Entropy **2002** , *4*, 65–127

entropy
ISSN 1099-4300
www.mdpi.org/entropy/

Extremal Black Holes in Supergravity and the Bekenstein–Hawking Entropy

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Received: 19 February 2002 / Accepted: 6 March 2002 / Published: 10 March 2002

Abstract: We review some results on the connection among supergravity central charges, BPS states and Bekenstein–Hawking entropy. In particular, $N = 2$ supergravity in four dimensions is studied in detail. For higher N supergravities we just give an account of the general theory specializing the discussion to the $N = 8$ case when one half of supersymmetry is preserved. We stress the fact that for extremal supergravity black holes the entropy formula is topological, that is the entropy turns out to be a moduli independent quantity and can be written in terms of invariants of the duality group of the supergravity theory.

Keywords: Supergravity; Black Holes; BPS states; Central charges

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1 Introduction: Extremal Black Holes from Classical General Relativity to String Theory

Black hole physics has many aspects of great interest to physicists with very different cultural backgrounds. These range from astrophysics to classical general relativity, to quantum field theory in curved space–times, particle physics and finally string theory and supergravity. This is not surprising since black holes are one of the basic consequences of a fundamental theory, namely Einstein general relativity. Furthermore black holes have fascinating thermodynamical properties that seem to encode the deepest properties of the so far unestablished fundamental theory of quantum gravity. Central in this context is the Bekenstein–Hawking entropy:

$$S_{BH} = \frac{k_B}{G\hbar}\,\frac{1}{4}\,\mathrm{Area}_H \tag{1}$$

where k_B is the Boltzman constant, G is Newton's constant, $\hbar$ is Planck's constant and Area_H denotes the area of the horizon surface.

This very precise relation between a thermodynamical quantity and a geometrical quantity such as the horizon area is a puzzle that stimulated the interest of theoretical phisicists for more than twenty years. Indeed a microscopic statistical explanation of the area law for the black hole entropy has been correctly regarded as possible only within a solid formulation of quantum gravity. Superstring theory is the most serious candidate for a theory of quantum gravity and as such should eventually provide such a microscopic explanation of the area law. Although superstrings have been around for more than twenty years, a significant progress in this direction came only recently [1], after the so called second string revolution (1995). Indeed black holes are a typical non–perturbative phenomenon and perturbative string theory could say very little about their entropy: only non perturbative string theory can have a handle on it. Progresses in this direction came after 1995 through the recognition of the role of string dualities. These dualities allow to relate the strong coupling regime of one superstring model to the weak coupling regime of another one and are all encoded in the symmetry group (the U–duality group) of the low energy *supergravity effective action.*

What we want to emphasize is that the first instance of a microscopic explanation of the area law within string theory has been limited to what in the language of general relativity would be an *extremal black hole.* Indeed the extremality condition, namely the coincidence of two horizons, obtains, in the context of a supersymmetric theory, a profound reinterpretation that makes extremal black holes the most interesting objects to study. To introduce the concept consider the usual Reissner Nordstrom metric describing a black–hole of mass m and electric (or magnetic) charge q:

$$ds^2 = -dt^2 \left(1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}\right) + d\rho^2 \left(1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}\right)^{-1} + \rho^2 \, d\Omega^2 \tag{2}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta \, d\phi^2)$ is the metric on a 2–sphere. As it is well known the metric (2) admits two Killing horizons where the norm of the Killing vector $\frac{\partial}{\partial t}$ changes sign. The horizons are at the two roots of the quadratic form $\Delta \equiv -2m\rho + q^2 + \rho^2$ namely at:

$$\rho_\pm = m \pm \sqrt{m^2 - q^2} \tag{3}$$

If $m < |q|$ the two horizons disappear and we have a naked singularity. For this reason in the context of classical general relativity the *cosmic censorship* conjecture was advanced that singularities should always be hidden inside horizon and this conjecture was formulated as the bound:

$$m \geq |q| \tag{4}$$

Of particular interest are the states that saturate the bound (4). If $m = |q|$ the two horizons coincide and, setting:

$$m = |q| \quad ; \quad \rho = r + m \quad ; \quad r^2 = \vec{x} \cdot \vec{x} \tag{5}$$

the metric (2) can be rewritten as:

$$\begin{aligned} ds^2 &= -dt^2 \left(1 + \frac{q}{r}\right)^{-2} + \left(1 + \frac{q}{r}\right)^2 \left(dr^2 + r^2 \, d\Omega^2\right) \\ &= -H^{-2}(\vec{x}) \, dt^2 + H^2(\vec{x}) \, d\vec{x} \cdot d\vec{x} \end{aligned} \tag{6}$$

where by:

$$H(\vec{x}) = \left(1 + \frac{q}{\sqrt{\vec{x} \cdot \vec{x}}}\right) \tag{7}$$

we have denoted a harmonic function in a three–dimensional space spanned by the three cartesian coordinates $\vec{x}$ with the boundary condition that $H(\vec{x})$ goes to 1 at infinity.

Moreover, the extremal Reissner–Nordström configuration is a soliton of classical general relativity, interpolating between the flat Minkowski space–time, asymptotically reached at spatial infinity $\rho \to \infty$, and the Bertotti–Robinson metric describing the conformally flat geometry near the horizon $r \to 0$ [2]:

$$ds^2_{BR} = -\frac{r^2}{M^2_{BR}} dt^2 + \frac{M^2_{BR}}{r^2} \left(dr^2 + r^2 d\Omega\right). \tag{8}$$

Last, let us note the the condition $m = |q|$ can be written as a *no force condition* between the gravitational interaction $F_g = \frac{m}{r^2}$ and the electric repulsion $F_q = -\frac{q}{r^2}$.

Extremal black-hole configurations are embedded in a natural way in supergravity theories. Indeed supergravity, being invariant under local super-Poincaré transformations, includes general relativity, that is it describes gravitation coupled to other fields in a supersymmetric framework. Therefore it admits, among its classical solutions, black holes.

Thinking of a black-hole configuration as a particular bosonic background of an N-extended locally supersymmetric theory gives a simple and natural understanding to the cosmic censorship conjecture. Indeed, in theories with extended supersymmetry the bound (5) is just a consequence of the supersymmetry algebra, and this ensures that in these theories the cosmic censorship conjecture is always verified, that is there are no naked singularities.
For extremal configurations, supersymmetry imposes that if the bound $m = |q|$ is saturated in the classical theory, the same must be true also when quantum corrections are taken into account. This is particularly relevant, since the quantum physics of black holes is described by Hawking theory, which states that quantum black holes are not stable, they radiate a termic radiation as a black–body, and correspondingly they lose their energy (mass). The only stable black-hole configurations are then the extremal ones, because they have the minimal possible energy compatible with the relation (4) and so they cannot radiate.

When the black hole is embedded in an N-extended supergravity background the solutions depend in general also on scalar fields. In this case, the electric charge q has to be replaced by the maximum eigenvalue of the central charge appearing in the supersymmetry algebra (depending on the expectation value of scalar fields and on the electric and magnetic charges). The Reissner–Nordström metric takes in general a more complicated form.
However, extremal black holes preserving some supersymmetries have a peculiar feature: the event horizon loses all information about scalar fields. Indeed, also in the scalar-dependent case, the near horizon geometry is still described by a conformally flat, Bertotti–Robinson-type geometry, with a mass parameter M_{BR} depending on the electric and magnetic charges but not on the scalars. [1]
This peculiarity has a counterpart in the fact that extremal solutions of supergravity have to satisfy a set of first order differential equations imposed by the existence of at least one supercovariant spinor, leaving invariant a fraction of the initial supersymmetry. First order differential equations $\frac{d\Phi}{dr} = f(\Phi)$ have in general fixed points, corresponding to the values of r for which $f(\Phi) = 0$.
It is possible to show [3] that the first order differential equations expressing the Killing spinors

[1]This in fact can be thought as one aspect of the more general fact (true also for non-supersymmetric black holes) that near the horizon black holes loose their 'hair' (no hair theorems). That is if one tries to perturb the black hole with whatever additional hair (some slight mass anisotropy, or a long-range field, like a scalar) all these features disappear near the horizon, except for those associated with the conserved quantities of general relativity, namely, for a non-rotating black hole, its mass and charge.
Again, as it happened for the BPS bound, supersymmetry gives a deeper and more natural understanding of this general feature.

equations for extremal black holes have as fixed point exactly the event horizon. The horizon is an attractor point [4]. Scalar fields, independently of their boundary conditions at spatial infinity, approaching the horizon flow to a fixed point given by a certain ratio of electric and magnetic charges.
Remembering now that the black-hole entropy is given by the area–entropy Bekenstein–Hawking relation (1), we see that the entropy of extremal black holes is a topological quantity, in the sense that it is fixed in terms of the quantized electric and magnetic charges while it does not depend on continuous parameters as scalars. The horizon mass parameter M_{BR} turns out to be given in this case (extremal configurations) by the maximum eigenvalue Z_M of the central charge appearing in the supersymmetry algebra, evaluated at the fixed point:

$$M_{BR} = M_{BR}(e,g) = Z_M(\phi_{fix}, e, g) \tag{9}$$

that gives, for the Bekenstein–Hawking entropy:

$$S_{B-H} = \frac{A_{BR}(e,g)}{4} = \pi |Z_M(\phi_{fix}, e, g)|^2 \tag{10}$$

Many efforts were spent in the course of the years to give an explanation for the large, topological entropy of extremal black holes in the context of a quantum theory of gravity, like string theory. In particular, one would like to give a microscopical, statistical mechanics interpretation of this thermodynamical quantity. Although we will not treat at all the microscopical point of view throughout this paper, it is important to mention that such an interpretation became possible after the introduction of D-branes in the context of string theory [5]. Following this approach, extremal black holes are interpreted as bound states of D-branes in a space–time compactified to four or five dimensions, and the different microstates giving rise to the Bekenstein–Hawking entropy come from the different ways of wrapping branes in the internal directions.
It is important to note that all calculations made in particular cases using this approach furnished values, for the Bekenstein–Hawking entropy, compatible with those performed with the supergravity, macroscopical techniques. The entropy formula turns out to be in all cases a U-duality invariant expression (homogeneous of degree two) built out of electric and magnetic charges and as such can be in fact also computed through certain (moduli independent) topological quantities which only depend on the nature of the U-duality groups and the appropriate representations of electric and magnetic charges. For example in the $N = 8$ theory the entropy can be shown to correspond to the unique quartic E_7 invariant built in terms of the 56 dimensional representation. Actually, one can derive for all $N \geq 2$ theories topological U-invariants constructed in terms of the (moduli dependent) central charges and matter charges and show that, as expected, they coincide with the squared ADM mass at fixed scalars.

In the next section we shall interpret black holes of this form as BPS saturated states namely as quantum states filling special irreducible representations of the supersymmetry algebra, the so called *short supermultiplets*, the shortening condition being precisely the saturation of the cosmic censorship bound (4). Indeed such a bound can be restated as the equality of the mass with the central charge which occurs when a certain fraction of the supersymmetry charges identically annihilate the state. The remaining supercharges applied to the BPS state build up a unitary irreducible representation of supersymmetry that is shorter than the typical one since it contains less states. As we stress in the next section it is precisely this interpretation what makes extremal black holes relevant to the string theory. Indeed these classical solutions of supergravity belong to the non perturbative particle spectrum of superstring theory, and are not accessible to perturbative string theory.

2 Extremal Black–Holes as quantum BPS states

In the previous section we have reviewed the idea of extremal black holes as it arises in classical general relativity. Extremal black–holes have become objects of utmost relevance in the context of superstrings after *the second string revolution* has taken place in 1995. Indeed supersymmetric extremal black–holes have been studied in depth in a vast recent literature [6, 2, 7]. This interest is just part of a more general interest in the p–brane classical solutions of supergravity theories in all dimensions $4 \leq D \leq 11$ [8, 9]. This interest streams from the interpretation of the classical solutions of supergravity that preserve a fraction of the original supersymmetries as the BPS non perturbative states necessary to complete the perturbative string spectrum and make it invariant under the many conjectured duality symmetries [10, 11, 12, 13, 14]. Extremal black–holes and their parent p–branes in higher dimensions are therefore viewed as additional *particle–like* states that compose the spectrum of a fundamental quantum theory. The reader should be advised that the holes we are discussing here are neither stellar–mass, nor mini–black holes: their mass is typically of the order of the Planck–mass:

$$M_{Black\ Hole} \sim M_{Planck} \tag{11}$$

The Schwarzschild radius is therefore microscopic.

Yet, as the monopoles in gauge theories, these non–perturbative quantum states originate from regular solutions of the classical field equations, the same Einstein equations one deals with in classical general relativity and astrophysics. The essential new ingredient, in this respect, is supersymmetry that requires the presence of *vector fields* and *scalar fields* in appropriate proportions. Hence the black–holes we are going to discuss are solutions of generalized Einstein–Maxwell–dilaton equations.

¿From an abstract viewpoint BPS saturated states are characterized by the fact that they preserve a fraction, 1/2 or 1/4 or 1/8 of the original supersymmetries. What this actually means is that there is a suitable projection operator $\mathbb{P}^2_{BPS} = \mathbb{P}_{BPS}$ acting on the supersymmetry charge Q_{SUSY}, such that:

$$(\mathbb{P}_{BPS}\, Q_{SUSY}) \mid \text{BPS state} >= 0 \tag{12}$$

Since the supersymmetry transformation rules of any supersymmetric field theory are linear in the first derivatives of the fields eq.(12) is actually a *system of first order differential equations.* This system has to be combined with the second order field equations of supergravity and the common solutions to both system of equations is a classical BPS saturated state. That it is actually an exact state of non–perturbative string theory follows from supersymmetry representation theory. The classical BPS state is by definition an element of a *short supermultiplet* and, if supersymmetry is unbroken, it cannot be renormalized to a *long supermultiplet.*

Translating eq. (12) into an explicit first order differential system requires knowledge of the supersymmetry transformation rules of supergravity. These latter have a rich geometrical structure that is the purpose of the present paper to illustrate. Indeed the geometrical structure of supergravity which originates in its scalar sector is transferred into the physics of extremal black holes by the BPS saturation condition.

To fulfill the above program, it is necessary first to review the formalism of $D = 4$ N-extended supergravity theories. We begin by recalling the algebraic definition of $D = 4$ BPS states in a theory with an even number of supercharges $N = 2\nu$ [2].

2.1 General definition of BPS states in a 4D theory with $N = 2 \times p$ supersymmetries

The $D = 4$ supersymmetry algebra with $N = 2 \times p$ supersymmetry charges is given by

$$\begin{aligned} \left\{\overline{Q}_{A\alpha}\,,\,\overline{Q}_{B\beta}\right\} &= \mathrm{i}\,(\mathbf{C}\,\gamma^{\mu})_{\alpha\beta}\; P_{\mu}\,\delta_{AB} \;-\; \mathbf{C}_{\alpha\beta}\,\mathbf{Z}_{AB} \\ & (A, B = 1, \ldots, 2p) \end{aligned} \tag{13}$$

where the SUSY charges $\overline{Q}_A \equiv Q^{\dagger}_A\gamma_0 = Q^T_A\,\mathbf{C}$ are Majorana spinors, $\mathbf{C}$ is the charge conjugation matrix, P_μ is the 4–momentum operator and the antisymmetric tensor $\mathbf{Z}_{AB} = -\mathbf{Z}_{BA}$ is the central charge operator. It can always be reduced to normal form

$$\mathbf{Z}_{AB} \quad = \begin{pmatrix} \epsilon Z_1 & 0 & \ldots & 0 \\ 0 & \epsilon Z_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \epsilon Z_p \end{pmatrix} \tag{14}$$

[2]The case with N = odd can be similarly treated but needs some minor modifications due to the fact the eigenvalues of an antisymmetric matrix in odd dimensions are $\{\pm\mathrm{i}\lambda_i, 0\}$.

where ϵ is the 2×2 antisymmetric matrix, (every zero is a 2×2 zero matrix) and the p skew eigenvalues Z_I of $\mathbf{Z}_{AB}$ are the central charges.

If we identify each index $A, B, \ldots$ with a pair of indices

$$A = (a, I) \quad ; \quad a, b, \cdots = 1, 2 \quad ; \quad I, J, \cdots = 1, \ldots, p \tag{15}$$

then the superalgebra (13) can be rewritten as:

$$\left\{ \overline{Q}_{aI|\alpha}\, ,\, \overline{Q}_{bJ|\beta} \right\} = \mathrm{i}\, (C\, \gamma^\mu)_{\alpha\beta}\, P_\mu\, \delta_{ab}\, \delta_{IJ} - C_{\alpha\beta}\, \epsilon_{ab} \times \mathbb{Z}_{IJ} \tag{16}$$

where the SUSY charges $\overline{Q}_{aI} \equiv Q^\dagger_{aI} \gamma_0 = Q^T_{aI}\, C$ are Majorana spinors, C is the charge conjugation matrix, P_μ is the 4–momentum operator, ϵ_{ab} is the two–dimensional Levi Civita symbol and the central charge operator is now represented by the *symmetric tensor* $\mathbb{Z}_{IJ} = \mathbb{Z}_{JI}$ which can always be diagonalized $\mathbb{Z}_{IJ} = \delta_{IJ}\, Z_J$. The p eigenvalues Z_J are the skew eigenvalues introduced in equation (14).

The Bogomolny bound on the mass of a generalized monopole state:

$$M \geq |\, Z_I| \qquad \forall Z_I\, ,\, I = 1, \ldots, p \tag{17}$$

is an elementary consequence of the supersymmetry algebra and of the identification between *central charges* and *topological charges*. To see this it is convenient to introduce the following reduced supercharges:

$$\overline{S}^{\pm}_{aI|\alpha} = \frac{1}{2} \left(\overline{Q}_{aI} \gamma_0 \pm \mathrm{i}\, \epsilon_{ab}\, \overline{Q}_{bI} \right)_\alpha \tag{18}$$

They can be regarded as the result of applying a projection operator to the supersymmetry charges:

$$\begin{aligned} \overline{S}^{\pm}_{aI} &= \overline{Q}_{bI}\, \mathbb{P}^{\pm}_{ba} \\ \mathbb{P}^{\pm}_{ba} &= \frac{1}{2} \left(\mathbf{1} \delta_{ba} \pm \mathrm{i} \epsilon_{ba} \gamma_0 \right) \end{aligned} \tag{19}$$

Combining eq.(16) with the definition (18) and choosing the rest frame where the four momentum is $P_\mu = (M, 0, 0, 0)$, we obtain the algebra:

$$\left\{ \overline{S}^{\pm}_{aI}\, ,\, \overline{S}^{\pm}_{bJ} \right\} = \pm \epsilon_{ac}\, C\, \mathbb{P}^{\pm}_{cb}\, (M \mp Z_I)\, \delta_{IJ} \tag{20}$$

By positivity of the operator $\left\{ \overline{S}^{\pm}_{aI}\, ,\, \overline{S}^{\pm}_{bJ} \right\}$ it follows that on a generic state the Bogomolny bound (17) is fulfilled. Furthermore it also follows that the states which saturate the bounds:

$$(M \pm Z_I)\, |\text{BPS state}, i\rangle = 0 \tag{21}$$

are those which are annihilated by the corresponding reduced supercharges:

$$\overline{S}^{\pm}_{aI}\,|\mathrm{BPS\ state},i\rangle = 0 \tag{22}$$

On one hand eq.(22) defines *short multiplet representations* of the original algebra (16) in the following sense: one constructs a linear representation of (16) where all states are identically annihilated by the operators $\overline{S}^{\pm}_{aI}$ for $I = 1, \ldots, n_{max}$. If $n_{max} = 1$ we have the minimum shortening, if $n_{max} = p$ we have the maximum shortening. On the other hand eq.(22) can be translated into a first order differential equation on the bosonic fields of supergravity.

Indeed, let us consider a configuration where all the fermionic fields are zero. In order for a configuration to be supersymmetric we have to impose that the supersymmetry variations of all the fields are zero in the background. Since the bosonic fields transform into spinors, they are automatically zero in the background; for the fermionic fields, instead, this condition gives a differential equation for the bosonic fields which is called "Killing spinor" equation. Indeed, the gravitino transformation law contains the covariant derivative of the supersymmetry parameter ϵ_A and the differential equation one obtains in this case determines the functional dependence of the parameter on the space–time coordinates. The corresponding solution for ϵ_A is called a Killing spinor.

Setting the fermionic SUSY rules appropriate to such a background equal to zero we find the following Killing spinor equation:

$$0 = \delta\mathrm{fermions} = \mathrm{SUSY\ rule}\,(\mathrm{bosons}, \epsilon_{AI}) \tag{23}$$

where the SUSY parameter satisfies the following conditions (ξ^μ denotes a time–like Killing vector):

$$\begin{array}{rcll} \xi^\mu\,\gamma_\mu\,\epsilon_{aI} & = & \mathrm{i}\,\varepsilon_{ab}\,\epsilon^{bI} & ; \quad I = 1, \ldots, n_{max} \\ \epsilon_{aI} & = & 0 & ; \quad I > n_{max} \end{array} \tag{24}$$

Hence eq.s (23) with a parameter satisfying the condition (24) will be our operative definition of BPS states.

