg(\rho)}\,\mathbf{R}\right), \tag{101}$$

$$\frac{\partial\rho}{\partial t}=D\nabla\cdot\left(h(\rho)\nabla\rho+\beta g(\rho)\nabla\Phi\right)+\nabla\cdot\left(\sqrt{2Dg(\rho)}\,\mathbf{R}\right), \tag{102}$$

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[g(\rho)\left(TC''(\rho)\nabla\rho+\nabla\Phi\right)\right]+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\mathbf{R}\right), \tag{103}$$

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left(g(\rho)\nabla\mu\right)+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\mathbf{R}\right), \tag{104}$$

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[\frac{g(\rho)}{\rho}\left(\nabla P+\rho\nabla\Phi\right)\right]+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\mathbf{R}\right), \tag{105}$$

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[g(\rho)\left(\nabla h+\nabla\Phi\right)\right]+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\mathbf{R}\right). \tag{106}$$

5. A New Form of Generalized Entropy

In [46], we have introduced a GFP equation of the form

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[T\nabla\rho^{\gamma}+\rho(1-K\rho/\sigma_0)\nabla\Phi\right]. \tag{107}$$

It corresponds to Equation (44) with $h(\rho)=\gamma\rho^{\gamma-1}$ and $g(\rho)=\rho(1-K\rho/\sigma_0)$. For $\gamma\neq 1$ and $\sigma_0<+\infty$, we get a GFP equation with a power law diffusion and a density-dependent mobility taking into account exclusion ($K>0$) or inclusion ($K<0$) constraints in position space. It is associated with the generalized entropy $S=-\int C(\rho)\,d\mathbf{r}$ with [see equation (51)]:

$$C(\rho)=\gamma\sigma_0^{\gamma}\int_0^{\rho/\sigma_0}\Phi_{\gamma-2}(t)\,dt, \tag{108}$$

where

$$\Phi_m(t)=\int_0^t\frac{x^m}{1-Kx}\,dx. \tag{109}$$

The generalized entropy can be expressed in terms of hypergeometric functions. Indeed, one has

$$\Phi_m(t)=\frac{t^{m+1}}{m+1}\,{}_2F_1(1,1+m,2+m,Kt) \tag{110}$$

and

$$C(\rho)=\frac{\rho^{\gamma}}{\gamma-1}\,{}_2F_1\left(1,\gamma-1,\gamma+1,\frac{K\rho}{\sigma_0}\right). \tag{111}$$

The pressure $P=T\rho^{\gamma}$ is that of a polytrope of index γ. The generalized Smoluchowski equation (71) with the same entropy as Equation (107) has an equation of state given by (see Equation (72)):

$$P_{GS}=\gamma T\sigma_0^{\gamma}\Phi_{\gamma-1}\left(\frac{\rho}{\sigma_0}\right). \tag{112}$$

Equation (107) generalizes many GFP equations introduced in the literature [45,46].

(i) For $\gamma=1$ and $\sigma_0\to+\infty$, we recover the Smoluchowski equation which is a FP equation with a normal diffusion and a normal mobility [4]. It is associated with the Boltzmann entropy

$$S=-\int\rho\ln\rho\,d\mathbf{r}. \tag{113}$$

The steady state of the Smoluchowski equation is the Boltzmann distribution

$$\rho=e^{-\beta\Phi-\alpha-1}. \tag{114}$$

(ii) For $\gamma\neq 1$ and $\sigma_0\to+\infty$, we get a GFP equation with an anomalous diffusion and a constant mobility [30,31]. It is associated with the Tsallis entropy

$$S=-\frac{1}{\gamma-1}\int(\rho^{\gamma}-\rho)\,d\mathbf{r}, \tag{115}$$

where the polytropic index γ plays the role of the usual Tsallis index q. The steady state of the Smoluchowski equation with an anomalous diffusion is the Tsallis distribution

$$\rho=\left[\frac{1}{\gamma}-\frac{\gamma-1}{\gamma}(\beta\Phi+\alpha)\right]_+^{1/(\gamma-1)} \tag{116}$$

with the notation $[x]_+ = x$ when $x > 0$ and $[x]_+ = 0$ when $x < 0$.

(iii) For $\gamma = 1$ and $\sigma_0 < +\infty$, we get the fermionic ($K = +1$) or bosonic ($K = -1$) Smoluchowski equation, which is a GFP equation with a normal diffusion and a variable mobility taking into account exclusion or inclusion constraints in position space [20,27–29,32,36,37,39,42,43,46,53]. It is associated with the Fermi-Dirac ($K = +1$) or Bose-Einstein ($K = -1$) entropy in position space

$$S = -\sigma_0 \int \left\{ \frac{\rho}{\sigma_0} \ln \frac{\rho}{\sigma_0} + \frac{1}{K} \left(1 - \frac{K\rho}{\sigma_0} \right) \ln \left(1 - \frac{K\rho}{\sigma_0} \right) \right\} d\mathbf{r}. \tag{117}$$

The steady state of the fermionic/bosonic Smoluchowski equation is the Fermi-Dirac/Bose-Einstein distribution

$$\rho = \frac{\sigma_0}{e^{\beta\Phi+\alpha} + K}. \tag{118}$$

For $K \neq \pm 1$, the distribution (118) describes intermediate statistics [24,46]. The generalized Smoluchowski equation (71) with the same entropy has an equation of state [46]:

$$P_{GS} = -T\frac{\sigma_0}{K} \ln(1 - K\rho/\sigma_0). \tag{119}$$

(iv) For $\gamma = 2$, Equation (107) reduces to [46]:

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T\nabla\rho^2 + \rho(1 - K\rho/\sigma_0)\nabla\Phi \right]. \tag{120}$$

In that case, the generalized entropy (108)-(109) has a simple explicit expression [46]:

$$S = -\frac{2\sigma_0^2}{K^2} \int \left(1 - \frac{K\rho}{\sigma_0} \right) \ln \left(1 - \frac{K\rho}{\sigma_0} \right) d\mathbf{r}. \tag{121}$$

It can be viewed as the difference between a Fermi-Dirac/Bose-Einstein-type entropy and the Boltzmann entropy. The steady state of Equation (120) is given by

$$\rho = \frac{\sigma_0}{K} \left[1 - e^{\frac{K}{2\sigma_0}(\beta\Phi+\alpha)} \right]_+. \tag{122}$$

The generalized Smoluchowski equation (71) with the same entropy as Equation (120) has an equation of state

$$P_{GS} = -2T\frac{\sigma_0}{K} \left[\rho + \frac{\sigma_0}{K} \ln(1 - K\rho/\sigma_0) \right]. \tag{123}$$

(v) For $\gamma \to 0$ and $T \to +\infty$ in such a way that γT is finite, and noted T again, Equation (107) becomes

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T\nabla \ln \rho + \rho(1 - K\rho/\sigma_0)\nabla\Phi \right]. \tag{124}$$

It corresponds to Equation (44) with $h(\rho) = 1/\rho$ and $g(\rho) = \rho(1 - K\rho/\sigma_0)$. It is associated with the generalized entropy

$$S = -\int \left[-\ln\rho + \frac{K\rho}{\sigma_0} \ln \left(\frac{\rho/\sigma_0}{1 - K\rho/\sigma_0} \right) + \ln(1 - K\rho/\sigma_0) \right] d\mathbf{r}. \tag{125}$$

The pressure $P = T \ln \rho$ is that of a logotrope [54]. The steady state of Equation (124) is given by

$$\rho = \frac{\sigma_0/K}{1 + W\left[\frac{1}{K} e^{\frac{\sigma_0}{K}(\beta\Phi+\alpha)-1} \right]}, \tag{126}$$

where $W(z)$ is the Lambert function defined implicitly by the equation $We^W = z$. The generalized Smoluchowski equation (71) with the same entropy as Equation (124) has an equation of state

$$P_{GS} = T \ln\left(\frac{\rho}{1 - K\rho/\sigma_0}\right). \tag{127}$$

For $\sigma_0 \to +\infty$, the generalized entropy (125) reduces to the log-entropy [54]:

$$S = \int \ln\rho \, d\mathbf{r}, \tag{128}$$

and the steady state of Equation (124) is given by the Lorentzian-type distribution

$$\rho = \frac{1}{\beta\Phi + \alpha}. \tag{129}$$

If we account for fluctuations, Equation (107) is replaced by the stochastic GFP equation

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[T\nabla\rho^\gamma + \rho(1 - K\rho/\sigma_0)\nabla\Phi\right] + \nabla\cdot\left[\sqrt{2\xi T\rho(1 - K\rho/\sigma_0)}\mathbf{R}(\mathbf{r},t)\right]. \tag{130}$$

For systems with long-range interactions, the potential $\Phi(\mathbf{r}, t)$ is given by Equation (13). Depending on the form of the potential of interaction, the steady states of Equation (107) exhibit a rich variety of phase transitions, as studied in [53] for self-gravitating systems and bacterial populations. When fluctuations are taken into account, the system exhibits random transitions from one phase to the other similarly to the study of [47].

