entropy

Article

A Functorial Construction of Quantum Subtheories

Ivan Contreras [1] and Ali Nabi Duman [2,*]

1 Department of Mathematics, University of Illinois at Urbana-Champain, Urbana, IL 61801, USA; icontrer@illinois.edu

2 Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* Correspondence: aliduman@kfupm.edu.sa; Tel.: +966-138-602-632

Academic Editors: Giacomo Mauro D'Ariano, Andrei Khrennikov and Massimo Melucci
Received: 2 March 2017; Accepted: 4 May 2017; Published: 11 May 2017

Abstract: We apply the geometric quantization procedure via symplectic groupoids to the setting of epistemically-restricted toy theories formalized by Spekkens (Spekkens, 2016). In the continuous degrees of freedom, this produces the algebraic structure of quadrature quantum subtheories. In the odd-prime finite degrees of freedom, we obtain a functor from the Frobenius algebra of the toy theories to the Frobenius algebra of stabilizer quantum mechanics.

Keywords: categorical quantum mechanics; geometric quantization; epistemically-restricted theories; Frobenius algebras

1. Introduction

The aim of geometric quantization is to construct, using the geometry of the classical system, a Hilbert space and a set of operators on that Hilbert space that give the quantum mechanical analogue of the classical mechanical system modeled by a symplectic manifold [1–3]. Starting with a symplectic space M corresponding to the classical phase space, the square integrable functions over M is the first Hilbert space in the construction, called the prequantum Hilbert space. In this case, the classical observables are mapped to the operators on this Hilbert space, and the Poisson bracket is mapped to the commutator. The desired quantum Hilbert space consists of the sections of the prequantum Hilbert space, which depends on the "position" variables. These "position" variables are obtained by splitting the phase space via the polarization P, which is the Lagrangian subspace (i.e., the maximal subspace where the symplectic form vanishes) of the phase space.

The space of functions on M is a commutative algebra under the operations of pointwise addition and multiplication. A bivector field on M determines a Poisson bracket so that M can be regarded as an approximation to a noncommutative algebra. The quantization approach due to Rieffel aims to obtain such a C^*-algebra, which is approximated by the Poisson algebra of the functions on M [4]. In this case, the algebra after quantization is a continuous field of C^*-algebras rather than a single algebra. On the other hand, Hawkins suggests a quantization recipe using symplectic groupoids to obtain a single C^*-algebra [5]. In this paper, we use the quantization formulation of Hawkins to investigate the epistemic toy theory due to Spekkens [6,7].

Recently, there has been a growing interest in quantum foundations in light of the quantum information revolution [8–10]. In this direction, Spekkens introduced this toy theory in support of the epistemic view of quantum mechanics [6]. The toy theory reproduces a large part of quantum theory by positing restrictions on the knowledge of an observer. The distinctively quantum phenomena arising in the toy theory include complementarity, no-cloning, no-broadcasting, teleportation, entanglement, Choi–Jamiolkowski isomorphism, Naimark extension, etc. On the other hand, the phenomena, such as Bell inequality violations, non-contextuality inequality violations and computational speed-up, do not arise in the toy theory.

The toy theory that we are interested in is the generalization of the original theory to the continuous and finite variables [7]. This is achieved by positing a restriction on what kind of statistical distributions over the space of physical states can be prepared. The new theory is called epistricted theory. In this way, quantum subtheories, the Gaussian subtheory of quantum mechanics, the stabilizer subtheory for qutrits and the Gaussian epistricted optics can be obtained from statistical classical theories, Liouville mechanics, statistical theory of trits and statistical optics, respectively.

The epistemic restriction defined on the classical phase-space states that an agent knows the values of a set of variables that commute relative to the Poisson bracket and maximally ignorant otherwise. Hence, a symplectic structure, which appears in the function space of the phase space, has mathematical correspondence with the ingredients of the quantization scheme. As a result, we conclude that the geometric quantization, via Hawkins' symplectic groupoid approach, produces a C^*-algebra that encodes the algebraic structure of the quadrature subtheories. Moreover, this construction gives us a functor from epistricted theories to the quantum subtheories.

In the second part of this paper, we construct a similar quantization functor of the toy theory for discrete degrees of freedom. In this case, the toy theory is defined precisely the same as the continuous case except that the finite dimensional symplectic vector space is over a finite field with odd prime characteristic. However, in order to apply groupoid quantization, we resort to the methods of categorical quantum mechanics pioneered by Abramsky and Coecke [11]. The categorical description of the toy theory is given in [12–14], where the toy theory is formulated as a subcategory of the dagger compact symmetric monoidal category of finite sets a **Rel**, and the toy observables correspond to dagger Frobenius algebras.

We start our construction with the dagger Frobenius algebras of the toy observables, which are functorially characterized as groupoids by Heunen, Catteneo and the first author in [15]. After equipping the resulting groupoid with a symplectic structure, we construct the pair groupoid to apply the quantization recipe of Hawkins. One can also obtain this pair groupoid from a different direction called CP^*-construction introduced in [16]. In the category of Hilbert spaces, Frobenius algebras correspond to finite dimensional C^*-algebras under this construction as as consequence of [17]. For the category **Rel**, the pair groupoids are the objects of $CP^*[\mathbf{Rel}]$. Hence, our main result establishes a functor from the dagger Frobenius algebra in **Rel** for epistricted theories to the Frobenius algebra in the category of Hilbert spaces.

The outline of this paper is as follows. We begin Section 2 with a brief summary of the geometric quantization procedure. We then discuss epistricted theories of continuous variables and their correspondence in the geometric quantization framework. We next briefly review Eli Hawkins' groupoid quantization recipe from which we obtain the usual Moyal quantization as a twisted group C^*-algebra from the geometric formulation of epistricted theories. We finally conclude that the resulting C^*-algebra contains phase-space formalism for quadrature subtheories. In Section 3, we follow the same quantization procedure for the odd-discrete degrees of freedom. We end the paper with the conclusion and discussions.

2. Continuous Degrees of Freedom

The main idea in this section is to describe the general framework of geometric quantization in the context of epistemically-restricted theories with continuous variables. We start with a quick overview of the standard literature on geometric quantization, and then, we move on to the interpretation for epistemically-restricted theories. We end the section with the algebraic counterpart of geometric quantization, introduction Hawkins' approach of quantization via symplectic groupoids. The outcome of this approach is a C^*-algebra for the epistricted theory.

2.1. Overview of Geometric Quantization

There are several ideas behind the construction of geometric quantization; however, the main objective is to produce quantum objects by using the geometry of the objects from the classical theory. In the sequel, we follow closely the approach of Bates and Weinstein [2].

2.1.1. The WKB Method

A basic technique for obtaining approximate solutions to the Schrödinger equation from classical motions is called the WKB method, after Wentzel, Kramers, and Brillouin. The WKB picture appears as an effort to describe quantum mechanics from a geometric viewpoint. It essentially approximates the solution of the time-independent Schrödingerequation, in the form:

$$\phi = e^{iS/\hbar},$$

where S is a solution of the Hamilton–Jacobi Equation:

$$H(x, \partial S/\partial x) = E.$$

We can then use the geometry of the phase space to realize the solution to the Schrödinger equation as a Lagrangian submanifold $\mathcal{L}$ of the level set $H^{-1}(E)$. More precisely, let us consider the semiclassical approximation for ϕ. From the transport equation:

$$a \triangle S + 2 \sum \frac{\partial a}{\partial x_j} \frac{\partial S}{\partial x_j} = 0$$

where a is a function on $\mathbb{R}^n$, and after multiplying both sides by a, we obtain that:

$$\mathrm{div}(a^2 \nabla S) = 0. \tag{1}$$

Now, if we consider the vector field:

$$X_{H|L} = \sum_j \frac{\partial S}{\partial x_j} \frac{\partial}{\partial q_j} - \frac{\partial V}{\partial q_j} \frac{\partial}{\partial p_j}$$

onto $\mathbb{R}^n$ where the Hamiltonian H is $H(q,p) = \sum p_i^2/2 + V(q)$ and $|dx| = |dx_1 \wedge \ldots \wedge dx_n|$ is the canonical density on $\mathbb{R}^n$, its projection $X_H^{(x)}$ onto $\mathbb{R}^n$ satisfies the following invariance condition:

$$\mathfrak{L}_{X_H^{(x)}}(a^2|dx|) = 0, \tag{2}$$

where $\mathfrak{L}$ denotes the Lie derivative, if we restrict to the Lagrangian submanifold $\mathcal{L} = im(dS)$. Since the vector field X_H is tangent to the manifold $\mathcal{L}$ and $\mathfrak{L}$ is diffeomorphisms invariant, Equation (2) implies that the pullback $\pi^*(a^2|dx|)$ is invariant under the flow of X_H, where $\pi : T^*\mathbb{R}^n \to \mathcal{L}$ denotes the projection onto $\mathcal{L}$.

