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Multitouch Options

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Abstract: In this article, the multitouch option, also called the $n-$touch option (or the "baseball" option when $n = 3$) is analyzed and valued in closed form. This is a kind of barrier option that has been traded for a long time on the markets, but that does not yet admit a known valuation formula. The multitouch option sets a gradual knock-out/knock-in mechanism based on the number of times the underlying asset has crossed a predefined barrier in various time intervals before expiry. The higher the number of predefined time intervals during which the barrier has been touched, the lower the value of a knock-out contract at expiry, and conversely for a knock-in one. Multitouch options can be viewed as an extension of step barrier options, preserving the ability of the latter to adjust the exposure to risk over time, while eliminating the notorious danger of "sudden death" that holders of step barrier options are faced with. They are thus less risky and more flexible than step barrier options, and all the more so when compared to standard barrier options. This article also provides closed-form valuation of multitouch options with nonstandard features such as an outside barrier or a barrier defined as a continuous function of time.

Keywords: multitouch option; n-touch option; baseball option; barrier option; step barrier option; outside barrier; moving barrier; first passage time; boundary crossing probability; dimension; multivariate Gaussian integral

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1. Introduction

Barrier options are the most heavily traded nonstandard European options in the financial markets, particularly in the foreign exchange ones. They are also embedded in a lot of popular structured derivatives in stock and interest rate markets (see, e.g., Bouzoubaa and Osseiran 2010). Moreover, as analytical tools, they are at the core of the modeling of major financial phenomena such as default risk, in the so-called "structural models" (see, e.g., Bielecki and Rutkowski 2004). The reader unacquainted with barrier options may refer to, e.g., Cont (2010) or to an online financial encyclopedia for basic facts and definitions.

Since their first appearance as traded contracts in the 1970s, there have been a huge number of variations in their payoff, leading to a wide variety of nonstandard barrier options. Among the most well-known of these are the partial-time, the outside and the step barrier options. The specificity of partial-time barrier options is that barrier crossing is not monitored during the entire option's lifetime. It may end before expiry ("early-ending" barrier) or start after the contract's inception ("forward-start" barrier). Heynen and Kat (1994a) and Carr (1995) were the first to publish exact formulae for early-ending and forward-start barrier options. More generally, barrier monitoring may start any time after the contract's inception and terminate any time before expiry. This flexible specification of the time during which a barrier is active, known as a "window", was handled by Armstrong (2001) for single barriers (also called one-sided barriers) and by Guillaume (2003) for double barriers (also called two-sided barriers) and combinations of one-sided and two-sided barriers. The knock-out or knock-in condition during the option's lifetime and the moneyness condition at expiry may also be defined w.r.t. two different underlying assets. This is what characterizes an outside option, which was first valued by Heynen and Kat (1994b). Finally, instead of being constant, the barrier may be piecewise constant, i.e., defined as a step function: the option's lifetime is divided into several time intervals in which the barrier takes different values. The exact analytical valuation of step barrier functions was first achieved by Guillaume (2001) when the barrier is one-sided and by Guillaume (2010) when the barrier is two-sided.

More recent contributions in the literature on barrier options primarily focus on numerical methods of approximation under models other than the standard geometric Brownian motion, such as stochastic volatility (e.g., Carr et al. 2020; Cao et al. 2023), stochastic volatility and jumps (e.g., Guardasoni and Sanfelici 2016), and Markov regime switching (e.g., Zhang and Li 2022), which are not the subject of this article.

A major reason for the success of barrier options is that they allow investors to choose the market scenarios they want to be insured against, i.e., only those that are adverse to their positions, unlike the vanilla option that hedges them against all possible scenarios, including those that are favorable to their positions. As such, barrier options are both more flexible and less expensive than vanilla options. In addition, partial-time barrier options also allow investors to choose the time intervals on which they want to be hedged, while step barrier options allow them to modulate the level of the barrier during the option's life. As for outside barrier options, they make it possible to manage the effect of volatility by combining a low volatility on the asset to which a knock-out barrier is assigned and a high volatility on the asset whose moneyness is tested at expiry. For more background on how to make an optimal use of all these instruments, the reader may refer to Das (2006).