6. Generalized Stochastic Cahn-Hilliard Equations

6.1. Short-Range Interactions

We assume that $u(|\mathbf{r} - \mathbf{r}'|)$ is a short-range potential of interaction but that the stochastic GFP equation (90) remains valid. This is the case for a potential $u(|\mathbf{r} - \mathbf{r}'|)$ that is screened on a distance that is short with respect to the system size but large with respect to the characteristic microscopic scale. Setting $\mathbf{q} = \mathbf{r}' - \mathbf{r}$ and writing

$$\Phi(\mathbf{r}, t) = \int u(q)\rho(\mathbf{r} + \mathbf{q}, t)d\mathbf{q}, \tag{131}$$

we can Taylor expand $\rho(\mathbf{r} + \mathbf{q}, t)$ to second order in $\mathbf{q}$ so that

$$\rho(\mathbf{r} + \mathbf{q}, t) = \rho(\mathbf{r}, t) + \sum_i \frac{\partial\rho}{\partial x_i} q_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2\rho}{\partial x_i \partial x_j} q_i q_j. \tag{132}$$

Substituting this expansion in Equation (131), we obtain

$$\Phi(\mathbf{r}, t) = -a\rho(\mathbf{r}, t) - \frac{b}{2}\Delta\rho(\mathbf{r}, t) \tag{133}$$

with $a = -S_d \int_0^{+\infty} u(q)q^{d-1}dq$ and $b = -\frac{1}{d}S_d \int_0^{+\infty} u(q)q^{d+1}dq$ (the coefficients a and b are positive in the case of attractive interactions and negative in the case of repulsive interactions). We note that $l = (b/a)^{1/2}$ has the dimension of a length corresponding to the range of the interaction. For the sake of generality, we assume that the particles are submitted, in addition to the self-interaction, to an external potential $\Phi_{\rm ext}(\mathbf{r})$. In that case, the previously derived GFP equations remain valid provided that Φ is replaced by $\Phi + \Phi_{\rm ext}$. On the other hand, the energy is $E = \frac{1}{2}\int \rho\Phi \, d\mathbf{r} + \int \rho\Phi_{\rm ext} \, d\mathbf{r}$ so that the free energy (47) becomes

$$F[\rho] = \frac{1}{2}\int \rho\Phi \, d\mathbf{r} + \int \rho\Phi_{\rm ext} \, d\mathbf{r} + T\int C(\rho) \, d\mathbf{r}. \tag{134}$$

Substituting Equation (133) in Equations (90) and (134), we obtain the stochastic partial differential equation

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[g(\rho)\nabla\frac{\delta F_C}{\delta\rho}\right] + \nabla\cdot\left[\sqrt{2\xi T g(\rho)}\mathbf{R}(\mathbf{r},t)\right] \tag{135}$$

where the free energy is given by

$$F_C[\rho] = \frac{b}{2}\int\left[\frac{1}{2}(\nabla\rho)^2 + V(\rho)\right] d\mathbf{r} \tag{136}$$

with the potential

$$V(\rho) = \frac{2T}{b}C(\rho) - \frac{a}{b}\rho^2 + \frac{2}{b}\rho\,\Phi_{\rm ext}. \tag{137}$$

At $T = 0$, the potential reduces to

$$V_0(\rho) = -\frac{a}{b}\rho^2 + \frac{2}{b}\rho\,\Phi_{\rm ext}, \tag{138}$$

which is purely quadratic. Equation (135) can be explicitly written as

$$\xi\frac{\partial\rho}{\partial t} = -\frac{b}{2}\nabla\cdot[g(\rho)\nabla(\Delta\rho - V'(\rho))] + \nabla\cdot\left[\sqrt{2\xi T g(\rho)}\,\mathbf{R}(\mathbf{r},t)\right]. \tag{139}$$

Without the noise term, the steady state of Equation (135) is given by

$$\frac{\delta F_C}{\delta\rho} = \mu, \tag{140}$$

where μ is a constant. This equilibrium condition can be explicitly written as

$$\Delta\rho - V'(\rho) = -\frac{2\mu}{b}, \tag{141}$$

or, equivalently, as

$$\Delta\rho - \frac{2T}{b}C'(\rho) + \frac{2a}{b}\rho - \frac{2}{b}\Phi_{\rm ext} = -\frac{2\mu}{b}. \tag{142}$$

At $T = 0$, we obtain an inhomogeneous Helmholtz equation

$$\Delta\rho + \frac{2a}{b}\rho = \frac{2}{b}\Phi_{\rm ext} - \frac{2\mu}{b}. \tag{143}$$

For a quadratic potential $\Phi_{\rm ext}(\mathbf{r}) = \Phi_0(r^2 - r_0^2)$, this equation takes the form $\Delta\rho + k^2\rho = Ar^2 + B$, where we have defined $k^2 = 2a/b$, $A = 2\Phi_0/b$, and $B = -(2/b)\Phi_0 r_0^2 - 2\mu/b$. The solution of this equation is

$$\rho(\mathbf{r}) = \frac{A}{k^2}r^2 + \left(\frac{B}{k^2} - \frac{2dA}{k^4}\right) + \rho_s(\mathbf{r}), \tag{144}$$

where $\rho_s(\mathbf{r})$ is the solution of the homogeneous Helmholtz equation $\Delta\rho_s + k^2\rho_s = 0$. In the spherically symmetric case, $\rho_s(r) = K\cos(kr)$ in $d = 1$, $\rho_s(r) = KJ_0(kr)$ in $d = 2$, and $\rho_s(r) = K\sin(kr)/r$ in $d = 3$.

6.2. Analogy with Cahn-Hilliard Equations

Morphologically, the stochastic equation (135)-(136), or equivalently Equation (139), obtained in the previous section is similar to the stochastic Cahn-Hilliard equation for model B (conserved dynamics) [55] recalled in Appendix B. There are, however, crucial differences between these two equations. First, the density $\rho(\mathbf{r}, t)$ appears explicitly in the deterministic current and in the noise term of Equations (135) and (139) through the function $g(\rho)$ while it is absent in the stochastic Cahn-Hilliard equation (270). Secondly, in the Cahn-Hilliard equation, the potential $V(\rho)$ has a double-well shape of the typical form $V(\rho) = A(\sigma^2 - \rho^2)^2$ leading to a phase separation while, in the present case, the potential (137) is more general. It is only in the particular case $g(\rho) = 1$ and $C(\rho) = \rho^4$, implying $h(\rho) = 12\rho^2$, that Equation (135)-(136), or Equation (139), reduces to a form that is formally equivalent to the Cahn-Hilliard equation with $V(\rho) = \frac{2T}{b}(\frac{a}{4T} - \rho^2)^2$. Finally, we note that the Cahn-Hilliard equation is a heuristic equation (to our knowledge it has not been derived from first principles or from a microscopic dynamics) while Equation (135)-(136), or Equation (139), is derived as a particular limit of the GFP equation (90). As a result, the potential $V(\rho)$ in Equation (137) is entirely determined by the function $C(\rho)$ entering in the generalized free energy defined by Equation (134).

6.3. Expanded Form of the Generalized Stochastic Cahn-Hilliard Equation

The generalized stochastic Cahn-Hilliard equation may be derived directly from the GFP equation

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot [T h(\rho) \nabla \rho + g(\rho) \nabla (\Phi + \Phi_{\text{ext}})] + \nabla \cdot \left(\sqrt{2\xi T g(\rho)} \mathbf{R} \right) \tag{145}$$

by substituting Equation (133). This yields

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T h(\rho) \nabla \rho - a g(\rho) \nabla \rho - \frac{b}{2} g(\rho) \nabla (\Delta \rho) + g(\rho) \nabla \Phi_{\text{ext}} \right] + \nabla \cdot \left(\sqrt{2\xi T g(\rho)} \mathbf{R} \right) \tag{146}$$

or, equivalently,

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ g(\rho) \left[T C''(\rho) \nabla \rho - a \nabla \rho - \frac{b}{2} \nabla (\Delta \rho) + \nabla \Phi_{\text{ext}} \right] \right\} + \nabla \cdot \left(\sqrt{2\xi T g(\rho)} \mathbf{R} \right). \tag{147}$$

The free energy associated with Equation (146) is

$$F_C[\rho] = \int \rho \Phi_{\text{ext}} \, d\mathbf{r} - \frac{a}{2} \int \rho^2 \, d\mathbf{r} + \frac{b}{4} \int (\nabla \rho)^2 \, d\mathbf{r} + T \int C(\rho) \, d\mathbf{r}. \tag{148}$$

Without the noise term, the steady state of Equation (147) is given by Equation (142). At $T = 0$, Equation (147) reduces to

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ g(\rho) \left[-a \nabla \rho - \frac{b}{2} \nabla (\Delta \rho) + \nabla \Phi_{\text{ext}} \right] \right\}. \tag{149}$$

In that case, the noise term disappears. The steady state of Equation (149) is determined by the inhomogeneous Helmholtz equation (143).

6.4. Particular Form: Anomalous Diffusion and Normal Mobility

If we assume $g(\rho) = \rho$ and $h(\rho) = \rho C''(\rho)$, the generalized stochastic Cahn-Hilliard equation (146) becomes

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[T\rho C''(\rho)\nabla\rho - a\rho\nabla\rho - \frac{b}{2}\rho\nabla(\Delta\rho) + \rho\nabla\Phi_{\rm ext}\right] + \nabla\cdot\left(\sqrt{2\xi T\rho}\mathbf{R}\right). \tag{150}$$

At $T = 0$, it reduces to

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[-a\rho\nabla\rho - \frac{b}{2}\rho\nabla(\Delta\rho) + \rho\nabla\Phi_{\rm ext}\right] \tag{151}$$

or, equivalently, to

$$\xi\frac{\partial\rho}{\partial t} = -\frac{a}{2}\Delta\rho^2 - \frac{b}{2}\nabla\cdot(\rho\nabla(\Delta\rho)) + \nabla\cdot(\rho\nabla\Phi_{\rm ext}). \tag{152}$$

Introducing the pressure defined by Equation (72), we can rewrite Equation (150) as

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\nabla P - a\rho\nabla\rho - \frac{b}{2}\rho\nabla(\Delta\rho) + \rho\nabla\Phi_{\rm ext}\right] + \nabla\cdot\left(\sqrt{2\xi T\rho}\mathbf{R}\right). \tag{153}$$

Without the noise term, the steady state of Equation (153) is given by

$$\nabla P - a\rho\nabla\rho - \frac{b}{2}\rho\nabla(\Delta\rho) + \rho\nabla\Phi_{\rm ext} = \mathbf{0} \tag{154}$$

which can be viewed as a condition of hydrostatic equilibrium.