This discussion implies that a semi-classical state can be defined geometrically as a Lagrangian submanifold $\mathcal{L}$ of $\mathbb{R}^{2n}$, equipped with a half density function a. This semi-classical state is stationary when $\mathcal{L}$ lies in the level set of the Hamiltonian, and the half density a is invariant under the Hamiltonian flow. Transformations of the state correspond to Hamiltonians on $\mathbb{R}^{2n}$. To summarize this geometric picture, Table 1 exhibits the correspondence between semi-classical objects (of geometric nature) and quantum objects (of algebraic nature) in this particular case.

Note that in Table 1, to the semi-classical space $(\mathbb{R}^{2n}, \omega)$, we associate the so-called intrinsic Hilbert space $\mathfrak{H}_{\mathbb{R}^{2n}}$, that is the Hilbert space of half densities on $\mathbb{R}^{2n}$, which must be introduced in order for the invariance condition in Equation (2) to make sense in terms of density functions.

Table 1. Correspondence between classical and quantum objects.

Object	Semi-Classical (Geometric) Version	Quantum (Algebraic) Version
Phase space	$(\mathbb{R}^{2n}, \omega)$	Hilbert space $\mathfrak{H}_{\mathbb{R}^{2n}}$
State	Lagrangian submanifold of $\mathbb{R}^{2n}$ with half-density	half-density on $\mathbb{R}^n$
Transformations	Hamiltonian H on $\mathbb{R}^{2n}$	operator $\hat{H}$ on smooth half densities
Stationary state	Lagrangian submanifold in level set of H with invariant half-density	eigenvector of $\hat{H}$

2.1.2. Basic Symplectic and Poisson Geometry

From now on, we consider finite dimensional vector spaces V to be symplectic, if they are equipped with a non-degenerate skew form ω. For a vector subspace W of V, its orthogonal complement is defined by $W^{\perp} = \{x \in V : \omega(x, y) = 0, \forall y \in W\}$. We have the following special cases for W:

- W is isotropic if $W \subseteq W^{\perp}$.
- W is coisotropic if $W^{\perp} \subseteq W$.
- W is symplectic if $W \cap W^{\perp} = \{0\}$.
- W is Lagrangian if $W = W^{\perp}$.

It can be easily checked that if W is Lagrangian, then $\dim W = \frac{1}{2} \dim V$.

Definition 1. *A manifold is called Lagrangian (resp. isotropic, coisotropic and symplectic) if its tangent space is a Lagrangian subspace at every point.*

We also consider Poisson algebras, which are commutative algebras $(P, +, \bullet)$ equipped with a Lie bracket $[,]$ that is a derivation for the commutative product. As a particular case in our discussion, the algebra of functions of a symplectic manifold (M, ω) is naturally a Poisson algebra.

2.1.3. Prequantum Line Bundle

In this section, we follow Dirac's approach to axiomatize the quantization procedure.

Definition 2. *A pre-quantization is a linear map $P \to \hat{P}_{\mathcal{H}}$ from a Poisson algebra (more precisely, the algebra of functions of a Poisson manifold M) into the set of operators on a (pre)-Hilbert space $\mathcal{H}$, satisfying the following properties:*

1. $\hat{Id}_P = Id_{\hat{P}_{\mathcal{H}}}$.
2. $\hat{[F,G]} = \frac{i}{\hbar}(\hat{F}\hat{G} - \hat{G}\hat{F})$.
3. $\hat{F^*} = (\hat{F})^*$, *where $*$ denotes complex conjugation on left side and adjunction on the right side.*

Definition 3. *A pre-quantization is called quantization if, in addition to the properties above, the following condition is satisfied:*

4. *For a complete set of functions $\{F_i\}$, its quantization $\{\hat{F}_i\}$ is also a complete set of operators.*

Proposition 1. *In the specific case where M is a cotangent bundle T^*N, a pre-quantization (referred to in the literature as the Koopman–Van Hove–Segal pre-quantization) can be constructed, and it has the following form:*

$$\hat{F} = F + \frac{\hbar}{2\pi i} X_F - \theta(X_F), \tag{3}$$

where X_F is a Hamiltonian vector field with generating function F and θ is a primitive of the Liouville form ω_{T^*N}.

In order to implement this pre-quantization for a arbitrary symplectic manifold (M, ω), we require a complex line bundle over M, equipped with a Hermitian structure and a Hermitian connection ∇, for which the pre-quantization formula 3 takes the following form:

$$\hat{F} = F + \frac{\hbar}{2\pi i}\nabla_{X_F}. \tag{4}$$

Provided a compatibility condition between $\mathrm{curv}(\nabla)$ and ω, this formula gives a pre-quantization for (M, ω).

2.1.4. Polarization

It is easy to realize in some examples that the Hilbert space of pre-quantization is too big for the completeness Condition 4 to hold. By using the ordinary viewpoint of quantum mechanics, only half of the coordinates of the classical phase space are required to write down the wave functions, depending on whether the coordinate or momentum representation is considered. In (symplectic) geometric terms, for general symplectic manifolds, a polarization is defined as follows:

Definition 4. *Let (M, ω) be a symplectic manifold. A polarization of M is a Lagrangian involutive distribution $\mathcal{P}$ of M.*

Thus, the quantization space consists of functions constant along the leaves of a the distribution $\mathcal{P}$ on M; more precisely, the quantization Hilbert space $\mathcal{H}$ is the space of sections s of the complex line bundle on M such that:

$$\nabla_{X_P} s = 0, \tag{5}$$

where X_P is a vector field tangent to the polarization $\mathcal{P}$.

2.2. Quadrature Epistricted Theories

We now introduce the quadrature epistricted theories for continuous variables [7]. The epistemic restrictions on classical variables are adopted from the condition of the joint measurability of quantum observables.

Definition 5. *A set of variables are jointly knowable if and only if it is commuting with respect to the Poisson bracket.*

The other restriction besides joint knowability is that an agent can know only the variables that are the linear combination of the position and momentum variables. Such variables are called quadrature variables. Hence, the valid epistemic states are the ones for which an agent knows the values of a set of quadrature variables that commute with respect to the Poisson bracket and that is maximally ignorant otherwise. This notion is termed classical complementarity.

Example 1 (Darboux coordinates)**.** *If we start with the phase space $\Omega = \mathbb{R}^{2n}$ where a point is denoted by $\mathbf{m} = (\mathbf{p}_1, \mathbf{q}_1, \ldots, \mathbf{p}_n, \mathbf{q}_n)$, epistemic restrictions imply that the functionals $f : \Omega \to \mathbb{R}$ are of the form:*

$$f = \mathbf{a_1} q_1 + \mathbf{b_1} p_1 + \ldots + \mathbf{a_n} q_n + \mathbf{b_n} p_n + \mathbf{c}$$

where $\mathbf{a_1}, \mathbf{b_1}, \ldots, \mathbf{a_n}, \mathbf{b_n}, \mathbf{c} \in \mathbb{R}$ and $p_i(\mathbf{m}) = \mathbf{p_i}$ and $q_i(\mathbf{m}) = \mathbf{q_i}$ are functionals associated with momentum and position, respectively. Hence, each functional f is associated with a vector $\mathbf{f} = (\mathbf{a_1}, \mathbf{b_1}, \ldots, \mathbf{a_n}, \mathbf{b_n})$. It is not hard to show that the value of the Poisson bracket over the phase space is uniform and equal to the symplectic inner product:

$$[f, g]_{PB}(\mathbf{m}) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right)(\mathbf{m}) = \langle \mathbf{f}, \mathbf{g} \rangle$$

where:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \mathbf{f}^T J \mathbf{g}$$

and J is the skew symmetric $2n \times 2n$ matrix with components $J_{ij} = \delta_{i,j+1} - \delta_{i+1,j}$. Hence, the vector space Ω becomes a symplectic vector space with the symplectic inner product $\omega = \langle \cdot, \cdot \rangle$. This allows us to give the geometric presentation of the quadrature variables.