3 Four-dimensional BPS black–holes and the general form of the supergravity action

In this section we begin the study of BPS black–hole solutions in four space-time dimensions. To this aim we first have to introduce the main features of four dimensional supergravities. Four dimensional supergravity theories contain in the bosonic sector, besides the metric, a number of vectors and scalars. The relevant bosonic action is known to have the following general form:

$$\begin{aligned} \mathcal{S} \;=\; & \int \sqrt{-g}\, d^4x \left(2R + \mathrm{Im}\,\mathcal{N}_{\Lambda\Gamma} F_{\mu\nu}{}^{\Lambda} F^{\Gamma|\mu\nu} + \frac{1}{6} g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J + \right. \\ + \; & \left. \frac{1}{2}\mathrm{Re}\,\mathcal{N}_{\Lambda\Gamma} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\mu\nu}{}^{\Lambda} F^{\Gamma}{}_{\rho\sigma} \right) \end{aligned} \tag{25}$$

Table 1: *Scalar Manifolds of Extended Supergravities*

N	U-Duality group	$\mathcal{M}_{scalar}$	n_V, m
1	$\mathcal{U} \subset Sp(2n, \mathbb{R})$	Kähler	n_V, m
2	$\mathcal{U} \subset Sp(2n+2, \mathbb{R})$	$\mathcal{M}^Q(n_H) \otimes \mathcal{M}^{SK}(n)$	$n+1, 2n+4n_H$
3	$SU(3,n) \subset Sp(2n+6, \mathbb{R})$	$\frac{SU(3,n)}{S(U(3)\times U(n))}$	$3+n,\ 6n$
4	$SU(1,1) \otimes SO(6,n) \subset Sp(2n+12, \mathbb{R})$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6)\times SO(n)}$	$6+n,\ 6n+2$
5	$SU(1,5) \subset Sp(20, \mathbb{R})$	$\frac{SU(1,5)}{S(U(1)\times U(5))}$	$10,\ 10$
6	$SO^\star(12) \subset Sp(32, \mathbb{R})$	$\frac{SO^\star(12)}{U(1)\times SU(6)}$	$16,\ 30$
$7, 8$	$E_{7(-7)} \subset Sp(56, \mathbb{R})$	$\frac{E_{7(-7)}}{SU(8)}$	$28,\ 70$

$\mathcal{M}^Q(n_H)$ denotes a quaternionic manifold of quaternionic dimension n_H and $\mathcal{M}^{SK}(n)$ a Special Kähler manifold of complex dimension n.

where $g_{IJ}(\phi)$ $(I, J, \cdots = 1, \cdots, m$ is the scalar metric on the σ-model described by the m–dimensional scalar manifold $\mathcal{M}_{scalar}$ and the vectors kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a complex, symmetric, $n_V \times n_V$ matrix depending on the scalar fields. The number of vectors and scalars, that is n_V and m, and the geometrical properties of the scalar manifold $\mathcal{M}_{scalar}$ depend on the number N of supersymmetries and are resumed in Table 1. The relation between this scalar geometry and the kinetic matrix $\mathcal{N}$ has a very general and universal form. Indeed it is related to the solution of a general problem, namely how to lift the action of the scalar manifold isometries from the scalar to the vector fields. Such a lift is necessary because of supersymmetry since scalars and vectors generically belong to the same supermultiplet and must rotate coherently under symmetry operations. This problem has been solved in a general (non supersymmetric) framework in reference [15] by considering the possible extension of the Dirac electric–magnetic duality to more general theories involving scalars. In the next subsection we review this approach and in particular we show how enforcing covariance with respect to such duality rotations leads to a determination of the kinetic matrix $\mathcal{N}$. The structure of $\mathcal{N}$ enters the black–hole equations in a crucial way so that the topological invariant associated with the hole, that is its *entropy*, is an invariant of the group of electro-magnetic duality rotations, the U–duality group.

3.1 Duality Rotations and Symplectic Covariance

Let us review the general structure of an abelian theory of vectors and scalars displaying covariance under a group of duality rotations. The basic reference is the 1981 paper by Gaillard and Zumino [15]. A general presentation in $D = 2p$ dimensions can be found in [18]. Here we fix $D = 4$.

We consider a theory of n_V gauge fields A^Λ_μ, in a $D = 4$ space–time with Lorentz signature.

They correspond to a set of n_V differential 1–forms

$$A^\Lambda \;\equiv\; A^\Lambda_\mu \, dx^\mu \qquad (\Lambda = 1, \dots, n_V) \tag{26}$$

The corresponding field strengths and their Hodge duals are defined by

$$\begin{aligned}
F^\Lambda &\equiv d\,A^\Lambda \equiv \mathcal{F}^\Lambda_{\mu\nu}\, dx^\mu \wedge dx^\nu \\
\mathcal{F}^\Lambda_{\mu\nu} &\equiv \frac{1}{2}\left(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu\right) \\
{}^\star F^\Lambda &\equiv \widetilde{\mathcal{F}}^\Lambda_{\mu\nu}\, dx^\mu \wedge dx^\nu \\
\widetilde{\mathcal{F}}^\Lambda_{\mu\nu} &\equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda|\rho\sigma}
\end{aligned} \tag{27}$$

In addition to the gauge fields let us also introduce a set of real scalar fields ϕ^I ($I = 1, \dots, m$) spanning an m–dimensional manifold $\mathcal{M}_{scalar}$ endowed with a metric $g_{IJ}(\phi)$. Utilizing the above field content we can write the following action functional:

$$\mathcal{S} = \int \left\{ -\,\gamma_{\Lambda\Sigma}(\phi)\, F^\Lambda_{\mu\nu}\, F^{\Sigma\mu\nu} + \theta_{\Lambda\Sigma}(\phi)\, F^\Lambda_{\mu\nu} \star F^{\Sigma\mu\nu} + \frac{1}{2}\, g_{IJ}(\phi)\, \partial_\mu\phi^I\, \partial^\mu\phi^J \right\} \mathrm{d}^4x \tag{28}$$

where the scalar fields dependent $n_V \times n_V$ matrix $\gamma_{\Lambda\Sigma}(\phi)$ generalizes the inverse of the squared coupling constant $\frac{1}{g^2}$ appearing in ordinary gauge theories. The field dependent matrix $\theta_{\Lambda\Sigma}(\phi)$ is instead a generalization of the *theta*–angle of quantum chromodynamics. Both γ and θ are symmetric matrices. Finally, we have introduced the operator $\star$ that maps a field strength into its Hodge dual

$$\left(\star\,\mathcal{F}^\Lambda\right)_{\mu\nu} \equiv \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathcal{F}^{\Lambda|\rho\sigma}. \tag{29}$$

Introducing self–dual and antiself–dual combinations

$$\begin{aligned}
\mathcal{F}^\pm &= \frac{1}{2}\left(\mathcal{F} \pm \mathrm{i} \star \mathcal{F}\right) \\
\star\,\mathcal{F}^\pm &= \mp \mathrm{i}\mathcal{F}^\pm
\end{aligned} \tag{30}$$

and the field–dependent symmetric matrices

$$\begin{aligned}
\mathcal{N} &= \theta - \mathrm{i}\gamma \\
\overline{\mathcal{N}} &= \theta + \mathrm{i}\gamma\,,
\end{aligned} \tag{31}$$

the vector part of the Lagrangian (28) can be rewritten as

$$\mathcal{L}_{vec} = \mathrm{i}\,\left[\mathcal{F}^{-T}\overline{\mathcal{N}}\mathcal{F}^- - \mathcal{F}^{+T}\mathcal{N}\mathcal{F}^+\right] \tag{32}$$

Introducing further the new tensors

$$\widetilde{\mathcal{G}}^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}} = \left(-\gamma_{\Lambda\Sigma} + \theta_{\Lambda\Sigma}\right)\mathcal{F}^{\Sigma}_{\mu\nu} \leftrightarrow \mathcal{G}^{\mp\Lambda}_{\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{\mp\Lambda}_{\mu\nu}} \tag{33}$$

the Bianchi identities and field equations associated with the Lagrangian (28) can be written as

$$\partial^{\mu}\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu} = 0 \tag{34}$$

$$\partial^{\mu}\widetilde{\mathcal{G}}^{\Lambda}_{\mu\nu} = 0 \tag{35}$$

or equivalently

$$\partial^{\mu}\mathrm{Im}\mathcal{F}^{\pm\Lambda}_{\mu\nu} = 0 \tag{36}$$

$$\partial^{\mu}\mathrm{Im}\mathcal{G}^{\pm\Lambda}_{\mu\nu} = 0\ . \tag{37}$$

This suggests that we introduce the $2n_V$ column vector

$$\mathbf{V} \equiv \begin{pmatrix} \star\mathcal{F} \\ \star\mathcal{G} \end{pmatrix} \tag{38}$$

and that we consider general linear transformations on such a vector

$$\begin{pmatrix} \star\mathcal{F} \\ \star\mathcal{G} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \star\mathcal{F} \\ \star\mathcal{G} \end{pmatrix} \tag{39}$$

For any matrix $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_V, \mathbb{R})$ the new vector $\mathbf{V}'$ of magnetic and electric field–strengths satisfies the same equations (35) as the old one. In a condensed notation we can write

$$\partial\,\mathbf{V} = 0 \quad \Longleftrightarrow \quad \partial\,\mathbf{V}' = 0 \tag{40}$$

Separating the self–dual and anti–self–dual parts

$$\mathcal{F} = \left(\mathcal{F}^{+} + \mathcal{F}^{-}\right) \qquad ; \qquad \mathcal{G} = \left(\mathcal{G}^{+} + \mathcal{G}^{-}\right) \tag{41}$$

and taking into account that we have

$$\mathcal{G}^{+} = \mathcal{N}\mathcal{F}^{+} \quad \mathcal{G}^{-} = \overline{\mathcal{N}}\mathcal{F}^{-} \tag{42}$$

the duality rotation of eq. (39) can be rewritten as

$$\begin{pmatrix} \mathcal{F}^{+} \\ \mathcal{G}^{+} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \mathcal{F}^{+} \\ \mathcal{N}\mathcal{F}^{+} \end{pmatrix} \qquad ; \qquad \begin{pmatrix} \mathcal{F}^{-} \\ \mathcal{G}^{-} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \mathcal{F}^{-} \\ \overline{\mathcal{N}}\mathcal{F}^{-} \end{pmatrix}. \tag{43}$$

Now, let us note that, since Φ^I are the scalar partners of A^Λ, when a duality rotation is performed on the vector field strengths and their duals, also the scalars get transformed correspondingly, through the action of some diffeomorphism on the scalar manifold $\mathcal{M}_{scal}$. In particular, also the kinetic matrix $\mathcal{N}(\Phi)$, that in supersymmetric theories is a function of scalars, transforms under a duality rotation. That is, a duality transformation ξ acts in the following way on the supersymmetric system:

$$\xi : \begin{cases} \Phi & \to & \Phi' = \xi(\Phi) \\ \mathcal{N}(\Phi) & \to & \mathcal{N}'\left(\xi(\Phi)\right) \\ V & \to & V'^{\mp} = S_\xi V^{\mp} \end{cases} \tag{44}$$

Thus, the transformation laws of the equations of motion and of $\mathcal{N}$, and so also the matrix $\mathcal{S}_\xi$, will be induced by a diffeomorphism of the scalar fields.

Focusing in particular on the third relation in (44), that explicitly reads:

$$\begin{pmatrix} \mathcal{F}^{\pm\prime} \\ \mathcal{G}^{\pm\prime} \end{pmatrix} = \begin{pmatrix} A_\xi \mathcal{F}^\pm + B_\xi \mathcal{G}^\pm \\ C_\xi \mathcal{F}^\pm + D_\xi \mathcal{G}^\pm \end{pmatrix} \tag{45}$$

let us note that it contains the magnetic field strength $\mathcal{G}_\Lambda^\mp$, which is defined as a variation of the kinetic lagrangian. Under the transformations (44) the lagrangian transforms in the following way:

$$\begin{aligned} \mathcal{L}' &= \mathrm{i}\Big[(A_\xi + B_\xi\mathcal{N})_\Gamma{}^\Lambda (A_\xi + B_\xi\mathcal{N})^\Sigma{}_\Delta \mathcal{N}'_{\Lambda\Sigma}(\Phi)\mathcal{F}^{+\Gamma}\mathcal{F}^{+\Delta} \\ &- \left(A_\xi + B_\xi\overline{\mathcal{N}}\right)_\Gamma{}^\Lambda \left(A_\xi + B_\xi\overline{\mathcal{N}}\right)^\Sigma{}_\Delta \overline{\mathcal{N}}'_{\Lambda\Sigma}(\Phi)\mathcal{F}^{-\Gamma}\mathcal{F}^{-\Delta}\Big]; \end{aligned} \tag{46}$$

Equations (44) must be consistent with the definition of $\mathcal{G}^\mp$ as a variation of the lagrangian (46):

$$\mathcal{G}'^{+}_\Lambda = (C_\xi + D_\xi\mathcal{N})_{\Lambda\Sigma} F^{+\Sigma} \equiv -\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}'}{\partial\mathcal{F}'^{+\Lambda}} = (A_\xi + B_\xi\mathcal{N})^\Delta_{\ \Sigma}\mathcal{N}'_{\Lambda\Delta}\mathcal{F}^{+\Sigma} \tag{47}$$

that implies:

$$\mathcal{N}'_{\Lambda\Sigma}(\Phi') = \left[(C_\xi + D_\xi\mathcal{N})\cdot(A_\xi + B_\xi\mathcal{N})^{-1}\right]_{\Lambda\Sigma}; \tag{48}$$

Recalling now that the matrix $\mathcal{N}$ is symmetric, and that this property must be true also in the duality transformed system, it follows that the matrix $\mathcal{S}_\xi$ must satisfy a constraint that allow to fix the duality groups.
Indeed, imposing that $\mathcal{N}$ and $\mathcal{N}'$ be both symmetric matrices, gives the constraint:

$$\mathcal{S} \in Sp(2n_V, \mathbb{R}) \subset GL(2n_V, \mathbb{R}). \tag{49}$$

This observation has important implications on the scalar manifold $\mathcal{M}_{scal}$. Indeed, all the above discussion implies that on the scalar manifold the following homomorphism is defined:

$$Diff(\mathcal{M}_{scal}) \to Sp(2n, \mathbb{R}) \tag{50}$$

In particular, the presence on the manifold of a function of scalars transforming with a fractional linear transformation under a duality rotation (that is a diffeomorphism) on scalars, induces the existence on $\mathcal{M}_{scal}$ of a linear structure (inherited from vectors). In particular, as we will see in section 3.2, for the $N=2$ four dimensional theory this implies that the scalar manifold be a *special manifold*, that is a Kähler–Hodge manifold endowed with a flat symplectic bundle.

As it is necessary for a duality rotation, the transformation (44), that is a duality symmetry on the system field-equations/Bianchi-identities, cannot be extended to a symmetry of the lagrangian. The scalar lagrangian $\mathcal{L}_{scal}$ is left invariant under the action of the isometry group of the metric g_{IJ}, but the vector part is in general not invariant. Indeed, the transformed lagrangian under the action of $\mathcal{S} \in Sp(2n_V, \mathbb{R})$ can be rewritten:

$$\begin{aligned} \mathrm{I}m\left(\mathcal{F}^{-\Lambda}\mathcal{G}^-_\Lambda\right) &\rightarrow \mathrm{I}m\left(\mathcal{F}'^{-\Lambda}\mathcal{G}'^-_\Lambda\right) \\ &= \mathrm{I}m\big[\mathcal{F}^{-\Lambda}\mathcal{G}^-_\Lambda + 2(C^TB)_\Lambda{}^\Sigma\mathcal{F}^{-\Lambda}\mathcal{G}^-_\Sigma + \\ &+ (C^TA)_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}\mathcal{F}^{-\Sigma} + (D^TB)^{\Lambda\Sigma}\mathcal{G}^-_\Lambda\mathcal{G}^-_\Sigma\big] \end{aligned} \tag{51}$$

It is evident from (51) that only the transformations with $B=C=0$ are symmetries.
If $C \neq 0$, $B=0$ the lagrangian varies for a topological term:

$$(C^TA)_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\mu\nu}\,{}^\star\mathcal{F}^{\Sigma|\mu\nu} \tag{52}$$

corresponding to a redefinition of the function $\theta_{\Lambda\Sigma}$; such a transformation leaves classical physics invariant, being a total derivative, but it is relevant in the quantum theory. It is a symmetry of the partition function only if $\Delta\theta = \frac{1}{2}(C^TA)$ is an integer multiple of 2π, and this implies that $S \in Sp(2n_V, \mathbb{Z}) \subset Sp(2n_V, \mathbb{R})$.
For $B \neq 0$ neither the action nor the perturbative partition function are invariant. Let us observe that, for $B \neq 0$, the transformation law of the kinetic matrix $\mathcal{N} = \theta - \mathrm{i}\gamma$ contains the transformation $\mathcal{N} \rightarrow -\frac{1}{\mathcal{N}}$ that is it exchanges the weak and strong coupling regimes of the theory. One can then think a supersymmetric quantum field theory being described by a collection of local lagrangians, each defined in a local patch. They are all equivalent once one defines for each of them what is *electric* and what is *magnetic*. Duality transformations map this set of lagrangians one into the other. At this point we observe that the supergravity bosonic lagrangian (25) is exactly of the form considered in this section as far as the matter content is concerned, so that we may apply the above considerations about duality rotations to the supergravity case. In particular, the U-duality acts in all theories with $N \geq 2$ supercharges, where the vector supermultiplets contain both vectors and scalars. For $N=1$ supergravity, instead, vectors and scalars are still present but they are not related by supersymmetry, and as a consequence they are not related by U-duality rotations, so that the previous formalism does not apply. In the next section we will discuss in a geometrical framework the structure of the supergravity theories for $N \geq 2$. In

particular, we will give the expression for the kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ in terms of the $Sp(2n_V)$ coset representatives embedding the U-duality group for theories whose σ-model is a coset space, namely $N > 2$. Furthermore we will show that the $N = 2$ case can be treated in a completely analogous way even if the σ-model of the scalars is not in general a coset space.

3.2 Duality symmetries and central charges in four dimensions

Let us restrict our attention to N-extended supersymmetric theories coupled to the gravitational field, that is to supergravity theories, whose bosonic action has been given in (25). For any theory we analyze the group theoretical structure and find the expression of the central charges, and the properties they obey. We will see that in each theory all fields are in some representation of the isometry group U of scalar fields or of its maximal compact subgroup H. This is just a consequence of the Gaillard-Zumino duality acting on the 2-forms and their duals, discussed in the preceding section, so that a restriction to the integers of U is the duality group.

As we have already mentioned, all $D = 4$ supergravity theories contain scalar fields whose kinetic Lagrangian is described by σ–models of the form U/H, with the exception of $D = 4$, $N = 1, 2$. We begin to examine the theories with $N > 2$, and then we will generalize the results to the $N = 2$ case (The $N = 1$ case, as explained before, is of no concern to us.).
Here U is a non compact group acting as an isometry group on the scalar manifold while H, the isotropy subgroup, is of the form:

$$H = H_{Aut} \otimes H_{matter} \tag{53}$$

H_{Aut} being the automorphism group of the supersymmetry algebra while H_{matter} is related to the matter multiplets. (Of course $H_{matter} = 1\!\!1$ in all cases where supersymmetric matter doesn't exist, namely $N > 4$). The coset manifolds U/H and the automorphism groups for various supergravity theories for any D and N can be found in the literature (see for instance [16], [17], [18], [19]). As it was discussed in the previous section, the group U acts linearly on the field strengths $\mathcal{F}^{\Lambda}_{\mu\nu}$ appearing in the gravitational and matter multiplets. Here and in the following the index Λ runs over the dimensions of some representation of the duality group U. The true duality symmetry (U–duality), acting on integral quantized electric and magnetics charges, is the restriction of the continuous group U to the integers [13]. The moduli space of these theories is $\mathrm{U}(\mathbb{Z})\backslash\mathrm{U}/H$.

All the properties of the given supergravity theories are completely fixed in terms of the geometry of U/H, namely in terms of the coset representatives L satisfying the relation:

$$L(\Phi') = gL(\Phi)h(g, \Phi) \tag{54}$$

where $g \in \mathrm{U}$, $h \in H$ and $\Phi' = \Phi'(\Phi)$, Φ being the coordinates of U/H. Note that the scalar fields in U/H can be assigned, in the linearized theory, to linear representations R_H of the local isotropy group H so that dim R_H = dim U $-$ dim H (in the full theory, R_H is the representation which the vielbein of U$/H$ belongs to).