6.5. Particular Form: Normal Diffusion and Anomalous Mobility

If we assume $h(\rho) = 1$ and $g(\rho) = 1/C''(\rho)$, the generalized stochastic Cahn-Hilliard equation (146) becomes

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[T\nabla\rho - \frac{a}{C''(\rho)}\nabla\rho - \frac{b}{2C''(\rho)}\nabla(\Delta\rho) + \frac{1}{C''(\rho)}\nabla\Phi_{\rm ext}\right] + \nabla\cdot\left(\sqrt{\frac{2\xi T}{C''(\rho)}}\mathbf{R}\right). \tag{155}$$

At $T = 0$, it reduces to

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left[-\frac{a}{C''(\rho)}\nabla\rho - \frac{b}{2C''(\rho)}\nabla(\Delta\rho) + \frac{1}{C''(\rho)}\nabla\Phi_{\rm ext}\right]. \tag{156}$$

6.6. Equivalent Forms of the Generalized Stochastic Cahn-Hilliard Equation

We list below different explicit forms of the generalized stochastic Cahn-Hilliard equation. Introducing the pressure defined by Equation (72), we can rewrite Equation (147) as

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left\{\frac{g(\rho)}{\rho}\left[\nabla P - a\rho\nabla\rho - \frac{b}{2}\rho\nabla(\Delta\rho) + \rho\nabla\Phi_{\rm ext}\right]\right\} + \nabla\cdot\left(\sqrt{2\xi T g(\rho)}\mathbf{R}\right). \tag{157}$$

Using Equations (75) and (148), the free energy associated with Equation (157) is

$$F_C[\rho] = \int\rho\Phi_{\rm ext}\,d\mathbf{r} - \frac{a}{2}\int\rho^2\,d\mathbf{r} + \frac{b}{4}\int(\nabla\rho)^2\,d\mathbf{r} + \int\rho\int^\rho\frac{P(\rho')}{\rho'^2}\,d\rho'\,d\mathbf{r}. \tag{158}$$

Without the noise term, the steady state of Equation (157) is given by Equation (154).

Introducing the enthalpy defined by Equation (78), we can rewrite Equation (147) as

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left\{g(\rho)\left[\nabla h-a\nabla\rho-\frac{b}{2}\nabla(\Delta\rho)+\nabla\Phi_{\rm ext}\right]\right\}+\nabla\cdot\left(\sqrt{2\xi T g(\rho)}\mathbf{R}\right). \tag{159}$$

Using Equations (80) and (148), the free energy associated with Equation (159) is

$$F_C[\rho]=\int\rho\Phi_{\rm ext}\,d\mathbf{r}-\frac{a}{2}\int\rho^2\,d\mathbf{r}+\frac{b}{4}\int(\nabla\rho)^2\,d\mathbf{r}+\int H(\rho)\,d\mathbf{r}. \tag{160}$$

Without the noise term, the steady state of Equation (159) is given by

$$\Delta\rho+\frac{2a}{b}\rho-\frac{2}{b}h(\rho)=\frac{2}{b}\Phi_{\rm ext}-\frac{2\mu}{b}. \tag{161}$$

6.7. Simple Systems: Normal Diffusion and Normal Mobility

For simple systems with a normal diffusion and a normal mobility, corresponding to $h(\rho)=1$ and $g(\rho)=\rho$, Equation (145) reduces to the stochastic Smoluchowski equation

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot[T\nabla\rho+\rho\nabla(\Phi+\Phi_{\rm ext})]+\nabla\cdot\left(\sqrt{2\xi T\rho}\,\mathbf{R}\right), \tag{162}$$

discussed in Section 4.1. Substituting Equation (133) in Equation (162), we get the generalized stochastic Cahn-Hilliard equation

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[T\nabla\rho-a\rho\nabla\rho-\frac{b}{2}\rho\nabla(\Delta\rho)+\rho\nabla\Phi_{\rm ext}\right]+\nabla\cdot\left(\sqrt{2\xi T\rho}\mathbf{R}\right) \tag{163}$$

or, equivalently,

$$\xi\frac{\partial\rho}{\partial t}=T\Delta\rho-\frac{a}{2}\Delta\rho^2-\frac{b}{2}\nabla\cdot(\rho\nabla(\Delta\rho))+\nabla\cdot(\rho\nabla\Phi_{\rm ext})+\nabla\cdot\left(\sqrt{2k_B T\xi\rho}\,\mathbf{R}\right). \tag{164}$$

Without the noise term, the steady state of Equation (163) is given by

$$\Delta\rho+\frac{2a}{b}\rho-\frac{2T}{b}\ln\rho=\frac{2}{b}\Phi_{\rm ext}-\frac{2\mu}{b}+\frac{2T}{b}. \tag{165}$$

7. Analogy with An Effective Generalized Thermodynamics

We still consider short-range interactions, as in the previous section, but we assume here that the second term in Equation (133) can be neglected, so that

$$\Phi(\mathbf{r},t)=-a\rho(\mathbf{r},t). \tag{166}$$

This amounts to taking $b=0$ in the previous equations. In that case, we can develop an analogy with an effective generalized thermodynamics even for simple systems described by the ordinary Boltzmann thermodynamics (see Sections 7.2 and 7.3).

7.1. General Results

Substituting Equation (166) in Equation (145), we obtain

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[Th(\rho)\nabla\rho-ag(\rho)\nabla\rho+g(\rho)\nabla\Phi_{\rm ext}\right]+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\,\mathbf{R}\right). \tag{167}$$

The free energy associated with Equation (167) is

$$F_C=\int\rho\Phi_{\rm ext}\,d\mathbf{r}-\frac{a}{2}\int\rho^2\,d\mathbf{r}+T\int C(\rho)\,d\mathbf{r}. \tag{168}$$

Without the noise term, the steady state of Equation (167) is determined by the condition

$$TC'(\rho)-a\rho+\Phi_{\rm ext}=\mu. \tag{169}$$

If we define

$$T_{\rm eff}=-a,\qquad h_{\rm eff}(\rho)=\frac{T}{T_{\rm eff}}h(\rho)+g(\rho), \tag{170}$$

we can rewrite Equation (167) as

$$\xi\frac{\partial\rho}{\partial t}=\nabla\cdot\left[T_{\rm eff}h_{\rm eff}(\rho)\nabla\rho+g(\rho)\nabla\Phi_{\rm ext}\right]+\nabla\cdot\left(\sqrt{2\xi Tg(\rho)}\mathbf{R}\right). \tag{171}$$

The deterministic term can be viewed as a GFP equation of the form of Equation (44) with a nonlinear mobility $g(\rho)$, a nonlinear diffusion $h_{\rm eff}(\rho)$, an external potential $\Phi_{\rm ext}(\mathbf{r})$, and an effective temperature $T_{\rm eff}$. We note that the effective temperature $T_{\rm eff}$ is positive when $a<0$ (repulsive interactions) and *negative* when $a>0$ (attractive interactions). The complete equation can be interpreted as a stochastic GFP equation. However, we note that the temperature appearing in the noise term is the thermodynamic temperature T, not the effective temperature $T_{\rm eff}$. The effective generalized entropy associated with Equation (171) is determined by the relation

$$C''_{\rm eff}(\rho)=\frac{h_{\rm eff}(\rho)}{g(\rho)}=\frac{T}{T_{\rm eff}}\frac{h(\rho)}{g(\rho)}+1=\frac{T}{T_{\rm eff}}C''(\rho)+1. \tag{172}$$

After integration, we obtain

$$C_{\rm eff}(\rho)=\frac{T}{T_{\rm eff}}C(\rho)+\frac{1}{2}\rho^2, \tag{173}$$

hence

$$S_{\rm eff}=\frac{T}{T_{\rm eff}}S-\frac{1}{2}\int\rho^2\,d\mathbf{r}. \tag{174}$$

The effective free energy is

$$F_{\rm eff}=E_{\rm ext}-T_{\rm eff}S_{\rm eff}, \tag{175}$$

where

$$E_{\rm ext}=\int\rho\Phi_{\rm ext}\,d\mathbf{r} \tag{176}$$

is the energy of the particles in the external potential. Introducing the pressure defined by Equation (72), we can rewrite Equation (167) as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{g(\rho)}{\rho} \left(\nabla P_{\text{eff}} + \rho \nabla \Phi_{\text{ext}} \right) \right] + \nabla \cdot \left(\sqrt{2\xi T g(\rho)} \mathbf{R} \right), \tag{177}$$

with an effective pressure

$$P_{\text{eff}}(\rho) = P(\rho) - \frac{1}{2} a \rho^2. \tag{178}$$

We see therefore the analogy with an effective generalized thermodynamics. The effective entropy (174) is the sum of the generalized entropy S plus a Tsallis entropy (115) of index $\gamma = 2$. The effective pressure is the sum of the pressure $P(\rho)$ plus the pressure of a polytrope of index $\gamma = 2$.

Remark 7. *When $b \neq 0$, we can rewrite Equation (157) as*

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ \frac{g(\rho)}{\rho} \left[\nabla P_{\text{eff}} - \frac{b}{2} \rho \nabla (\Delta \rho) + \rho \nabla \Phi_{\text{ext}} \right] \right\} + \nabla \cdot \left(\sqrt{2\xi T g(\rho)} \mathbf{R} \right). \tag{179}$$

The free energy associated with Equation (179) is

$$F_C[\rho] = \int \rho \Phi_{\text{ext}} \, d\mathbf{r} + \frac{b}{4} \int (\nabla \rho)^2 \, d\mathbf{r} + \int \rho \int^{\rho} \frac{P_{\text{eff}}(\rho')}{\rho'^2} \, d\rho' \, d\mathbf{r}. \tag{180}$$

Without the noise term, the steady state of Equation (179) is given by

$$\nabla P_{\text{eff}} - \frac{b}{2} \rho \nabla (\Delta \rho) + \rho \nabla \Phi_{\text{ext}} = \mathbf{0}. \tag{181}$$

7.2. Simple Systems

We consider a simple system with a normal diffusion and a normal mobility described by the stochastic Smoluchowski equation (162). We stress that this system is described by standard thermodynamics associated with the Boltzmann entropy (113).