The only set of variables jointly knowable are the ones that are Poisson commuting. In symplectic geometry, this set corresponds to the subspace V of vectors whose symplectic inner product vanishes, i.e., $\forall \mathbf{f}, \mathbf{g} \in V$ $\langle \mathbf{f}, \mathbf{g} \rangle = 0$. For a $2n$-dimensional phase space, the maximum possible dimension of such a V is n. Such a maximal space is a Lagrangian space as defined above, and it corresponds to the maximal possible knowledge an agent can have. In order to specify an epistemic state, one should also set the values of the variables on V. The linear functional v acting on a quadrature functional corresponds to the set of vectors in $\mathbf{v} \in V$, which is determined via $v(f) = \mathbf{f}^T \mathbf{v}$. That is, for every vector $\mathbf{v} \in V$, we obtain distinct value assignment.

In summary, a pure state in the epistricted theory consists of a Lagrangian subspace $V \in \mathbb{R}^{2n}$ and a valuation functional $v : \mathbb{R}^{2n} \to \mathbb{R}$. In geometric quantization, the half density function can be regarded as this valuation function.

On the other hand, the valid transformations are the symplectic transformations that map the quadrature variables to itself. These transformations map a phase space vector $\mathbf{m} \in \Omega$ to $S\mathbf{m} + \mathbf{a}$ where $\mathbf{a}$ is a displacement vector and S is $2n \times 2n$ a symplectic matrix. The group formed by these transformations is called the affine symplectic group, which is subgroup of the Hamiltonian symplectomorphism group. Thus, each of these transformations can be obtained from a Hamiltonian. Finally, the sharp measurements are parametrized by Poisson commuting sets of quadrature variables (isotropic subspaces V), and the outcomes are indexed by the vectors in V.

We summarize the correspondence between geometric quantization and epistricted theories in Table 2.

Table 2. Correspondence between geometric quantization and epistricted theories.

Object	Semi-Classical Version in Quantization	Epistricted Theories
Phase space	$(\mathbb{R}^{2n}, \omega)$	$(\mathbb{R}^{2n}, \omega)$
State	Lagrangian submanifold of $\mathbb{R}^{2n}$ with half-density $a : \mathbb{R}^{2n} \to \mathbb{R}$	Lagrangian subspace with a valuation function $v : \mathbb{R}^{2n} \to \mathbb{R}$
Transformations	Hamiltonian H on $\mathbb{R}^{2n}$	affine symplectic transformation

2.3. Hawkins' Groupoid Quantization

The aim of this section is to point out that the epistricted theories can be quantized by a twisted polarized convolution C^*-algebra of a symplectic groupoid in the sense of Hawkins. The main idea in this method is to find a C^*-algebra that is approximated by a Poisson algebra of functions on a manifold. C^*-algebra quantization is mainly developed by the work of Rieffel, where the quantization is stated as a continuous field of C^*-algebras $\{\mathcal{A}_\hbar\}$. Hawkins' construction gives a single algebra $\mathcal{A}_1$ by involving additional structures on the symplectic groupoid. In his approach, it is possible to reinterpret geometric quantization for a broader class of examples, coming from deformation quantization of Poisson algebras. This gives a rigorous treatment to the dictionary strategy of Weinstein relating the symplectic category and its geometrically quantized counterpart [2].

2.3.1. Symplectic Groupoids

We start with the definition of symplectic groupoid, arising from the usual definition of the Lie groupoid, requiring compatibility conditions with a symplectic structure on the space of arrows.

Definition 6. *A topological groupoid Σ is a groupoid object in the category of topological spaces, that is Σ consists of a space of Σ_0 of objects and a space Σ_2 of arrows, together with five continuous structure maps:*

- *The source map $s : \Sigma_2 \to \Sigma_0$ assigns to each arrow $g \in \Sigma_2$ its source $s(g)$.*
- *The target map $t : \Sigma_2 \to \Sigma_0$ assigns to each arrow $g \in \Sigma_2$ its target $t(g)$. For two objects $x, y \in \Sigma_0$, one writes $g : x \to y$ to indicate that $g \in \Sigma_2$ is an arrow with $s(g) = x$ and $t(g)$.*
- *If g and h are arrows with $s(h) = t(g)$, one can form their composition, denoted hg, with $s(hg) = s(g)$ and $t(hg) = t(h)$. If $g : x \to y$ and $h : y \to z$, then hg is defined, and $hg : x \to z$. The composition map m is defined by $m(h, g) = hg$, and it is a well-defined map $m : \Sigma_m \to \Sigma_2$, where $\Sigma_m := \{(h, g) : s(h) = t(g)\}$.*
- *The unit map $u : \Sigma_0 \to \Sigma_2$ is a two-sided unit for composition.*
- *The involution map $-^* : \Sigma_2 \to \Sigma_2$. Here, if $g : x \to y$, then $g^* : y \to x$ is two-sided inverse for composition.*

Σ is said to be a groupoid over Σ_0

Definition 7. *A Lie groupoid is a topological groupoid Σ where Σ_0 and Σ_2 are smooth manifolds and such that the structure maps s, t, m, u and $-^*$ are smooth. Moreover, s and t are required to be submersions, so that the domain of m is a smooth manifold.*

Definition 8. *A Lie groupoid Σ is called a symplectic groupoid if Σ_2 is a symplectic manifold with symplectic form ω, and the graph multiplication relation $\mathfrak{m} = \{(xy, x, y) : (x, y) \in \Sigma_2\}$ is a Lagrangian submanifold of $\Sigma_2 \oplus \overline{\Sigma}_2 \oplus \overline{\Sigma}_2$, where $\overline{\Sigma}$ is the symplectic manifold $(\Sigma_2, -\omega)$.*

This definition is equivalent to saying that the symplectic form ω is multiplicative, i.e., it satisfies the following compatibility conditions with the multiplication and projection maps:

$$m^*\omega = pr_1^*\omega + pr_2^*\omega, \tag{6}$$

where pr_1 and pr_2 are the projections of Σ_m onto the first and second component, respectively. As $\mathfrak{m}$ is Lagrangian, one can find a unique Poisson structure on Σ_0 of a symplectic groupoid, such that s is a Poisson map, and t is anti-Poisson. Hence, we have the following definition.

Definition 9. *A symplectic groupoid Σ is said to integrate a Poisson manifold Ω if there exists a Poisson isomorphism from Σ_0 onto Ω.*

The following are the basic examples of symplectic groupoids, the first one being of central importance for the geometric quantization procedure in epistricted theories.

Example 2 (Pair groupoid of a symplectic manifold)**.** *As we will describe in more detail later in the paper, given a smooth manifold M, the manifold $M \times M$ is naturally the space of arrows for a Lie groupoid, called the pair groupoid. In the case where M is equipped with a symplectic structure ω, then the Lie groupoid $Pair(M)$ is a symplectic groupoid with symplectic structure $\omega \oplus \omega$.*

Example 3 (Cotangent bundle)**.** *If M is a manifold, any vector bundle E over M is a Lie groupoid over M; the multiplication is given by fiber addition; the source and target maps are projection onto the base; whereas the unit is given by the zero section of the bundle. In the particular case that $E = T^*M$ and that ω is the Liouville form on the cotangent bundle, it it easy to verify that T^*M is a symplectic groupoid over M.*

Here is Hawkins' strategy for geometric quantization of a manifold Ω. For a detailed discussion, one can refer to [5]:

- Construct an symplectic groupoid Σ over Ω.
- Construct a pre-quantization (σ, L, ∇) of Σ.

- Choose a symplectic groupoid polarization P of Σ, which satisfies both symplectic and groupoid polarization.
- Construct a "half form" bundle.
- Ω is quantized by twisted, polarized convolution algebra $C_P^*(\Sigma, \sigma)$.

Proposition 2. *Hawkins' geometric quantization of the symplectic space $\Omega = \mathbb{R}^{2n}$ and Darboux coordinates (Example 1) is the Moyal quantization of the Poisson algebra of the symplectic vector space.*

Proof. In the particular case that the symplectic manifold is a vector space $\Omega = \mathbb{R}^{2n}$ with symplectic form ω, which is the context of the epistricted theories, we have the symplectic groupoid $\Omega \oplus \Omega^*$ integrating the symplectic vector space Ω, where the multiplication is given by fiber addition on $\Omega^* = \{(p^1, p^2, \cdots, p^{2n})\}$, i.e., the symplectic integration comes equipped with Darboux coordinates.

