Uncertainty Relation Based on Wigner–Yanase–Dyson Skew Information with Quantum Memory

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Abstract: We present uncertainty relations based on Wigner–Yanase–Dyson skew information with quantum memory. Uncertainty inequalities both in product and summation forms are derived. It is shown that the lower bounds contain two terms: one characterizes the degree of compatibility of two measurements, and the other is the quantum correlation between the measured system and the quantum memory. Detailed examples are given for product, separable and entangled states.

Keywords: uncertainty relation; Wigner–Yanase–Dyson skew information; quantum memory

1. Introduction

The uncertainty principle is an essential feature of quantum mechanics, characterizing the experimental measurement incompatibility of non-commuting quantum mechanical observables in the preparation of quantum states. Heisenberg first introduced variance-based uncertainty [1]. Later, Robertson [2] proposed the well-known formula of the uncertainty relation, $V(\rho, R)V(\rho, S) \geq \frac{1}{4}|Tr\rho[R, S]|^2$, for arbitrary observables R and S, where $[R, S] = RS - SR$ and $V(\rho, R)$ is the standard deviation of R. Schrödinger gave a further improved uncertainty relation [3]:

$$V(\rho, R)V(\rho, S) \geq \frac{1}{4}|\langle[R, S]\rangle|^2 + |\frac{1}{2}\langle\{R, S\}\rangle - \langle R\rangle\langle S\rangle|^2$$

where $\langle R\rangle = Tr(\rho R)$, and $\{R, S\} = RS + SR$ is the anti-commutator. Since then many kinds of uncertainty relations have been presented [4–11]. In addition to the uncertainty of the standard deviation, entropy can be used to quantify uncertainties [12]. The first entropic uncertainty relation was given by Deutsch [13] and was then improved by Maassen and Uffink [14]:

$$H(R) + H(S) \geq \log_2 \frac{1}{c}$$

where $R = \{|u_j\rangle\}$, and $S = \{|v_k\rangle\}$ are two orthonormal bases on d-dimensional Hilbert space H, and $H(R) = -\Sigma_j p_j log p_j$ ($H(S) = -\Sigma_k q_k log q_k$) is the Shannon entropy of the probability distribution $p_j = \langle u_j|\rho|u_j\rangle$ ($q_k = \langle v_k|\rho|v_k\rangle$) for state ρ of H. The number c is the largest overlap among all $c_{jk} = |\langle u_j|v_k\rangle|^2$ between the projective measurements R and S. Berta et al. [15] bridged the gap between cryptographic scenarios and the uncertainty principle and derived this landmark uncertainty relation for measurements R and S in the presence of quantum memory B:

$$H(R|B) + H(R|B) \geq \log_2 \frac{1}{c} + H(A|B)$$

where $H(R|B) = H(\rho_{RB}) - H(\rho_B)$ is the conditional entropy with $\rho_{RB} = \Sigma_j(|u_j\rangle\langle u_j| \otimes I)\rho_{AB}(|u_j\rangle\langle u_j| \otimes I)$ (similarly for $H(S|B)$), and d is the dimension of the subsystem A. The term

$H(A|B) = H(\rho_{AB}) - H(\rho_B)$ appearing on the right-hand side is related to the entanglement between the measured particle A and the quantum memory B. The bound of Berta et al. has been further improved [16–18]. Moreover, there are also some uncertainty relations given by the generalized entropies, such as the Rényi entropy [19–21] and the Tsallis entropy [22–24], and even more general entropies such as the (h, Φ) entropies [25]. These uncertainty relations not only manifest the physical implications of the quantum world but also play roles in entanglement detection [26,27], quantum spin squeezing [28,29] and quantum metrology [30,31].