Substituting Equation (166) in Equation (162), we obtain

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(T \nabla \rho - \frac{a}{2} \nabla \rho^2 + \rho \nabla \Phi_{\text{ext}} \right) + \nabla \cdot \left(\sqrt{2\xi T \rho} \mathbf{R} \right). \tag{182}$$

The corresponding free energy is

$$F = \int \rho \Phi_{\text{ext}} \, d\mathbf{r} + T \int \rho \ln \rho \, d\mathbf{r} - \frac{a}{2} \int \rho^2 \, d\mathbf{r}. \tag{183}$$

Without the noise term, the steady state of Equation (182) is determined by the equation

$$-T \ln \rho + a\rho = \Phi_{\text{ext}} - \mu + T. \tag{184}$$

The solution of Equation (184) can be written as

$$\rho(\mathbf{r}) = \frac{T}{|a|} W \left[\frac{|a|}{T} e^{-\frac{1}{T} \Phi_{\text{ext}}(\mathbf{r}) + \frac{\mu}{T} - 1} \right]. \tag{185}$$

In the case of repulsive interactions ($a < 0$), $W(z)$ is the Lambert function defined implicitly by the equation $We^W = z$. In the case of attractive interactions ($a > 0$), $W(z)$ is a new function defined implicitly by the equation $W^{-1}e^W = z$.

If we define

$$T_{\text{eff}} = -a, \qquad h_{\text{eff}}(\rho) = \frac{T}{T_{\text{eff}}} + \rho, \tag{186}$$

we can rewrite Equation (182) as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T_{\text{eff}} h_{\text{eff}}(\rho) \nabla \rho + \rho \nabla \Phi_{\text{ext}} \right] + \nabla \cdot \left(\sqrt{2\xi T \rho} \mathbf{R} \right). \tag{187}$$

This equation can be interpreted as a GFP equation of the form of Equation (44) with a normal mobility, a nonlinear diffusion $h_{\text{eff}}(\rho)$, an external potential, and an effective temperature T_{eff}. The effective generalized entropy associated with Equation (187) is

$$S_{\text{eff}} = -\frac{T}{T_{\text{eff}}} \int \rho \ln \rho \, d\mathbf{r} - \frac{1}{2} \int \rho^2 \, d\mathbf{r}. \tag{188}$$

This is the sum of the Boltzmann entropy (113) and Tsallis entropy (115) with $\gamma = 2$. The effective free energy is given by Equation (175) with Equation (188). On the other hand, Equation (182) can be rewritten as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P_{\text{eff}} + \rho \nabla \Phi_{\text{ext}}) + \nabla \cdot \left(\sqrt{2\xi T \rho} \mathbf{R} \right) \tag{189}$$

with an effective pressure

$$P_{\text{eff}}(\rho) = \rho T - \frac{1}{2} a \rho^2. \tag{190}$$

This is the sum of an isothermal (Boltzmann) equation of state and a polytropic (Tsallis) equation of state of index $\gamma = 2$.

We emphasize that, because of the relation (166), a notion of generalized thermodynamics has emerged although the system is intrinsically described by ordinary thermodynamics based on the Boltzmann entropy. This shows that the relation with generalized thermodynamics in the present context is purely effective or coincidental.

7.3. Simple Systems at $T = 0$

At $T = 0$, Equation (182) reduces to

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(-\frac{a}{2} \nabla \rho^2 + \rho \nabla \Phi_{\text{ext}} \right). \tag{191}$$

The corresponding free energy is

$$F = \int \rho \Phi_{\text{ext}} \, d\mathbf{r} - \frac{a}{2} \int \rho^2 \, d\mathbf{r}. \tag{192}$$

The steady state of Equation (191) is given by

$$\rho(\mathbf{r}) = \frac{1}{a} \Phi_{\text{ext}}(\mathbf{r}) - \frac{\mu}{a}. \tag{193}$$

If we define $T_{\rm eff} = -a$ and $h_{\rm eff}(\rho) = \rho$, we can rewrite Equation (191) as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[T_{\rm eff} h_{\rm eff}(\rho) \nabla \rho + \rho \nabla \Phi_{\rm ext} \right]. \tag{194}$$

This can be interpreted as a GFP equation of the form of Equation (44) with a normal mobility, a nonlinear diffusion $h_{\rm eff}(\rho) = \rho$, an external potential $\Phi_{\rm ext}(\mathbf{r})$, and an effective temperature $T_{\rm eff}$. The effective generalized entropy associated with Equation (194) is

$$S_{\rm eff} = -\frac{1}{2} \int \rho^2 \, d\mathbf{r}. \tag{195}$$

This corresponds to a Tsallis entropy (115) with $\gamma = 2$. The effective free energy is given by Equation (175) with Equation (195). On the other hand, Equation (191) can be rewritten as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P_{\rm eff} + \rho \nabla \Phi_{\rm ext}) \tag{196}$$

with an effective pressure

$$P_{\rm eff}(\rho) = -\frac{1}{2} a \rho^2. \tag{197}$$

This is the equation of state of a polytrope of index $\gamma = 2$. We again emphasize that, in the present context, the notion of generalized thermodynamics is purely effective since the system is intrinsically described by ordinary thermodynamics. Furthermore, at $T = 0$, the microscopic evolution is deterministic, not stochastic, so there is not even a notion of thermodynamics in a strict sense.

Remark 8. *The situation is very similar to the one encountered in the case of the Gross-Pitaevskii (GP) equation (see* [56] *and Section II of* [57]*). Indeed, the GP equation can be derived from the mean field Schrödinger equation with a potential* $\Phi(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}', t) \, d\mathbf{r}'$ *by using Equation (166) valid for systems with short-range interactions. Actually, Equation (166) corresponds to a pair contact potential of the form* $u = -a\delta(\mathbf{r} - \mathbf{r}')$*. The coefficient* a *is related to the scattering length* a_s *of the bosons by the relation* $a = -4\pi a_s \hbar / m^3$*. We have* $a < 0$ *for repulsive self-interactions (*$a_s > 0$*) and* $a > 0$ *for attractive self-interactions (*$a_s < 0$*). Using the Madelung transformation, the GP equation can be written in the form of hydrodynamic equations involving a quantum potential and a pressure associated with a polytropic equation of state of index* $\gamma = 2$ *equivalent to Equation (197).*

8. Application to Systems of Physical Interest

In this section, we discuss different systems of physical interest to which our general formalism applies.

8.1. Self-Gravitating Brownian Particles

Chavanis and Sire [58,59] have studied a system of overdamped Brownian particles in gravitational interaction in a space of dimension d. Their evolution is described by the N-body Langevin equations

$$\frac{d\mathbf{r}_i}{dt} = -\chi G m \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^d} + \sqrt{2D} \, \mathbf{R}_i(t), \tag{198}$$

where G is the gravitational constant. In the mean field approximation, we obtain the Smoluchowski-Poisson system

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right), \tag{199}$$

$$\Delta \Phi = S_d G \rho, \tag{200}$$

where S_d is the surface of a d-dimensional hypersphere. If we take fluctuations into account, the evolution of the smooth density field is described by the stochastic Smoluchowski equation

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) + \nabla \cdot \left[\sqrt{2 \xi k_B T \rho}\, \mathbf{R}(\mathbf{r}, t) \right]. \tag{201}$$

Related models have been introduced in the literature by slightly changing the potential of interaction in order to have a homogeneous phase. For example, we can consider a screened gravitational interaction with a screening length $\sim k^{-1}$. In that case, the Poisson equation (200) is replaced by the screened Poisson equation

$$\Delta \Phi - k^2 \Phi = S_d G \rho. \tag{202}$$

We can also introduce a sort of "neutralizing" background with a constant density $-\overline{\rho}$ like in the Jellium model of plasma physics. In that case, the Poisson equation (200) is replaced by the modified Poisson equation

$$\Delta \Phi = S_d G (\rho - \overline{\rho}). \tag{203}$$

The Smoluchowski-Poisson system has been studied in [16,58–62]. The screened Smoluchowski-Poisson system and the modified Smoluchowski-Poisson system have been studied in [63,64]. Finally, the stochastic modified Smoluchowski-Poisson system (which takes fluctuations into account) has been studied in [47].

The various generalizations introduced in the present paper can be applied to this system of self-gravitating Brownian particles. For example, the generalized stochastic Smoluchowski equation writes

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P + \rho \nabla \Phi) + \nabla \cdot \left[\sqrt{2 \xi T \rho}\, \mathbf{R}(\mathbf{r}, t) \right], \tag{204}$$

where $P(\rho)$ is an arbitrary barotropic equation of state. The generalized Smoluchowski-Poisson system has been studied in [65] for a polytropic equation of state. More generally, we can propose a model of self-gravitating Brownian particles of the form

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{g(\rho)}{\rho} (\nabla P + \rho \nabla (\Phi + \Phi_{\rm ext})) \right] + \nabla \cdot \left[\sqrt{2 \xi T g(\rho)}\, \mathbf{R}(\mathbf{r}, t) \right]. \tag{205}$$

In the case of short-range interactions (or in the limit of strong screening $k^2 \to +\infty$), we can make the approximation $\Phi = -(S_d G/k^2)\rho - (S_d G/k^4)\Delta\rho$ as in Section 6, and we get the generalized stochastic Cahn-Hilliard equation

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{g(\rho)}{\rho} \left(\nabla P - \frac{S_d G}{k^2} \rho \nabla \rho - \frac{S_d G}{k^4} \rho \nabla (\Delta \rho) + \rho \nabla \Phi_{\rm ext} \right) \right] + \nabla \cdot \left[\sqrt{2 \xi T g(\rho)}\, \mathbf{R}(\mathbf{r}, t) \right]. \tag{206}$$

Remark 9. *The Smoluchowski-Poisson system (199)-(200) describes the evolution of Brownian particles that interact through the attractive gravitational force ($G > 0$). The case of Brownian charges that interact through the repulsive electric force leads to the Debye-Hückel* [66]*, or Nernst-Planck* [67–69]*, model. It corresponds to Equations (199)-(200) with $G < 0$. Similar Fokker-Planck equations have been introduced by Chavanis* [70,71] *for two-dimensional point vortices. The interaction between point vortices is "repulsive" at positive temperatures and "attractive" at negative temperatures* [72]*. We refer to* [70,71] *for more details about these systems and their analogies.*

8.2. Colloid Particles at a Fluid Interface

Dominguez *et al.* [73] have investigated the dynamics of colloids at a fluid interface driven by attractive capillary interactions. They showed that the capillary attraction is formally analogous to two-dimensional gravity. The evolution of the areal number density profile $\rho(\mathbf{r}, t)$ of colloids is governed by the system of equations

$$\frac{\partial \rho}{\partial t} = \Gamma \nabla \cdot \left[\nabla P(\rho) - f \rho \nabla U \right], \tag{207}$$

$$\Delta U - \frac{U}{\lambda^2} = -\frac{f}{\gamma}\rho, \tag{208}$$

where Γ is a mobility coefficient of the particles at the interface, P is a pressure taking short-range interactions into account, f is the capillary monopole associated with a single particle, $U(\mathbf{r}, t)$ is the ensemble-averaged interfacial deformation, γ is the surface tension, and λ is the capillary length.