More explicitly, $\hat{\omega}(u) : v \mapsto \omega(u, v)$ gives a map $\hat{\omega} : \mathbb{R}^{2n} \to \mathbb{R}^{2n*}$. One obtains a symplectic structure:

$$\sigma((x,y),(z,w)) = \omega(x,z) - \omega(y,w)$$
$$= \hat{\omega}(x-y)[\frac{z+w}{2}] - \hat{\omega}(z-w)[\frac{x+y}{2}].$$

We identify $\mathbb{R}^{2n} \oplus \bar{\mathbb{R}}^{2n}$ with the cotangent bundle $T^*(\mathbb{R}^{2n})$ as follows: for the local coordinates of covectors (u, ξ), (v, η) in $T^*(\mathbb{R}^{2n})$, the cotangent symplectic structure is

$$\sigma^*((u,\xi),(v,\eta)) = \xi(u) - \eta(v).$$

This gives us a symplectomorphism $\Phi : \mathbb{R}^{2n} \oplus \bar{\mathbb{R}}^{2n} \to T^*(\mathbb{R}^{2n})$ such that:

$$\Phi : (x,y) \mapsto (1/2(x+y), \hat{\omega}(x-y))$$

where $\Phi^*\sigma^* = \sigma$ (this example has also been studied by Hawkins (see Example 6.2 of [5])).

One can obtain the the Darboux coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ of $T^*(\mathbb{R}^{2n})$ from the symplectomorphism Φ. The projection of $T^*(\mathbb{R}^{2n})$ to $\mathbb{R}^{2n*}$ is a fibration of groupoids whose fibers are Lagrangian. Thus, this is a polarization of the symplectic groupoid given by:

$$P = span\{\partial/\partial p_1, \dots, \partial/\partial p_n\}$$

The symplectic potential, which vanishes on P, can be chosen as $\theta_P = -p^i dq_i$.

This polarization gives us the half-form pairing, which enables quantizable observables to be represented as operators on the Hilbert space $L^2(\mathbb{R}^{2n})$. Hence, this yields the correspondence between the kernels of operators on $L^2(\mathbb{R}^{2n})$ and Weyl symbols of these operators. This kernel T of an operator f is given by:

$$Tf(p,q) = C \int f(\frac{p+q}{2}, \zeta) e^{i\zeta(q-p)/\hbar} d\zeta.$$

The quantization procedure gives the twisted group algebra $C^*(\Omega^*, \sigma)$ where $\sigma : \Omega^* \times \Omega^* \to \mathbb{T}$, $\sigma(x,y) = e^{\frac{-i}{\{q,p\}}}$. This is the usual Moyal quantization of a Poisson vector space (see [18]). In this setting, the observables correspond to functions in classical phase-space, and the Moyal product of functions is derived from the product of the pair of observables. In this case, the position and momentum operators correspond to the generators of the Heisenberg group, and they are related to each other by a Fourier transform. □

Theorem 1. *Quadrature quantum subtheories and the Moyal quantization from Proposition 2 coincide.*

Proof. To be consistent with the formalism of [7], we work with projector valued measures (PVM) rather than Hermitian operators. PVMs are used in quantum information and quantum foundations to represent measurements, as eigenvalues of Hermitian operators are operationally insignificant and serve as labels of outcomes. A projector-valued measure with outcome set K is a set of projectors $\{\Pi_k : k \in K\}$ such that $\Pi_k^2 = \Pi_k$, $\forall k \in K$ and $\sum_k \Pi_k = I$. Hence, the position (momentum) observables are the set of projectors onto position (momentum) eigenstates (in the continuous case, one can also use Hermitian operators corresponding to the real valued functionals, but the commutation relation of Hermitian operators does not have a finite counterpart. Therefore, Spekkens preferred to use PVMs in order to cover finite and continuous cases simultaneously):

$$\mathcal{O}_q = \{\hat{\Pi}_q(\mathbf{q}) : \mathbf{q} \in \mathbb{R}\}$$

where

$$\hat{\Pi}_q(\mathbf{q}) = |\mathbf{q}\rangle_q\langle\mathbf{q}|.$$

We now define a unitary representation of symplectic affine transformation to introduce the other quadrature observables. The projective unitary representation $\hat{V}$ of the symplectic group acting on the phase space Ω satisfies $\hat{V}(S)\hat{V}(S') = e^{i\phi}\hat{V}(SS')$ for every symplectic matrix $S : \Omega \to \Omega$ and where $e^{i\phi}$ is a phase factor. The action of this unitary is defined by the conjugation:

$$\mathcal{V}(S)(\cdot) = \hat{V}(S)(\cdot)\hat{V}^{\dagger}(S).$$

For a single degree of freedom, let S_f be the symplectic matrix that takes the position functional q to a quadrature functional f, such that $S_f\mathbf{q} = \mathbf{f}$. Then, the quadrature observable associated with f is defined as follows:

$$\mathcal{O}_f = \{\hat{\Pi}_f(\mathbf{f}) : \mathbf{f} \in \mathbb{R}\}$$

where:

$$\hat{\Pi}_f(\mathbf{f}) = \mathcal{V}(S_f)(\hat{\Pi}_q(\mathbf{f})).$$

For the n degrees of freedom $\Omega = \mathbb{R}^{2n}$, the quadrature observable associated with f is given by:

$$\mathcal{O}_f = \{\hat{\Pi}_f(\mathbf{f}) : \mathbf{f} \in \mathbb{R}^{2n}\}$$

where:

$$\hat{\Pi}_f(\mathbf{f}) = \mathcal{V}(S_f)(I \otimes \cdots \otimes \hat{\Pi}_{q_i}(\mathbf{f}) \otimes \cdots \otimes I)$$

for $S_f\mathbf{q}_i = \mathbf{f}$. We also know that the set of quadrature observables $\{\mathcal{O}_{f_i}\}$ commutes if and only if the corresponding functionals $\{f_i\}$ are Poisson-commuting (see [6]). Hence, the commuting set of quadrature observables can be labeled by isotropic subspaces of Ω. This set defines a single quadrature observable:

$$\mathcal{O}_V = \{\hat{\Pi}_V(\mathbf{v}) : \mathbf{v} \in V\}$$

where:

$$\hat{\Pi}_V(\mathbf{v}) = \prod_{\mathbf{f}^{(i)}} \hat{\Pi}_{f^{(i)}}(f^{(i)}\mathbf{v}).$$

On the other hand, in the geometric quantization procedure, any functional f on Ω is mapped to a Hermitian operator $\hat{f}$ in a prequantum Hilbert space, which corresponds to the observable $\mathcal{O}_f = \{\hat{\Pi}_f(\mathbf{f}) : \mathbf{f} \in \mathbb{R}^{2n}\}$. Moreover, the commutation relation for the observables in both quadrature subtheories and geometric quantization is implied by the Poisson commutation relation of the classical observables. As the polarization is the commuting set of these Hermitian operators, the state that is obtained after quantization is the operator $\hat{\Pi}_V(\mathbf{v})$. The choice of the vertical polarization for the groupoid $\Omega \oplus \Omega^*$ is the responsible of the correspondence between the two quantum states. The half-form pairing defined above can be computed in terms of the integral kernel of the projection operator $\hat{\Pi}_f$, which has Weyl symbol f. This establishes a correspondence between phase-space formalism and quantum mechanics, and the Moyal product is deduced from this correspondence. $\square$

In [6], the operational equivalence quantum subtheories and epistricted theories are proven using Wigner representation, which maps operators in Hilbert space to the functions in the phase-space formulation of quantum mechanics. It is also a well-known fact that the Wigner representation of an operator product is given by the Moyal product. As a result, geometric quantization with an appropriate choice of polarization is operationally equivalent to epistricted theories. We can also conclude that group algebra $C^*(H) = C^*(\Omega^*, \sigma)$, which is the Hilbert space considered as a group representation of the Heisenberg group H, contains the algebraic structure of quadrature subtheories.

This discussion leads to the following theorem:

Theorem 2 (Main result in the continuous case)**.** *The geometric quantization, via Hawkins' symplectic groupoid approach, of the Spekkens toy theory of continuous degrees of freedom produces a C^*-algebra that is a group representation for the Heisenberg group H, and it encodes the algebraic structure of the quadrature subtheories, via Moyal quantization.*

2.4. Functoriality

The functoriality of geometric quantization is a delicate issue, and it has been proven that the quantization that fits with the Schröedinger picture is in fact not functorial. There are several problems even before quantization, in particular that the symplectic category is not quite a category, since the composition of Lagrangian correspondences is not in general well defined, and also that, when it is defined, the composition is not continuous with the standard topology in the Lagrangian Grassmannian. The failure of geometric quantization to functorially represent Schröedinger's picture is given, e.g., in Gotay's work [19].

