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Correction

Correction: Naudts, J. Quantum Statistical Manifolds. *Entropy* 2018, *20*, 472

Jan Naudts

Departement Fysica, Universiteit Antwerpen, Universiteitsplein 1, 2610 Wilrijk Antwerpen, Belgium; jan.naudts@uantwerpen.be

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Abstract: Section 4 of "Naudts J. Quantum Statistical Manifolds. *Entropy* **2018**, *20*, 472" contains errors. They have limited consequences for the remainder of the paper. A new version of this Section is found here. Some smaller shortcomings of the paper are taken care of as well. In particular, the proof of Theorem 3 was not complete, and is therefore amended. Also, a few missing references are added.

Theorem 1.

Theorem 2.

1. Corrections in Section 3

The display on top of page 5 should read

$$\begin{aligned}
||f_{\rho,K}|| &= \sup_{A\in\mathcal{A}} \{f_{\rho,K}(A) : ||A|| \le 1\} \\
&= \sup_{A\in\mathcal{A}} \{(\pi(A)K\Omega_\rho, \Omega_\rho) : ||A|| \le 1\} \\
&= ||\,|K|^{1/2}\Omega_\rho||^2 \\
&\le ||\,|K|^{1/2}||^2 = ||K||.
\end{aligned}$$

The operator K is replaced by $|K|$ because K need not be positive.

The sentence "This is a prerequisite for proving in the next Theorem that this map is the Fréchet derivative of the chart ξ_ρ." should read "This is a prerequisite for proving in the next Theorem that this map is the Fréchet derivative of the inverse of the chart ξ_ρ."

The proof of the following Theorem is amended.

Theorem 3. *The inverse of the map $\xi_\rho : \mathbb{M} \mapsto \mathcal{B}_\rho$, defined in Theorem 2, is Fréchet-differentiable at $\omega = \omega_\rho$. The Fréchet derivative is denoted F_ρ. It maps K to $f_{\rho,K}$, where the latter is defined by (10).*

Proof. Let $K = \xi_\rho(\omega_\sigma)$. One calculates

$$\begin{aligned}
||\omega_\sigma - \omega_\rho - F_\rho K|| &= \sup_{A\in\mathcal{A}} \{|\omega_\sigma(A) - \omega_\rho(A) - F_\rho K(A)| : ||A|| \le 1\} \\
&= \sup_{A\in\mathcal{A}} \left\{|(\pi(A)\Omega_\rho, [e^{K-\alpha(K)} - \mathbb{I} - K]\Omega_\rho)| : ||A|| \le 1\right\} \\
&\le ||e^{K-\alpha(K)} - \mathbb{I} - K|| \\
&\le |\alpha(K)| + \mathrm{o}(||K - \alpha(K)||). \qquad (11)
\end{aligned}$$

Note that

$$|\alpha(K)| \le \log ||e^K|| \le ||K||$$

and

$$||K - \alpha(K)|| \leq 2||K||.$$

In addition, if $||K|| < 1$ then one has

$$\alpha_\rho(K) \leq \log(1 + ||K\Omega_\rho||^2) \leq ||K\Omega_\rho||^2.$$

This holds because $\lambda \leq 1$ implies $\exp(\lambda) \leq 1 + \lambda + \lambda^2$. One concludes that (11) converges to 0 faster than linearly as $||K||$ tends to 0. This proves that $F_\rho K$ is the Fréchet derivative of $\xi_\rho(\omega_\sigma) \mapsto \omega_\sigma$ at $\sigma = \rho$. □

2. New Version of Section 4

Propositions 1 and 2 of [1] are not correct. This only has consequences for one sentence in the Introduction of [1] and for the results reported in Section 4 of [1]. The text in the Introduction "Next, an atlas is introduced which contains a multitude of charts, one for each element of the manifold. Theorem 4 proves that the manifold is a Banach manifold and that the cross-over maps are linear operators." should be changed to "Next, an atlas is introduced which contains a multitude of charts, one for each element of the manifold. Theorem 4 proves that the manifold is a Banach manifold and that the cross-over maps are continuous."

A new version of Section 4 follows below:

4. The Atlas

Following the approach of Pistone and collaborators [1,3,4,24], we build an atlas of charts ξ_ρ, one for each strictly positive density matrix ρ. The compatibility of the different charts requires the study of the cross-over map $\xi_{\rho_1}(\sigma) \mapsto \xi_{\rho_2}(\sigma)$, where ρ_1, ρ_2, σ are arbitrary strictly positive density matrices.

Simplify notations by writing ξ_1 and ξ_2 instead of ξ_{ρ_1}, respectively ξ_{ρ_2}. Similarly, write Ω_1 and Ω_2 instead of Ω_{ρ_1}, respectively Ω_{ρ_2}, and F_1, F_2 instead of F_{ρ_1}, respectively F_{ρ_2}.

Proposition 1. *RETRACTED*

Continuity of the cross-over map follows from the continuity of the exponential and logarithmic functions and from the following result.

Proposition 2. *Fix strictly positive density matrices ρ_1 and ρ_2. There exists a linear operator Y such that for any strictly positive density matrix σ and corresponding positive operators X_1, X_2 in the commutant $\mathcal{A}'$ one has $X_2 = YX_1Y^*$.*

Proof. Using the notations of the Appendix of [1], one has

$$X_i \quad = \quad J_i(\rho_i^{-1/2}\sigma\rho_i^{-1/2} \otimes \mathbb{I})J_i^*, \quad i = 1, 2.$$

Note that the isometry J depends on the reference state with density matrix ρ. Therefore, it carries an index i. The above expression for X_i implies that

$$X_2 = YX_1Y^* \quad \text{with} \quad Y = J_2(\rho_2^{-1/2}\rho_1^{1/2} \otimes \mathbb{I})J_1^*.$$

□

Theorem 4. *The set $\mathbb{M}$ of faithful states on the algebra $\mathcal{A}$ of square matrices, together with the atlas of charts ξ_ρ, where ξ_ρ is defined by Theorem 1, is a Banach manifold. For any pair of strictly positive density matrices ρ_1 and ρ_2, the cross-over map $\xi_2 \circ \xi_1^{-1}$ is continuous.*

Proof. The continuity of the map $X_1 \mapsto X_2$ follows from the previous Proposition. The continuity of the maps $K_1 \mapsto X_1$ and $X_2 \mapsto K_2$ follows from the continuity of the exponential and logarithmic functions and the continuity of the function α. □

3. Corrections in Section 9

In the proof of Proposition 4, the symbol Ω_ρ is missing five times in obvious places. It has been added.

4. Added References

In the overview of papers devoted to the study of the quantum statistical manifold in the finite-dimensional case, the references [2,3] should be added. A quantum version of the work of Pistone and Sempi [4], alternative to [5], is found in [6]. Reference [7] to the work of Ciaglia et al. has been updated.

References

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