In [32], an uncertainty relation based on Wigner–Yanase skew information $I(\rho, H)$ has been obtained with quantum memory, where $I(\rho, H) = \frac{1}{2}Tr[(i[\sqrt{\rho}, H])^2] = Tr(\rho H^2) - Tr(\sqrt{\rho}H\sqrt{\rho}H)$ quantifies the degree of non-commutativity between a quantum state ρ and an observable H, which is reduced to the variance $V(\rho, H)$ when ρ is a pure state. In fact, the Wigner–Yanase skew information $I(\rho, H)$ is generalized to Wigner–Yanase–Dyson skew information $I_\alpha(\rho, H)$, $\alpha \in [0,1]$ (see [33]):

$$
\begin{aligned}
I_\alpha(\rho, H) &= \tfrac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\
&= Tr(\rho H^2) - Tr(\rho^\alpha H \rho^{1-\alpha} H) \qquad \alpha \in [0,1]
\end{aligned}
\tag{1}
$$

Here the Wigner–Yanase–Dyson skew information $I_\alpha(\rho, H)$ reduces to the Wigner–Yanase skew information $I(\rho, H)$ when $\alpha = \frac{1}{2}$. The Wigner–Yanase–Dyson skew information $I_\alpha(\rho, H)$ reduces to the standard deviation $V(\rho, H)$ when ρ is a pure state.

The convexity of $I_\alpha(\rho, H)$ with respect to ρ has been proven by Lieb in [34]. In [35], Kenjiro introduced another quantity:

$$
\begin{aligned}
J_\alpha(\rho, H) &= \tfrac{1}{2}Tr[(\{\rho^\alpha, H_0\})(\{\rho^{1-\alpha}, H_0\})] \\
&= Tr(\rho H_0^2) + Tr(\rho^\alpha H_0 \rho^{1-\alpha} H_0) \qquad \alpha \in [0,1]
\end{aligned}
\tag{2}
$$

where $H_0 = H - Tr(\rho H)I$ with I being the identity operator.

For a quantum state ρ and observables R, S and $0 \le \alpha \le 1$, the following inequality holds [35]:

$$
U_\alpha(\rho, R)U_\alpha(\rho, S) \ge \alpha(1-\alpha)|Tr\rho[R, S]|^2
\tag{3}
$$

where $U_\alpha(\rho, R) = \sqrt{I_\alpha(\rho, R)J_\alpha(\rho, R)}$ can be regarded as a kind of measure for quantum uncertainty, in the sense given by [35]. For a pure state, a standard deviation-based relation is recovered from Equation (3). When $\alpha = \frac{1}{2}$, it is reduced to the result of [36].

Inspired by the works [32,35], in this paper, we study the uncertainty relations based on Wigner–Yanase–Dyson skew information in the presence of quantum memory, which generalize the results in [32] to the case of Wigner–Yanase–Dyson skew information, and the results in [35], which generalize to the case with the presence of quantum memory. We present uncertainty inequalities both in product and summation forms, and show that the lower bounds contain two terms: one concerns the compatibility of two measurement observables, and the other concerns the quantum correlations between the measured system and the quantum memory. We compare the lower bounds for product, separable and entangled states by detailed examples.

2. Results

Let $\phi_k = |\phi_k\rangle\langle\phi_k|$ and $\psi_k = |\psi_k\rangle\langle\psi_k|$ be the rank 1 spectral projectors of two non-degenerate observables R and S with the eigenvectors $|\phi_k\rangle$ and $|\psi_k\rangle$, respectively. Similarly to [32], we define $UN_\alpha(\rho, \phi) = \sum_k U_\alpha(\rho, \phi_k) = \sum_k \sqrt{I_\alpha(\rho, \phi_k)J_\alpha(\rho, \phi_k)}$ as the uncertainty of ρ associated to the projective measurement $\{\phi_k\}$, and $U_\alpha(\rho, \psi)$ to $\{\psi_k\}$.

Let ρ_{AB} be a bipartite state on $H_A \otimes H_B$, where H_A and H_B denote the Hilbert space of subsystems A and B, respectively. Let V be any orthogonal basis space on H_A and $|\phi_k\rangle$ be an orthogonal basis of H_A. We define a quantum correlation of ρ_{AB} as

$$\tilde{D}_\alpha(\rho_{AB}) = \min_V \sum_k [I_\alpha(\rho_{AB}, \phi_k \otimes I_B) - I_\alpha(\rho_A, \phi_k)] \qquad (4)$$

where the minimum is taken over all the orthogonal bases on H_A, $\rho_A = Tr_B \rho_{AB}$.