As explained in the following, the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ for the 2–forms $\mathcal{F}^\Lambda$ is fixed in terms of L and the physical field strengths of the interacting theories are "dressed" with scalar fields in terms of the coset representatives. This allows us to write down the central charges associated to the vectors in the gravitational multiplet in a neat way in terms of the geometrical structure of the moduli space. In an analogous way also the vectors of the matter multiplets give rise to charges which, as we will see, are closely related to the central charges.

To any field–strength $\mathcal{F}^\Lambda$ we may associate a magnetic charge g^Λ and an electric charge e_Λ given respectively by:

$$g^\Lambda = \int_{S^2} \mathcal{F}^\Lambda \qquad\qquad e_\Lambda = \int_{S^2} \mathcal{G}_\Lambda \tag{55}$$

These charges however are not the physical charges of the interacting theory; the latter ones can be computed by looking at the transformation laws of the fermion fields, where the physical field–strengths appear dressed with the scalar fields [19],[20]. Let us first introduce the central charges: they are associated to the dressed 2–form T_{AB} appearing in the supersymmetry transformation law of the gravitino 1-form. We have indeed:

$$\delta\psi_A = \nabla\epsilon_A + \alpha T_{AB|\mu\nu}\gamma^a\gamma^{\mu\nu}\epsilon^B V_a + \cdots \tag{56}$$

Here ∇ is the covariant derivative in terms of the space–time spin connection and the composite connection of the automorphism group H_{Aut}, α is a coefficient fixed by supersymmetry, V^a is the space–time vielbein, $A = 1, \cdots, N$ is the index acted on by the automorphism group. Here and in the following the dots denote trilinear fermion terms which are characteristic of any supersymmetric theory but do not play any role in the following discussion. The field-strength T_{AB} will be constructed by dressing the bare field-strengths $\mathcal{F}^\Lambda$ with the coset representative $L(\Phi)$ of U$/H$, Φ denoting a set of coordinates of U$/H$.

Note that the same field strength T_{AB} which appears in the gravitino transformation laws is also present in the dilatino transformation laws in the following way:

$$\delta\chi_{ABC} = P_{ABCD,\ell}\partial_\mu\phi^\ell\gamma^\mu\epsilon^D + \beta T_{[AB|\mu\nu}\gamma^{\mu\nu}\epsilon_{C]} + \cdots \tag{57}$$

In an analogous way, when vector multiplets are present, the matter vector field strengths appearing in the transformation laws of the gaugino fields are dressed with the scalars:

$$\delta\lambda_{IA} = \mathrm{i}P_{IAB,i}\partial_\mu\Phi^i\gamma^\mu\epsilon^B + \gamma T_{I|\mu\nu}\gamma^{\mu\nu}\epsilon_A + \cdots \tag{58}$$

where $P_{ABCD} = P_{ABCD,\ell}d\phi^\ell$ and $P^I_{AB} = P^I_{AB,i}d\Phi^i$ are the vielbein of the scalar manifolds spanned by the scalar fields of the gravitational and vector multiplets respectively (more precise definitions are given below), and β and γ are constants fixed by supersymmetry.

In order to give the explicit dependence on scalars of T_{AB}, T^I, it is necessary to recall from the previous subsection that, according to the Gaillard–Zumino construction, the isometry group U of the scalar manifold acts on the vector $(F^{-\Lambda}, \mathcal{G}^-_\Lambda)$ (or its complex conjugate) as a subgroup of $Sp(2n_V, \mathbb{R})$ (n_V is the number of vector fields) with duality transformations interchanging electric and magnetic field–strengths:

$$\mathcal{S}\begin{pmatrix}\mathcal{F}^{-\Lambda}\\ \mathcal{G}^-_\Lambda\end{pmatrix} = \begin{pmatrix}\mathcal{F}^{-\Lambda}\\ \mathcal{G}^-_\Lambda\end{pmatrix}' \tag{59}$$

according to the discussion in the previous subsection.
If $L(\Phi)$ is the coset representative of U in some representation, $\mathcal{S}$ represents the embedded coset representative belonging to $Sp(2n_V, \mathbb{R})$ and in each theory, A, B, C, D can be constructed in terms of $L(\Phi)$. Using a complex basis in the vector space of $Sp(2n_V)$, we may rewrite the symplectic matrix as a pseudo-unitary symplectic matrix of the following form:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h}\\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h}\end{pmatrix} = \mathcal{A}^{-1}\mathcal{S}\mathcal{A} \tag{60}$$

where:

$$\begin{aligned} f &= \frac{1}{\sqrt{2}}(A - \mathrm{i}B)\\ h &= \frac{1}{\sqrt{2}}(C - \mathrm{i}D)\\ \mathcal{A} &= \begin{pmatrix} 1 & 1\\ -\mathrm{i} & \mathrm{i}\end{pmatrix} \end{aligned} \tag{61}$$

We will denote the $Sp(2n_V)$ group in the pseudo-unitary basis as $Usp(n_V, n_V)$. The requirement that $U \in Usp(n_V, n_V)$ must satisfy is:

$$U^t \mathbb{C} U = \mathbb{C}\,, \quad \mathbb{C}^t = -\mathbb{C} \quad (U \text{ symplectic}) \tag{62}$$

$$U^\dagger \eta U = \eta\,, \quad \eta = \begin{pmatrix} \mathbb{1}_{n_V\times n_V} & 0\\ 0 & -\mathbb{1}_{n_V\times n_V}\end{pmatrix} \quad (U \text{ pseudo-unitary}) \tag{63}$$

which implies, on the sub-blocks f and h:

$$\begin{cases} \mathrm{i}(f^\dagger h - h^\dagger f) = \mathbb{1}\\ (f^t h - h^t f) = 0 \end{cases} \tag{64}$$

The $n_V \times n_V$ subblocks of U are submatrices f, h which can be decomposed with respect to the isotropy group $H_{Aut} \times H_{matter}$ as:

$$\begin{aligned} f &= (f^\Lambda_{AB}, f^\Lambda_I)\\ h &= (h_{\Lambda AB}, h_{\Lambda I}) \end{aligned} \tag{65}$$

where AB are indices in the antisymmetric representation of $H_{Aut} = SU(N) \times U(1)$ and I is an index of the fundamental representation of H_{matter}. Upper $SU(N)$ indices label objects in the complex conjugate representation of $SU(N)$: $(f^\Lambda_{AB})^* = f^{\Lambda AB}$ etc.

Note that we can consider $(f^\Lambda_{AB}, h_{\Lambda AB})$ and $(f^\Lambda_I, h_{\Lambda I})$ as symplectic sections of a $Sp(2n_V, \mathbb{R})$ bundle over U$/H$. We will see in the following that this bundle is actually flat. The real embedding given by S is appropriate for duality transformations of $\mathcal{F}^\pm$ and their duals $\mathcal{G}^\pm$, according to equations (43), while the complex embedding in the matrix U is appropriate in writing down the fermion transformation laws and supercovariant field–strengths. The kinetic matrix $\mathcal{N}$, according to Gaillard–Zumino [15], can be written in terms of the sub-blocks f, h, and turns out to be:

$$\mathcal{N} = hf^{-1}, \qquad \mathcal{N} = \mathcal{N}^t \tag{66}$$

transforming projectively under $Sp(2n_V, \mathbb{R})$ duality rotations as already shown in the previous section. By using (64)and (66) we find that

$$(f^t)^{-1} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})\overline{f} \tag{67}$$

that is

$$\begin{aligned} f_{AB\Lambda} &\equiv (f^{-1})_{AB\Lambda} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\overline{f}^\Sigma_{AB} && (68)\\ f_{I\Lambda} &\equiv (f^{-1})_{I\Lambda} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\overline{f}^\Sigma_I && (69)\end{aligned}$$

It can be shown [19] that the dressed graviphotons and matter self–dual field–strengths appearing in the transformation law of gravitino (56), dilatino (57) and gaugino (58) can be constructed as a symplectic invariant using the f and h matrices, as follows:

$$\begin{aligned} T^-_{AB} &= \mathrm{i}(\overline{f}^{-1})_{AB\Lambda}F^{-\Lambda} = f^\Lambda_{AB}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma} = h_{\Lambda AB}F^{-\Lambda} - f^\Lambda_{AB}\mathcal{G}^-_\Lambda \\ T^-_I &= \mathrm{i}(\overline{f}^{-1})_{I\Lambda}F^{-\Lambda} = f^\Lambda_I(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma} = h_{\Lambda I}F^{-\Lambda} - f^\Lambda_I\mathcal{G}^-_\Lambda \\ \overline{T}^{+AB} &= (T^-_{AB})^* \\ \overline{T}^{+I} &= (T^-_I)^* \end{aligned} \tag{70}$$

(For $N > 4$, supersymmetry does not allow matter multiplets and $f_{\Lambda I} = T_I = 0$). To construct the dressed charges one integrates $T_{AB} = T^+_{AB} + T^-_{AB}$ and (for $N = 3, 4$) $T_I = T^+_I + T^-_I$ on a large 2-sphere. For this purpose we note that

$$\begin{aligned} T^+_{AB} &= h_{\Lambda AB}F^{+\Lambda} - f^\Lambda_{AB}\mathcal{G}^+_\Lambda = 0 && (71)\\ T^+_I &= h_{\Lambda I}F^{+\Lambda} - f^\Lambda_I\mathcal{G}^+_\Lambda = 0 && (72)\end{aligned}$$

as a consequence of eqs. (66), (43). Therefore we can introduce the "dressed" charges:

$$Z_{AB}(\Phi_0) = \int_{S^2} T_{AB} = \int_{S^2} (T^+_{AB} + T^-_{AB}) = \int_{S^2} T^-_{AB} = h_{\Lambda AB} g^\Lambda - f^\Lambda_{AB} e_\Lambda \tag{73}$$

$$Z_I(\Phi_0) = \int_{S^2} T_I = \int_{S^2} (T^+_I + T^-_I) = \int_{S^2} T^-_I = h_{\Lambda I} g^\Lambda - f^\Lambda_I e_\Lambda \quad (N \leq 4) \tag{74}$$

where:

$$e_\Lambda = \int_{S^2} \mathcal{G}_\Lambda, \quad g^\Lambda = \int_{S^2} F^\Lambda \tag{75}$$

and the sections (f^Λ, h_Λ) on the right hand side now depend on the *v.e.v.*'s $\Phi_0 \equiv \Phi(r = \infty)$ of the scalar fields Φ^I. We see that the central and matter charges are given in this case by symplectic invariants and that the presence of dyons in $D = 4$ is related to the symplectic embedding.

The scalar field dependent combinations of fields strengths appearing in the fermion supersymmetry transformation rules have a profound meaning and play a key role in the physics of BPS black–holes. The combination $T_{AB\mu\nu}$ is named the graviphoton field strength and its integral over a 2–sphere at infinity gives the value of *the central charge* Z_{AB} of the $N = 2$ supersymmetry algebra. The combination $T^I_{\mu\nu}$ is named the matter field strength. Evaluating its integral on a 2–sphere at infinity one obtains the so called *matter charges* Z^I.

We are now able to derive some differential relations among the central and matter charges using the Maurer–Cartan equations obeyed by the scalars through the embedded coset representative U. Indeed, let $\Gamma = U^{-1}dU$ be the $Usp(n_V, n_V)$ Lie algebra left invariant one form satisfying:

$$d\Gamma + \Gamma \wedge \Gamma = 0 \tag{76}$$

In terms of (f, h) Γ has the following form:

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \mathrm{i}(f^\dagger dh - h^\dagger df) & \mathrm{i}(f^\dagger d\overline{h} - h^\dagger d\overline{f}) \\ -\mathrm{i}(f^t dh - h^t df) & -\mathrm{i}(f^t d\overline{h} - h^t d\overline{f}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega}^{(H)} \end{pmatrix} \tag{77}$$

where the $n_V \times n_V$ subblocks $\Omega^{(H)}$ and $\mathcal{P}$ embed the H connection and the vielbein of U$/H$ respectively . This identification follows from the Cartan decomposition of the $Usp(n_V, n_V)$ Lie algebra. Explicitly, if we define the $H_{Aut} \times H_{matter}$–covariant derivative of a vector $V = (V_{AB}, V_I)$ as:

$$\nabla V = dV - V\omega, \quad \omega = \begin{pmatrix} \omega^{AB}{}_{CD} & 0 \\ 0 & \omega^I{}_J \end{pmatrix} \tag{78}$$

we have:

$$\Omega^{(H)} = \mathrm{i}[f^\dagger(\nabla h + h\omega) - h^\dagger(\nabla f + f\omega)] = \omega \mathbb{1} \tag{79}$$

where we have used:

$$\nabla h = \overline{\mathcal{N}} \nabla f; \quad h = \mathcal{N} f \tag{80}$$

and the fundamental identity (64). Furthermore, using the same relations, the embedded vielbein $\mathcal{P}$ can be written as follows:

$$\mathcal{P} = -\mathrm{i}(f^t \nabla h - h^t \nabla f) = \mathrm{i} f^t (\mathcal{N} - \overline{\mathcal{N}}) \nabla f \tag{81}$$

¿From (60) and (77), we obtain the $(n_V \times n_V)$ matrix equation:

$$\begin{aligned} \nabla(\omega)(f + \mathrm{i}h) &= (\overline{f} + \mathrm{i}\overline{h})\mathcal{P} \\ \nabla(\omega)(f - \mathrm{i}h) &= (\overline{f} - \mathrm{i}\overline{h})\mathcal{P} \end{aligned} \tag{82}$$

together with their complex conjugates. Using further the definition (65) we have:

$$\begin{aligned} \nabla(\omega) f^\Lambda_{AB} &= \overline{f}^\Lambda_I P^I_{AB} + \frac{1}{2} \overline{f}^{\Lambda CD} P_{ABCD} \\ \nabla(\omega) f^\Lambda_I &= \frac{1}{2} \overline{f}^{\Lambda AB} P_{ABI} + \overline{f}^{\Lambda J} P_{JI} \end{aligned} \tag{83}$$

where we have decomposed the embedded vielbein $\mathcal{P}$ as follows:

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{ICD} & P_{IJ} \end{pmatrix} \tag{84}$$

the subblocks being related to the vielbein of U/H, $P = L^{-1} \nabla^{(H)} L$, written in terms of the indices of $H_{Aut} \times H_{matter}$. In particular, the component P_{ABCD} is completely antisymmetric in its indices. Note that, since f belongs to the unitary matrix U, we have: $(f^\Lambda_{AB}, f^\Lambda_I)^\star = (\overline{f}^{\Lambda AB}, \overline{f}^{\Lambda I})$. Obviously, the same differential relations that we wrote for f hold true for the dual matrix h as well.

Using the definition of the charges (73), (74) we then get the following differential relations among charges:

$$\begin{aligned} \nabla(\omega) Z_{AB} &= \overline{Z}_I P^I_{AB} + \frac{1}{2} \overline{Z}^{CD} P_{ABCD} \\ \nabla(\omega) Z_I &= \frac{1}{2} \overline{Z}^{AB} P_{ABI} + \overline{Z}_J P^J_I \end{aligned} \tag{85}$$

Depending on the coset manifold, some of the subblocks of (84) can be actually zero. For example in $N = 3$ the vielbein of $\mathrm{U}/H = \frac{SU(3,n)}{SU(3) \times SU(n) \times U(1)}$ [21] is P_{IAB} (AB antisymmetric), $I = 1, \cdots, n; A, B = 1, 2, 3$ and it turns out that $P_{ABCD} = P_{IJ} = 0$.

In $N = 4$, $\mathrm{U}/H = \frac{SU(1,1)}{U(1)} \times \frac{O(6,n)}{O(6) \times O(n)}$ [22], and we have $P_{ABCD} = \epsilon_{ABCD} P$, $P_{IJ} = \overline{P} \delta_{IJ}$, where P is the Kählerian vielbein of $\frac{SU(1,1)}{U(1)}$, ($A, \cdots, D$ $SU(4)$ indices and I, J $O(n)$ indices) and P_{IAB} is the vielbein of $\frac{O(6,n)}{O(6) \times O(n)}$.

For $N > 4$ (no matter indices) we have that $\mathcal{P}$ coincides with the vielbein P_{ABCD} of the relevant U/H.

For the purpose of comparison of the previous formalism with the $N = 2$ supergravity case, where the σ-model is in general not a coset, it is interesting to note that, if the connection $\Omega^{(H)}$ and the vielbein $\mathcal{P}$ are regarded as data of U/H, then the Maurer–Cartan equations (83) can be interpreted as an integrable system of differential equations for a section $V = (V_{AB}, V_I, \overline{V}^{AB}, \overline{V}^I)$ of the symplectic fiber bundle constructed over U/H. Namely the integrable system:

$$\nabla \begin{pmatrix} V_{AB} \\ V_I \\ \overline{V}^{AB} \\ \overline{V}^I \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}P_{ABCD} & P_{ABJ} \\ 0 & 0 & \frac{1}{2}P_{ICD} & P_{IJ} \\ \frac{1}{2}P^{ABCD} & P^{ABJ} & 0 & 0 \\ \frac{1}{2}P^{ICD} & P^{IJ} & 0 & 0 \end{pmatrix} \begin{pmatrix} V_{CD} \\ V_J \\ \overline{V}^{CD} \\ \overline{V}^J \end{pmatrix} \tag{86}$$

has $2n$ solutions given by $V = (f^\Lambda{}_{AB}, f^\Lambda_I), (h_{\Lambda AB}, h_{\Lambda_I})$, $\Lambda = 1, \cdots, n$. The integrability condition (76) means that Γ is a flat connection of the symplectic bundle. In terms of the geometry of U/H this in turn implies that the $\mathbb{H}$–curvature (and hence, since the manifold is a symmetric space, also the Riemannian curvature) is constant, being proportional to the wedge product of two vielbein.

Besides the differential relations (85), the charges also satisfy sum rules.

The sum rule has the following form:

$$\frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{87}$$

where $\mathcal{M}(\mathcal{N})$ and P are:

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & -Re\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -Re\mathcal{N} & \mathbb{1} \end{pmatrix} \tag{88}$$

$$P = \begin{pmatrix} g^\Lambda \\ e_\Lambda \end{pmatrix} \tag{89}$$

In order to obtain this result we just need to observe that from the fundamental identities (64) and from the definition of the kinetic matrix given in (66) it follows:

$$ff^\dagger = -\mathrm{i}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1} \tag{90}$$

$$hh^\dagger = -\mathrm{i}\left(\overline{\mathcal{N}}^{-1} - \mathcal{N}^{-1}\right)^{-1} \equiv -\mathrm{i}\mathcal{N}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1}\overline{\mathcal{N}} \tag{91}$$

$$hf^\dagger = \mathcal{N} ff^\dagger \tag{92}$$

$$fh^\dagger = ff^\dagger \overline{\mathcal{N}}. \tag{93}$$

3.2.1 The $N=2$ theory

The formalism we have developed so far for the $D=4$, $N>2$ theories is completely determined by the embedding of the coset representative of U/H in $Sp(2n,\mathbb{R})$ and by the embedded Maurer–Cartan equations (83). We want now to show that this formalism, and in particular the identities (64), the differential relations among charges (85) and the sum rules (87) of $N=2$ matter-coupled supergravity [23],[24] can be obtained in a way completely analogous to the coset space σ-model cases discussed in the previous subsection. This follows essentially from the fact that, though the scalar manifold $\mathcal{M}_{N=2}$ of the $N=2$ theory is not in general a coset manifold, nevertheless it has a symplectic structure identical to the $N>2$ theories, as a consequence of the Gaillard–Zumino duality.

In the case of $N=2$ supergravity the requirements imposed by supersymmetry on the scalar manifold $\mathcal{M}_{scalar}$ of the theory is that it should be the following direct product: $\mathcal{M}_{scalar} = \mathcal{M}^{SK} \otimes \mathcal{M}^{Q}$ where $\mathcal{M}^{SK}$ is a *special Kähler* manifold of complex dimension n and $\mathcal{M}^{Q}$ a *quaternionic* manifold of real dimension $4n_H$. Note that n and n_H are respectively the number of *vector multiplets* and *hypermultiplets* contained in the theory. The direct product structure imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets. In the construction of BPS black–holes it turns out that the hyperscalars are spectators playing no dynamical role. Hence we do not discuss here the hypermultiplets any further and we confine our attention to an $N=2$ supergravity where the graviton multiplet, containing, besides the graviton $g_{\mu\nu}$, also a graviphoton A^0_μ, is coupled to n *vector multiplets*. Such a theory has an action of type (25) where the number of gauge fields is $n_V = 1+n$ and the number of scalar fields is $m = 2\,n$. Correspondingly the indices have the following ranges

$$\begin{aligned} \Lambda, \Sigma, \Gamma, \ldots &= 0, 1, \ldots, n \\ I, J, K, \ldots &= 1, \ldots, 2\,n \end{aligned} \tag{94}$$

To make the action (25) fully explicit, we need to discuss the geometry of the vector multiplets scalars, namely special Kähler geometry. We refer to [25] for a detailed analysis. A special Kähler manifold is a Kähler-Hodge manifold endowed with an extra symplectic structure. A Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a $U(1)$ bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form K:

$$c_1(\mathcal{L}) = [\,K\,] \tag{95}$$

In local terms this means that there is a holomorphic section $W(z)$ such that we can write

$$K = \mathrm{i}\, g_{i\overline{\jmath}}\, dz^i \wedge d\overline{z}^{j^\star} \tag{96}$$

where z^i are n holomorphic coordinates on $\mathcal{M}^{SK}$ and $g_{i\bar{\jmath}}$ its metric.
In this case the $U(1)$ Kähler connection is given by

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\partial_i \mathcal{K} dz^i - \partial_{\bar{\imath}} \mathcal{K} d\bar{z}^{\bar{\imath}}\right) \tag{97}$$

where $\mathcal{K}$ is the Kähler potential, so that $K = d\mathcal{Q}$.