Apart from a redefinition of the parameters, these equations are equivalent to the generalized screened Smoluchowski-Poisson system (see Section 8.1). We have the correspondences $\xi = 1/\Gamma$, $\Phi = -fU$, $k^2 = 1/\lambda^2$, and $2\pi G = f^2/\gamma$. Using the formalism developed in the present paper, we can propose a generalized model of colloid particles at a fluid interface of the form

$$\frac{\partial \rho}{\partial t} = \Gamma \nabla \cdot \left\{ \frac{g(\rho)}{\rho} \left[\nabla P(\rho) - f \rho \nabla (U + U_{\rm ext}) \right] \right\} + \nabla \cdot \left[\sqrt{2\Gamma T g(\rho)}\, \mathbf{R}(\mathbf{r}, t) \right]. \tag{209}$$

In the case of short-range interactions (or in the limit of strong screening $\lambda \to 0$), we can make the approximation $U = (f\lambda^2/\gamma)\rho + (f\lambda^4/\gamma)\Delta\rho$ as in Section 6, and we get the generalized stochastic Cahn-Hilliard equation

$$\frac{\partial \rho}{\partial t} = \Gamma \nabla \cdot \left\{ \frac{g(\rho)}{\rho} \left[\nabla P(\rho) - \frac{f^2 \lambda^2}{\gamma} \rho \nabla \rho - \frac{f^2 \lambda^4}{\gamma} \rho \nabla (\Delta \rho) - f \rho \nabla U_{\rm ext} \right] \right\} + \nabla \cdot \left[\sqrt{2\Gamma T g(\rho)}\, \mathbf{R}(\mathbf{r}, t) \right]. \tag{210}$$

8.3. Superconductor of Type-II

Zapperi *et al.* [74] have introduced a model of disordered superconductors of type II. In an infinitely long cylinder, flux lines can be modeled as a set of interacting particles performing an overdamped motion in a random pinning landscape. The equation of motion for each flux line i can be written as

$$\Gamma \frac{d\mathbf{r}_i}{dt} = \sum_j \mathbf{J}(\mathbf{r}_i - \mathbf{r}_j) + \sum_p \mathbf{G}\left[(\mathbf{R}_p - \mathbf{r}_i)/l \right] + \sqrt{2\Gamma k_B T} \mathbf{R}_i(t), \tag{211}$$

where the effective viscosity can be expressed in terms of material parameters as $\Gamma = \Phi_0 H_{c2}/\rho_n c^2$. Here, Φ_0 is the magnetic quantum flux, c is the speed of light, ρ_n is the resistivity of the normal phase, and H_{c2} is the upper critical field. The first term on the right hand side of Equation (211) represents the vortex-vortex interaction. It is given by $\mathbf{J}(\mathbf{r}) = [\Phi_0^2/(8\pi\lambda^3)]K_1(|\mathbf{r}|/\lambda)\hat{\mathbf{r}}$, where the function K_1 is a Bessel function decaying exponentially for $|\mathbf{r}| > \lambda$, and λ is the London penetration length. The interaction is repulsive. The second term on the right hand side of Equation (211) accounts for the interaction between pinning centers, modeled as localized traps, and flux lines. Here $\mathbf{G}$ is the force due to a pinning center located at $\mathbf{R}_p$, l is the range of the wells (typically $l \ll \lambda$), and $p = 1, ..., N_p$ (N_p is the total number of pinning centers). We shall regard this term as a (given) external force deriving from a potential $U_{\rm ext}$. Finally, the last term in Equation (211) is an uncorrelated thermal noise.

The mean field Smoluchowski equation associated with Equation (211) is

$$\Gamma\frac{\partial\rho}{\partial t} = \nabla\cdot(k_B T\nabla\rho + \rho\nabla U + \rho\nabla U_{\rm ext}), \tag{212}$$

$$\Delta U - \frac{1}{\lambda^2}U = -\frac{\Phi_0^2}{4\lambda^2}\rho. \tag{213}$$

Apart from a redefinition of the parameters, these equations are equivalent to the screened Smoluchowski-Poisson system (see Section 8.1). We have the correspondences $\xi = \Gamma$, $\Phi = U$, $k^2 = 1/\lambda^2$, and $2\pi G = -\Phi_0^2/(4\lambda^2)$. Using the formalism developed in the present paper, we can propose a generalized model of superconductors of type II of the form

$$\Gamma\frac{\partial\rho}{\partial t} = \nabla\cdot\left\{\frac{g(\rho)}{\rho}\left[\nabla P(\rho) + \rho\nabla(U + U_{\rm ext})\right]\right\} + \nabla\cdot\left[\sqrt{2\Gamma T g(\rho)}\,\mathbf{R}(\mathbf{r},t)\right]. \tag{214}$$

In the case of short-range interactions (or in the limit of strong screening $\lambda \to 0$), we can make the approximation $U = (\Phi_0^2/4)\rho + (\lambda^2\Phi_0^2/4)\Delta\rho$ as in Section 6, and we get the generalized stochastic Cahn-Hilliard equation

$$\Gamma\frac{\partial\rho}{\partial t} = \nabla\cdot\left\{\frac{g(\rho)}{\rho}\left[\nabla P(\rho) + \frac{1}{4}\Phi_0^2\rho\nabla\rho + \frac{1}{4}\lambda^2\Phi_0^2\rho\nabla(\Delta\rho) + \rho\nabla U_{\rm ext}\right]\right\} + \nabla\cdot\left[\sqrt{2\Gamma T g(\rho)}\,\mathbf{R}(\mathbf{r},t)\right]. \tag{215}$$

For simple systems where $P = \rho k_B T$ and $g(\rho) = \rho$, and when the term $(\lambda^2\Phi_0^2/4)\Delta\rho$ is neglected in the expansion of U (see Section 7), as well as the noise term, we recover the equation

$$\Gamma\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\frac{1}{4}\Phi_0^2\rho\nabla\rho + \rho\nabla U_{\rm ext}\right) + k_B T\Delta\rho \tag{216}$$

considered by Zapperi *et al.* [74]. Andrade *et al.* [75] discussed this equation in the framework of Tsallis generalized thermodynamics. However, as clarified in Section 7.2, the relation with generalized thermodynamics is here purely effective, or coincidental, since the system is fundamentally described by Boltzmann's thermodynamics.

8.4. Dynamical Theory of Nucleation

Lutsko [76] has developed a dynamical theory of nucleation based on fluctuating hydrodynamics appropriate to colloids and macro-molecules in solution. The starting point of his theory is the stochastic Smoluchowski equation for Brownian particles in interaction derived in [15,52]. It can be written as

$$\frac{\partial \rho}{\partial t} = D\nabla \cdot \left(\rho \nabla \frac{\delta \beta F}{\delta \rho}\right) + \nabla \cdot \left[\sqrt{2D\rho}\mathbf{R}(\mathbf{r},t)\right], \tag{217}$$

where $\rho(\mathbf{r},t)$ is the local density field, D is the tracer diffusion constant for the large molecules in solution, $F[\rho]$ is the free energy which is a functional of the local density (see Appendix C), $\beta = 1/k_B T$ where T is the temperature, and $\mathbf{R}(\mathbf{r},t)$ is a delta-correlated white noise. Equation (217) is equivalent to the generalized stochastic Smoluchowski equation (92) with a normal mobility. Using the theory developed in the present paper, we can propose a generalized model of nucleation of the form

$$\frac{\partial \rho}{\partial t} = D\nabla \cdot \left(g(\rho) \nabla \frac{\delta \beta F}{\delta \rho}\right) + \nabla \cdot \left[\sqrt{2Dg(\rho)}\mathbf{R}(\mathbf{r},t)\right] \tag{218}$$

involving a density-dependent mobility. For short-range interactions, we obtain the generalized stochastic Cahn-Hilliard equation of Section 6.

8.5. Chemotaxis of Bacterial Populations

Keller and Segel [77] have introduced a simple mathematical model to describe the chemotaxis of bacterial populations. It consists in two coupled differential equations

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D\nabla\rho - \chi\rho\nabla c), \tag{219}$$

$$\frac{\partial c}{\partial t} = D_c \Delta c - kc + h\rho, \tag{220}$$

that govern the evolution of the density of bacteria $\rho(\mathbf{r},t)$ and the evolution of the secreted chemical $c(\mathbf{r},t)$. The bacteria diffuse with a diffusion coefficient D and they also move along the gradient of the chemical (chemotactic drift). The coefficient χ is a measure of the strength of the influence of the chemical gradient on the flow of bacteria. The interaction can be attractive ($\chi > 0$) or repulsive ($\chi < 0$). On the other hand, the chemical is produced by the bacteria with a rate h and is degraded with a rate k. It also diffuses with a diffusion coefficient D_c. Equations (219)–(220) can be derived from N-body stochastic Langevin equations in a mean field approximation (see [78] and Appendix A of [79]). If we take fluctuations into account, we obtain the stochastic KS model

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D\nabla\rho - \chi\rho\nabla c) + \nabla \cdot \left[\sqrt{2D\rho}\,\mathbf{R}(\mathbf{r},t)\right] \tag{221}$$

introduced by Chavanis [18]. When the diffusivity of the chemical D_c is large, the term $\partial c/\partial t$ in Equation (220) can be neglected. In the case where there is no degradation of the chemical ($k = 0$), writing $h = \lambda D_c$ and taking the limit $D_c \to +\infty$ with $\lambda = O(1)$, we get the modified Poisson equation (see Appendix C of [79]):

$$\Delta c = -\lambda(\rho - \overline{\rho}). \tag{222}$$

The concentration of the chemical is given by a Poisson equation which incorporates a sort of "neutralizing background" played by the term $-\overline{\rho}$. We now consider the case of a finite degradation rate ($k \neq 0$). Writing $h = \lambda D_c$ and $k = k_0^2 D_c$, and taking the limit $D_c \to +\infty$ with $\lambda = O(1)$ and $k_0 = O(1)$, we get the screened Poisson equation (see Appendix C of [79]):

$$\Delta c - k_0^2 c = -\lambda \rho. \tag{223}$$

The interaction is shielded on a typical distance k_0^{-1} [42]. The Keller-Segel equations (219)-(223) are isomorphic to the Smoluchowski-Poisson equations (199)-(203) governing the evolution of self-gravitating Brownian particles. In this analogy, the concentration $c(\mathbf{r}, t)$ of the chemical plays the same role as the gravitational potential $\Phi(\mathbf{r}, t)$. We note that the field equation (220) in the KS model is more complicated than the Poisson equation (200) in the original SP system. When the term $\partial c/\partial t$ is taken into account in Equation (220), the proper expressions of the free energy and of the H-theorem are given in [46] and in Appendix E of [79]. On the other hand, in the framework of the KS model, it is possible to rigorously justify the modified Poisson equation (222) and the screened Poisson equation (223) that were introduced heuristically in the SP system (see Section 8.1). As a result, the KS model admits spatially homogeneous distributions while the original SP system does not. As emphasized in [63], this removes the so-called *Jeans swindle* [80] appearing in astrophysics.