For any bipartite state ρ_{AB} and any observable X_A on H_A, we have $I_\alpha(\rho_{AB}, X_A \otimes I_B) \geq I_\alpha(\rho_A, X_A)$, which follows from Corollary 1.3 in [34] and Lemma 2 in [37]. Therefore, $\tilde{D}_\alpha(\rho_{AB}) \geq 0$. Furthermore, $\tilde{D}_\alpha(\rho_{AB}) = 0$ when ρ_{AB} is a classical quantum correlated state, which follows from the proof in Theorem 1 of [38]. $\tilde{D}_\alpha(\rho_{AB})$ has a measurement on subsystem A, which gives an explicit physical meaning: it is the minimal difference of incompatibility of the projective measurement on the bipartite state ρ_{AB} and on the local reduced state ρ_A. $\tilde{D}_\alpha(\rho_{AB})$ quantifies the quantum correlations between the subsystems A and B. We have the following.

Theorem 1. *Let ρ_{AB} be a bipartite quantum state on $H_A \otimes H_B$ and $\{\phi_k\}$ and $\{\psi_k\}$ be two sets of rank 1 projective measurements on H_A. Then*

$$UN_\alpha(\rho_{AB}, \phi \otimes I)UN_\alpha(\rho_{AB}, \psi \otimes I) \geq \sum_k L^2_{\alpha,\rho_A}(\phi_k, \psi_k) + \tilde{D}^2_\alpha(\rho_{AB}) \qquad (5)$$

where $L_{\alpha,\rho_A}(\phi_k, \psi_k) = \alpha(1 - \alpha)\dfrac{|Tr\rho_A[\phi_k,\psi_k]|^2}{\sqrt{J_\alpha(\rho_A,\phi_k) \cdot J_\alpha(\rho_A,\psi_k)}}$.

Proof of Theorem 1. By definition, we have

$$
\begin{aligned}
&UN_\alpha(\rho_{AB}, \phi \otimes I)UN_\alpha(\rho_{AB}, \psi \otimes I) \\
&= \sum_k \sqrt{I_\alpha(\rho_{AB}, \phi_k \otimes I) \cdot J_\alpha(\rho_{AB}, \phi_k \otimes I)} \cdot \sum_k \sqrt{I_\alpha(\rho_{AB}, \psi_k \otimes I) \cdot J_\alpha(\rho_{AB}, \psi_k \otimes I)} \\
&\geq \sum_k I_\alpha(\rho_{AB}, \phi_k \otimes I) \cdot \sum_k I_\alpha(\rho_{AB}, \psi_k \otimes I) \\
&= [\sum_k (I_\alpha(\rho_{AB}, \phi_k \otimes I) - I_\alpha(\rho_A, \phi_k)) + \sum_k I_\alpha(\rho_A, \phi_k)] \\
&\quad \cdot [\sum_k (I_\alpha(\rho_{AB}, \psi_k \otimes I) - I_\alpha(\rho_A, \psi_k)) + \sum_k I_\alpha(\rho_A, \psi_k)] \\
&\geq [\tilde{D}_\alpha(\rho_{AB}) + \sum_k I_\alpha(\rho_A, \phi_k)] \cdot [\tilde{D}_\alpha(\rho_{AB}) + \sum_k I_\alpha(\rho_A, \psi_k)] \\
&\geq \tilde{D}^2_\alpha(\rho_{AB}) + \sum_k I_\alpha(\rho_A, \phi_k)I_\alpha(\rho_A, \psi_k) \\
&\geq \tilde{D}^2_\alpha(\rho_{AB}) + \sum_k \frac{\alpha^2(1-\alpha)^2|Tr\rho_A[\phi_k,\psi_k]|^4}{J_\alpha(\rho_A,\phi_k)J_\alpha(\rho_A,\psi_k)} \\
&\triangleq \tilde{D}^2_\alpha(\rho_{AB}) + \sum_k L^2_{\alpha,\rho_A}(\phi_k, \psi_k)
\end{aligned}
\qquad (6)
$$

where the first inequality is due to $J_\alpha(\rho, H) \geq I_\alpha(\rho, H)$ [35], and the last inequality follows from Equation (3). □