Let now $\Phi(z, \bar{z})$ be a section of the $U(1)$ bundle. By definition its covariant derivative is

$$\nabla \Phi = (d + ip\mathcal{Q})\Phi \tag{98}$$

or, in components,

$$\nabla_i \Phi \;=\; (\partial_i + \tfrac{1}{2} p \partial_i \mathcal{K})\Phi \quad ; \quad \nabla_{\bar{\imath}} \Phi \;=\; (\partial_{\bar{\imath}} - \tfrac{1}{2} p \partial_{\bar{\imath}} \mathcal{K})\Phi \tag{99}$$

A covariantly holomorphic section is defined by the equation: $\nabla_{\bar{\imath}} \Phi = 0$. Setting:

$$\widetilde{\Phi} = e^{-p\mathcal{K}/2}\Phi\,. \tag{100}$$

we get:

$$\nabla_i \widetilde{\Phi} \;=\; (\partial_i + p\partial_i \mathcal{K})\widetilde{\Phi} \quad ; \quad \nabla_{i^*} \widetilde{\Phi} \;=\; \partial_{i^*} \widetilde{\Phi} \tag{101}$$

so that under this map covariantly holomorphic sections Φ become truly holomorphic sections.

There are several equivalent ways of defining what a special Kähler manifold is. An intrinsic definition is the following. A special Kähler manifold can be given by constructing a $2n+2$-dimensional symplectic bundle over the Kähler–Hodge manifold whose generic sections (with weight $p = 1$)

$$V = (f^\Lambda, h_\Lambda) \qquad \Lambda = 0, \ldots, n\ , \tag{102}$$

are covariantly holomorphic

$$\nabla_{\bar{\imath}} V = (\partial_{\bar{\imath}} - \frac{1}{2}\partial_{\bar{\imath}}\mathcal{K})V = 0 \tag{103}$$

and satisfy the further condition

$$i < V, \overline{V} >= i(\overline{f}^\Lambda h_\Lambda - \overline{h}_\Lambda f^\Lambda) = 1\ , \tag{104}$$

where $<\ ,\ >$ denotes a symplectic inner product with metric chosen to be $\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

Defining $U_i = D_i V = (f^\Lambda_i, h_{\Lambda i})$, and introducing a symmetric three-tensor C_{ijk} by

$$D_i U_j = i C_{ijk} g^{k\bar{k}} \overline{U}_{\bar{k}}\ , \tag{105}$$

the set of differential equations

$$\begin{aligned} D_i V &= U_i \\ D_i U_j &= i C_{ijk} g^{k\bar{k}} \overline{U}_{\bar{k}} \\ D_i U_{\bar{\jmath}} &= g_{i\bar{\jmath}} \overline{V} \\ D_i \overline{V} &= 0 \end{aligned} \tag{106}$$

defines a symplectic connection. Requiring that the differential system (106) is integrable is equivalent to require that the symplectic connection is flat. Since the integrability condition of (106) gives constraints on the base Kähler–Hodge manifold, we define special-Kähler a manifold whose associated symplectic connection is flat. At the end of this section we will give the restrictions on the manifold imposed by the flatness of the connection.

It must be noted that, for special Kähler manifolds, the Kähler potential can be computed as a symplectic invariant from eq. (104). Indeed, introducing also the holomorphic sections

$$\begin{aligned} \Omega &= e^{-\mathcal{K}/2} V = e^{-\mathcal{K}/2} (f^\Lambda, h_\Lambda) = (X^\Lambda, F_\Lambda) \\ \partial_{\bar{\imath}} \Omega &= 0 \end{aligned} \tag{107}$$

eq. (104) gives

$$\mathcal{K} = -\ln \mathrm{i} < \Omega, \overline{\Omega} >= -\ln i (\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda) \ . \tag{108}$$

If we introduce the complex symmetric $(n+1)\times(n+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}$ defined through the relations

$$h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma \quad , \quad h_{i^\star\Lambda} = \mathcal{N}_{\Lambda\Sigma} f^\Sigma_{i^\star} \ , \tag{109}$$

then we have:

$$< V, \overline{V} > \; = \; \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma} f^\Lambda \overline{f}^\Sigma = -\mathrm{i} \quad \rightarrow \quad \mathcal{K} = -\ln[\mathrm{i}(\overline{X}^\Lambda \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma} X^\Sigma)] \tag{110}$$

$$g_{i\bar{\jmath}} = -i < U_i, U_{\bar{\jmath}} >= -2 f^\Lambda_i \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f^\Sigma_{\bar{\jmath}} \ , \tag{111}$$

$$C_{ijk} = < D_i U_j, U_k >= 2\mathrm{i}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f^\Lambda_i \nabla_j f^\Sigma_k \ . \tag{112}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}$ turns out to be the matrix appearing in the kinetic lagrangian of the vectors in $N=2$ supergravity. Under coordinate transformations, the sections Ω transform as

$$\widetilde{\Omega} = e^{-f_S(z)} \mathcal{S} \Omega \ , \tag{113}$$

where $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp(2n_V, \mathrm{I\!R})$ and the factor $e^{-f_S(z)}$ is a $U(1)$ Kähler transformation. We also note that, from the definition of $\mathcal{N}$, eq. (109):

$$\widetilde{\mathcal{N}}(\widetilde{X}, \widetilde{F}) = (C + D\mathcal{N}(X,F))(A + B\mathcal{N}(X,F))^{-1} \ , \tag{114}$$

Let us now set $n_V = n+1$ and define the $n_V \times n_V$ matrices:

$$f^\Lambda_\Sigma \equiv \left(f^\Lambda, \overline{f}^{\Lambda i}\right); \quad h_{\Lambda\Sigma} \equiv \left(h_\Lambda, \overline{h}_\Lambda^{\ i}\right) \tag{115}$$

where $\overline{f}^{\Lambda i} \equiv \overline{f}^\Lambda_{\ \bar{\jmath}} g^{\bar{\jmath}i}$, $\overline{h}_\Lambda^{\ i} \equiv \overline{h}_{\Lambda\bar{\jmath}} g^{\bar{\jmath}i}$, the set of algebraic relations of special geometry can be written in matrix form as:

$$\begin{cases} \mathrm{i}(f^\dagger h - h^\dagger f) & = & \mathbb{1} \\ (f^t h - h^t f) & = & 0 \end{cases} \tag{116}$$

Recalling equations (64) we see that the previous relations imply that the matrix U:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix} \tag{117}$$

is a pseudo-unitary symplectic matrix. In fact if we set $f^\Lambda \to f^\Lambda \epsilon_{AB} \equiv f^\Lambda_{AB}$ and flatten the world-indices of $(f^\Lambda_i, \overline{f}^\Lambda_{\bar{\imath}})$ (or $(\overline{h}_i, h_{\bar{\imath}})$) with the Kählerian vielbein $P^I_i, \overline{P}^{\overline{I}}_{\bar{\imath}}$:

$$(f^\Lambda_I, \overline{f}^\Lambda_{\overline{I}}) = (f^\Lambda_i P^i_I, \overline{f}^\Lambda_{\bar{\imath}} \overline{P}^{\bar{\imath}}_{\overline{I}}), \qquad P^I_i \overline{P}^{\overline{J}}_{\bar{\jmath}} \eta_{I\overline{J}} = g_{i\bar{\jmath}} \tag{118}$$

where $\eta_{I\overline{J}}$ is the flat Kählerian metric and $P^i_{\ I} = (P^{-1})^I_{\ i}$, the relations (116) are just a particular case of equations (64) since, for $N=2$, $H_{Aut} = SU(2) \times U(1)$, so that f^Λ_{AB} is actually an $SU(2)$ singlet.

Let us now consider the analogous of the embedded Maurer–Cartan equations of U/H. We introduce, as before, the matrix one–form $\Gamma = U^{-1}dU$ satisfying the relation $d\Gamma + \Gamma \wedge \Gamma = 0$. We further introduce the covariant derivative of the symplectic section $(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I)$ with respect to the $U(1)$–Kähler connection $\mathcal{Q}$ and the spin connection ω^{IJ} of $\mathcal{M}_{N=2}$:

$$\begin{aligned} &\nabla(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I) = \\ &d(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, f^\Lambda, \overline{f}^\Lambda_I) + (f^\Lambda, \overline{f}^\Lambda_{\overline{J}}, \overline{f}^\Lambda, f^\Lambda_J) \begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 & 0 & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^{\overline{I}}_{\overline{J}} + \omega^{\overline{I}}_{\ \overline{J}} & 0 & 0 \\ 0 & 0 & \mathrm{i}\mathcal{Q} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\mathcal{Q}\delta^I_J + \omega^I_{\ J} \end{pmatrix} \end{aligned} \tag{119}$$

the Kähler weight of $(f^\Lambda, \overline{f}^\Lambda_I)$ and $(\overline{f}^\Lambda, \overline{f}^\Lambda_{\overline{I}})$ being $p=1$ and $p=-1$ respectively. Using the same decomposition as in equation (77) and eq.s (78), (79) we have in the $N=2$ case:

$$\begin{aligned} \Gamma &= \begin{pmatrix} \Omega & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega} \end{pmatrix}, \\ \Omega &= \omega = \begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^I_J + \overline{\omega}^I_{\ J} \end{pmatrix} \end{aligned} \tag{120}$$

For the subblock $\mathcal{P}$ we obtain:

$$\mathcal{P} = -\mathrm{i}(f^t \nabla h - h^t \nabla f) = \mathrm{i} f^t (\mathcal{N} - \overline{\mathcal{N}}) \nabla f = \begin{pmatrix} 0 & P_{\overline{I}} \\ P^J & P^J_{\overline{I}} \end{pmatrix} \tag{121}$$

where $\overline{P}^J \equiv \eta^{J\overline{I}} P_{\overline{I}}$ is the $(1,0)$–form Kählerian vielbein while $P^J_{\overline{I}} \equiv \mathrm{i}\left(f^t(\mathcal{N}-\overline{\mathcal{N}})\nabla f\right)^J_{\ \overline{I}}$ is a one–form which in general cannot be expressed in terms of the vielbein P^I, since the manifold is in general not a coset, and therefore represents a new geometrical quantity on $\mathcal{M}_{N=2}$. Note that we get zero in the first entry of equation (121) by virtue of the fact that the identity (116) implies $f^\Lambda(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma} f^\Sigma_i = 0$ and that f^Λ is covariantly holomorphic. If Ω and $\mathcal{P}$ are considered as data on $\mathcal{M}_{N=2}$ then we may interpret $\Gamma = U^{-1}dU$ as an integrable system of differential equations, namely:

$$\nabla(V, \overline{U}_{\overline{I}}, \overline{V}, U_I) = (V, \overline{U}_{\overline{J}}, \overline{V}, U_J) \begin{pmatrix} 0 & 0 & 0 & \overline{P}_I \\ 0 & 0 & \overline{P}^J & \overline{P}^J_{\ I} \\ 0 & P_{\overline{I}} & 0 & 0 \\ P^J & P^J_{\overline{I}} & 0 & 0 \end{pmatrix} \tag{122}$$

where the flat Kähler indices $I, \overline{I}, \cdots$ are raised and lowered with the flat Kähler metric $\eta_{I\overline{J}}$. Note that the equation (122) coincides with the set of relations (106) if we trade world indices $i, \overline{\imath}$ with flat indices $I, \overline{I}$, provided we also identify:

$$\overline{P}^{\overline{J}}_{\ I} = \overline{P}^{\overline{J}}_{\ Ik} dz^k = P^{\overline{J},i} P_I^{\ j} C_{ijk} dz^k. \tag{123}$$

Then, the integrability condition $d\Gamma + \Gamma \wedge \Gamma = 0$ is equivalent to the flatness of the special Kähler symplectic connection and it gives the following three constraints on the Kähler base manifold:

$$\begin{aligned} d(\mathrm{i}\mathcal{Q}) + \overline{P}_I \wedge P^I &= 0 \to \partial_{\overline{\jmath}} \partial_i \mathcal{K} = P^I_{\ ,i} \overline{P}_{I,\overline{\jmath}} = g_{i\overline{\jmath}} && (124) \\ (d\omega + \omega \wedge \omega)^{\overline{J}}_{\ \overline{I}} &= P_{\overline{I}} \wedge \overline{P}^{\overline{J}} - \mathrm{i} d\mathcal{Q} \delta^{\overline{J}}_{\overline{I}} - \overline{P}^{\overline{J}}_{\ L} \wedge P^L_{\overline{I}} && (125) \\ \nabla P^J_{\overline{I}} &= 0 && (126) \\ \overline{P}_J \wedge P^J_{\overline{I}} = 0 & && (127) \end{aligned}$$

Equation (124) implies that $\mathcal{M}_{N=2}$ is a Kähler–Hodge manifold. Equation (125), written with holomorphic and antiholomorphic curved indices, gives:

$$R_{\overline{\imath} j \overline{k} l} = g_{\overline{\imath} l} g_{j\overline{k}} + g_{\overline{k} l} g_{\overline{\imath} j} - \overline{C}_{\overline{\imath}\overline{k}\overline{m}} C_{jln} g^{\overline{m}n} \tag{128}$$

which is the usual constraint on the Riemann tensor of the special geometry. The further special geometry constraints on the three tensor C_{ijk} are then consequences of equations (126), (127), which imply:

$$\begin{aligned} \nabla_{[l} C_{i]jk} &= 0 \\ \nabla_{\bar{l}} C_{ijk} &= 0 \end{aligned} \tag{129}$$

In particular, the first of eq. (129) also implies that C_{ijk} is a completely symmetric tensor.

In summary, we have seen that the $N = 2$ theory and the higher N theories have essentially the same symplectic structure, the only difference being that since the scalar manifold of $N = 2$ is not in general a coset manifold the symplectic structure allows the presence of a new geometrical quantity which physically corresponds to the anomalous magnetic moments of the $N = 2$ theory. It goes without saying that, when $\mathcal{M}_{N=2}$ is itself a coset manifold [26], then the anomalous magnetic moments C_{ijk} must be expressible in terms of the vielbein of U/H.

To complete the analogy between the $N = 2$ theory and the higher N theories in $D = 4$, we also give for completeness the supersymmetry transformation laws, the central and matter charges, the differential relations among them and the sum rules.

The transformation laws for the chiral gravitino ψ_A and gaugino λ^{iA} fields are:

$$\delta\psi_{A\mu} = D_\mu \epsilon_A + \epsilon_{AB} T_{\mu\nu} \gamma^\nu \epsilon^B + \cdots \tag{130}$$

$$\delta\lambda^{iA} = \mathrm{i}\partial_\mu z^i \gamma^\mu \epsilon^A + \frac{\mathrm{i}}{2} T_{\bar{\jmath}\mu\nu} \gamma^{\mu\nu} g^{i\bar{\jmath}} \epsilon^{AB} \epsilon_B + \cdots \tag{131}$$

where:

$$T \equiv h_\Lambda F^\Lambda - f^\Lambda \mathcal{G}_\Lambda \tag{132}$$

$$T_{\bar{\imath}} \equiv \overline{h}_{\Lambda\bar{\imath}} F^\Lambda - \overline{f}^\Lambda_{\bar{\imath}} \mathcal{G}_\Lambda \tag{133}$$

are respectively the graviphoton and the matter-vectors, z^i $(i = 1, \cdots, n)$ are the complex scalar fields and the position of the $SU(2)$ automorphism index A (A,B=1,2) is related to chirality (namely (ψ_A, λ^{iA}) are chiral, $(\psi^A, \lambda^{\bar{\imath}}_A)$ antichiral). In principle only the (anti) self dual part of F and $\mathcal{G}$ should appear in the transformation laws of the (anti)chiral fermi fields; however, exactly as in eqs. (71),(72) for $N > 2$ theories, from equations (106) it follows that :

$$T^+ = h_\Lambda F^{+\Lambda} - f^\Lambda \mathcal{G}^+_\Lambda = 0 \tag{134}$$

so that $T = T^-$ (and $\overline{T} = \overline{T}^+$). Note that both the graviphoton and the matter vectors are symplectic invariant according to the fact that the fermions do not transform under the duality group (except for a possible R-symmetry phase). To define the physical charges let us note that in presence of electric and magnetic sources we can write:

$$\int_{S^2} F^\Lambda = g^\Lambda, \quad \int_{S^2} \mathcal{G}_\Lambda = e_\Lambda. \tag{135}$$

The central charges and the matter charges are now defined as the integrals over a S^2 of the physical graviphoton and matter vectors:

$$Z = \int_{S^2} T = \int_{S^2} (h_\Lambda F^\Lambda - f^\Lambda \mathcal{G}_\Lambda) = (h_\Lambda(z,\overline{z}) g^\Lambda - f^\Lambda(z,\overline{z}) e_\Lambda) \tag{136}$$

where $z^i, \overline{z}^{\overline{\imath}}$ denote the v.e.v. of the moduli fields in a given background. Owing to eq. (106) we get immediately:

$$Z_i = \nabla_i Z \tag{137}$$

As a consequence of the symplectic structure, one can derive two sum rules for Z and Z_i:

$$|Z|^2 + |Z_i|^2 \equiv |Z|^2 + Z_i g^{i\overline{\jmath}} \overline{Z}_{\overline{\jmath}} = -\frac{1}{2} P^t \mathcal{M} P \tag{138}$$

where:

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & -Re\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -Re\mathcal{N} & \mathbb{1} \end{pmatrix} \tag{139}$$

and:

$$P = \left(g^\Lambda, e_\Lambda\right) \tag{140}$$

Equation (139) is obtained by using exactly the same procedure as in (88).

4 $N = 2$ BPS black holes: general discussion

Recently [27],[28], S. Ferrara and R. Kallosh gave a general rule for finding the values of fixed scalars, and then the Bekenstein–Hawking entropy, in $N = 2$ theories through an *extremum principle*. It states that the fixed scalars Φ_{fix} are the ones (among all the possible values taken by scalar fields) that extremize the ADM mass of the black hole in moduli space:

$$\Phi_{fix} \ : \quad \frac{\partial M_{ADM}(\Phi)}{\partial \Phi_i} |_{\Phi_{fix}} = 0 \tag{141}$$

Correspondingly, the Bekenstein–Hawking entropy is given in terms of that extremum among the possible ADM masses (given by all possible boundary conditions that one can impose on scalars at spatial infinity), identified with the Bertotti–Robinson mass M_{BR}:

$$M_{BR} \equiv M_{ADM}(\Phi_{fix}) = \text{extr}\{M_{ADM}(\Phi(\infty))\} \tag{142}$$

The *extremum principle* (141) can be explained for the $N = 2$ theory [27],[28] by means of the special geometry relations on the Killing spinor equations near the horizon.

Indeed, the Killing spinor equations expressing the existence of unbroken supersymmetries is obtained, for the gauginos in the $N=2$ case [25], setting equal to zero the r.h.s. of equation (131) that is, using flat indices:

$$\delta\lambda_{IA} = P_{I,i}\partial_\mu z^i \gamma^\mu \varepsilon_{AB}\epsilon^B + T_{I|\mu\nu}\gamma^{\mu\nu}\epsilon_A + \cdots = 0. \tag{143}$$

Approaching the black-hole horizon, the scalars z^i reach their fixed values so that

$$\partial_\mu z^i = 0 \tag{144}$$

and equation (143) is satisfied for

$$T_I = 0 \tag{145}$$

that is, using integrated quantities:

$$Z_I = Z_i P^i_I = \int_{S^2_\infty} T_I = h_{\Lambda I}(z(\infty))g^\Lambda - f^\Lambda_I(z(\infty))e_\Lambda = 0. \tag{146}$$

That is, the Killing spinor equation imposes the vanishing of the matter charges near the horizon. Now, eq. (146) shows that the matter charges Z_I are linear in the scalar functions $f^\Lambda_I(z(\infty))$, $h_{\Lambda I}(z(\infty))$ so that, remembering eq. (137), we then have, near the horizon:

$$Z_I = \nabla_I Z = 0 \tag{147}$$

where Z is the central charge appearing in the $N=2$ supersymmetry algebra, so that:

$$\partial_i |Z| = 0 \tag{148}$$

which, for an extremal black hole ($|Z| = M_{ADM}$), coincides with eq. (141) giving the fixed scalars $\Phi_{fix} \equiv z_{fix}$. We see that the entropy of the black–hole is related to the central charge, namely to the integral of the graviphoton field strength evaluated for very special values of the scalar fields z^i. These special values, the *fixed scalars* z^i_{fix}, are functions solely of the electric and magnetic charges $\{q_\Sigma, p^\Lambda\}$ of the black hole and are attained by the scalars $z^i(r)$ at the black-hole horizon $r=0$.