Using the formalism developed in the present paper, we can propose a generalized model of chemotaxis of the form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [Dh(\rho)\nabla\rho - \chi g(\rho)\nabla c] + \nabla \cdot \left[\sqrt{2Dg(\rho)}\,\mathbf{R}(\mathbf{r}, t)\right]. \tag{224}$$

In the case of short-range interactions (*i.e.*, for a strong degradation of the chemical $k_0^2 \to +\infty$), we can make the approximation $c = (\lambda/k_0^2)\rho + (\lambda/k_0^4)\Delta\rho$ as in Section 6, and we get the generalized stochastic Cahn-Hilliard equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[Dh(\rho)\nabla\rho - \frac{\lambda\chi}{k_0^2} g(\rho)\nabla\rho - \frac{\lambda\chi}{k_0^4} g(\rho)\nabla(\Delta\rho)\right] + \nabla \cdot \left[\sqrt{2Dg(\rho)}\,\mathbf{R}(\mathbf{r}, t)\right]. \tag{225}$$

Remark 10. *As emphasized in* [46]*, the chemotaxis of bacterial population is an important physical model described by generalized stochastic FP equations. It is associated with a notion of generalized thermodynamics because, in the case of complex systems (like those occurring in biology), the coefficients of diffusion and drift depend on the density. This is an heuristic approach to take into account microscopic constraints that affect the dynamics of particles at small scales and lead to non-Boltzmannian equilibrium distributions. Indeed, it is not surprising that the mobility or the diffusive properties of a bacterium depend on its environment. For example, in a dense medium, its motion can be hampered by the presence of the other bacteria so that its mobility is reduced.*

8.6. Application to 2D Turbulence

Two-dimensional turbulence has the striking property of organizing spontaneously into long-lived coherent structures (vortices or jets) [81]. A statistical theory of 2D turbulence has been developed by Miller [82] and Robert and Sommeria [83] for isolated systems. We consider here the case where the system is forced at small scales and follow the heuristic approach of Ellis *et al.* [84] complemented by Chavanis [85–88]. We assume that the small-scale forcing is encoded in a prior vorticity distribution $\chi(\sigma)$. This prior vorticity distribution determines a generalized entropy of the form [87]:

$$S[\overline{\omega}] = -\int C(\overline{\omega})\, d\mathbf{r}, \tag{226}$$

with

$$C(\overline{\omega}) = -\int^{\overline{\omega}} [(\ln \hat{\chi})']^{-1}(-x)\, dx, \tag{227}$$

where $\hat{\chi}(E) = \int_{-\infty}^{+\infty} \chi(\sigma) e^{-\sigma E}\, d\sigma$. Explicit examples of prior vorticity distributions, and of the corresponding generalized entropies, are given in [85–88]. For example, when the prior vorticity distribution is a Gaussian, $\chi(\sigma) = (2\pi\Omega_2)^{-1/2}\exp(-\sigma^2/2\Omega_2)$, the generalized entropy is proportional to minus the enstrophy $S = -(1/2\Omega_2)\int \overline{\omega}^2\, d\mathbf{r}$ (see [87] and Section 5 of [88]). The functional $S[\overline{\omega}]$ has the status of an entropy in the sense of the theory of large deviations [84]. Indeed, the probability of the coarse-grained vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium is given by

$$P[\overline{\omega}] \propto e^{nS[\overline{\omega}]}\delta(E[\overline{\omega}] - E)\delta(\Gamma[\overline{\omega}] - \Gamma), \tag{228}$$

where $E[\overline{\omega}] = \frac{1}{2}\int \overline{\omega}\psi\, d\mathbf{r}$ and $\Gamma[\overline{\omega}] = \int \overline{\omega}\, d\mathbf{r}$ are the energy and the circulation that are approximately conserved by the flow (robust constraints), ψ is the stream function that is related to the vorticity $\overline{\omega}$ by the screened Poisson equation $\Delta\psi - \psi/R^2 = -\overline{\omega}$ where R is the Rossby length, and $\epsilon = 1/n$ is a small number that measures the importance of fluctuations. According to Equation (228), the most probable vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium is the one that maximizes the generalized entropy (226)–(227) at fixed energy and circulation. Writing the variational problem in the form $\delta S - \beta\delta E - \alpha\delta\Gamma = 0$ where β (inverse temperature) and α (chemical potential) are Lagrange multipliers, we find that the coarse-grained flow at statistical equilibrium is given by

$$-\Delta\psi + \frac{1}{R^2}\psi = \overline{\omega} = (C')^{-1}(-\beta\psi - \alpha). \tag{229}$$

This is the fundamental mean field equation of the statistical theory of 2D turbulence. This equation must be solved for a given C, and the Lagrange multipliers α and β must be related to the constraints Γ and E. One must select only stable states (entropy maxima, not minima or saddle points). If several stable states exist for the same values of the constraints, one must compare their entropy to distinguish fully stable states (global entropy maxima) from metastable states (local entropy maxima). One can then study phase transition from one state to the other by changing the values of the robust constraints (Γ, E).

The previous treatment corresponds to the microcanonical ensemble (MCE) where the energy and the circulation are fixed. We can also consider the canonical ensemble (CE) where the temperature and the circulation are fixed. In the canonical ensemble, the probability of the coarse-grained vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium is given by

$$P[\overline{\omega}] \propto e^{nJ[\overline{\omega}]}\delta(\Gamma[\overline{\omega}] - \Gamma), \tag{230}$$

where $J[\overline{\omega}] = S[\overline{\omega}] - \beta E[\overline{\omega}]$ is the generalized free energy. According to Equation (230), the most probable vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium in the canonical ensemble is the one that maximizes the generalized free energy at fixed circulation. We note that the free energy J differs from the usual free energy $F = E - TS$ by a factor $-\beta$. The inverse temperature β can be positive or negative in 2D turbulence [72]. As a result, the equilibrium state always corresponds to a maximum of J while it corresponds to a minimum of F when $\beta > 0$ and to a maximum of F when $\beta < 0$.

We can finally consider the grand canonical ensemble (GCE) where the temperature and the chemical potential are fixed. In the grand canonical ensemble, the probability of the coarse-grained vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium is given by

$$P[\overline{\omega}] \propto e^{nG[\overline{\omega}]}, \tag{231}$$

where $G[\overline{\omega}] = S[\overline{\omega}] - \beta E[\overline{\omega}] - \alpha\Gamma[\overline{\omega}]$ is the generalized grand potential. According to Equation (231), the most probable vorticity field $\overline{\omega}(\mathbf{r})$ at statistical equilibrium in the grand canonical ensemble is the one that maximizes the generalized grand potential (without constraint).

We note that the critical points of these different maximization problems (microcanonical, canonical, and grand canonical) are all given by Equation (229). However, the stability of the solutions (whether they are true maxima or saddle points of the thermodynamical potential) may differ because the ensembles can be inequivalent [84,89]. In [89], we have introduced different types of relaxation equations associated with these maximization problems. Although being phenomenological, these equations may describe the dynamical evolution of the system towards equilibrium. In any case, they can be used as numerical algorithms to compute the statistical equilibrium state and guarantee that we have selected a stable solution among all the solutions of Equation (229). When the system possesses several stable equilibrium states, the choice of equilibrium depends on a notion of basin of attraction.

In MCE, where the energy and the circulation are fixed, we have proposed to describe the relaxation of the system towards statistical equilibrium by an equation of the form

$$\frac{\partial\overline{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\overline{\omega} = \nabla\cdot\left\{D\left[\nabla\overline{\omega} + \frac{\beta(t)}{C''(\overline{\omega})}\nabla\psi\right]\right\}, \tag{232}$$

with

$$\beta(t) = -\frac{\int D\nabla\overline{\omega}\cdot\nabla\psi\, d\mathbf{r}}{\int D\frac{(\nabla\psi)^2}{C''(\overline{\omega})}\, d\mathbf{r}}. \tag{233}$$

We have also proposed the alternative equation

$$\frac{\partial\overline{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\overline{\omega} = -[C'(\overline{\omega}) + \beta(t)\psi + \alpha(t)], \tag{234}$$

where the coefficients $\alpha(t)$ and $\beta(t)$ are determined by the system of linear equations

$$\langle C'(\overline{\omega})\psi\rangle + \beta(t)\langle\psi^2\rangle + \alpha(t)\langle\psi\rangle = 0, \tag{235}$$

$$\langle C'(\overline{\omega})\rangle + \beta(t)\langle\psi\rangle + \alpha(t)A = 0, \tag{236}$$

where $\langle X\rangle = \int X\, d\mathbf{r}$ and A is the domain area. By construction, these equations conserve the energy and the circulation and increase the generalized entropy until the equilibrium state is reached (H-theorem).