Theorem 1 gives a product form of the uncertainty relation. Comparing the results (Equation (3)) without quantum memory with those (Equation (5)) with quantum memory, one finds that if the observables A and B satisfy $[A, B] = 0$, the bound is trivial in Equation (3), while in Equation (5), even if the projective measurements ϕ_k and ψ_k satisfy $[\phi_k, \psi_k] = 0$, that is, $L_{\alpha,\rho_A}(\phi_k, \psi_k) = 0$, $\tilde{D}_\alpha(\rho_{AB})$ may still not be trivial because of correlations between the system and the quantum memory.

Corresponding to the product form of the uncertainty relation, we can also derive the sum form of the uncertainty relation:

Theorem 2. *Let ρ_{AB} be a quantum state on $H_A \otimes \tilde{H}_B$ and $\{\phi_k\}$ and $\{\psi_k\}$ be two sets of rank 1 projective measurements on H_A. Then*

$$UN_\alpha(\rho_{AB}, \phi \otimes I) + UN_\alpha(\rho_{AB}, \psi \otimes I) \geq 2 \sum_k L_{\alpha, \rho_A}(\phi_k, \psi_k) + 2\tilde{D}_\alpha(\rho_{AB}) \qquad (7)$$

Proof of Theorem 2. By definition and taking into account the fact that $J_\alpha(\rho, H) \geq I_\alpha(\rho, H)$ [35], we have

$$UN_\alpha(\rho_{AB}, \phi \otimes I) + UN_\alpha(\rho_{AB}, \psi \otimes I)$$
$$= \sum_k \sqrt{I_\alpha(\rho_{AB}, \phi_k \otimes I) \cdot J_\alpha(\rho_{AB}, \phi_k \otimes I)} + \sum_k \sqrt{I_\alpha(\rho_{AB}, \psi_k \otimes I) \cdot J_\alpha(\rho_{AB}, \psi_k \otimes I)}$$
$$\geq \sum_k I_\alpha(\rho_{AB}, \phi_k \otimes I) + \sum_k I_\alpha(\rho_{AB}, \psi_k \otimes I)$$

While

$$\sum_k I_\alpha(\rho_{AB}, \phi_k \otimes I) + \sum_k I_\alpha(\rho_{AB}, \psi_k \otimes I)$$
$$= \sum_k I_\alpha(\rho_A, \phi_k) + \sum_k I_\alpha(\rho_A, \psi_k) + \sum_k [I_\alpha(\rho_{AB}, \phi_k \otimes I) - I_\alpha(\rho_A, \phi_k)]$$
$$+ \sum_k [I_\alpha(\rho_{AB}, \psi_k \otimes I) - I_\alpha(\rho_A, \psi_k)]$$
$$\geq \sum_k I_\alpha(\rho_A, \phi_k) + \sum_k I_\alpha(\rho_A, \psi_k) + 2\tilde{D}_\alpha(\rho_{AB})$$

where the inequality follows from Equation (4). By using the inequality $a + b \geq 2\sqrt{ab}$ for positive $a = I_\alpha(\rho_A, \phi_k)$ and $b = I_\alpha(\rho_A, \psi_k)$, we further obtain

$$UN_\alpha(\rho_{AB}, \phi \otimes I) + UN_\alpha(\rho_{AB}, \psi \otimes I)$$
$$\geq 2 \sum_k \sqrt{I_\alpha(\rho_A, \phi_k) \cdot I_\alpha(\rho_A, \psi_k)} + 2\tilde{D}_\alpha(\rho_{AB})$$
$$\geq 2 \sum_k \alpha(1 - \alpha) \frac{|Tr\rho_A[\phi_k, \psi_k]|^2}{\sqrt{J_\alpha(\rho_A, \phi_k) \cdot J_\alpha(\rho_A, \psi_k)}} + 2\tilde{D}_\alpha(\rho_{AB}) \qquad (8)$$
$$\triangleq 2 \sum_k L_{\alpha, \rho_A}(\phi_k, \psi_k) + 2\tilde{D}_\alpha(\rho_{AB})$$

where the second inequality follows from Equation (3). □

We note that Equation (7) reduces to an inequality that agrees with the result of [32] when $\alpha = \frac{1}{2}$. Theorem 2 is a generalization of the theorem in [32].