Let us discuss the explicit solution of the Killing spinor equation and the general properties of BPS saturated black–holes in the context of $N=2$ supergravity. As our analysis will reveal, these properties are completely encoded in the special Kähler geometric structure of the mother theory. To illustrate in more detail what happens, let us consider a black-hole ansatz[3] for the metric:

$$ds^2 = e^{2U(r)}\,dt^2 - e^{-2U(r)}\,d\vec{x}^2 \quad ; \quad \left(r^2 = \vec{x}^2\right) \tag{149}$$

[3]This ansatz is dictated by the general p-brane solution of supergravity bosonic equations in any dimensions [8].

and for the vector field strengths:

$$F^{\Lambda} = \frac{p^{\Lambda}}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \frac{\ell^{\Lambda}(r)}{r^3}e^{2\mathcal{U}}dt \wedge \vec{x}\cdot d\vec{x}. \tag{150}$$

It is convenient to rephrase the same ansatz in the complex formalism well-adapted to the $N = 2$ theory. To this effect we begin by constructing a 2–form which is *anti–self–dual* in the background of the metric (149) and whose integral on the 2–sphere at infinity S^2_∞ is normalized to 2π. A short calculation yields:

$$\begin{aligned} E^- &= \mathrm{i}\frac{e^{2U(r)}}{r^3}\,dt \wedge \vec{x}\cdot d\vec{x} + \frac{1}{2}\frac{x^a}{r^3}\,dx^b \wedge dx^c \epsilon_{abc} \\ 2\,\pi &= \int_{S^2_\infty} E^- \end{aligned} \tag{151}$$

from which one obtains:

$$E^-_{\mu\nu}\,\gamma^{\mu\nu} = 2\,\mathrm{i}\frac{e^{2U(r)}}{r^3}\,\gamma_a x^a\,\gamma_0\,\frac{1}{2}\,[\mathbf{1} + \gamma_5] \tag{152}$$

which will prove of great help in the unfolding of the supersymmetry transformation rules.

Next, introducing the following complex combination of the magnetic charge p^{Λ} and of the radial function $\ell^{\Sigma}(r)$ defined by eq. (150):

$$t^{\Lambda}(r) = 2\pi(p^{\Lambda} + \mathrm{i}\ell^{\Lambda}(r)) \tag{153}$$

we can rewrite the ansatz (150) as:

$$F^{-|\Lambda} = \frac{t^{\Lambda}}{4\pi}E^- \tag{154}$$

and we retrieve the original formulae from:

$$\begin{aligned} F^{\Lambda} &= 2\mathrm{Re}F^{-|\Lambda} = \tfrac{p^{\Lambda}}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \tfrac{\ell^{\Lambda}(r)}{r^3}e^{2\mathcal{U}}dt \wedge \vec{x}\cdot d\vec{x} \\ \widetilde{F}^{\Lambda} &= -2\mathrm{Im}F^{-|\Lambda} = -\tfrac{\ell^{\Lambda}(r)}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \tfrac{p^{\Lambda}}{r^3}e^{2\mathcal{U}}dt \wedge \vec{x}\cdot d\vec{x}. \end{aligned} \tag{155}$$

Before proceeding further it is convenient to define the electric and magnetic charges of the black hole as it is appropriate in any electromagnetic theory. Recalling the general form of the field equations and of the Bianchi identities as given in (35) we see that the field strengths $F_{\mu\nu}$ and $G_{\mu\nu}$ are both closed 2-forms, since their duals are divergenceless. Hence, for Gauss theorem, their integral on a closed space–like 2–sphere does not depend on the radius of the sphere. These integrals are the electric and magnetic charges of the hole that, in a quantum theory, we expect to be quantized. We set:

$$q_\Lambda \equiv \frac{1}{4\pi}\int_{S^2_\infty} G_{\Lambda|\mu\nu}\, dx^\mu \wedge dx^\nu \tag{156}$$
$$p^\Sigma \equiv \frac{1}{4\pi}\int_{S^2_\infty} F^\Sigma_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{157}$$

If rather than the integral of G_Λ we were to calculate the integral of $\widetilde{F}^\Lambda$, which is not a closed form, we would obtain a function of the radius:

$$4\pi\ell^\Lambda(r) = -\int_{S^2_r} \widetilde{F}^\Lambda = 2\mathrm{Im}\, t^\Lambda. \tag{158}$$

Consider now the Killing spinor equations obtained from the supersymmetry transformations rules (130), (131):

$$0 = \nabla_\mu \xi_A + \epsilon_{AB}\, T^-_{\mu\nu}\, \gamma^\nu \xi^B \tag{159}$$
$$0 = \mathrm{i}\,\nabla_\mu\, z^i\, \gamma^\mu \xi^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\xi_B\epsilon^{AB} \tag{160}$$

where the killing spinor $\xi_A(r)$ is of the form of a single radial function times a constant spinor satisfying:

$$\begin{aligned} \xi_A(r) &= e^{f(r)}\chi_A \qquad\qquad \chi_A = \text{constant} \\ \gamma_0\chi_A &= \pm\mathrm{i}\epsilon_{AB}\chi^B \end{aligned} \tag{161}$$

We observe that the condition (161) halves the number of supercharges preserved by the solution. Inserting eq.s (132),(133),(161) into eq.s(159), (160) and using the result (152), with a little work we obtain the first order differential equations:

$$\begin{aligned} \frac{dz^i}{dr} &= \mp\left(\frac{e^{U(r)}}{4\pi r^2}\right) g^{ij^\star}\overline{f}^\Lambda_{j^\star}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}t^\Sigma = \\ &\mp\left(\frac{e^{U(r)}}{4\pi r^2}\right) g^{ij^\star}\nabla_{j^\star}\overline{Z}(z,\overline{z},p,q) \end{aligned} \tag{162}$$

$$\frac{dU}{dr} = \mp\left(\frac{e^{U(r)}}{r^2}\right)(M_\Sigma p^\Sigma - L^\Lambda q_\Lambda) = \mp\left(\frac{e^{U(r)}}{r^2}\right) Z(z,\overline{z},p,q) \tag{163}$$

where $\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})$ is the kinetic matrix of special geometry defined by eq.(66), the vector $V = \big(L^\Lambda(z,\overline{z}), M_\Sigma(z,\overline{z})\big)$ is the covariantly holomorphic section of the symplectic bundle entering the definition of a Special Kähler manifold,

$$Z(z,\overline{z},p,q) \equiv \big(M_\Sigma p^\Sigma - L^\Lambda q_\Lambda\big) \tag{164}$$

is the local realization on the scalar manifold $\mathcal{SM}$ of the central charge of the $N = 2$ superalgebra,

$$Z^i(z, \overline{z}, p, q) \equiv g^{ij^\star} \nabla_{j^\star} \overline{Z}(z, \overline{z}, p, q) \tag{165}$$

are the central charges associated with the matter vectors, the so–called matter central charges.

To obtain eqs. (162),(163) we made use of the properties :

$$\begin{aligned} 0 &= \overline{h}_{j^\star|\Lambda} t^{\star\Sigma} - \overline{f}^{\Lambda}_{j^\star} \mathcal{N}_{\Lambda\Sigma} t^{\star\Sigma} \\ 0 &= M_\Sigma t^{\star\Sigma} - L^\Lambda \mathcal{N}_{\Lambda\Sigma} t^{\star\Sigma} \end{aligned} \tag{166}$$

which are a direct consequence of the definition (66) of the kinetic matrix. The electric charges $\ell^\Lambda(r)$ defined in (158) are *moduli dependent* charges which are functions of the radial direction through the moduli z^i. On the other hand, the *moduli independent* electric charges q_Λ in eqs. (163),(162) are those defined by eq.(156) which, together with p^Λ fulfil a Dirac quantization condition. Their definition allows them to be expressed in terms of of $t^\Lambda(r)$ as follows:

$$q_\Lambda = \frac{1}{2\pi} \mathrm{Re}(\mathcal{N}(z(r), \overline{z}(r)) t(r))_\Lambda \tag{167}$$

Equation (167) may be inverted in order to find the moduli dependence of $\ell_\Lambda(r)$. The independence of q_Λ on r is a consequence of one of the Maxwell's equations:

$$\partial_a \left(\sqrt{-g} \widetilde{G}^{a0|\Lambda}(r) \right) = 0 \Rightarrow \partial_r \mathrm{Re}(\mathcal{N}(z(r), \overline{z}(r)) t(r))^\Lambda = 0 \tag{168}$$

In this way we have reduced the condition that the black-hole should be a BPS saturated state to a pair of first order differential equations for the metric scale factor $U(r)$ and for the scalar fields $z^i(r)$. To obtain explicit solutions one should specify the special Kähler manifold one is working with, namely the specific Lagrangian model. There are, however, some very general and interesting conclusions that can be drawn in a model–independent way. They are just consequences of the fact that these BPS conditions are *first order differential equations*. Because of that there are fixed points (see the papers [29, 27, 30]) namely values either of the metric or of the scalar fields which, once attained in the evolution parameter r (= the radial distance) persist indefinitely. The fixed point values are just the zeros of the right hand side in either of the coupled eq.s (163) and (162). The fixed point for the metric equation is $r = \infty$, which corresponds to its asymptotic flatness. The fixed point for the moduli is $r = 0$. So, independently from the initial data at $r = \infty$ that determine the details of the evolution, the scalar fields flow into their fixed point values at $r = 0$, which, as we will show, turns out to be a horizon. Indeed in the vicinity of $r = 0$ also the metric takes the universal form of an $AdS_2 \times S^2$, Bertotti Robinson metric.

Let us see this more closely.

To begin with we consider the equations determining the fixed point values for the moduli and the universal form attained by the metric at the moduli fixed point:

$$
\begin{aligned}
0 &= -g^{ij^\star}\,\overline{f}^{\Gamma}_{j^\star}\,(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\;t^{\Lambda}(0) && (169)\\
\frac{dU}{dr} &\cong \mp\left(\frac{e^{U(r)}}{r^2}\right)Z\left(z_{fix},\overline{z}_{fix},p,q\right) && (170)
\end{aligned}
$$

Multiplying eq.(169) by f_i^{Σ} using the identity (111) and the definition (164) of the central charge we conclude that at the fixed point the following condition is true:

$$
0 = -\frac{1}{2}\,\frac{t^{\Lambda}}{4\pi} - \frac{Z_{fix}\,\overline{L}^{\Lambda}_{fix}}{8\pi} \tag{171}
$$

In terms of the previously defined electric and magnetic charges (see eq.s (156),(157), (167)) eq.(171) can be rewritten as:

$$
\begin{aligned}
p^{\Lambda} &= \mathrm{i}\left(Z_{fix}\,\overline{L}^{\Lambda}_{fix} - \overline{Z}_{fix}\,L^{\Lambda}_{fix}\right) && (172)\\
q_{\Sigma} &= \mathrm{i}\left(Z_{fix}\,\overline{M}^{fix}_{\Sigma} - \overline{Z}_{fix}\,M^{fix}_{\Sigma}\right) && (173)\\
Z_{fix} &= M^{fix}_{\Sigma}\,p^{\Sigma} - L^{\Lambda}_{fix}\,q_{\Lambda} && (174)
\end{aligned}
$$

which can be regarded as algebraic equations determining the value of the scalar fields at the fixed point as functions of the electric and magnetic charges p^{Λ}, q_{Σ}:

$$
L^{\Lambda}_{fix} = L^{\Lambda}(p,q) \;\longrightarrow\; Z_{fix} = Z(p,q) = \mathrm{const} \tag{175}
$$

In the vicinity of the fixed point the differential equation for the metric becomes:

$$
\pm\frac{dU}{dr} = \frac{Z(p,q)}{r^2}\,e^{U(r)} \tag{176}
$$

which has the approximate solution:

$$
\exp[U(r)] \stackrel{r\to 0}{\longrightarrow} \mathrm{const} + \frac{Z(p,q)}{r} \tag{177}
$$

Hence, near $r = 0$ the metric (149) becomes of the Bertotti Robinson type (see eq.(8)) with Bertotti Robinson mass given by:

$$
m^2_{BR} = |Z(p,q)|^2 \tag{178}
$$

In the metric (8) the surface $r = 0$ is light–like and corresponds to a horizon since it is the locus where the Killing vector generating time translations $\frac{\partial}{\partial t}$, which is time–like at spatial infinity $r = \infty$, becomes light–like. The horizon $r = 0$ has a finite area given by:

$$
\mathrm{Area}_H = \int_{r=0}\sqrt{g_{\theta\theta}\,g_{\phi\phi}}\,d\theta\,d\phi \;=\; 4\pi\,m^2_{BR} \tag{179}
$$

Hence, independently from the details of the considered model, the BPS saturated black–holes in an N=2 theory have a Bekenstein–Hawking entropy given by the following horizon area:

$$\frac{\mathrm{Area}_H}{4\pi} = |Z(p,q)|^2 \tag{180}$$

the value of the central charge being determined by eq.s (174). Such equations can also be seen as the variational equations for the minimization of the horizon area as given by (180), if the central charge is regarded as a function of both the scalar fields and the charges:

$$\begin{aligned} \mathrm{Area}_H(z,\overline{z}) &= 4\pi\,|Z(z,\overline{z},p,q)|^2 \\ \frac{\delta \mathrm{Area}_H}{\delta z} &= 0 \longrightarrow z = z_{fix} \end{aligned} \tag{181}$$

4.1 Extension to the $N > 2$ case

Let us now observe that the extremum principle just described, although shown to be true for $N = 2$ four dimensional black holes, has however a more general validity, being true for all N-extended supergravities in four dimensions (where the Bekenstein–Hawking entropy for black holes is in general different from zero) [19].

Indeed, the general discussion of section 3.2 has shown that the coset structure of extended supergravities in four dimensions (with $N > 2$) induces the existence, in every theory, of differential relations among central and matter charges that generalize eq. (147) given in equation (85). Furthermore, the Killing spinor equations for gauginos and dilatinos, analogous to eq. (85), are obtained by setting equal to zero the r.h.s of equations (57), (58). Correspondingly, at the fixed point $\partial_\mu \Phi^i = 0$ one gets again some conditions that allow to find the value of fixed scalars and hence of the Bekenstein–Hawking entropy.

The first condition, from the gaugino transformation law, is as before:

$$T_I = 0 \to Z_I = 0 \tag{182}$$

Moreover there is a further condition, from the dilatino equation:

$$T_{[AB}\epsilon_{C]} = 0 \to Z_{[AB}\epsilon_{C]} = 0 \tag{183}$$

Inserting these relations in eq. (85) one has that fixed scalars are found by solving:

$$Z_{[AB}\epsilon_{C]} = 0 \tag{184}$$

$$Z_I = 0 \Rightarrow \nabla Z_{AB} = \frac{1}{2} P_{ABCD}\overline{Z}^{CD} \tag{185}$$

¿From a case by case analysis of equations (184), (185) the explicit form of fixed scalars and then of the entropy is easily obtained for each theory.

Let us now look first at equation (184). We work in the normal frame, where the central charge matrix Z_{AB} is written in terms of its skew diagonal eigenvalues (we analyze the $N = 8$ case, that includes all lower N theories):

$$Z^{(N)}_{AB} = \begin{pmatrix} Z_1\epsilon & 0 & 0 & 0 \\ 0 & Z_2\epsilon & 0 & 0 \\ 0 & 0 & Z_3\epsilon & 0 \\ 0 & 0 & 0 & Z_4\epsilon \end{pmatrix} \quad ; \quad Z_i\epsilon = \begin{pmatrix} 0 & Z_i \\ -Z_i & 0 \end{pmatrix} \tag{186}$$

If only two Killing spinors, say ϵ_1, ϵ_2 are different from zero, then eq. (184) implies that the three central charge eigenvalues Z_2, Z_3, Z_4 must be zero, the only non vanishing eigenvalue being Z_1, and we are left with an $N = 2$ unbroken theory. On the other hand, if one more Killing spinor, say ϵ_3, is different from zero, then from eq. (183) we get that all the central charge eigenvalues are zero, so that this becomes the same Minkowski vacuum background that describes spatial infinity. Let us then consider the case $Z_{12} \neq 0$, $Z_{AB} = 0$ for $A, B \neq 1, 2$. Equation (185) now reduces to $\nabla Z_{12} = 0$, which gives the fixed scalars as an extremum of $(Z_{AB}\overline{Z}^{AB})^{\frac{1}{2}} \equiv |Z_{12}|$.

4.2 The geodesic potential

The results of the previous section can be retrieved in an alternative way, that has the advantage of being covariant, not referring explicitly to the horizon properties for finding the entropy [31],[32]. Let us consider the field equations for the metric components e^U (see eq. (149)) and for the scalar fields Φ^i, written in terms of the evolution parameter $\tau = \frac{1}{\rho} = \frac{1}{r - r_H}$ [4]:

$$\begin{cases} \frac{d^2U}{d\tau^2} = 2V(\Phi, e, g)e^{2U} \\ \frac{D^2\Phi_i}{D\tau^2} = \frac{1}{2}\frac{\partial V(\Phi,e,g)}{\partial \Phi^i}e^{2U} \\ \left(\frac{dU}{d\tau}\right)^2 + G_{ij}\frac{d\Phi^i}{d\tau}\frac{d\Phi^j}{d\tau} - V(\Phi, e, g)e^{2U} = 0 \end{cases} \tag{187}$$

Here G_{ij} is the metric of the sigma-model described by scalars while $V(\Phi, e, g)$ is a function of scalars and of the electric and magnetic charges of the theory defined by:

$$V = -\frac{1}{2}P^t\mathcal{M}(\mathcal{N})P \tag{188}$$

[4] The equations in (187) are valid for extremal black holes. For non extremal ones similar relations hold, where however in the third eq. in (187) there is one further contribution proportional to the surface gravity κ (that is to the black-hole temperature, which is zero only for extremal configurations).

where P is the symplectic vector $P = (g^\Lambda, e_\Lambda)$ of quantized electric and magnetic charges and $\mathcal{M}(\mathcal{N})$ is a symplectic $2n_V \times 2n_V$ matrix, whose $n_V \times n_V$ blocks are given in terms of an $n_V \times n_V$ matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} 1\!\!1 & -Re\mathcal{N} \\ 0 & 1\!\!1 \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} 1\!\!1 & 0 \\ -Re\mathcal{N} & 1\!\!1 \end{pmatrix} \tag{189}$$

The real and imaginary components of the matrix $\mathcal{N}$ appear in the vector kinetic terms of the supergravity lagrangian describing the black hole:

$$\mathcal{L} = -\frac{1}{2}R + \frac{1}{2}G_{ij}\partial_\mu \Phi^i \partial^m u \Phi^j - \frac{1}{4}Re\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\mu\nu|\Sigma} + \frac{1}{4}Im\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{*\mu\nu|\Sigma} + \text{fermions} \tag{190}$$

Let us note that the field equations (187) can be extracted from the effective 1–dimensional lagrangian:

$$\mathcal{L}_{eff} = \left(\frac{dU}{d\tau}\right)^2 + G_{ij}\frac{d\Phi^i}{d\tau}\frac{d\Phi^j}{d\tau} + V(\Phi, e, g)e^{2U}. \tag{191}$$

From equation (191) we see that the properties of extreme black holes are completely encoded in the metric of the scalar manifold G_{ij} and on the scalar effective potential V, known as *geodesic potential* [31],[32]. In particular it was shown in [31],[32] that the area of the event horizon is proportional to the value of V at the horizon:

$$\frac{A}{4\pi} = V(\Phi_h, e, g) \tag{192}$$

where Φ_h denotes the value taken by scalar fields at the horizon. To see this, let us consider the set of equations (187): it is possible to show that the field equations for the scalars give, near the horizon, the solution:

$$\Phi^i = \left(\frac{2\pi}{A}\right)\frac{\partial V}{\partial \Phi^i}\log\tau + \Phi^i_h. \tag{193}$$

From eq. (193) we see that the request that the horizon is a fixed point ($\frac{d\Phi^i}{d\tau} = 0$) implies that the geodesic potential is extremized in moduli space:

$$\Phi_h \ : \quad \frac{d\Phi^i}{d\tau} = 0 \quad \leftrightarrow \quad \frac{\partial V}{\partial \Phi^i}|_{\Phi_h} = 0. \tag{194}$$

Furthermore, let us consider the third of (187). Near the horizon, introducing the results (194), it becomes:

$$\left(\frac{dU}{d\tau}\right)^2 \sim V(\Phi_h(e, g), e, g)e^{2U} \tag{195}$$

from which it follows, for the metric components near the horizon:

$$e^{2U} \sim \frac{1}{\tau^2 V(\Phi_h)} = \frac{\rho^2}{V(\Phi_h)}, \tag{196}$$

that is:

$$ds^2_{hor} = \frac{\rho^2}{V(\Phi_h)} dt^2 - \frac{V(\Phi_h)}{\rho^2}\left(d\rho^2 + \rho^2 d\Omega\right). \tag{197}$$

By comparing eq. (197) with eq. (8) we see that $V(\Phi_h) = M^2_{BR}$ and therefore, remembering (10), we get the result (192).