However, the geometric quantization picture for symplectic groupoids turns out to be functorial with respect to the choices, i.e., the polarizations (the groupoid one), the half line bundle. The fact that the choices of polarizations are affine means that there is a higher structure for our C*-algebra quantization, namely the objects are symplectic manifolds; one-morphisms are Lagrangian polarizations; and two-morphisms are affine transformations between Lagrangian polarizations. These two-morphisms are reflected in C*-algebra automorphisms after quantization.

3. Finite Degrees of Freedom

We now discuss how the geometric quantization relates the epistricted theories to quadrature quantum subtheories for odd-prime discrete degrees of freedom. In [7], the operational equivalence of these two theories for continuous and odd-prime discrete cases was proven using Wigner representation. Here, we aim to construct a functor from a subcategory of the category of groupoids to the category of C^*-algebras. This corresponds to a functor from Frobenius algebras in the category **FRel** (Frobenius algebras in the category of sets and relations) to Frobenius algebras in the category of Hilbert spaces **FHilb**. Here is the sketch of our discrete quantization:

- We start with the special dagger Frobenius algebra of epistricted theories, **Spek**, which is a subcategory of finite sets and relations, **FRel**.
- We then construct the groupoid $\mathcal{G}$ corresponding to **Spek** via the explicit equivalence in Heunen et al. [15].
- We next obtain the pair groupoid from $\mathcal{G}$ and introduce the symplectic structure on it, which is compatible with the pair groupoid structure. In this case, each polarization corresponds to a Lagrangian subspace in epistricted theories.
- We then apply the geometric quantization procedure via Hawkins on the pair groupoid.
- Finally, we end up with the finite dimensional C^*-algebra from which one can construct special dagger Frobenius algebra over **FHilb** via [17].

We begin this section by reviewing the epistricted theories in the discrete case.

3.1. Quadrature Epistricted Theories

The formalism in the finite case is defined over the finite fields with prime order d. These fields are isomorphic to the integers modulo d, denoted by $\mathbb{Z}_d$. Hence, the configuration space and associated phase-space are $(\mathbb{Z}_d)^n$, $\Omega = (\mathbb{Z}_d)^{2n}$, respectively. The linear functionals are also in the form:

$$f = \mathbf{a_1} q_1 + \mathbf{b_1} p_1 + \ldots + \mathbf{a_n} q_n + \mathbf{b_n} p_n + \mathbf{c}$$

where $\mathbf{a_1}, \mathbf{b_1}, \ldots, \mathbf{a_n}, \mathbf{b_n}, c \in \mathbb{Z}_d$. Hence, a vector $\mathbf{f} = (\mathbf{a_1}, \mathbf{b_1}, \ldots, \mathbf{a_n}, \mathbf{b_n})$ specifies the position and momentum dependence of the quadrature functional f. The dual space $\Omega^* = (\mathbb{Z}_d)^n$ consists of these vectors associated with the functionals. The Poisson bracket, unlike the continuous case, is defined in terms of finite differences:

Definition 10. *The Poisson bracket in the finite case is given by:*

$$[f,g]_{PB}(\mathbf{m}) = \sum_{i=1}^{n} [(f(\mathbf{m}+\mathbf{q}_i) - f(\mathbf{m}))(g(\mathbf{m}+\mathbf{p}_i) - g(\mathbf{m})) - (f(\mathbf{m}+\mathbf{p}_i) - f(\mathbf{m}))(g(\mathbf{m}+\mathbf{q}_i) - g(\mathbf{m}))],$$

where the operations are in modulo d. The Poisson bracket, $[f,g]_{PB}(\mathbf{m})$, is also equal to symplectic inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ on the discrete phase space.

Like in the continuous case, an epistemic state is determined by the set of quadrature variables that are known to that agent and the values of these variables. This corresponds to the pair $(V^*, \mathbf{v})$, where V^* is an isotropic subspace of the phase space Ω^*, and $\mathbf{v}$ is a valuation vector in $V^{**} = V$. Similarly, the valid transformations are symplectic transformations, which preserve the symplectic inner product, and they form the affine symplectic group over the finite field $\mathbb{Z}_d$. Note that these transformations over a finite field are discrete in time; hence, they cannot be generated from a Hamiltonian unlike the continuous case.

Example 4. *As an example, we consider the quadrature epistricted theory of trits [7] for a single system. The configuration space and the phase space are $\mathbb{Z}_3$ and $\mathbb{Z}_3^2$, respectively. The quadrature functionals in this system are of the form $f = \mathbf{a}q + \mathbf{b}p + \mathbf{c}$ where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{Z}_3$. There are four inequivalent quadrature functionals:*

$$q, p, q + p, q + 2p.$$

Since none of these functionals Poisson commute, an agent can know at most one of them. This implies that there are twelve epistemic states, as the valuation vectors are chosen from $V = \mathbb{Z}_3$. These states are depicted in Figure 1 as 3×3 grids:

The valid transformations, which form the affine symplectic group over $\mathbb{Z}_3$, correspond to a certain subset of permutations of the functionals (See Figure 2 for an example).

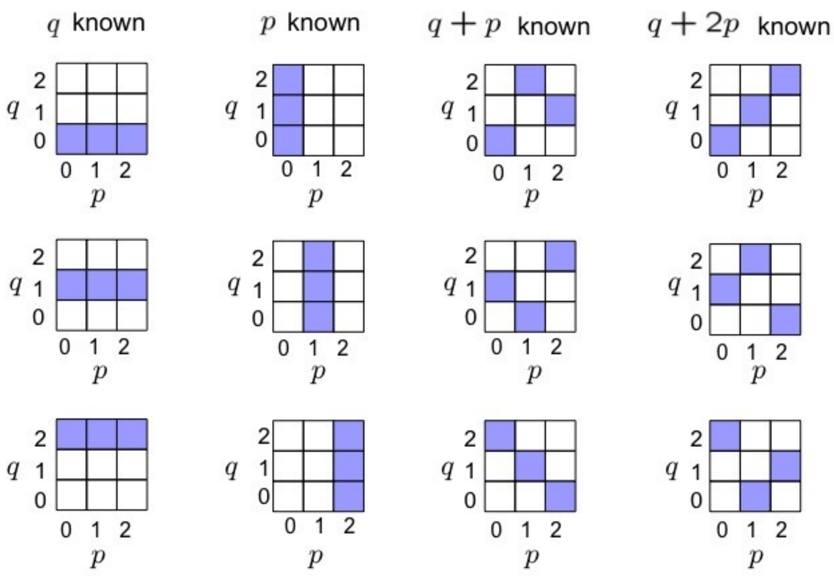


Figure 1. Twelve epistemic states.

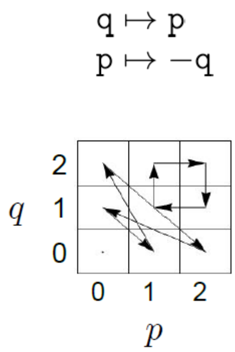


Figure 2. The valid transformations.

3.2. The Category of Epistricted Theories

We now turn to the category of the epistricted theory of trits. The arguments can easily be generalized to the epistricted theories for other odd primes. We start with the category of **FRel**, whose objects are sets and whose morphisms $X \to Y$ are relations $r \subseteq X \times Y$ and $s \circ r = \{(x,z) | \exists y, (x,y) \in r, (y,z) \in s\}$. **FRel** is a dagger symmetric monoidal category when the tensor product is chosen as a Cartesian product, the single element set $1 = \{\bullet\}$ as the identity and the relational converse as the dagger morphism $\dagger$.

Definition 11. *An object X in **FRel** with a morphism $m : X \times X \to X$ is called special dagger Frobenius algebra if and only if m has the following properties:*

- $(1 \times m) \circ (m^\dagger \times 1) = m^\dagger \circ m = (m \times 1) \circ (1 \times m^\dagger)$ *(F)*
- $m \circ m^\dagger = 1$ *(M)*
- $m \circ (1 \times m) = m \circ (m \times 1)$ *(A)*
- *there is $e : 1 \to X$ with $m \circ (e \times 1) = 1 = m \circ (1 \times e)$ (U).*

The conditions of Frobenius algebras are presented graphically in Figure 3. These diagrams encode composition by drawing morphisms on top of each other, and the monoidal product is the drawing morphism next to each. The dagger is a vertical reflection.