From Theorems 1 and 2, we obtain uncertainty relations in the form of the product and sum of skew information, which are different from the uncertainty of [39], which only deals with the single partite state. However, we treat the bipartite case with quantum memory B. It is shown that the lower bound contains two terms: one is the quantum correlation $\tilde{D}_\alpha(\rho_{AB})$, and the other is $\sum_k L_{\alpha, \rho_A}(\phi_k, \psi_k)$, which characterizes the degree of compatibility of the two measurements, just as for the meaning of $\log_2 \frac{1}{c}$ in the entropy uncertainty relation [15].

Example 1. *We consider the 2-qubit Werner state $\rho = \frac{2-p}{6}I + \frac{2p-1}{6}V$, where $p \in [-1, 1]$ and $V = \sum_{kl} |kl\rangle\langle lk|$.*

Let the Pauli matrices σ_x and σ_z be the two observables and $\{|\psi_k\rangle\}$ and $\{|\varphi_k\rangle\}$ be the eigenvectors of σ_x and σ_z, respectively, which satisfy $|\langle\psi_i|\varphi_j\rangle|^2 = \frac{1}{2}$, $i, j = 1, 2$. For all k, we have $Tr\rho_A[\psi_k, \varphi_k] = 0$, that is, $L_{\alpha, \rho_A}(\psi_k, \varphi_k) = 0$. The values of the left- and right-hand sides of Equation (5) are given by

$$4\left(\frac{2-p}{12} - \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{24}\right)$$
$$\times \left(\frac{4+p}{12} + \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{24}\right)$$

and

$$\left(\frac{2-p}{6} - \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{12}\right)^2$$

respectively; see Figure 1a for the uncertainty relations with different values of α.
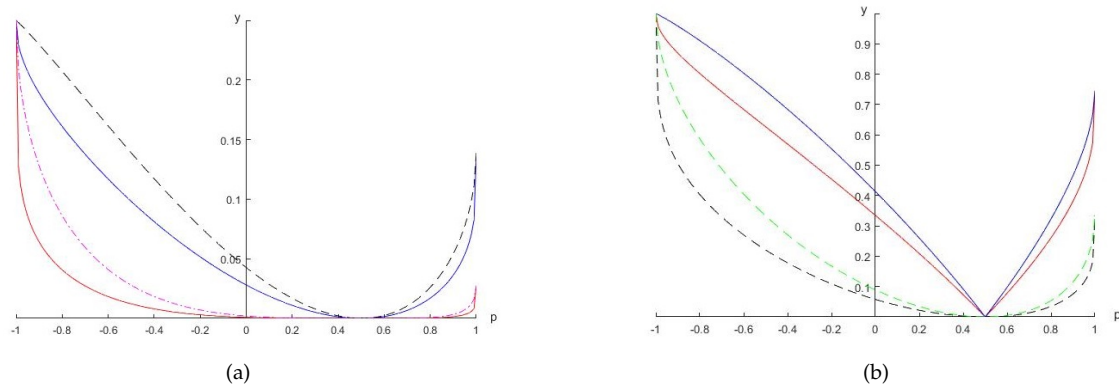
Figure 1. The *y*-axis shows the uncertainty and its lower bounds. (**a**) Blue (red) solid line for the value of the left (right)-hand side of Equation (5) with $\alpha = 0.2$; black dotted (red dot-dashed) line represents the value of the left (right)-hand side of Equation (5) with $\alpha = 0.5$. (**b**) Red solid (black dotted) line represents the value of the left (right)-hand side of Equation (7) with $\alpha = 0.2$; blue solid (green dotted) line represents the value of the left (right)-hand side of Equation (7) with $\alpha = 0.5$, which corresponds to Figure 1 in [32].