To summarize, we have just found that the area of the event horizon (and hence the Bekenstein–Hawking entropy of the black hole) is given by the geodesic potential evaluated at the horizon, and we also gave a tool for finding this value: the geodesic potential gets an extremum at the horizon.

However, the geodesic potential $V(\Phi, e, g)$ defined in eq.s (188) and (189) has a particular meaning in supergravity theories, that allows to find its extremum in an easy way. Indeed, an expression exactly coinciding with (188) has been found in section 3 in an apparently different context, as the result of a sum rule among central and matter charges in supergravity theories (88). So, in every supergravity theory, the geodesic potential has the general form:

$$V \equiv -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I \tag{198}$$

Then, to find the extremum of V we can apply the differential relations among central and matter charges found in Section 3, that in general read:

$$\begin{aligned} \nabla Z_{AB} &= \overline{Z}_I P^I_{AB} + \frac{1}{2} \overline{Z}^{CD} P_{ABCD} \\ \nabla Z_I &= \frac{1}{2} \overline{Z}^{AB} P_{ABI} + \overline{Z}^J P_{JI} \end{aligned} \tag{199}$$

where the matrices P_{ABCD}, P_{ABI}, P_{IJ} are the subblocks of the vielbein of the scalar manifolds U/H [19], already defined in Section 3:

$$\mathcal{P} \equiv L^{-1} \nabla L = \begin{pmatrix} P_{ABCD} & P_{ABI} \\ P_{IAB} & P_{IJ} \end{pmatrix} \tag{200}$$

written in terms of the indices of $H = H_{Aut} \times H_{matter}$.

Applying eq.s (199) to the geodesic potential, we find that the extremum is given by:

$$\begin{aligned} dV &= \frac{1}{2}\nabla Z_{AB}\overline{Z}^{AB} + \nabla Z_I \overline{Z}^I + c.c. = \\ &= \frac{1}{2}\left(\frac{1}{2}\overline{Z}^{CD}P_{ABCD} + \overline{Z}_I P^I_{AB}\right)\overline{Z}^{AB} \\ &+ \left(\frac{1}{2}\overline{Z}^{AB}P_{ABI} + \overline{Z}^J P_{JI}\right)\overline{Z}^I + c.c. = 0 \end{aligned} \tag{201}$$

that is $dV = 0$ for:

$$Z_I = 0\,;\, \overline{Z}^{AB}\overline{Z}^{CD}P_{ABCD} = 0 \tag{202}$$

Let us note that the conditions (202), defining the extremum of the geodesic potential and so the fixed scalars, have the same content, and are therefore completely equivalent, to the former relations found in the previous subsection from the Killing spinor conditions. However, with this latter procedure it is not necessary to specify explicitly horizon parameters (like the metric and the fixed values of scalars at that point), V being a well defined quantity over all the space–time.

As an exemplification of the method, let us analyze in detail the $D = 4$, $N = 4$ pure supergravity. The field content is given by the gravitational multiplet, that is by the graviton $g_{\mu\nu}$, four gravitini $\psi_{\mu A}$, $A = 1, \cdots, 4 \in SU(4)$, six vectors $A^{[AB]}_\mu$, four dilatini $\chi^{[ABC]}$ and a complex scalar $\phi = a + \mathrm{i}e^\varphi$ parametrizing the coset manifold $U/H = SU(1,1)/U(1)$. The symplectic $Sp(12)$–sections $(f^\Lambda_{AB}, h_{\Lambda AB})$ $(\Lambda \equiv [AB] = 1, \cdots, 6)$ over the scalar manifold are given by:

$$\begin{aligned} f^\Lambda_{AB} &= e^{-\varphi/2}\delta^\Lambda_{AB} \\ h_{\Lambda AB} &= \phi e^{-\varphi/2}\delta_{\Lambda AB} \end{aligned} \tag{203}$$

so that:

$$\mathcal{N}_{\Lambda\Sigma} = (h \cdot f^{-1})_{\Lambda\Sigma} = \phi\delta_{\Lambda\Sigma} \tag{204}$$

The central charge matrix is then given by:

$$Z_{AB} = h_{\Lambda AB}g^\Lambda - f^\Lambda_{AB}e_\Lambda = e^{-\varphi/2}(\phi g_{AB} - e_{AB}) \tag{205}$$

where:

$$g^\Lambda = \int F^\Lambda \equiv \int dA^\Lambda \quad e_\Lambda = \int \mathcal{N}_{\Lambda\Sigma}F^\Sigma \tag{206}$$

The geodesic potential is therefore:

$$\begin{aligned} V(\phi, e, g,) &= \frac{1}{2}e^{-\varphi}(\phi g_{AB} - e_{AB})(\overline{\phi}g^{AB} - e^{AB}) \\ &= \frac{1}{2}(a^2 e^{-\varphi} + e^\varphi)g_{AB}g^{AB} + e^{-\varphi}e_{AB}e^{AB} - 2ae^{-\varphi}e_{AB}g^{AB} \\ &\equiv \frac{1}{2}(g, e)\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} e^\varphi & 0 \\ 0 & e^{-\varphi} \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g \\ e \end{pmatrix} \end{aligned} \tag{207}$$

By extremizing the potential in the moduli space we get:

$$\begin{aligned} \frac{\partial V}{\partial a} &= 0 \quad \to \quad a_h = \frac{e_{AB} g^{AB}}{g_{AB} g^{AB}} \\ \frac{\partial V}{\partial \varphi} &= 0 \quad \to \quad e^{\varphi_h} = \frac{\sqrt{e_{AB} e^{AB} g_{CD} g^{CD} - (e_{AB} g^{AB})^2}}{g_{AB} g^{AB}} \end{aligned} \tag{208}$$

from which it follows that the entropy is:

$$S_{B-H} = 4\pi V(\phi_h, e, g) = 4\pi \sqrt{e_{AB} e^{AB} g_{CD} g^{CD} - (e_{AB} g^{AB})^2} \tag{209}$$

As a final observation, let us note, following [32], that the extremum reached by the geodesic potential at the horizon is in particular a minimum, unless the metric of the scalar fields change sign, corresponding to some sort of phase transitions, where the effective lagrangian description (190) of the theory breaks down. This can be seen from the properties of the Hessian of the geodesic potential. It was shown in [32] for the $N = 2$, $D = 4$ case that at the critical point $\Phi = \Phi_{fix} \equiv \Phi_h$, from the special geometry properties it follows:

$$\left(\overline{\partial}_{\bar{\imath}} \partial_j |Z|\right)_{fix} = \frac{1}{2} G_{\bar{\imath} j} |Z|_{fix} \tag{210}$$

and then, remembering, from the above discussion, that $V_{fix} = |Z_{fix}|^2$:

$$\left(\overline{\partial}_{\bar{\imath}} \partial_j V\right)_{fix} = \frac{1}{2} G_{\bar{\imath} j} V_{fix} \tag{211}$$

From eq. (211) it follows, for the $N = 2$ theory, that the minimum is unique. In the next section we will see that a result similar to (211) still hold for higher N theories, but that in general the Hessian of V has some degenerate directions.
Moreover, in the next subsection we will show one more technique for finding the entropy, exploiting the fact that it is a 'topological quantity' not depending on scalars. This last procedure is particularly interesting because it refers only to group theoretical properties of the coset manifolds spanned by scalars, and do not need the knowledge of any details of the black-hole horizon.

4.3 Central charges, U-invariants and entropy

Extremal black-holes preserving one supersymmetry correspond to N-extended multiplets with

$$M_{ADM} = |Z_1| > |Z_2| \cdots > |Z_{[N/2]}| \tag{212}$$

where Z_α, $\alpha = 1, \cdots, [N/2]$, are the proper values of the central charge antisymmetric matrix written in normal form [33]. The central charges $Z_{AB} = -Z_{BA}$, $A, B = 1, \cdots, N$, and matter

charges Z_I, $I = 1, \cdots, n$ are those (moduli-dependent) symplectic invariant combinations of field strenghts and their duals (integrated over a large two-sphere) which appear in the gravitino and gaugino supersymmetry variations respectively [34],[19],[20]. Note that the total number of vector fields is $n_V = N(N-1)/2 + n$ (with the exception of $N = 6$ in which case there is an extra singlet graviphoton)[16].
As we discussed in the previous section, at the attractor point, where M_{ADM} is extremized, supersymmetry requires that Z_α, $\alpha > 1$, vanish toghether with the matter charges Z_I, $I = 1, \cdots, n$ (n is the number of matter multiplets, which can exist only for $N = 3, 4$)

This result can be used to show that for "fixed scalars", corresponding to the attractor point, the scalar "potential" of the geodesic action [35],[31],[32]

$$V = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{213}$$

is extremized in moduli space. The main purpose of this subsection is to provide particular expressions which give the entropy formula as a moduli–independent quantity in the entire moduli space and not just at the critical points. Namely, we are looking for quantities $S\left(Z_{AB}(\phi), \overline{Z}^{AB}(\phi), Z_I(\phi), \overline{Z}^I(\phi)\right)$ such that $\frac{\partial}{\partial \phi^i} S = 0$, ϕ^i being the moduli coordinates [5]. To this aim, let us first consider invariants I_α of the isotropy group H of the scalar manifold U/H, built with the central and matter charges. We will take all possible H-invariants up to quartic ones for four dimensional theories (except for the $N = 3$ case, where the invariants of order higher than quadratic are not irreducible). Then, let us consider a linear combination $S^2 = \sum_\alpha C_\alpha I_\alpha$ of the H-invariants, with arbitrary coefficients C_α. Now, let us extremize S in the moduli space $\frac{\partial S}{\partial \Phi^i} = 0$, for some set of $\{C_\alpha\}$. Since $\Phi^i \in U/H$, the quantity found in this way (which in all cases turns out to be unique) is a U-invariant, and is indeed proportional to the Bekenstein–Hawking entropy.

These formulae generalize the quartic $E_{7(-7)}$ invariant of $N = 8$ supergravity [36] to all other cases. We will show in the appendix how these invariants can be computed in an almost trivial fashion by using the (non compact) Cartan elements of U/H. [6]

Let us first consider the theories $N = 3, 4$, where matter can be present [21],[37].

The U–duality groups [7] are, in these cases, $SU(3, n)$ and $SU(1, 1) \times SO(6, n)$ respectively. The central and matter charges Z_{AB}, Z_I transform in an obvious way under the isotropy groups

$$H = SU(3) \times SU(n) \times U(1) \qquad (N = 3) \tag{214}$$

$$H = SU(4) \times O(n) \times U(1) \qquad (N = 4) \tag{215}$$

[5]The Bekenstein-Hawking entropy $S_{BH} = \frac{A}{4}$ is actually πS in our notation.

[6]Our analysis is based on general properties of scalar coset manifolds. As a consequence, it can be applied straightforwardly also to the $N = 2$ cases, whenever one considers special coset manifolds.

[7]Here we denote by U-duality group the isometry group U acting on the scalars, although only a restriction of it to integers is the proper U-duality group [13].

Under the action of the elements of U/H the charges get mixed with their complex conjugate. The infinitesimal transformation can be read from the differential relations satisfied by the charges (199) [19] .

For $N = 3$:

$$P^{ABCD} = P_{IJ} = 0, \quad P_{ABI} \equiv \epsilon_{ABC} P_I^C \quad Z_{AB} \equiv \epsilon_{ABC} Z^C \tag{216}$$

Then the variations are:

$$\delta Z^A = \xi_I^A \overline{Z}^I \tag{217}$$

$$\delta Z_I = \xi_I^A \overline{Z}_A \tag{218}$$

where ξ_I^A are infinitesimal parameters of $K = U/H$.

The possible quadratic H-invariants are:

$$\begin{aligned} I_1 &= Z^A \overline{Z}_A \\ I_2 &= Z_I \overline{Z}^I \end{aligned} \tag{219}$$

So, the U-invariant expression is:

$$S = Z^A \overline{Z}_A - Z_I \overline{Z}^I \tag{220}$$

In other words, $\nabla_i S = \partial_i S = 0$, where the covariant derivative is defined in ref. [19].

Note that at the attractor point ($Z_I = 0$) it coincides with the moduli-dependent potential (213) computed at its extremum.

For $N = 4$

$$P_{ABCD} = \epsilon_{ABCD} P, \quad P_{IJ} = \eta_{IJ} P, \quad P_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} \overline{P}^{CDJ} \tag{221}$$

and the transformations of $K = \frac{SU(1,1)}{U(1)} \times \frac{O(6,n)}{O(6)\times O(n)}$ are:

$$\delta Z_{AB} = \frac{1}{2} \xi \epsilon_{ABCD} \overline{Z}^{CD} + \xi_{ABI} \overline{Z}^I \tag{222}$$

$$\delta Z_I = \xi \eta_{IJ} \overline{Z}^J + \frac{1}{2} \xi_{ABI} \overline{Z}^{AB} \tag{223}$$

with $\overline{\xi}^{ABI} = \frac{1}{2} \eta^{IJ} \epsilon^{ABCD} \xi_{CDJ}$.

The possible H-invariants are:

$$\begin{aligned} I_1 &= Z_{AB} \overline{Z}^{AB} \\ I_2 &= Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} \\ I_3 &= \epsilon^{ABCD} Z_{AB} Z_{CD} \\ I_4 &= Z_I Z^I \end{aligned} \tag{224}$$

There are three $O(6,n)$ invariants given by S_1, S_2, $\overline{S}_2$ where:

$$S_1 = \frac{1}{2} Z_{AB} \overline{Z}^{AB} - Z_I Z_I \tag{225}$$

$$S_2 = \frac{1}{4} \epsilon^{ABCD} Z_{AB} Z_{CD} - \overline{Z}_I Z_I \tag{226}$$

and the unique $SU(1,1) \times O(6,n)$ invariant S, $\nabla S = 0$, is given by:

$$S = \sqrt{(S_1)^2 - |S_2|^2} \tag{227}$$

At the attractor point $Z_I = 0$ and $\epsilon^{ABCD} Z_{AB} Z_{CD} = 0$ so that S reduces to the square of the BPS mass.
Note that, in absence of matter multiplets, one recovers the expression found in the previous subsecion by extremizing the geodesic potential.

For $N = 5, 6, 8$ the U-duality invariant expression S is the square root of a unique invariant under the corresponding U-duality groups $SU(5,1)$, $O^*(12)$ and $E_{7(-7)}$. The strategy is to find a quartic expression S^2 in terms of Z_{AB} such that $\nabla S = 0$, i.e. S is moduli-independent.

As before, this quantity is a particular combination of the H quartic invariants.

For $SU(5,1)$ there are only two $U(5)$ quartic invariants. In terms of the matrix $A_A^{\ B} = Z_{AC} \overline{Z}^{CB}$ they are: $(TrA)^2$, $Tr(A^2)$, where

$$TrA = Z_{AB} \overline{Z}^{BA} \tag{228}$$

$$Tr(A^2) = Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} \tag{229}$$

As before, the relative coefficient is fixed by the transformation properties of Z_{AB} under $\frac{SU(5,1)}{U(5)}$ elements of infinitesimal parameter ξ^C:

$$\delta Z_{AB} = \frac{1}{2} \xi^C \epsilon_{CABPQ} \overline{Z}^{PQ} \tag{230}$$

It then follows that the required invariant is:

$$S = \frac{1}{2} \sqrt{4Tr(A^2) - (TrA)^2} \tag{231}$$

For $N = 8$ the $SU(8)$ invariants are [8]:

$$I_1 = (TrA)^2 \tag{232}$$

$$I_2 = Tr(A^2) \tag{233}$$

$$I_3 = Pf\, Z = \frac{1}{2^4 4!} \epsilon^{ABCDEFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH} \tag{234}$$

[8] The Pfaffian of an $(n \times n)$ (n even) antisymmetric matrix is defined as $PfZ = \frac{1}{2^n n!} \epsilon^{A_1 \cdots A_n} Z_{A_1 A_2} \cdots Z_{A_{N-1} A_N}$, with the property: $|PfZ| = |detZ|^{1/2}$.

The $\frac{E_{7(-7)}}{SU(8)}$ transformations are:

$$\delta Z_{AB} = \frac{1}{2}\xi_{ABCD}\overline{Z}^{CD} \tag{235}$$

where ξ_{ABCD} satisfies the reality constraint:

$$\xi_{ABCD} = \frac{1}{24}\epsilon_{ABCDEFGH}\overline{\xi}^{EFGH} \tag{236}$$

One finds the following $E_{7(-7)}$ invariant [36]:

$$S = \frac{1}{2}\sqrt{4Tr(A^2) - (TrA)^2 + 32Re(Pf\,Z)} \tag{237}$$

The $N = 6$ case is the more complicated because under $U(6)$ the left-handed spinor of $O^*(12)$ splits into:

$$32_L \to (15, 1) + (\overline{15}, -1) + (1, -3) + (1, 3) \tag{238}$$

The transformations of $\frac{O^*(12)}{U(6)}$ are:

$$\delta Z_{AB} = \frac{1}{4}\epsilon_{ABCDEF}\xi^{CD}\overline{Z}^{EF} + \xi_{AB}\overline{X} \tag{239}$$

$$\delta X = \frac{1}{2}\xi_{AB}\overline{Z}^{AB} \tag{240}$$

where we denote by X the $SU(6)$ singlet.

The quartic $U(6)$ invariants are:

$$I_1 = (TrA)^2 \tag{241}$$

$$I_2 = Tr(A^2) \tag{242}$$

$$I_3 = Re(Pf\,ZX) = \frac{1}{2^3 3!}Re(\epsilon^{ABCDEF}Z_{AB}Z_{CD}Z_{EF}X) \tag{243}$$

$$I_4 = (TrA)X\overline{X} \tag{244}$$

$$I_5 = X^2\overline{X}^2 \tag{245}$$

The unique $O^*(12)$ invariant is:

$$S = \frac{1}{2}\sqrt{4I_2 - I_1 + 32I_3 + 4I_4 + 4I_5} \tag{246}$$

$$\nabla S = 0 \tag{247}$$

Note that at the attractor point $Pf\,Z = 0$, $X = 0$ and S reduces to the square of the BPS mass.

4.3.1 A simple determination of the U-invariants

In order to determine the quartic U-invariant expressions S^2 , $\nabla S = 0$, of the $N > 4$ theories, it is useful to use, as a calculational tool, transformations of the coset which preserve the normal form of the Z_{AB} matrix. It turns out that these transformations are certain Cartan elements in $K = U/H$ [38], that is they belong to $O(1,1)^p \in K$, with $p = 1$ for $N = 5$, $p = 3$ for $N = 6, 8$.

These elements act only on the Z_{AB} (in normal form), but they uniquely determine the U-invariants since they mix the eigenvalues e_i $(i = 1, \cdots, [N/2])$.

For $N = 5$, $SU(5,1)/U(5)$ has rank one (see ref. [39]) and the element is:

$$\delta e_1 = \xi e_2; \quad \delta e_2 = \xi e_1 \tag{248}$$

which is indeed a $O(1,1)$ transformation with unique invariant

$$|(e_1)^2 - (e_2)^2| = \frac{1}{2}\sqrt{8\left((e_1)^4 + (e_2)^4\right) - 4\left((e_1)^2 + (e_2)^2\right)^2} \tag{249}$$

For $N = 6$, we have $\xi_1 \equiv \xi_{12}; \xi_2 \equiv \xi_{34}; \xi_3 \equiv \xi_{56}$ and we obtain the 3 Cartan elements of $O^*(12)/U(6)$, which has rank 3, that is it is a $O(1,1)^3$ in $O^*(12)/U(6)$. Denoting by e the singlet charge, we have the following $O(1,1)^3$ transformations:

$$\begin{aligned} \delta e_1 &= \xi_2 e_3 + \xi_3 e_2 + \xi_1 e && (250)\\ \delta e_2 &= \xi_1 e_3 + \xi_3 e_1 + \xi_2 e && (251)\\ \delta e_3 &= \xi_1 e_2 + \xi_2 e_1 + \xi_3 e && (252)\\ \delta e &= \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 && (253) \end{aligned}$$

these transformations fix uniquely the $O^*(12)$ invariant constructed out of the five $U(6)$ invariants displayed in (241-245).