However, all the aforementioned barrier option contracts have one common limitation, i.e., the crossing of the barrier is designed as an "all or nothing" triggering mechanism. Indeed, a single passage at any moment that the barrier is active is enough to deprive a knock-out contract of all its value or to transform a knock-in contract into a vanilla option. For knock-out barriers, this is known as the "sudden death" risk. It is definitely an unattractive feature for investors in markets where a short-term volatility spike may entail a temporary breach of the barrier while the underlying asset has spent the vast majority of its time inside the authorized fluctuation range. It also makes hedging more difficult for traders, who are faced with discontinuous deltas and gammas going to infinity in the vicinity of the barrier. Various solutions to this problem have already been put forward. One of the oldest and simplest ones is the "soft barrier" (Hart and Ross 1994), in which the knock-out or knock-in provision is defined as a range between an upper level and a lower level, and different percentages of the option's payoff at expiry are paid out to the option's holder according to the highest or lowest point reached in this range during the option's lifetime. Another approach consists in defining the option's payoff as a function of the time spent above or below the barrier. The corresponding contract is known as "occupation-time derivatives". This approach was pioneered by Chesney et al. (1997) under the name of the "Parisian option" and by Linetsky (1999) under the name of "step option" (which is not to be confused with a step barrier option).

Multitouch options develop an alternative way of dealing with the "all or nothing" problem associated with traditional barrier options, which consists in setting a gradual knock-out/knock-in mechanism, based neither on the location of the maximum or minimum observed value of the underlying asset price within a range, nor on a measure of the occupation time of the underlying asset within an authorized fluctuation range, but rather on the number of times the underlying asset has crossed a predefined barrier in various time intervals before expiry. The higher the number of predefined time intervals during which the barrier has been touched, the lower the value of a knock-out contract at expiry, and conversely for a knock-in one. The $n-$touch option allows investors to weigh different knock-out or knock-in scenarios according to the number of passages to the barrier, whereas standard barrier options do not allow for distinguishing among these scenarios. This makes the multitouch barrier option a more flexible instrument that can better adapt to the investors' expectations or needs. Compared with a standard knock-out barrier option, an $n-$touch knock-out option not only makes it possible to adjust the exposure to risk over time in the same way as a step barrier option, but it also provides a multichance game, allowing its holder to receive a positive payoff at expiry even if the knock-out barrier has been breached.

The number of crossings on a finite time interval is a stochastic process that can be called the crossing counting process. Unlike other existing contracts, the multitouch barrier option is based on a measure of the frequency of barrier crossings or, equivalently, on a measure of the intensity of the crossing counting process defined as the mean number of crossings per time unit. For instance, with a standard barrier, or a step barrier, or a partial-time barrier, a process may cross the barrier once and then never cross it again until expiry. With an occupation-time contract, a process may spend some time within the required barrier range (i.e., below an up-and-out barrier and above a down-and-out barrier), and then spend all the time left until expiry outside this range. Meanwhile, in a multitouch setting, if the process has crossed the barrier at least once in each of the time intervals that partition the option's lifetime, and the number of these time intervals is large enough, then there cannot be any significant period of time during which the process has been continuously out of the barrier range. With this new instrument, what matters is not whether the process has hit the barrier range once, nor how long the process has stayed inside the barrier range, but how often it has visited this range, even for a very short period of time.

Despite the attractive features of multitouch options and the fact they have been traded for a long time io the markets, there is currently no available valuation formula for such instruments. To the best of our knowledge, there is not even a single published paper on this important topic among the vast literature on barrier options. The main contribution of this article is to show that a no-arbitrage exact value of a multitouch barrier option can be analytically computed in a standard geometric Brownian motion model, at least for a moderate number of barrier crossings. A few extensions to the more general payoffs and shapes of the barrier are also tackled for the first time, including an outside barrier and a barrier defined as a continuous function of time. Moreover, the resulting formulae are closed-form and easy to evaluate numerically, and can thus be directly implemented. This article is organized as follows: Section 2 provides a detailed description of the contracts under consideration, as well as a number of numerical results aimed at comparing multitouch barrier option prices with standard barrier option and step barrier option prices; Section 3 provides a proof of the valuation formula for a standard multitouch barrier option; Section 4 shows how to value an outside multitouch barrier option, as well as a multitouch barrier option, with a barrier defined as a piecewise exponential affine function of time, and discusses the possibility of an analytical valuation of multitouch barrier options with large numbers of barrier crossings.

2. Detailed Payoff and First Series of Numerical Results

The specificity of multitouch barrier options is to set a gradual knock-out/knock-in mechanism according to the number of times the underlying asset has hit a predefined barrier in various time intervals before expiry. In contrast with standard barrier options and their usual variants such as partial-time or outside barrier options, the knock-out/knock-in mechanism is not triggered once and for all by a single passage to the barrier. Instead, several levels of deactivation/activation are defined, depending on the number of hits by the underlying asset during the option's life. A fraction of the standard call or put's payoff is assigned to each number of hits. This fraction is a decreasing function of the number of hits if the option is of the knock-out type, while it is increasing if the option is of the knock-in type. Thus, a knock-out multitouch option does not expose the option's holder to the notorious risk of "sudden death" which is typical of a standard knock-out barrier option, whereby they lose the entirety of their claim the moment the underlying asset crosses the barrier before the option's expiry.