Similarly, we can obtain the values of the left- and right-hand sides of Equation (7):

$$4\sqrt{\left(\frac{2-p}{12} - \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{24}\right)}$$
$$\times \sqrt{\left(\frac{4+p}{12} + \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{24}\right)}$$

and

$$\frac{2-p}{3} - \frac{(3-3p)^\alpha(1+p)^{1-\alpha} + (1+p)^\alpha(3-3p)^{1-\alpha}}{6}$$

respectively; see Figure 1b.

Here we see explicitly that, just as for the Shannon entropy, Rényi entropy, Tsallis entropy, (h, Φ) entropies and Wigner–Yanase skew information, the Wigner–Yanase–Dyson skew information characterizes a special kind of information of a system or measurement outcomes, which needs to satisfy certain restrictions for given measurements and correlations between the system and the memory. Different α parameter values give rise to different kinds of information. From Figure 1, we see that for a given state and measurements, the differences between the left- and right-hand sides of the inequalities given by Equation (5) or (7) vary with the parameter α. Moreover, the degree of compatibility of the two measurements, $L_{\alpha, \rho_A}(\phi_k, \psi_k)$, vanishes for $\alpha = 0$ or 1, which is a fact in accordance with Equation (3), the case without quantum memory. For $p = 1/2$, the state ρ is maximally mixed. In this case, both sides of the inequalities given by Equations (5) and (7) vanish for any α.

Example 2. *Consider a separable bipartite state, $\rho^{AB} = \frac{1}{2}[|+\rangle\langle+| \otimes |0\rangle\langle0| + |-\rangle\langle-| \otimes |1\rangle\langle1|]$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$*

We still choose σ_x and σ_z to be the two observables. By calculation we obtain the following: For product states $|+\rangle\langle+| \otimes |0\rangle\langle0|$ and $|-\rangle\langle-| \otimes |1\rangle\langle1|$, both the left- and right-hand sides of Equation (5) are zero, and the right-hand side of Equation (7) is zero. For the separable bipartite state ρ^{AB}, the left- and right-hand sides of Equation (5) are $\frac{1}{2}$ and 0, respectively. Both the left- and right-hand sides of Equation (7) are zero.

Example 3. *For the Werner state $\rho_w^{AB} = (1-p)\frac{I}{4} + p|\varphi\rangle\langle\varphi|$, where $|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the Bell state, $p \in [0,1]$, and the state is separable when $p \leq \frac{1}{3}$.*

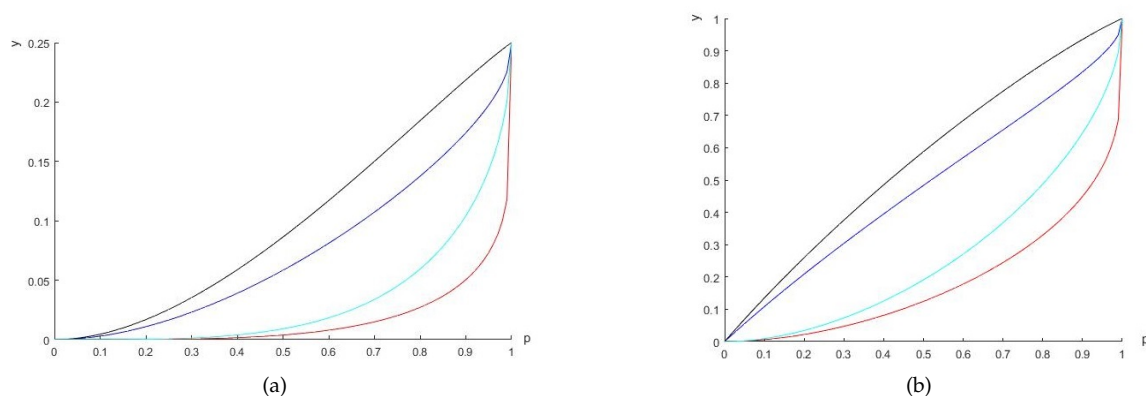*We have the values of the left- and right-hand sides of Equation (5), respectively:*

$$4(\frac{1+p}{8} - \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{16})$$
$$\times (\frac{3-p}{8} + \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{16})$$

and

$$4(\frac{1+p}{8} - \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{16})^2$$

See Figure 2a for a comparison with different values of α.