In CE, where the temperature and the circulation are fixed, we have proposed the relaxation equation

$$\frac{\partial\overline{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\overline{\omega} = \nabla\cdot\left\{D\left[\nabla\overline{\omega} + \frac{\beta}{C''(\overline{\omega})}\nabla\psi\right]\right\}. \tag{237}$$

This equation is similar to the generalized Smoluchowski equation. It can be rewritten as

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=-\nabla\cdot\left[\frac{D}{C''(\overline{\omega})}\nabla\frac{\delta J}{\delta\overline{\omega}}\right], \tag{238}$$

where $J=S-\beta E$ is the generalized free energy. We have also proposed the alternative equation

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=-[C'(\overline{\omega})+\beta\psi+\alpha(t)], \tag{239}$$

where the coefficient $\alpha(t)$ is determined by

$$\langle C'(\overline{\omega})\rangle+\beta\langle\psi\rangle+\alpha(t)A=0. \tag{240}$$

These equations conserve the circulation and increase the generalized free energy until the equilibrium state is reached (H-theorem).

In GCE, where the temperature and the chemical potential are fixed, we have proposed the relaxation equation

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=-[C'(\overline{\omega})+\beta\psi+\alpha]. \tag{241}$$

It can be rewritten as

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=\frac{\delta G}{\delta\overline{\omega}}, \tag{242}$$

where $G=S-\beta E-\alpha\Gamma$ is the generalized grand potential. This equation increases the generalized grand potential until the equilibrium state is reached (H-theorem). Equation (241) or (242), first introduced in [89], has been recently considered by other authors [90]. It is based on the maximization of a functional $G[\overline{\omega}]$, interpreted as an energy-Casimir functional [89], with no constraint (GCE). However, it may be important to take into account the conservation of energy and circulation, so the relaxation equations (232)–(240) in CE and MCE are also useful.

We now propose to take fluctuations into account by applying the general results of this paper. In MCE, Equation (232) is replaced by

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=\nabla\cdot\left\{D\left[\nabla\overline{\omega}+\frac{\beta(t)}{C''(\overline{\omega})}\nabla\psi\right]\right\}+\nabla\cdot\left[\sqrt{\frac{2D}{nC''(\overline{\omega})}}\mathbf{R}(\mathbf{r},t)\right]. \tag{243}$$

On the other hand, Equation (234) is replaced by

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=-[C'(\overline{\omega})+\beta(t)\psi+\alpha(t)]+\sqrt{\frac{2}{n}}\zeta(\mathbf{r},t). \tag{244}$$

We explicitly see on these expressions that $\epsilon = 1/n$ is a measure of the strength of the noise (fluctuations). In CE, Equation (237) is replaced by

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=\nabla\cdot\left\{D\left[\nabla\overline{\omega}+\frac{\beta}{C''(\overline{\omega})}\nabla\psi\right]\right\}+\nabla\cdot\left[\sqrt{\frac{2D}{nC''(\overline{\omega})}}\mathbf{R}(\mathbf{r},t)\right]. \tag{245}$$

This equation is similar to the generalized stochastic Smoluchowski equation. It can be rewritten as

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=-\nabla\cdot\left[\frac{D}{C''(\overline{\omega})}\nabla\frac{\delta J}{\delta\overline{\omega}}\right]+\nabla\cdot\left[\sqrt{\frac{2D}{nC''(\overline{\omega})}}\mathbf{R}(\mathbf{r},t)\right]. \tag{246}$$

On the other hand, Equation (239) is replaced by

$$\frac{\partial \overline{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\omega} = -[C'(\overline{\omega}) + \beta\psi + \alpha(t)] + \sqrt{\frac{2}{n}}\zeta(\mathbf{r}, t) \tag{247}$$

Finally, in GCE, Equation (241) is replaced by

$$\frac{\partial \overline{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\omega} = -[C'(\overline{\omega}) + \beta\psi + \alpha] + \sqrt{\frac{2}{n}}\zeta(\mathbf{r}, t). \tag{248}$$

It can be rewritten as

$$\frac{\partial \overline{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\omega} = \frac{\delta G}{\delta \overline{\omega}} + \sqrt{\frac{2}{n}}\zeta(\mathbf{r}, t). \tag{249}$$

In the limit of short-range interactions (*i.e.*, for a small Rossby radius $R \to 0$), we can replace ψ by $\psi = R^2\overline{\omega} + R^4\Delta\overline{\omega}$ in the foregoing equations (see also [85]) and obtain generalized stochastic Cahn-Hilliard and Ginzburg-Landau equations similar to those described in Section 6 and in Appendix B.

We note that for a Gaussian prior vorticity distribution, leading to a generalized entropy proportional to minus the enstrophy ($C(\overline{\omega}) = \overline{\omega}^2/2\Omega_2$) [88], the noise term in Equations (243), (245) and (246) is independent on the vorticity since $C''(\overline{\omega}) = 1/\Omega_2$ is a constant.

When the system possesses different stable equilibrium states for a given entropy S, these stochastic equations describe random transition from one state to the other, similarly to the study of [47]. For a recent application of this type of stochastic equations in 2D geophysical flows, see [90]. However, the real problem of forced 2D turbulence is more complicated [91] because, in general, we do not know the entropy S a priori. This is due to the fact that the 2D Euler equation admits an infinity of conserved quantities (Casimirs) .

9. Conclusion

We have introduced a new class of generalized stochastic Fokker-Planck equations associated with a notion of generalized thermodynamics. These equations take into account small-scale constraints, long-range interactions, and fluctuations. In the case of short-range interactions, they reduce to generalized stochastic Cahn-Hilliard equations. While these equations are very rich, they have not been studied in great detail until now. In the absence of forcing, they reduce to generalized mean field Fokker-Planck equations. They display a rich variety of phase transitions between different types of solutions as we change the control parameter (e.g. the temperature). Furthermore, they can relax towards metastable states with very long lifetimes that are as much, or even more, relevant than fully stable states. The selection between a metastable or a fully stable state depends on a complicated notion of basin of attraction. In the presence of forcing, these generalized stochastic Fokker-Planck equations exhibit random transitions from one state to another, as illustrated in [47]. The applications of these equations are huge, including self-gravitating Brownian particles, colloid particles at a fluid interface, superconductors of type II, nucleation, the chemotaxis of bacterial populations, and two-dimensional turbulence.

Until now, systems with long-range interactions and generalized thermodynamics have been studied by different communities (see, for example, the references in the books [10] and [19]) with, sometimes, violent polemics between them. The present contribution tries to make the link between these two topics by showing how a class of generalized Fokker-Planck equations can describe complex systems experiencing both small-scale constraints (generalized thermodynamics) and long-range interactions.

A. Application of the Landau-Lifshitz Theory of Fluctuations

In this Appendix, we derive the generalized stochastic Smoluchowski equation (90) from the theory of fluctuations developed by Landau and Lifshitz (see Chapter XVII in [51]).

We write the equation for the density in the conservative form

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}, \tag{250}$$

where $\mathbf{J}$ is the current

$$\mathbf{J} = -\frac{1}{\xi} g(\rho) \nabla \frac{\delta F}{\delta \rho} - \mathbf{q}(\mathbf{r}, t). \tag{251}$$

The first term in Equation (251) is the deterministic Smoluchowski current (see Equation (50)) and the second term is a stochastic term that takes fluctuations into account. The problem at hand consists in characterizing the stochastic term $\mathbf{q}(\mathbf{r}, t)$. In order to use the general theory of fluctuations [51], we divide the fluid volume in small elements ΔV and take the average of each quantity in each element. The continuum limit $\Delta V \to 0$ will be performed in the final expressions. Equations (250) and (251) correspond to the equations

$$\dot{x}_a = -\sum_b \gamma_{ab} X_b + y_a, \tag{252}$$

of the general theory [51] provided that we make the identifications $\dot{x}_a \to -J_\alpha$ and $y_a \to q_\alpha$. The X_a can be obtained from the expression of the rate of production of entropy. In fact, since we are working in the canonical ensemble, the proper thermodynamical potential is the free energy $F = E - TS$. Taking the time derivative of the free energy functional, using Equation (250), and integrating by parts, we obtain

$$\dot{F} = \int \nabla \frac{\delta F}{\delta \rho} \cdot \mathbf{J}\, d\mathbf{r}. \tag{253}$$

Note that for $\mathbf{q} = \mathbf{0}$ (no noise) we recover the appropriate form of the H-theorem valid in the canonical ensemble

$$\dot{F} = -\frac{1}{\xi} \int g(\rho) \left(\nabla \frac{\delta F}{\delta \rho} \right)^2 d\mathbf{r} \leq 0. \tag{254}$$

If we replace the integral in Equation (253) by a summation on ΔV, we obtain

$$\dot{F} = \sum \nabla \frac{\delta F}{\delta \rho} \cdot \mathbf{J}\, \Delta V. \tag{255}$$

According to the general theory [51], we must also have

$$\dot{F} = k_B T \sum_a X_a \dot{x}_a. \tag{256}$$

Comparing Equation (255) with Equation (256), we find that the X_a are given by

$$X_a \to -\nabla \left(\frac{\delta F}{\delta \rho}\right) \frac{\Delta V}{k_B T}. \tag{257}$$

It is now easy to find the expression of the coefficients γ_{ab} that appear in Equation (252). Comparing Equations (251), (252) and (257), we find that

$$\gamma_{ab} = 0 \quad (\text{if} \quad a \neq b); \qquad \gamma_{aa} = \frac{k_B T g(\rho)}{\xi \Delta V}. \tag{258}$$

Now, the general theory of fluctuations [51] gives

$$\langle y_a(t_1) y_b(t_2)\rangle = (\gamma_{ab} + \gamma_{ba})\delta(t_1 - t_2). \tag{259}$$

Therefore, the correlation function of the stochastic field $q(\mathbf{r}, t)$ satisfies

$$\langle q_\alpha(\mathbf{r}, t) q_\beta(\mathbf{r}', t')\rangle = 0 \quad (\text{if} \quad \mathbf{r} \neq \mathbf{r}'), \tag{260}$$

$$\langle q_\alpha(\mathbf{r}, t) q_\beta(\mathbf{r}, t')\rangle = \frac{2 k_B T g(\rho)}{\xi \Delta V} \delta_{\alpha\beta} \delta(t - t'). \tag{261}$$

Taking the limit $\Delta V \to 0$, we can condense the above formulae under the form

$$\langle q_\alpha(\mathbf{r}, t) q_\beta(\mathbf{r}', t')\rangle = \frac{2 k_B T g(\rho)}{\xi} \delta_{\alpha\beta} \delta(t - t') \delta(\mathbf{r} - \mathbf{r}'). \tag{262}$$

This leads to the expression of the stochastic term appearing in Equation (90).