More precisely, let us denote as S, K and T the underlying asset, the strike price and the option's expiry, respectively, and let us divide the option's lifetime into n intervals $[t_0 = 0, t_1], \ldots, [t_{n-1}, t_n = T]$. A knock-out or knock-in barrier is defined, the standard form of which is a piecewise constant function (also called a step function), i.e., a constant barrier $H_i > 0$ is associated with each time sub-interval $[t_{i-1}, t_i], \forall i \in \{1, \ldots, n\}$. However, other shapes can be specified for the barrier. For example, an extension of the valuation method to exponentially curved barriers is introduced in Section 4.

Then, a multitouch barrier call option of order n or, to put it more simply, an $n-$touch call option, provides its holder with the following payoff:

$$\sum_{i=0}^{n} \omega_i \mathbf{1}_{\{\eta(S(t), t \in [0,T]) = i\}} (S(T) - K)^+ \tag{1}$$

where $\eta(S(t), t \in [0,T]) \in \mathbb{N}$ is the number of predefined time intervals in which the barrier has been hit at least once, each $\omega_i \in \mathbb{R}_+$ represents a rate of participation in the payoff at expiry and $\mathbf{1}_{\{.\}}$ is the indicator function taking value 1 if its argument is true and zero otherwise.

An $n-$touch put option's payoff is defined similarly. A standard knock-out step barrier call is retrieved by setting $\omega_0 = 1$ and $\omega_i = 0$ for all $i \neq 0$, because then the indicator function in (1) takes value zero for any value of $\eta(S(t), t \in [0,T])$ other than zero, and also because any value of ω_0 other than 1 would result in a higher or a lower option value relative to that of the knock-out step barrier call. In the case $n = 3$, the $n-$touch option is sometimes called a "baseball" option. The name is derived from baseball game parlance, "three strikes and you are out".

There can be various ways to choose the ω_i's. The simplest choice is to fix each ω_i in the option's contract. However, you might want to make the ω_i's path-dependent, e.g., define them as functions of the maximum or minimum values of the underlying asset observed in each time interval $[t_{i-1}, t_i]$. In the remainder of this article, analytical results will be provided under the assumption that the $\omega_i's$ are simply a sequence of participation rates fixed in the option's contract.

In a standard $n-$touch barrier option, the predefined time intervals $[t_0 = 0, t_1], \ldots,$ $[t_{n-1}, t_n = T]$ form a partition of $[0, T]$. When the length of the union of nonintersecting predefined time intervals is smaller than the length of $[0, T]$, the $n-$touch barrier option is of a partial-time type.

Let us now provide a few illustrations of how payoffs can be formulated in more detail. For instance, the payoff on a standard 2-touch up-and-out put with expiry $T = t_2$ can be expanded as follows:

$$(K - S(t_2)) \left\{ \begin{array}{l} \omega_0 \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, S(t_2) < K\}} + \omega_1 \left(\mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, S(t_2) < K\}} + \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 \geq H_2, S(t_2) < K\}} \right) \\ + \omega_2 \mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, S(t_2) < K\}} \end{array} \right\} \tag{2}$$

where $\overline{S}_i^j = \sup_{t_i \leq t \leq t_j} S(t)$.

Likewise, the payoff on a 3-touch up-and-out put with expiry $T = t_3$ is given by:

$$(K - S(t_3)) \times I'(\omega_0 I_1 + \omega_1(I_2 + I_3 + I_4) + \omega_2(I_5 + I_6 + I_7) + \omega_3 I_8) \tag{3}$$

where:

$$I' = \mathbf{1}_{\{S(t_3) < K\}}, \; I_1 = \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3\}}, \; I_8 = \mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 \geq H_3\}} \tag{4}$$

$$I_2 = \mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3\}}, \; I_3 = \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3\}}, \; I_4 = \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 \geq H_3\}}$$

$$I_5 = \mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3\}}, I_6 = \mathbf{1}_{\{\overline{S}_0^1 < H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 \geq H_3\}}, I_7 = \mathbf{1}_{\{\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 \geq H_3\}}$$