(a)

(b)

Figure 2. The y-axis shows the uncertainty and the lower bounds. (**a**) Blue (red) solid line is the value of the left (right)-hand side of Equation (5) for $\alpha = 0.2$; black (blue-green) solid line represents the value of the left (right)-hand side of Equation (5) for $\alpha = 0.5$. (**b**) Blue (red) solid line represents value of the left (right)-hand side of Equation (7) for $\alpha = 0.2$; black (blue-green) solid line represents the value of the left (right)-hand side of Equation (7) for $\alpha = 0.5$.

We can also obtain the values of the left- and right-hand sides of Equation (7):
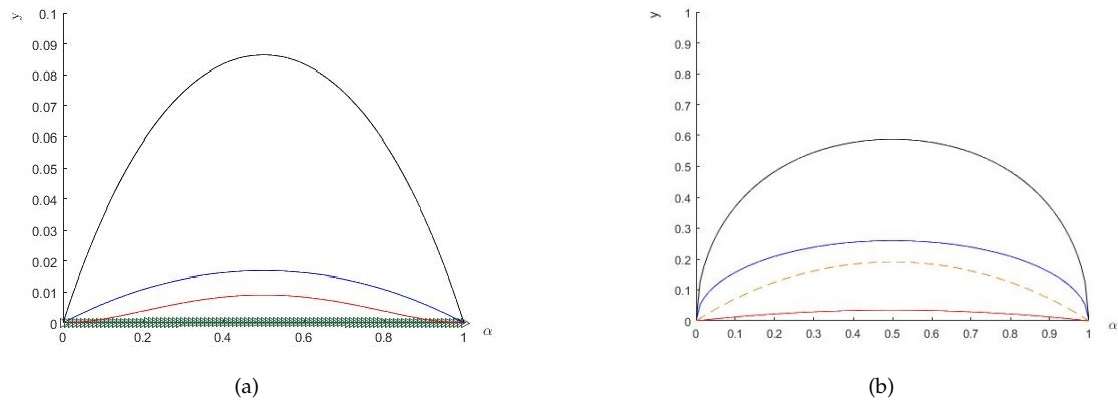
$$4\sqrt{\frac{1+p}{8} - \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{16}}$$
$$\times \sqrt{\frac{3-p}{8} + \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{16}}$$

and

$$\frac{1+p}{2} - \frac{(1-p)^\alpha(1+3p)^{1-\alpha} + (1-p)^{1-\alpha}(1+3p)^\alpha}{4}$$

respectively; see Figure 2b.

Moreover, when ρ_w^{AB} is separable, namely, $p \leq \frac{1}{3}$, the differences between the left- and right-hand sides of the inequalities are smaller than those of the entangled states. Figure 3 shows the differences for different values of p.

(a)   (b)

Figure 3. The *y*-axis shows the uncertainty and its lower bound; (**a**) $p = 0.2$ (ρ_w^{AB} is a separable state): blue solid line represents the value of the left-hand side of Equation (5), and the line (very near the *x*-axis) marked by triangles represents the corresponding lower bound; $p = 0.5$ (ρ_w^{AB} is an entangled state): the black (red) solid line represents the value of the left (right)-hand side of Equation (5). (**b**) Blue (red) solid line represents the value of the left (right)-hand side of Equation (7) for $p = 0.2$; black solid (red dashed) line represents the value of the left (right)-hand side of Equation (7) for $p = 0.5$.

3. Conclusions

We have investigated the uncertainty relations both in product and summation forms in terms of the Wigner–Yanase–Dyson skew information with quantum memory. It has been shown that the lower bounds contain two terms: one is the quantum correlation $\tilde{D}_\alpha(\rho_{AB})$, and the other is $\sum_k L_{\alpha,\rho_A}(\phi_k, \psi_k)$, which characterizes the degree of compatibility of the two measurements. By detailed examples, we have compared the lower bounds for product, separable and entangled states.

Author Contributions: Shao-Ming Fei guided the research; Jun Li finished the calculations and wrote the paper. Both authors have read and approved the final manuscript.

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