For $N = 8$ the infinitesimal parameter is ξ_{ABCD} and, using the reality condition, we get again a $O(1,1)^3$ in $E_{7(-7)}/SU(8)$. Setting $\xi_{1234} = \xi_{5678} \equiv \xi_{12}$, $\xi_{1256} = \xi_{3478} \equiv \xi_{13}$, $\xi_{1278} = \xi_{3456} \equiv \xi_{14}$, we have the following set of transformations:

$$\begin{aligned} \delta e_1 &= \xi_{12} e_2 + \xi_{13} e_3 + \xi_{14} e_4 && (254)\\ \delta e_2 &= \xi_{12} e_1 + \xi_{13} e_4 + \xi_{14} e_3 && (255)\\ \delta e_3 &= \xi_{12} e_4 + \xi_{13} e_1 + \xi_{14} e_2 && (256)\\ \delta e_4 &= \xi_{12} e_3 + \xi_{13} e_2 + \xi_{14} e_1 && (257) \end{aligned}$$

These transformations fix uniquely the relative coefficients of the three $SU(8)$ invariants:

$$
\begin{aligned}
I_1 &= e_1^4 + e_2^4 + e_3^4 + e_4^4 && (258)\\
I_2 &= (e_1^2 + e_2^2 + e_3^2 + e_4^2)^2 && (259)\\
I_3 &= e_1 e_2 e_3 e_4 && (260)\\
& && (261)
\end{aligned}
$$

It is easy to see that the transformations (250-253) and (254-257) correspond to three commuting matrices (with square equal to $\mathbb{1}$):

$$
\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{262}
$$

which are proper non compact Cartan elements of K. The reason we get the same transformations for $N = 6$ and $N = 8$ is because the extra singlet e of $N = 6$ can be identified with the fourth eigenvalue of the central charge of $N = 8$.

4.3.2 Extrema of the BPS mass and fixed scalars

In this subsection we would like to extend the analysis of the extrema of the black-hole induced potential

$$
V = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I \tag{263}
$$

which was performed in ref [32] for the $N = 2$ case to all $N > 2$ theories.

We recall that, in the case of $N = 2$ special geometry with metric $g_{i\bar{\jmath}}$, at the fixed scalar critical point $\partial_i V = 0$ the Hessian matrix reduces to:

$$
\begin{aligned}
(\nabla_i \nabla_{\bar{\jmath}} V)_{fixed} &= (\partial_i \partial_{\bar{\jmath}} V)_{fixed} = 2 g_{i\bar{\jmath}} V_{fixed} && (264)\\
(\nabla_i \nabla_j V)_{fixed} &= 0 && (265)
\end{aligned}
$$

The Hessian matrix is strictly positive-definite if the critical point is not at the singular point of the vector multiplet moduli-space. This matrix was related to the Weinhold metric earlier introduced in the geometric approach to thermodynamics and used for the study of critical phenomena [32].

For N-extended supersymmetry, a form of this matrix was also given and shown to be equal to [9]:

$$
V_{ij} = (\partial_i \partial_j V)_{fixed} = Z_{CD} Z^{AB} \left(\frac{1}{2} P^{CDPQ}_{\ \ \ \ ,j} P_{ABPQ,i} + P^{CD}_{I,j} P^{I}_{AB,j}\right). \tag{266}
$$

[9] Generically the indices i, j refer to real coordinates, unless the manifold is Kählerian, in which case we use holomorphic coordinates and formula (266) reduces to the hermitean $i\bar{\jmath}$ entries of the Hessian matrix.

It is our purpose to further investigate properties of the Weinhold metric for fixed scalars.

Let us first observe that the extremum conditions $\nabla_i V = 0$, using the relation between the covariant derivatives of the central charges, reduce to the conditions:

$$\epsilon^{ABCDL_1\cdots L_{N-4}} Z_{AB} Z_{CD} = 0, \quad Z_I = 0 \tag{267}$$

These equations give the fixed scalars in terms of electric and magnetic charges and also show that the topological invariants of the previous section reduce to the extremum of the square of the ADM mass since, when the above conditions are fulfilled, $(TrA)^2 = 2Tr(A^2)$, where $A_A^{\ B} = Z_{AB}\overline{Z}^{BC}$.

On the other hand, when these conditions are fulfilled, it is easy to see that the Hessian matrix is degenerate. To see this, it is sufficient to go, making an H transformation, to the normal frame in which these conditions imply $Z_{12} \neq 0$ with the other charges vanishing. Then we have:

$$\partial_i \partial_j V|_{fixed} = 4|Z_{12}|^2 (\frac{1}{2} P_j^{12ab} P_{12ab,i} + P_{,j}^{12I} P_{12I,i}), \quad a, b \neq 1, 2 \tag{268}$$

To understand the pattern of degeneracy for all N, we observe that when only one central charge in not vanishing the theory effectively reduces to an $N = 2$ theory. Then the actual degeneracy respects $N = 2$ multiplicity of the scalars degrees of freedom in the sense that the degenerate directions will correspond to the hypermultiplet content of $N > 2$ theories when decomposed with respect to $N = 2$ supersymmetry.

Note that for $N = 3$, $N = 4$, where P_{ABI} is present, the Hessian is block diagonal.

For $N = 3$, referring to eq. (216), since the scalar manifold is Kähler, P_{ABI} is a (1,0)-form while $P^{ABI} = \overline{P}_{ABI}$ is a (0,1)-form.

The scalars appearing in the $N = 2$ vector multiplet and hypermultiplet content of the vielbein are P_{3I} for the vector multiplets and P_{aI} $(a = 1, 2)$ for the hypermultiplets. From equation (268), which for the $N = 3$ case reads

$$\partial_{\bar{j}} \partial_i V|_{fixed} = 2|Z_{12}|^2 P_{3I,\bar{j}} P_{,i}^{3I} \tag{269}$$

we see that the metric has $4n$ real directions corresponding to n hypermultiplets which are degenerate.

For $N = 4$, referring to (221), P is the $SU(1,1)/U(1)$ vielbein which gives one matter vector multiplet scalar while P_{12I} gives n matter vector multiplets. The directions which are hypermultiplets correspond to P_{1aI}, P_{2aI} $(a = 3, 4)$. Therefore the "metric" V_{ij} is of rank $2n + 2$.

For $N > 4$, all the scalars are in the gravity multiplet and correspond to P_{ABCD}.

The splitting in vector and hypermultiplet scalars proceeds as before. Namely, in the $N = 5$ case we set $P_{ABCD} = \epsilon_{ABCDL} P^L$ $(A, B, C, D, L = 1, \cdots 5)$. In this case the vector multiplet scalars are P^a $(a = 3, 4, 5)$ while the hypermultiplet scalars are P^1, P^2 $(n_V = 3,\ n_h = 1)$.

For $N=6$, we set $P_{ABCD}=\frac{1}{2}\epsilon_{ABCDEF}P^{EF}$. The vector multiplet scalars are now described by P^{12}, P^{ab} $(A, B, ... = 1, ..., 6;\ a, b = 3, \cdots 6)$, while the hypermultiplet scalars are given in terms of P^{1a}, P^{2a}. Therefore we get $n_V = 6+1 = 7$, $n_h = 4$.

This case is different from the others because, besides the hypermultiplets P^{1a}, P^{2a}, also the vector multiplet direction P^{12} is degenerate.

Finally, for $N=8$ we have P_{1abc}, P_{2abc} as hypermultiplet scalars and P_{abcd} as vector multiplet scalars, which give $n_V = 15$, $n_h = 10$ (note that in this case the vielbein satisfies a reality condition: $P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDPQRS}\overline{P}^{PQRS}$). We have in this case 40 degenerate directions.

In conclusion we see that the rank of the matrix V_{ij} is $(N-2)(N-3)+2n$ for all the four dimensional theories.

5 BPS black holes in $N=8$ supergravity

In this section we consider BPS extremal black–holes in the context of $N=8$ supergravity.

$N=8$ supergravity is the 4–dimensional effective lagrangian of both type IIA and type IIB superstrings compactified on a torus T^6. Alternatively it can be viewed as the $4D$ effective lagrangian of 11–dimensional M–theory compactified on a torus T^7. For this reason its U–duality group $E_{7(7)}(\mathbb{Z})$, which is defined as the discrete part of the isometry group of its scalar manifold:

$$\mathcal{M}^{(N=8)}_{scalar} = \frac{E_{7(7)}}{SU(8)}, \tag{270}$$

unifies all superstring dualities relating the various consistent superstring models. The *non perturbative BPS states* one needs to adjoin to the string states in order to complete linear representations of the U–duality group are, generically, *BPS black–holes*.

These latter can be viewed as intersections of several p–brane solutions of the higher dimensional theory *wrapped* on the homology cycles of the T^6 (or T^7) torus. Depending on how many p–branes intersect, the residual supersymmetry can be:

1. $\frac{1}{2}$ of the original supersymmetry
2. $\frac{1}{4}$ of the original supersymmetry
3. $\frac{1}{8}$ of the original supersymmetry

The distinction between these three kinds of BPS solutions can be considered directly in a 4–dimensional setup and it is related to the structure of the central charge eigenvalues and to the behaviour of the scalar fields at the horizon. BPS black–holes with a finite horizon area are those for which the scalar fields are regular at the horizon and reach a fixed value there. These can only be the 1/8–type black holes, whose structure is that of $N = 2$ black–holes embedded into the $N = 8$ theory. For 1/2 and 1/4 black–holes the scalar fields always diverge at the horizon and the entropy is zero.

The nice point, in this respect, is that we can make a complete classification of all BPS black holes belonging to the three possible types. Indeed the distinction of the solutions into these three classes can be addressed a priori and, as we are going to see, corresponds to a classification into different orbits of the possible **56**–dimensional vectors $Q = \{p^\Lambda, q_\Sigma\}$ of magnetic-electric charges of the hole. Indeed $N = 8$ supergravity contains **28** gauge fields A^Λ_μ and correspondingly the hole can carry **28** magnetic p^Λ and **28** electric q_Σ charges. Through the symplectic embedding of the scalar manifold (270) it follows that the field strengths $F^\Lambda_{\mu\nu}$ plus their duals $G_{\Sigma|\mu\nu}$ transform in the fundamental **56** representation of $E_{7(7)}$ and the same is true of their integrals, namely the charges.

The Killing spinor equation that imposes preservation of either 1/2, or 1/4, or 1/8 of the original supersymmetries enforces two consequences different in the three cases:

1. a different decomposition of the scalar field manifold into two sectors:
   - a sector of *dynamical scalar fields* that evolve in the radial parameter r
   - a sector of *spectator scalar fields* that do not evolve in r and are constant in the BPS solution.
2. a different orbit structure for the charge vector Q

Then, up to U–duality transformations, for each case one can write a fully general *generating solution* that contains the minimal necessary number of excited dynamical fields and the minimal necessary number of non vanishing charges. All other solutions of the same supersymmetry type can be obtained from the generating one by the action of $E_{7(7)}$–rotations.

Such an analysis is clearly group–theoretical and requires the use of appropriate techniques.

5.1 Summary of $N = 8$ supergravity.

Let us now summarize the structure of $N = 8$ supergravity. The action is of the general form (25) with g_{IJ} being the invariant metric of $E_{7(7)}/SU(8)$ and the kinetic matrix $\mathcal{N}$ being determined via the appropriate symplectic embedding of $E_{7(7)}$ into $Sp(56, \mathbb{R})$ (see table 1). Hence, according to the general formalism discussed in section 3 and to eq.(60) we introduce the coset representative $\mathbb{L}$ of $\frac{E_{7(7)}}{SU(8)}$ in the **56** representation of $E_{7(7)}$:

$$\mathbb{L} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ \hline f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{array} \right) \tag{271}$$

where the submatrices (h, f) are 28×28 matrices labeled by antisymmetric pairs Λ, Σ, A, B ($\Lambda, \Sigma = 1, \ldots, 8$, $A, B = 1, \ldots, 8$) the first pair transforming under $E_{7(7)}$ and the second one under $SU(8)$:

$$(h, f) = \left(h_{\Lambda\Sigma|AB}, f^{\Lambda\Sigma}{}_{AB}\right) \tag{272}$$

As expected from the general formalism we have $\mathbb{L} \in Usp(28, 28)$. The vielbein P_{ABCD} and the $SU(8)$ connection $\Omega_A^{\ B}$ of $\frac{E_{7(7)}}{SU(8)}$ are computed from the left invariant 1-form $\mathbb{L}^{-1} d\mathbb{L}$:

$$\mathbb{L}^{-1} d\mathbb{L} = \left(\begin{array}{c|c} \delta^{[A}{}_{[C}\Omega^{B]}{}_{D]} & \overline{P}^{ABCD} \\ \hline P_{ABCD} & \delta_{[A}{}^{[C}\overline{\Omega}_{B]}{}^{D]} \end{array}\right) \tag{273}$$

where $P_{ABCD} \equiv P_{ABCD,i} d\Phi^i \quad (i = 1, \ldots, 70)$ is completely antisymmetric and satisfies the reality condition

$$P_{ABCD} = \frac{1}{24} \epsilon_{ABCDEFGH} \overline{P}^{EFGH} \tag{274}$$

The bosonic lagrangian of $N = 8$ supergravity is [40]

$$\begin{aligned} \mathcal{L} \;&=\; \int \sqrt{-g}\, d^4x \left(2R + \mathrm{Im}\, \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} F_{\mu\nu}{}^{\Lambda\Sigma} F^{\Gamma\Delta|\mu\nu} + \frac{1}{6} P_{ABCD,i} \overline{P}_j^{ABCD} \partial_\mu \Phi^i \partial^\mu \Phi^j + \right. \\ &+\; \left. \frac{1}{2} \mathrm{Re}\, \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\mu\nu}{}^{\Lambda\Sigma} F^{\Gamma\Delta}{}_{\rho\sigma} \right) \end{aligned} \tag{275}$$

where the curvature two-form is defined as

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}. \tag{276}$$

and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}$ is given by the usual general formula:

$$\mathcal{N} = h f^{-1} \quad \rightarrow \quad \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} = h_{\Lambda\Sigma|AB} f^{-1\ AB}{}_{\Gamma\Delta}. \tag{277}$$

Note that the 56 dimensional (anti)self-dual vector $\left(F^{\pm\ \Lambda\Sigma}, G^{\pm}_{\Lambda\Sigma}\right)$ transforms covariantly under $U \in Sp(56, \mathbb{R})$

$$\begin{aligned} & U \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F' \\ G' \end{pmatrix} \quad ; \quad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ & A^t C - C^t A = 0 \\ & B^t D - D^t B = 0 \\ & A^t D - C^t B = \mathbf{1} \end{aligned} \tag{278}$$

The matrix transforming the coset representative $\mathbb{L}$ from the $Usp(28,28)$ basis, eq.(271), to the real $Sp(56,\mathbb{R})$ basis is the Cayley matrix:

$$\mathbb{L}_{Usp} = \mathcal{C}\mathbb{L}_{Sp}\mathcal{C}^{-1} \qquad \mathcal{C} = \begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}. \tag{279}$$

Having established our definitions and notations, let us now write down the Killing spinor equations obtained by equating to zero the SUSY transformation laws of the gravitino $\psi_{A\mu}$ and dilatino χ_{ABC} fields of $N = 8$ supergravity in a purely bosonic background:

$$\delta\chi_{ABC} = 4\mathrm{i}\, P_{ABCD|i}\partial_\mu\Phi^i\gamma^\mu\epsilon^D - 3T^{(-)}_{[AB|\rho\sigma}\gamma^{\rho\sigma}\epsilon_{C]} = 0 \tag{280}$$

$$\delta\psi_{A\mu} = \nabla_\mu\epsilon_A - \frac{1}{4}T^{(-)}_{AB|\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon^B = 0 \tag{281}$$

where ∇_μ denotes the derivative covariant both with respect to Lorentz and $SU(8)$ local transformations

$$\nabla_\mu\epsilon_A = \partial_\mu\epsilon_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\epsilon_A - \Omega_A^{\ B}\epsilon_B \tag{282}$$

and where $T^{(-)}_{AB}$ is the "dressed graviphoton"$2-$form, defined according to the general formulae (70)

$$T^{(-)}_{AB} = \left(h_{\Lambda\Sigma AB}(\Phi)\, F^{-\Lambda\Sigma} - f^{\Lambda\Sigma}_{\ \ AB}(\Phi)\, G^{-}_{\Lambda\Sigma}\right) \tag{283}$$

¿From equations (277), (64) we have the following identities that are the particular $N = 8$ instance of eq.(71):

$$T^{+}_{AB} = 0 \rightarrow T^{-}_{AB} = T_{AB} \qquad \overline{T}^{-}_{AB} = 0 \rightarrow \overline{T}^{+}_{AB} = \overline{T}_{AB}$$

Following the general procedure indicated by eq.(73) we can define the central charge:

$$Z_{AB} = \int_{S^2} T_{AB} = h_{\Lambda\Sigma|AB}p^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{\ \ AB}q_{\Lambda\Sigma} \tag{284}$$

which in our case is an antisymmetric tensor transforming in the **28** irreducible representation of $SU(8)$. In eq.(284) the integral of the two-form T_{AB} is evaluated on a large two-sphere at infinity and the quantized charges $(p_{\Lambda\Sigma},\ q^{\Lambda\Sigma})$ are defined, following the general eq.s (75) by

$$\begin{aligned} p^{\Lambda\Sigma} &= \int_{S^2} F^{\Lambda\Sigma} \\ q_{\Lambda\Sigma} &= \int_{S^2} \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} \star F^{\Gamma\Delta}. \end{aligned} \tag{285}$$

5.2 The Killing spinor equation and its covariance group

In order to translate eq.(280) and (281) into first order differential equations on the bosonic fields of supergravity we consider a configuration where all the fermionic fields are zero and a SUSY parameter that satisfies the following conditions:

$$\begin{array}{rcl} \chi^\mu \, \gamma_\mu \, \epsilon_A &=& \mathrm{i}\mathbb{C}_{AB} \, \epsilon^B \quad ; \quad A, B = 1, \dots, n_{max} \\ \epsilon_A &=& 0 \qquad\qquad\; ; \quad A > n_{max} \end{array} \tag{286}$$

Here χ^μ is a time–like Killing vector for the space–time metric (in the following we just write $\chi^\mu \gamma_\mu = \gamma^0$) and ϵ_A, ϵ^A denote the two chiral projections of a single Majorana spinor: $\gamma_5 \, \epsilon_A = \epsilon_A$, $\gamma_5 \, \epsilon^A = -\epsilon^A$. We name such an equation the *Killing spinor equation* and the investigation of its group–theoretical structure is the main task we face in order to derive the three possible types of BPS black–holes, those preserving 1/2 or 1/4 or 1/8 of the original supersymmetry. To appreciate the distinction among the three types of $N = 8$ black–hole solutions we need to recall the results of [41] where a classification was given of the **56**–vectors of quantized electric and magnetic charges $\vec{Q}$ characterizing such solutions. The basic argument is provided by the reduction of the central charge skew–symmetric tensor $\mathbb{Z}_{AB}$ to normal form. The reduction can always be obtained by means of local $SU(8)$ transformations, but the structure of the skew eigenvalues depends on the orbit–type of the **56**–dimensional charge vector which can be described by means of its stabilizer subgroup $G_{stab}(\vec{Q}) \subset E_{7(7)}$:

$$g \, \in \, G_{stab}(\vec{Q}) \subset E_{7(7)} \quad \Longleftrightarrow \quad g \, \vec{Q} = \vec{Q} \tag{287}$$

There are three possibilities:

$$\begin{array}{cccc} \text{SUSY} & \text{Central Charge} & \text{Stabilizer} \equiv G_{stab} & \text{Normalizer} \equiv G_{norm} \\ 1/2 & Z_1 = Z_2 = Z_3 = Z_4 & E_{6(6)} & O(1,1) \\ 1/4 & Z_1 = Z_2 \neq Z_3 = Z_4 & SO(5,5) & SL(2,\mathbb{R}) \times O(1,1) \\ 1/8 & Z_1 \neq Z_2 \neq Z_3 \neq Z_4 & SO(4,4) & SL(2,\mathbb{R})^3 \end{array} \tag{288}$$

where the normalizer $G_{norm}(\vec{Q})$ is defined as the subgroup of $E_{7(7)}$ that commutes with the stabilizer:

$$[G_{norm} \, , \, G_{stab}] = 0 \tag{289}$$

The main result of [42] is that the most general 1/8 black–hole solution of $N = 8$ supergravity is related to the normalizer group $SL(2, \mathbb{R})^3$. In the subsequent paper [43] the 1/2 and 1/4 cases were completely worked out. Finally in [44] the explicit form of the generating solutions was discussed for the 1/8 case. In this paper we review the simlest case, corresponding to 1/2 preserved supersymmetry. For the other cases, we refer the reader to references [44],[45].