B. Stochastic Ginzburg-Landau and Cahn-Hilliard Equations

In this Appendix, we recall the stochastic Ginzburg-Landau equation and the stochastic Cahn-Hilliard equation in order to emphasize the analogies and the differences with the stochastic equation (135)–(137) introduced in this paper.

The stochastic Ginzburg-Landau equation for model A (non-conserved dynamics) [55] writes

$$\xi \frac{\partial \rho}{\partial t} = -\frac{\delta F}{\delta \rho} + \sqrt{2 \xi k_B T} \zeta(\mathbf{r}, t), \tag{263}$$

where $\zeta(\mathbf{r}, t)$ is a one-dimensional Gaussian white noise. $F[\rho]$ can be an arbitrary functional of ρ, but it is usually written in the form

$$F[\rho] = \int \left[\frac{1}{2}(\nabla \rho)^2 + V(\rho)\right] d\mathbf{r}. \tag{264}$$

The potential $V(\rho)$ can also be arbitrary, but it is oftentimes replaced by its normal form close to a critical point according to the Landau theory of phase transitions. For a functional of the form of Equation (264), the stochastic Ginzburg-Landau equation can be rewritten as

$$\xi \frac{\partial \rho}{\partial t} = \Delta \rho - V'(\rho) + \sqrt{2 \xi k_B T} \zeta(\mathbf{r}, t). \tag{265}$$

In the absence of noise, this equation satisfies an H-theorem:

$$\dot{F} = \int \frac{\delta F}{\delta \rho}\frac{\partial \rho}{\partial t}\, d\mathbf{r} = -\frac{1}{\xi}\int \left(\frac{\delta F}{\delta \rho}\right)^2 d\mathbf{r} \leq 0. \tag{266}$$

The equilibrium state is given by

$$\frac{\delta F}{\delta \rho} = -\Delta\rho + V'(\rho) = 0. \tag{267}$$

In the presence of noise, the FP equation for the probability density $P[\rho, t]$ of the density field $\rho(\mathbf{r}, t)$ at time t is

$$\xi \frac{\partial P}{\partial t}[\rho, t] = \int \frac{\delta}{\delta \rho(\mathbf{r}', t)} \left\{ \left[k_B T \frac{\delta}{\delta \rho} + \frac{\delta F}{\delta \rho} \right] P[\rho, t] \right\} d\mathbf{r}'. \tag{268}$$

Its stationary solution is

$$P[\rho] = \frac{1}{Z(\beta)} e^{-\beta F[\rho]}. \tag{269}$$

The stochastic Cahn-Hilliard equation for model B (conserved dynamics) [55] writes

$$\xi \frac{\partial \rho}{\partial t} = \Delta \frac{\delta F}{\delta \rho} + \sqrt{2\xi k_B T}\nabla \cdot \mathbf{R}, \tag{270}$$

where $\mathbf{R}(\mathbf{r}, t)$ is a d-dimensional Gaussian white noise. We note that $M = \int \rho\, d\mathbf{r}$ is conserved. For a functional of the form of Equation (264), Equation (270) can be rewritten as

$$\xi \frac{\partial \rho}{\partial t} = -\Delta(\Delta\rho - V'(\rho)) + \sqrt{2\xi k_B T}\nabla \cdot \mathbf{R}. \tag{271}$$

In the absence of noise, this equation satisfies an H-theorem:

$$\dot{F} = \int \frac{\delta F}{\delta \rho}\frac{\partial \rho}{\partial t}\, d\mathbf{r} = -\frac{1}{\xi}\int \left(\nabla\frac{\delta F}{\delta \rho}\right)^2 d\mathbf{r} \leq 0. \tag{272}$$

The equilibrium state is given by

$$\frac{\delta F}{\delta \rho} = -\Delta\rho + V'(\rho) = \mu, \tag{273}$$

where μ is a constant. In the presence of noise, the FP equation for the probability density $P[\rho, t]$ of the density field $\rho(\mathbf{r}, t)$ at time t is

$$\xi \frac{\partial P}{\partial t}[\rho, t] = -\int \frac{\delta}{\delta \rho(\mathbf{r}', t)} \left\{ \Delta \left[k_B T \frac{\delta}{\delta \rho} + \frac{\delta F}{\delta \rho} \right] P[\rho, t] \right\} d\mathbf{r}'. \tag{274}$$

Its stationary solution is

$$P[\rho] = \frac{1}{Z(\beta)} e^{-\beta F[\rho]} \delta(M[\rho] - M). \tag{275}$$

C. Long and Short-Range Interactions

The generalized FP equation (42) can be introduced phenomenologically by allowing the diffusion coefficient and the mobility of the particles in the usual FP equation to depend on the local density [45,46]. It can also be derived from a kinetic theory in which the transition probability from one site to another depends on the occupancy of these sites (see [24] and Section 2.11 of [46]). In this Appendix, we show that the generalized Smoluchowski equation (71) can also be derived from the dynamic density functional theory (DDFT) used in the theory of simple liquids when the particles experience short-range interactions [92]. In that case, the nonlinear pressure is due to the correlations induced by the short-range interactions, and the drift is due to the long-range interactions. We expose here only the main arguments and refer to [93] and to Appendix C of [62] for a more detailed discussion.

We consider a system of N Brownian particles in interaction described by the coupled stochastic equations (1). The evolution of the N-body distribution function is governed by the N-body Smoluchowski equation (3) and the evolution of the one-body distribution function is given by Equation (35). This equation is exact but it is not closed. For a system with purely long-range interactions, we can make the mean field approximation and obtain the mean field Smoluchowski equation (36). More generally, we assume that the potential of interaction u is the sum of a long-range potential u_{LR} and a short-range potential u_{SR}, so that $u = u_{LR} + u_{SR}$. Concerning the long-range potential, we make the mean field approximation $\rho_2(\mathbf{r},\mathbf{r}',t) = \rho(\mathbf{r},t)\rho(\mathbf{r}',t)$ leading to

$$\int \rho_2(\mathbf{r},\mathbf{r}',t)\nabla u_{LR}(|\mathbf{r}-\mathbf{r}'|)\,d\mathbf{r}' = \rho(\mathbf{r},t)\nabla\Phi(\mathbf{r},t), \tag{276}$$

where $\Phi(\mathbf{r},t)$ is the mean field potential defined by Equation (13) with u replaced by u_{LR}. To evaluate the integral corresponding to the short-range interactions, we use an approximation that has become standard in the DDFT of fluids [92] and take

$$\int \rho_2(\mathbf{r},\mathbf{r}',t)\nabla u_{SR}(|\mathbf{r}-\mathbf{r}'|)\,d\mathbf{r}' = \rho(\mathbf{r},t)\nabla\frac{\delta F_{\rm ex}}{\delta\rho}[\rho(\mathbf{r},t)], \tag{277}$$

where $F_{\rm ex}[\rho]$ is the excess free energy calculated at equilibrium. Equation (277) is exact at equilibrium (see, e.g., [93]), and the approximation consists in extending it out-of-equilibrium with the actual density $\rho(\mathbf{r},t)$ calculated at each time. This closure is equivalent to assuming that the two-body dynamic correlations are the same as those in an equilibrium fluid with the same one-body density profile. With the approximations (276) and (277), Equation (35) becomes

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\frac{k_B T}{m}\nabla\rho + \rho\nabla\frac{\delta F_{\rm ex}}{\delta\rho} + \rho\nabla\Phi\right), \tag{278}$$

where we have used the Einstein relation (2).

If we assume that the excess free energy $F_{\rm ex}$ depends only on the density (and on the temperature T that is fixed in the canonical ensemble), we can write

$$\frac{k_B T}{m}\nabla\rho + \rho\nabla\frac{\delta F_{\rm ex}}{\delta\rho} = \nabla P_{\rm id} + \nabla P_{\rm ex} = \nabla P(\rho), \tag{279}$$

where $P_{\rm id}(\mathbf{r},t) = \rho(\mathbf{r},t)k_BT/m$ is the ideal gas law, $P_{\rm ex} = P_{\rm ex}(\rho)$ is the excess pressure due to short-range interactions, and $P = P_{\rm id} + P_{\rm ex}$ is the total pressure given by a barotropic equation of state $P = P(\rho)$. Substituting Equation (279) in Equation (278), we obtain the generalized Smoluchowski equation

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\nabla P + \rho\nabla\Phi\right). \tag{280}$$

In this approach, the long-range interactions generate a mean field term $\rho\nabla\Phi$ and the short-range interactions generate a nonlinear pressure term ∇P. For ideal systems without short-range interactions ($F_{\rm ex} = 0$), the pressure reduces to the perfect gas law $P = \rho k_BT/m$, and we recover the mean field Smoluchowski equation (37).

We conclude on a subtle issue (see [93] for more details). The generalized Smoluchowski equation can be obtained from the generalized Kramers equation in a strong friction limit [46,94]. The generalized Kramers equation arises from the existence of small-scale constraints in velocity space. In that case, the velocity distribution of the particles is non-Boltzmannian and gives rise to a nonlinear pressure in the Smoluchowski equation. In the DDFT theory presented in this Appendix, the velocity distribution of the particles is Boltzmannian and the nonlinear pressure arises from the correlations due to the short-range interactions. Finally, in the approach developed in the main part of this paper, the velocity distribution of the particles is Boltzmannian and the pressure comes from small-scale constraints in position space. These remarks, that will be developed elsewhere [49], show that the generalized Smoluchowski equation can have different physical justifications and interpretations and can be derived in different manners.

Conflicts of Interest

The author declares no conflict of interest.

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