The category **FRel** has morphisms $\eta : 1 \to X \times X$ satisfying:

- $(\eta^\dagger \times 1) \circ (1 \times \eta) = 1 = (1 \times \eta^\dagger \circ (\eta \times 1))$(C).

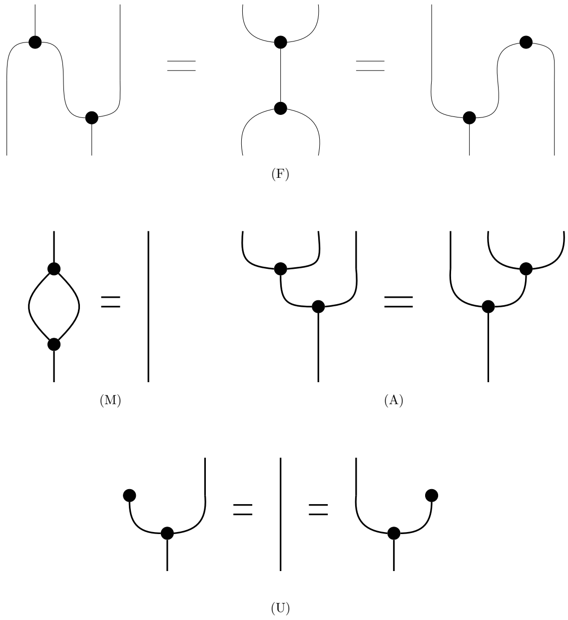


Figure 3. String diagrams of the properties for the objects in **FRel** .

Proposition 3 ([15]). ***FRel*** *is a compact closed category.*

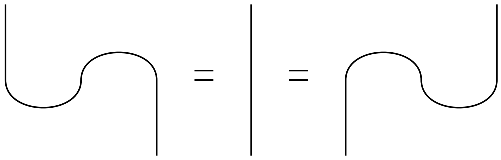
Remark 1. *Frobenius algebras with some additional properties on the category of finite dimensional Hilbert spaces* ***FHilb*** *correspond to quantum observables [20]. They are called classical structures in this category theoretical context. In [21], the graphical formulization of complementarity is given using the string diagrams. This results in complete graphical calculus for stabilizer quantum mechanics [22] and Spekkens' toy theory [12].*

The compact structure can be induced from the Frobenius algebra by $\eta = m^\dagger \circ e$. As a result of the compact structure, we can define transposes of morphism $r : X \to Y$ by $\lceil r \rceil = (1 \times r) \circ \eta : 1 \to X \times Y$. The category of Frobenius algebras in **FRel** with the following morphism is a well-defined category (see Proposition 14 of [15]).

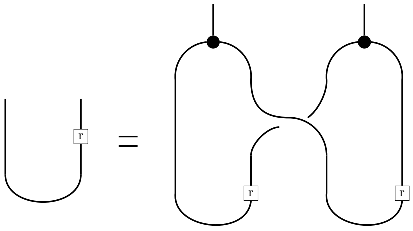
Definition 12. *A morphism $(X, m_X) \to (Y, m_Y)$ in the category of Frobenius algebras in* ***FRel*** *is a morphism $r : X \to Y$ satisfying:*

- $(m_X \times m_Y) \circ (1 \times \sigma \times 1) \circ (\lceil r \rceil \times \lceil r \rceil) = \lceil r \rceil$ *(R)*
- $(r \times \eta^\dagger) \circ (m_X^\dagger \times 1) \circ (e_X \times 1) = (e_Y^\dagger \times 1) \circ (m_Y \times 1) \circ (r \times \eta)$ *(I) where $\sigma : X \times Y \to Y \times X$ is a natural swap isomorphism.*

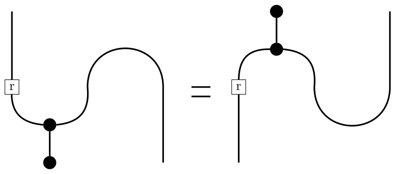
These morphisms are depicted in Figure 4



(C)



(R)



(I)

Figure 4. String diagram of the properties for the morphisms in **FRel**.

Proposition 4. *The category **Spek** for the toy theory of trits is a subcategory of **FRel**.*

Proof. The category **Spek** for the toy theory of trits is defined as the category whose objects are the single element one and n-fold Cartesian product of the nine-element set $IX := \{1, 2, \ldots, 9\}$. The morphisms of **Spek** can be constructed by composition, the Cartesian product and the relational converse from the following relations:

- The unit (deleting) relation $e : IX \to 1$ defined by $\{1, 4, 7\} \sim \bullet$
- The relation $m : IX \to IX \times IX$ defined as:

1	2	3						
3	1	2						
2	3	1						
			4	5	6			
			6	4	5			
			5	6	4			
						7	8	9
						9	7	8
						8	9	7

  For example, $\{1\} \sim \{(1,1), (2,2), (3,3)\}$, $\{2\} \sim \{(1,2), (2,3), (3,1)\}$, etc.
- The permutations $\sigma_i : IX \to IX$ that correspond to affine symplectic maps on the phase-space.
- The relevant unit, associativity and symmetry natural isomorphisms.

Twelve epistemic states for a single system are given by the following relations:

$$q \text{ known: } \bullet \sim \{1,2,3\}, \bullet \sim \{4,5,6\}, \bullet \sim \{7,8,9\}.$$
$$p \text{ known: } \bullet \sim \{1,4,7\}, \bullet \sim \{2,5,8\}, \bullet \sim \{3,6,9\}.$$
$$p+q \text{ known: } \bullet \sim \{1,6,8\}, \bullet \sim \{2,4,9\}, \bullet \sim \{3,5,7\}.$$
$$p+2q \text{ known: } \bullet \sim \{1,5,9\}, \bullet \sim \{2,6,7\}, \bullet \sim \{3,4,6\}.$$

It is straightforward to verify that (IX, m, e) is the special dagger Frobenius algebra. $\square$

Remark 2. *This structure corresponds to the observable for which q is known. Hence, the relations* $\bullet \sim \{1,2,3\}$, $\bullet \sim \{4,5,6\}$, $\bullet \sim \{7,8,9\}$ *are the copyable (classical) states for this observable. The other observables can be found by composing m with various valid permutations.*

3.3. Frobenius Algebras as Groupoids

We start our procedure of the discrete geometric quantization with constructing the groupoid corresponding to the Frobenius algebra (IX, m, e). The groupoid characterization of dagger Frobenius algebras is given in [15]. We now give the groupoid following [15].

Definition 13. *The following objects and morphisms in* ***Rel*** *obtained from the Frobenius algebra* (IX, m, e) *form a groupoid* Σ *in the category of sets and functions* ***Set*** *(see Theorem 7 of [15]).*

- $\Sigma_1 = IX$
- $\Sigma_2 = Image(m) = \bigcup_{k=0}^{2}\{(3k+1, 3k+1), (3k+1, 3k+2), (3k+1, 3k+3), (3k+2, 3k+3), (3k+2, 3k+2), (3k+2, 3k+1), (3k+3, 3k+1), (3k+3, 3k+2), (3k+3, 3k+3)\}$
- $\Sigma_0 = U = Domain(e) = \{1, 4, 7\}$
- $u = U \times U : \Sigma_0 \to \Sigma_1$
- $s = \{(f, x) \in \Sigma_1 \times \Sigma_0 | (f, x) \in \Sigma_2\} : \Sigma_1 \to \Sigma_0$
- $t = \{(f, y) \in \Sigma_1 \times \Sigma_0 | (y, f) \in \Sigma_2\} : \Sigma_1 \to \Sigma_0$
- $-^* = \{(g, f) \in \Sigma_2 | m(g, f) \in U, m(f, g) \in U\} : \Sigma_1 \to \Sigma_1$

Remark 3. *As proven in [15], this assignment is functorial, if we consider morphisms of groupoids to be sub-groupoids.*

Considering the set IX as the finite field $\mathbb{Z}_3^2$, one can equip IX with the symplectic product $\omega = \langle \cdot, \cdot \rangle$.

Lemma 1. *The graph of the multiplication* $\mathfrak{m} = \{(xy, x, x | (x, y) \in \Sigma_2)\}$ *is a Lagrangian subspace of* $\Sigma_2 \oplus \Sigma_2 \oplus \Sigma_2$.