In all three cases the Killing spinor equation breaks the original $SU(8)$ automorphism group of the supersymmetry algebra to the subgroup $Usp(2\, n_{max}) \times SU(8 - 2\, n_{max}) \times U(1)$

We then have to decompose $N = 8$ supergravity into multiplets of the lower supersymmetry $N' = 2\, n_{max}$. This is easily understood by recalling that close to the horizon of the black hole one doubles the supersymmetries holding in the bulk of the solution. Hence the near horizon supersymmetry is precisely $N' = 2\, n_{max}$ and the black solution can be interpreted as a soliton that interpolates between *ungauged* $N = 8$ supergravity at infinity and some form of N' supergravity at the horizon.

5.3 The 1/2 SUSY case

Here we have $n_{max} = 8$ and correspondingly the covariance subgroup of the Killing spinor equation is $Usp(8) \subset SU(8)$. Indeed condition (286) can be rewritten as follows:

$$\gamma^0\, \epsilon_A = \mathrm{i}\mathbb{C}_{AB}\, \epsilon^B \quad ; \quad A, B = 1, \dots, 8 \tag{290}$$

where $\mathbb{C}_{AB} = -\mathbb{C}_{BA}$ denotes an 8×8 antisymmetric matrix satisfying $\mathbb{C}^2 = -\mathbb{1}$. The group $Usp(8)$ is the subgroup of unimodular, unitary 8×8 matrices that are also symplectic, namely that preserve the matrix $\mathbb{C}$. Relying on eq. (288) we see that in the present case $G_{stab} = E_{6(6)}$ and $G_{norm} = O(1, 1)$. Furthermore we have the following decomposition of the **70** irreducible representation of $SU(8)$ into irreducible representations of $Usp(8)$:

$$\mathbf{70} \stackrel{Usp(8)}{\longrightarrow} \mathbf{42} \oplus \mathbf{1} \oplus \mathbf{27} \tag{291}$$

Furthermore, we also decompose the **56** charge representation of $E_{7(7)}$ with respect to $O(1, 1) \times E_{6(6)}$ obtaining

$$\mathbf{56} \stackrel{Usp(8)}{\longrightarrow} (\mathbf{1}, \mathbf{27}) \oplus (\mathbf{1}, \mathbf{27}) \oplus (\mathbf{2}, \mathbf{1}) \tag{292}$$

In order to single out the content of the first order Killing spinor equations we need to decompose them into irreducible $Usp(8)$ representations. The gravitino equation (281) is an **8** of $SU(8)$ that remains irreducible under $Usp(8)$ reduction. On the other hand the dilatino equation (280) is a **56** of $SU(8)$ that reduces as follows:

$$\mathbf{56} \stackrel{Usp(8)}{\longrightarrow} \mathbf{48} \oplus \mathbf{8} \tag{293}$$

Hence altogether we have that 3 Killing spinor equations in the representations **8**, **8**$'$, **48** constraining the scalar fields parametrizing the three subalgebras **42**, **1** and **27**. In the sequel we will work out the consequences of these constraints and find which scalars are set to constants, which are instead evolving and how many charges are different from zero. As we will explicitly see, the content of the Killing spinor equations after $Usp(8)$ decomposition, is such as to set to a constant 69 scalar fields: indeed in this case $G_{norm} = O(1,1)$ and $H_{norm} = \mathbf{1}$, so that there is just one surviving field parametrizing $G_{norm} = O(1,1)$. Moreover, the same Killing spinor equations tell us that the 54 belonging to the two $(\mathbf{1}, \mathbf{27})$ representation of eq. (292) are actually zero, leaving only two non–vanishing charges transforming as a doublet of $O(1,1)$. Let us now discuss the explicit solution. The $N = 1/2$ SUSY preserving black hole solution of $N = 8$ supergravity has 4 equal skew eigenvalues in the normal frame for the central charges. The stabilizer of the normal form is $E_{6(6)}$ and the normalizer of this latter in $E_{7(7)}$ is $O\left(1,1\right)$:

$$E_{7(7)} \supset E_{6(6)} \times O\left(1,1\right) \tag{294}$$

According to our previous discussion, the relevant subgroup of the $SU\left(8\right)$ holonomy group is $Usp\left(8\right)$, since the BPS Killing spinor conditions involve supersymmetry parameters ϵ_A, ϵ^A satisfying eq.(290). As discussed in the introduction, it is natural to guess that modulo $U-$duality transformations the complete solution is given in terms of a single scalar field parametrizing $O\left(1,1\right)$. Indeed, we can now demonstrate that according to the previous discussion there is just one scalar field, parametrizing the normalizer $O\left(1,1\right)$, which appears in the final lagrangian, since the Killing spinor equations imply that 69 out of the 70 scalar fields are actually constants. In order to achieve this result, we have to decompose the $SU\left(8\right)$ tensors appearing in the equations (280),(281) with respect to $Usp\left(8\right)$ irreducible representations. According to the decompositions

$$\begin{array}{ccc} \mathbf{70} & \stackrel{Usp(8)}{=} & \mathbf{42} \oplus \mathbf{27} \oplus \mathbf{1} \\ \mathbf{28} & \stackrel{Usp(8)}{=} & \mathbf{27} \oplus \mathbf{1} \end{array} \tag{295}$$

we have

$$\begin{aligned} P_{ABCD} &= \overset{\circ}{P}_{ABCD} + \frac{3}{2} C_{[AB} \overset{\circ}{P}_{CD]} + \frac{1}{16} C_{[AB} C_{CD]} P \\ T_{AB} &= \overset{\circ}{T}_{AB} + \frac{1}{8} C_{AB} T \end{aligned} \tag{296}$$

where the notation $\overset{\circ}{t}_{A_1 \ldots, A_n}$ means that the antisymmetric tensor is $Usp\left(8\right)$ irreducible, namely has vanishing C-traces: $C^{A_1 A_2} \overset{\circ}{t}_{A_1 A_2 \ldots, A_n} = 0$.

Starting from equation (280) and using equation (290) we easily find:

$$4P_{,a} \gamma^a \gamma^0 - 6T_{ab} \gamma^{ab} = 0\,, \tag{297}$$

where we have twice contracted the free $Usp(8)$ indices with the $Usp(8)$ metric C_{AB}. Next, using the decomposition (296), eq. (280) reduces to

$$-4\left(\overset{\circ}{P}_{ABCD,a}+\frac{3}{2}\,\overset{\circ}{P}_{[CD,a}\,C_{AB]}\right)C^{DL}\gamma^{a}\gamma^{0}-3\,\overset{\circ}{T}_{[AB}\,\delta^{L}_{C]}\gamma^{ab}=0\,. \tag{298}$$

Now we may alternatively contract equation (298) with C^{AB} or δ^{L}_{C} obtaining two relations on $\overset{\circ}{P}_{AB}$ and $\overset{\circ}{T}_{AB}$ which imply that they are separately zero:

$$\overset{\circ}{P}_{AB}=\overset{\circ}{T}_{AB}=0\,, \tag{299}$$

which also imply, taking into account (298)

$$\overset{\circ}{P}_{ABCD}=0\,. \tag{300}$$

Thus we have reached the conclusion

$$\begin{aligned} \overset{\circ}{P}_{ABCD|i}\,\partial_{\mu}\Phi^{i}\gamma^{\mu}\epsilon^{D} &= 0 \\ \overset{\circ}{P}_{AB|i}\,\partial_{\mu}\Phi^{i}\gamma^{\mu}\epsilon^{B} &= 0 \qquad (301)\\ \overset{\circ}{T}_{AB} &= 0 \qquad (302) \end{aligned}$$

implying that 69 out the 70 scalar fields are actually constant, while the only surviving central charge is that associated with the singlet two-form T. Since T_{AB} is a complex combination of the electric and magnetic field strengths (283), it is clear that eq. (302) implies the vanishing of 54 of the quantized charges $p^{\Lambda\Sigma}, q_{\Lambda\Sigma}$, the surviving two charges transforming as a doublet of $O(1,1)$ according to eq. (292). The only non-trivial evolution equation relates P and T as follows:

$$\left(\widehat{P}\partial_{\mu}\Phi\gamma^{\mu}-\frac{3}{2}\mathrm{i}\,T^{(-)}_{\rho\sigma}\gamma^{\rho\sigma}\gamma^{0}\right)\epsilon_{A}=0 \tag{303}$$

where we have set $P=\widehat{P}d\Phi$ and Φ is the unique non trivial scalar field parametrizing $O(1,1)$.

In order to make this equation explicit we perform the usual static ansätze. For the metric we set the ansatz (149). The scalar fields are assumed to be radial dependent and for the vector field strengths we assume the ansatz of eq.(154) which adapted to the $E_{7(7)}$ notation reads as follows:

$$\begin{aligned} F^{-\Lambda\Sigma} &= \frac{1}{4\pi}t^{\Lambda\Sigma}\,(r)\,E^{(-)} \qquad (304)\\ t^{\Lambda\Sigma}\,(r) &= 2\pi\,(g+\mathrm{i}\ell\,(r))^{\Lambda\Sigma} \qquad (305) \end{aligned}$$

The anti self dual form $E^{(-)}$ was defined in eq. (151). Using (283), (304) we have

$$T^{-}_{ab}=\mathrm{i}\ t^{\Lambda\Sigma}(r)E^{-}_{ab}C^{AB}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}{}_{AB}\,. \tag{306}$$

A simple gamma matrix manipulation gives further

$$\gamma_{ab}\, E^{\mp}_{ab} = 2\mathrm{i}\frac{e^{2U}}{r^3} x^i \gamma^0 \gamma^i \left(\frac{\pm 1 + \gamma_5}{2}\right) \tag{307}$$

and we arrive at the final equation

$$\frac{d\Phi}{dr} = -\frac{\sqrt{3}}{4}\ell(r)^{\Lambda\Sigma}\, \mathrm{Im}\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}\, f^{\Gamma\Delta}{}_{AB}\, \frac{e^U}{r^2}\,. \tag{308}$$

In eq. (308), we have set $p^{\Lambda\Sigma} = 0$ since reality of the l.h.s. and of $f^{\Gamma\Delta}_{AB}$(see eq. (323)) imply the vanishing of the magnetic charge. Furthermore, we have normalized the vielbein component of the $Usp(8)$ singlet as follows

$$\widehat{P} = 4\sqrt{3} \tag{309}$$

which corresponds to normalizing the $Usp(8)$ vielbein as

$$P^{(singlet)}_{ABCD} = \frac{1}{16} P C_{[AB} C_{CD]} = \frac{\sqrt{3}}{4} C_{[AB} C_{CD]}\, d\Phi\,. \tag{310}$$

This choice agrees with the normalization of the scalar fields existing in the current literature. Let us now consider the gravitino equation (281). Computing the spin connention $\omega^a{}_b$ from equation (149), we find

$$\begin{aligned} \omega^{0i} &= \frac{dU}{dr}\frac{x^i}{r} e^{U(r)} V^0 \\ \omega^{ij} &= 2\frac{dU}{dr}\frac{x_k}{r}\,\eta^{k[i}\, V^{j]}\, e^U \end{aligned} \tag{311}$$

where $V^0 = e^U\, dt$, $V^i = e^{-U}\, dx^i$. Setting $\epsilon_A = e^{f(r)}\zeta_A$, where ζ_A is a constant chiral spinor, we get

$$\begin{aligned} &\left\{ \frac{df}{dr}\frac{x^i}{r} e^{f+U} \delta^B_A V^i + \Omega_{A,\alpha}{}^{B} \partial_i \Phi^\alpha e^f V^i \right. \\ &\left. -\frac{1}{4}\left(2\frac{dU}{dr}\frac{x^i}{r} e^U e^f \left(\gamma^0\gamma^i V^0 + \gamma^{ij} V_j\right)\right) \delta^B_A \;+\; \delta^B_A\, T^-_{ab}\gamma^{ab}\gamma^c\gamma^0 V_c \right\} \zeta_B = 0 \end{aligned} \tag{312}$$

where we have used eqs.(281),(282), (296). This equation has two sectors; setting to zero the coefficient of V^0 or of $V^i\gamma^{ij}$ and tracing over the A, B indices we find two identical equations, namely:

$$\frac{dU}{dr} = -\frac{1}{8}\ell(r)^{\Lambda\Sigma}\frac{e^U}{r^2} C^{AB} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta} f^{\Gamma\Delta}{}_{AB}. \tag{313}$$

Instead, if we set to zero the coefficient of V^i, we find a differential equation for the function $f\,(r)$, which is uninteresting for our purposes. Comparing now equations (308) and (313) we immediately find

$$\Phi = 2\sqrt{3}\, U \tag{314}$$

In order to compute the l.h.s. of eq.s (308), (313) and the lagrangian of the 1/2 model, we need the explicit form of the coset representative $\mathbb{L}$ given in equation (271). This will also enable us to compute explicitly the r.h.s. of equations (308), (313). In the present case the explicit form of $\mathbb{L}$ can be retrieved by exponentiating the $Usp\,(8)$ singlet generator. As stated in equation (273), the scalar vielbein in the $Usp\,(28,28)$ basis is given by the off diagonal block elements of $\mathbb{L}^{-1}d\mathbb{L}$, namely

$$\mathbb{P} = \begin{pmatrix} 0 & \overline{P}_{ABCD} \\ P_{ABCD} & 0 \end{pmatrix}. \tag{315}$$

¿From equation (310), we see that the $Usp\,(8)$ singlet corresponds to the generator

$$\mathbb{K} = \frac{\sqrt{3}}{4}\left(\begin{array}{c|c} 0 & C^{[AB}C^{HL]} \\ \hline C_{[CD}C_{RS]} & 0 \end{array}\right) \tag{316}$$

and therefore, in order to construct the coset representative of the $O\,(1,1)$ subgroup of $E_{7(7)}$, we need only to exponentiate $\Phi\mathbb{K}$. Note that $\mathbb{K}$ is a $Usp\,(8)$ singlet in the **70** representation of $SU\,(8)$, but it acts non-trivially in the **28** representation of the quantized charges $\left(q_{AB}, p^{AB}\right)$. It follows that the various powers of $\mathbb{K}$ are proportional to the projection operators onto the irreducible $Usp(8)$ representations **1** and **27** of the charges:

$$\mathbb{P}_1 = \frac{1}{8}C^{AB}C_{RS} \tag{317}$$

$$\mathbb{P}_{27} = (\delta^{AB}_{RS} - \frac{1}{8}C^{AB}C_{RS}). \tag{318}$$

Straightforward exponentiation gives

$$\exp(\Phi\mathbb{K}) = \cosh\left(\frac{1}{2\sqrt{3}}\Phi\right)\mathbb{P}_{27} + \frac{3}{2}\sinh\left(\frac{1}{2\sqrt{3}}\Phi\right)\mathbb{P}_{27}\mathbb{K}\mathbb{P}_{27} + \tag{319}$$

$$+\cosh\left(\frac{\sqrt{3}}{2}\Phi\right)\mathbb{P}_1 + \frac{1}{2}\sinh\left(\frac{\sqrt{3}}{2}\Phi\right)\mathbb{P}_1\mathbb{K}\mathbb{P}_1 \tag{320}$$

Since we are interested only in the singlet subspace

$$\mathbb{P}_1\exp[\Phi\mathbb{K}]\mathbb{P}_1 = \cosh(\frac{\sqrt{3}}{2}\Phi)\mathbb{P}_1 + \frac{1}{2}\sinh(\frac{\sqrt{3}}{2}\Phi)\mathbb{P}_1\mathbb{K}\mathbb{P}_1 \tag{321}$$

$$\mathbb{L}_{singlet} = \frac{1}{8}\left(\begin{array}{c|c} \cosh(\frac{\sqrt{3}}{2}\Phi)C^{AB}C_{CD} & \sinh(\frac{\sqrt{3}}{2}\Phi)C^{AB}C^{FG} \\ \hline \sinh(\frac{\sqrt{3}}{2}\Phi)C_{CD}C_{LM} & \cosh(\frac{\sqrt{3}}{2}\Phi)C_{CM}C^{FG} \end{array}\right). \tag{322}$$

Comparing (322) with the equation (271), we find [10]:

$$f = \frac{1}{8\sqrt{2}} e^{\frac{\sqrt{3}}{2}\Phi} C^{AB} C_{CD} \tag{323}$$

$$h = -\mathrm{i}\,\frac{1}{8\sqrt{2}}\, e^{-\frac{\sqrt{3}}{2}\Phi} C_{AB} C_{CD} \tag{324}$$

and hence, using $\mathcal{N} = hf^{-1}$, we find

$$\mathcal{N}_{AB\,CD} = -\,\mathrm{i}\,\frac{1}{8}\, e^{-\sqrt{3}\Phi} C_{AB} C_{CD} \tag{325}$$

so that we can compute the r.h.s. of (308), (313). Using the relation (314) we find a single equation for the unknown functions $U\,(r)$, $\ell\,(r) = C_{\Lambda\Sigma}\ell^{\Lambda\Sigma}\,(r)$

$$\frac{dU}{dr} = \frac{1}{8\sqrt{2}} \frac{\ell\,(r)}{r^2} \exp\left(-2U\right) \tag{326}$$

At this point to solve the problem completely we have to consider also the second order field equation obtained from the lagrangian. The bosonic supersymmetric lagrangian of the 1/2 preserving supersymmetry case is obtained from equation (275) by substituting the values of P_{ABCD} and $\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}$ given in equations (310) and (277) into equation (275). We find

$$\mathcal{L} = 2R - e^{-\sqrt{3}\Phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi \tag{327}$$

Note that this action has the general form of 0–brane action in $D = 4$. According to this we expect a solution where:

$$\begin{aligned}
U &= -\frac{1}{4}\log H(r) \\
\Phi &= -\frac{\sqrt{3}}{2}\log H(r) \\
\ell &= 2r^3 \frac{d}{dr}\left(H(r)\right)^{-\frac{1}{2}} = k \times \left(H(r)\right)^{-\frac{3}{2}}
\end{aligned} \tag{328}$$

where $H(r) = 1 + k/r$ denotes a harmonic function.

The resulting field equations are

<u>*Einstein equation:*</u>

$$U'' + \frac{2}{r}U' - (U)^2 = \frac{1}{4}(\Phi')^2 \tag{329}$$

[10] Note that we are we are writing the coset matrix with the same pairs of indices $AB, CD, \ldots$ without distinction between the pairs $\Lambda\Sigma$ and AB as was done in sect. (5.1)

Maxwell equation:

$$\frac{d}{dr}(e^{-\sqrt{3}\Phi}\ell(r)) = 0 \tag{330}$$

Dilaton equation:

$$\Phi'' + \frac{2}{r}\Phi' = -e^{-\sqrt{3}\Phi+2U}\ell(r)^2\frac{1}{r^4}\,. \tag{331}$$

¿From Maxwell equations one immediately finds

$$\ell\,(r) = e^{\sqrt{3}\Phi(r)}. \tag{332}$$

Taking into account (314), the second order field equation and the first order Killing spinor equation have the common solution

$$\begin{aligned} U &= -\frac{1}{4}\log\, H(x) \\ \Phi &= -\frac{\sqrt{3}}{2}\log\, H(x) \\ \ell &= H(x)^{-\frac{3}{2}} \end{aligned} \tag{333}$$

where:

$$H(x) \equiv 1 + \sum_i \frac{k_i}{\vec{x} - \vec{x}_i^0} \tag{334}$$

is a harmonic function describing 0–branes located at $\vec{x}_\ell^0$ for $\ell = 1, 2, \ldots,$ each brane carrying a charge k_i. In particular for a single 0–brane we have:

$$H(x) = 1 + \frac{k}{r} \tag{335}$$

and the solution reduces to the expected form (328).

6 Conclusions

In this paper we have given an account of the deep connection due to supersymmetry among the central charges of supergravity theory, BPS states and the Bekenstein–Hawking entropy of extremal black holes. One of the most relevant results concerns the structure of the entropy formula, which turns out to depend only on the quantized electric and magnetic charges of the theory. Actually, the entropy is proportional to some group theoretical invariants which can be constructed out of the duality group of the corresponding supergravity theory.

An important point, not discussed in this paper, is the fact that four-dimensional black–hole configurations can be interpreted as the four-dimensional appearence of more general configurations named "black–p-branes" (namely p dimensional extended objects in D dimensions interpolating between flat Minkowski space at spatial infinity and the horizon geometry) in higher D dimensional supergravities. When the extra D−4 spatial dimensions are compactified, the p-brane configuration can be suitably wrapped over some p-dimensional homology cycles of the compact space, giving rise to a point-like configuration, namely the four-dimensional black hole. This approach has been very fruitful and has a wide range of applications, among which we mention the fact that it allows a statistical interpretation of the black-hole entropy. The reader interested in the subject is referred to the literature [5],[6],[8],[9].

Acknowledgements

Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime, in which L. Andrianopoli and R. D'Auria are associated to Torino University.

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