Proof.

$$\begin{aligned}\mathfrak{m} = \bigcup_{k=0}^{2} \{&(3k+1, 3k+1, 3k+1), (3k+2, 3k+1, 3k+2), (3k+3, 3k+1, 3k+3),\\ &(3k+3, 3k+2, 3k+3), (3k+1, 3k+2, 3k+2), (3k+2, 3k+2, 3k+1),\\ &(3k+2, 3k+3, 3k+1), (3k+3, 3k+3, 3k+2), (3k+1, 3k+3, 3k+3)\}.\end{aligned}$$

Equipped with the symplectic product $\omega = \langle \cdot, \cdot \rangle$, $\mathfrak{m}$ becomes the Lagrangian subspace of $\Sigma_2 \oplus \Sigma_2 \oplus \Sigma_2 = \mathbb{Z}_3^6$ with the basis

$$\{((0,0)(0,1), (0,1)), ((0,1), (0,0), (0,1)), ((1,0), (1,0), (1,0))\}$$

where $(a, b) \in \mathbb{Z}_3^2$. $\square$

3.4. Weyl Correspondence and Pair Groupoid

In order to apply geometric quantization, we need a notion of differential forms suitable for the symplectic finite vector space. As noticed in [23], Kahler differentials are the ideal tool in this setting [24].

We now briefly review the algebraic geometry that we are going to use. Let:

$$\mathbb{Z}_3[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

be the algebra of polynomials in two variables over $\mathbb{Z}_3$. The formal derivatives of the these polynomials are evaluated using the same rules for polynomial functions.

The algebra of Kahler differential $\Omega(\mathbb{Z}_3^{2n})$ is defined as the $\mathbb{Z}_3$-linear combinations of the following terms:

$$f_{i_1,\ldots,i_k,j_1,\ldots,j_l} dx_1 \wedge \ldots dx_k \wedge dy_1 \wedge \cdots \wedge dy_l.$$

One can also define the vector space of Kahler j-forms $\Omega^j(\mathbb{Z}_3^{2n})$ for which there is also a differential:

$$d : \Omega^j(\mathbb{Z}_3^{2n}) \longrightarrow \Omega^{j+1}(\mathbb{Z}_3^{2n}).$$

The symplectic product ω defined in Section 2 corresponds to the following Kahler j-form:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

which satisfies $d\omega = 0$. From now on, we will take $n = 1$ for brevity.

We can now define the pair groupoid and polarization necessary for geometric quantization that will give us the Weyl operator in the discrete case. We first define a skew-symmetric invertible map $\hat{\omega} : \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3^{2*}$ as $\hat{\omega}(u) : v \mapsto \omega(u, v)$.

In the discrete geometric quantization procedure for the symplectic space $M = (\mathbb{Z}_3^2, \omega)$, the groupoid associated with M consists of $G = M \times \overline{M}$, where $\overline{M} = (M, -\omega)$. G is endowed with the multiplication $(x, y) \cdot (y, z) = (x, z)$. M embeds in $M \times \overline{M}$ as the diagonal $\{(x, x) | x \in M\}$, and s and t are the projections $s(x, y) = (y, y)$ and $t(x, y) = (x, x)$. In this groupoid, there is exactly one arrow from any object to another.

Starting with the groupoid G, one can define a symplectomorphism Φ from G to the cotangent bundle $T^*(M) = \mathbb{Z}_3^2 \times \mathbb{Z}_3^{2*}$ as:

$$\phi : (x, y) \mapsto (\frac{1}{2}(x + y), \hat{\omega}(x - y)).$$

It is then clear that such a groupoid is symplectic, and it integrates the symplectic space $M = (\mathbb{Z}_3^2, \omega)$. ϕ is explicitly given as:

$$\phi(x_1, x_2; y_1, y_2) = (\frac{x_1 + y_1}{2}, x_1 - y_1; x_2 - y_2, \frac{x_2 + y_2}{2}),$$

where $(x_1, x_2; y_1, y_2) \in G$ and $(q_1, q_2; p_1, p_2) \in T^*(M)$.

Now, we consider two real polarizations of G:

$$F = span\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\}$$

$$P = span\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_2}\}.$$

The symplectic potentials that vanish on F and P may be taken as:

$$\Phi_F = -x_2 dx_1 + y_2 dy_1 = -p_1 dq_1 - p_2 dq_2$$

$$\Phi_P = -p_1 dq_1 + q_2 dp_2.$$

We then obtain $\Phi_F - \Phi_P = -d(q_2p_2)$. Hence, the inner product polarized sections of line bundles is:

$$\sum_{q_1,q_2,p_3} f(x_1,y_1)g(q_1,p_2)e^{-iq_2p_2/3}$$
$$= \sum_{p_2,x_1,y_1} f(x_1,y_1)g((x_1+y_1)/2,p_2)e^{i(y_1-x_1)p_2/3}$$
$$= (f, Wg)$$

where:

$$Wg(x_1,y_1) = \sum_p g((x_1+y_1)/2,p)e^{i(y_1-x_1)p/3}$$

can be considered as the integral kernel of the Weyl operator. From Hawkins' perspective, the corresponding algebra is the twisted group algebra $C^*(\mathbb{Z}_3^{2*},\sigma)$. As the Weyl operator is the representation of the finite Heisenberg group H, as shown in [25], $C^*(\mathbb{Z}_3^{2*},\sigma)$ is isomorphic to the group algebra of $C^*(H)$.

Remark 4. *Note that we cannot apply the same procedure to the toy bits, i.e.,* $\Omega = \mathbb{Z}_2$*, as the symplectomorphism* Φ *and other steps of quantization include division by two.*

Our main result produces a functorial quantization via symplectic groupoids, in the case of epistricted theories with an odd prime number of degrees of freedom.

Theorem 3 (Main result for the finite case)**.** *The discrete geometric quantization procedure is a functor from the Frobenius algebra in* ***Rel*** *for epistricted theories to the Frobenius algebra for stabilizer quantum mechanics in the odd prime discrete case.*

Proof. $\bar{\Sigma}$ can be equipped with a symplectic structure so that it becomes the symplectic groupoid where the polarization is $P = span\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_2}\}$ corresponding to $\bullet \to U \times U$ in $(IX \times IX, \bar{m}, \bar{e})$. Hence, the quantization gives us a subalgebra of $C^*(\mathbb{Z}_3^{2*},\sigma)$ as we only consider the linear combination of position and momentum operators. The resulting operator algebra is a projective representation of the finite Heisenberg group given by the above discrete Weyl transform W. The stabilizer states are joint eigenstates of commuting Weyl operators. In [7], it has been shown that the stabilizer states is equivalent to quadrature states of epistricted theories.

The resulting finite dimensional algebra $C^*(\mathbb{Z}_3^{2*},\sigma) \cong C^*(H)$ is equivalent to a dagger Frobenius algebra in **Hilb** (see Theorem 4.7 of [17]). By the functoriality of quantization in this specific case and the functoriality of the above embedding into $End(IX)$ (see Corollary 4.4 of [16]), we obtain a functor from the dagger Frobenius algebras in **Rel** to the dagger Frobenius algebras in **Hilb**. The affine symplectic transformations of the epistricted theories are mapped to the group representations of the affine symplectic group, which acts as a superoperator in the resulting C^*-algebra. □

We now construct a pair groupoid M from the dagger Frobenius algebra (IX, m, e). We start with the monoid structure $(IX \times IX, id_{IX} \times \eta^\dagger \times id_{IX}, \eta)$ in **Rel**, where $\eta := m^\dagger \circ e$. This monoid is a specific example of endomorphism monoids in [17], which is an analogue of algebras of bounded linear operators. Note that the new monoid multiplication $m' = id_{IX} \times \eta^\dagger \times id_{IX}$ is precisely the multiplication in $m'((x,y),(y,z)) = (x,z)$ in the pair groupoid, and the unit is the diagonal $\eta = e \circ m$: $\bullet \to \{(a,a) | a \in IX\}$. The abstract polarization P in this context can be cast as $\bullet \to U \times U$. We denote this monoid as $End(IX)$

The algebra (IX, m, e) can be embedded into endomorphism monoid $End(IX)$ similar to the fact that every algebra has a homomorphism in the algebra of operators. The embedding homomorphism $h : (IX, m, e) \to End(IX)$ is defined by:

$$h := m.$$

It is easy to show that h preserves multiplication and the unit. One can also refer to Lemma 3.19 in [17] for a more general case. Let $(IX \times IX, \bar{m}, \bar{e} = \eta)$ denote the image of h in the endomorphism monoid. We now can construct the groupoid $\bar{\Sigma}$ from the dagger Frobenius algebra $(IX \times IX, \bar{m}, \bar{e})$ following the construction in [15] one more time:

- $\bar{\Sigma}_1 = IX \times IX$
- $\bar{\Sigma}_2 = Image(\bar{m})$
- $\bar{\Sigma}_0 = \bar{U} = Image(\bar{e})$
- $\bar{u} = \bar{U} \times \bar{U}$
- $\bar{s} = \{(f, x) \in \bar{\Sigma}_1 \times \bar{\Sigma}_0 | (f, x) \in \bar{\Sigma}_2\} : \bar{\Sigma}_1 \to \bar{\Sigma}_0$
- $\bar{t} = \{(f, y) \in \bar{\Sigma}_1 \times \bar{\Sigma}_0 | (y, f) \in \bar{\Sigma}_2\} : \bar{\Sigma}_1 \to \bar{\Sigma}_0$
- $-^* = \{(g, f) \in \bar{\Sigma}_2 | m(g, f) \in \bar{U}, m(f, g) \in \bar{U}\} : \bar{\Sigma}_1 \to \bar{\Sigma}_1$

4. Conclusions and Further Work

We have established the relationship between geometric quantization and quadrature subtheories for the continuous degrees of freedom. We conclude that the group algebra $C^*(H)$ for Heisenberg group H contains the quadrature subtheories as a result of groupoid quantization procedure. One can use this fact to give the operator algebraic approach to quantum optics.

4.1. C^*-Quantization

This construction also suggests that there is a "geometric quantization" functor, from a subcategory of the category of groupoids to the category of C^*-algebras. Following [15], this corresponds to a functor from Frobenius algebras in the category **FRel** (Frobenius algebras in the category of sets and relations) to Frobenius algebras in the category of Hilbert spaces **FHilb**. The functor has to be defined in the subcategory of Frobenius algebras arising from symplectic groupoids, and the morphisms have to be adapted in order to obtain functoriality.

4.2. The Even Case

We investigate discrete degrees of freedom. The variables in this case are chosen from a finite field instead of real numbers. Even though Spekkens' original toy theory [7] is contained in the case where the finite field is $\mathbb{Z}/2$, we consider odd degrees freedom. The reason is that for $\Omega = (\mathbb{Z}/2)^n$, the discrete Wigner representation can take negative values, and therefore, the epistricted theory does not coincide with the quadrature subtheories [6]. Our main result is to give a discrete version of groupoid quantization. The resulting algebra is $C^*(H)$ for the finite Heisenberg group H. This finite C^*-algebra corresponds to a Frobenius structure via the construction of Vicary [17]. Thus, one can study quantum phenomena such as complementarity in quadrature theories in this algebraic framework.

4.3. Geometric Quantization Over Finite Fields

In the work of Gurevich and Hadani [26], a functorial description of geometric quantization is developed for vector spaces over fields with positive characteristics. The odd prime case is resemblant of the discrete geometric quantization procedure we have described in this paper. We expect to have a more explicit comparison in the future between our quantization procedure for the odd finite case and this geometric quantization program.

4.4. Quantum Circuit Dynamics via Path Integrals

For Clifford circuits, Penney et al. define the relative phases of different discrete-time paths in terms of classical action [23]. They show that for each gate, one can associate a symplectomorphism on the phase-space, and for each symplectomorphism, one can define a generating function on two copies of the configuration space. The action functional for a sequence of gates is defined using the sum of the generating functions. This approach can be cast using discrete geometric quantization used by the paper. Using our method, one can extend the results in [23] to different kinds of quantum circuits. Similarly, geometric quantization of physical theories, where space-time is discrete (e.g., cellular automata, discrete mechanical systems), will be treated in future work.

Acknowledgments: We thank Chris Heunen for bringing the question of complementarity in Spekkens' toy theory to our attention and also for the helpful discussions during the course of this research. We are indebted to the reviewers for their thorough reviews and highly appreciate the comments and suggestions. We are also grateful to Rob Spekkens for his permission to use the figures in Example 4. We acknowledge King Fahd University of Petroleum and Minerals (KFUPM) for funding this work through Project No. SR141007. IC was partially supported by the SNF grant PBZHP2-147294. The string diagrams were created using the PGF/TikZ package for LaTeX and the TikZiT software.

Author Contributions: Ivan Contreras and Ali Nabi Duman wrote the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ali, S.T.; Rudolph, T.; Englis, M. Quantization methods: A guide for physicist and analyst. *Rev. Math. Phys.* **2005**, *17*, 391–490.
2. Bates, S.; Weinstein, A. *Lectures on Geometry of Quantization*; American Mathematical Society: Providence, RI, USA, 1997.
3. Woodhouse, N.M.J. *Geometric Quantization*; Oxford University Press: Oxford, UK, 1992.
4. Rieffel, M.A. Quantization and $C*$-algebras. *Contemp. Math.* **1994**, *167*, 67.
5. Hawkins, E. A groupoid approach to quantization. *J. Symplect. Geom.* **2008**, *6*, 61–125.
6. Spekkens, R.W. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A* **2007**, *75*, 032110.
7. Spekkens, R.W. Quasi-quantization: Classical statistical theories with an epistemic restriction. *Quantum Theory Inf. Found. Foils* **2016**, 83–135, doi:10.1007/978-94-017-7303-4_4.
8. Bengtsson, I.; Khrennikov, A. Preface. *Found. Phys.* **2011**, *41*, 281, doi:10.1007/s10701-010-9524-1.
9. D'Ariano, G.M.; Jaeger, G.; Khrennikov, A.; Plotnitsky, A. Preface of the special issue quantum theory: Advances and problems. *Phys. Scr.* **2014**, *163*, 010301.
10. Khrennikov, A.; Weihs, G. Preface of the Special Issue Quantum Foundations: Theory and Experiment. *Found. Phys.* **2012**, *42*, 721–724.
11. Abramsky, S.; Coecke, B. A categorical semantics of quantum protocols. In Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, Turku, Finland, 13–17 July 2004; pp. 415–425.
12. Backens, M.; Duman, A.N. A Complete Graphical Calculus for Spekkens' Toy Bit Theory. *Found. Phys.* **2016**, *46*, 70–103.
13. Coecke, B.; Edwards, B. Toy quantum categories. *Electron. Notes Theor. Comput. Sci.* **2011**, *270*, 29–40.
14. Coecke, B.; Edwards, B.; Spekkens, R.W. Phase Groups and the Origin of Non-locality for Qubits. *Electron. Notes Theor. Comput. Sci.* **2011**, *270*, 15–36.
15. Cattaneo, A.; Contreras, I.; Heunen, C. Relative Frobenius Algebras are Groupoids. *J. Pure Appl. Algebr.* **2013**, *217*, 114–124.
16. Coecke, B.; Heunen, C.; Kissinger, A. Categories of Quantum and Classical Channels. *Quantum Inf. Process.* **2016**, *15*, 5179–5209.
17. Vicary, J. Categorical Formulation of Finite-Dimensional Quantum Algebras. *Commun. Math. Phys.* **2011**, *304*, 765–796.
18. Rieffel, M.A. *Deformation Quantization for Actions of* $\mathbb{R}^d$; American Mathematical Society: Providence, RI, USA, 1993; p. 506.
19. Gotay, M.J. Functorial Geometric Quantization and Van Hove's Theorem. *Int. J. Theor. Phys.* **1980**, *19*, 139–161.
20. Coecke, B.; Paquette, É.O.; Pavlovic, D. Classical and quantum structuralism. *Semantic Techniques in Quantum Computation*; Cambridge University Press: Cambridge, UK, 2008.
21. Coecke, B.; Duncan, R. Interacting quantum observables: Categorical algebra and diagrammatics. *New J. Phys.* **2011**, *13*, 043016.
22. Backens, M. The ZX-calculus is complete for stabilizer quantum mechanics. *New J. Phys.* **2014**, *16*, 093021.

23. Penney, M.D.; Koh, D.E.; Spekkens, R.W. Quantum circuit dynamics via path integrals: Is there a classical action for discrete time paths? *arXiv* **2016**, arXiv:1604.07452.
24. Hartshorne, R. *Algebraic Geometry*; Springer: Berlin, Germany, 2013; p. 52.
25. Gross, D. Hudson's Theorem for finite-dimensional quantum systems. *J. Math. Phys.* **2006**, 47, 122107.
26. Gurevich, S.; Hadani, R. Quantization of symplectic vector spaces over finite fields. *J. Symplect. Geom.* **2009**, *7*, 475–502.