Other knock-out or knock-in payoffs can be easily expanded in a similar manner by using the law of total probability. For instance, the payoff on a 3-touch down-and-in call writes:

$$(S(t_3) - K) \times J'(\omega_0 J_1 + \omega_1(J_2 + J_3 + J_4) + \omega_2(J_5 + J_6 + J_7) + \omega_3 J_8) \tag{5}$$

where:

$$J' = \mathbf{1}_{\{S(t_3)>K\}}, J_1 = \mathbf{1}_{\{\underline{S}_0^1 \le H_1, \underline{S}_1^2 \le H_2, \underline{S}_2^3 \le H_3\}}, J_8 = \mathbf{1}_{\{\underline{S}_0^1 > H_1, \underline{S}_1^2 > H_2, \underline{S}_2^3 > H_3\}} \tag{6}$$

$$J_2 = \mathbf{1}_{\{\underline{S}_0^1 \le H_1, \underline{S}_1^2 > H_2, \underline{S}_2^3 > H_3\}}, \ J_3 = \mathbf{1}_{\{\underline{S}_0^1 > H_1, \underline{S}_1^2 \le H_2, \underline{S}_2^3 > H_3\}}, \ J_4 = \mathbf{1}_{\{\underline{S}_0^1 > H_1, \underline{S}_1^2 > H_2, \underline{S}_2^3 \le H_3\}}$$

$$J_5 = \mathbf{1}_{\{\underline{S}_0^1 \le H_1, \underline{S}_1^2 \le H_2, \underline{S}_2^3 > H_3\}}, \ J_6 = \mathbf{1}_{\{\underline{S}_0^1 > H_1, \underline{S}_1^2 \le H_2, \underline{S}_2^3 \le H_3\}}, \ J_7 = \mathbf{1}_{\{\underline{S}_0^1 \le H_1, \underline{S}_1^2 > H_2, \underline{S}_2^3 \le H_3\}}$$

$$\underline{S}_i^j = \inf_{t_i \le t \le t_j} S(t)$$

It is clear that any multitouch barrier option can be decomposed into a portfolio of nonstandard step barrier options combining various up-and-in, up-and-out, down-and-in, and down-and-out steps.

Let us focus on the valuation of a 3-touch up-and-out put with expiry t_3. Following the martingale equivalent method of option pricing, the no-arbitrage value of this option in a Black–Scholes model is given by:

$$\exp(-rt_3)E_Q\left[(K - S(t_3))^+ \sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right] \tag{7}$$

$$= \exp(-rt_3)\left\{KE_Q\left[\mathbf{1}_{\{I'\}} \times \sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right] - E_Q\left[S(t_3) \times \mathbf{1}_{\{I'\}} \times \sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right]\right\} \tag{8}$$

where:

$$\sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}} = \omega_0 I_1 + \omega_1(I_2 + I_3 + I_4) + \omega_2(I_5 + I_6 + I_7) + \omega_3 I_8 \tag{9}$$

- Q is the classical "risk-neutral" measure (i.e., the unique equivalent martingale measure in the Black–Scholes model) under which the stochastic differential of S writes:

$$dS(t) = rS(t)dt + \sigma S(t)dB(t) \tag{10}$$

in which r is the riskless rate, $\sigma \in \mathbb{R}_+$ and $B(t)$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$.

After an elementary application of the Cameron–Martin–Girsanov theorem, the value of the 3-touch up-and-out put becomes:

$$\sum_{i=1}^{3} \omega_i \left\{ e^{-rt_3} KE_Q\left(I' \times \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right) - S(0)E_{Q^{(S)}}\left(I' \times \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right)\right\} \tag{11}$$

where $Q^{(S)}$ is the classical forward-neutral measure whose Radon–Nikodym derivative w.r.t. Q is given by:

$$\frac{dQ^{(S)}}{dQ}\bigg|\mathcal{F}_t = \exp\left(\sigma B(t) - \frac{\sigma^2}{2}t\right) \tag{12}$$

Therefore, it suffices to compute each $E_Q\left(I' \times \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right), i \in \{0, \ldots, 3\}$. Each $E_{Q^{(S)}}\left(I' \times \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right)$ will then be inferred by a mere change in drift in the stochastic differential of S. The detailed computation of each $E_Q\left(I' \times \mathbf{1}_{\{\eta(S(t), t\in[0,T])=i\}}\right)$ is provided in Section 3. Meanwhile, we proceed with some numerical results. In Tables 1–6, the prices of four different types of options are compared as functions of the underlying asset's volatility: vanilla put, standard UOP (up-and-out put), 3-step UOP, and 3-touch UOP. We focus on an up-and-out barrier, since this is the consistent and most widespread form of insurance against adverse movements in the market on a long spot position. The inputs of the tables vary according to the direction of the steps (increasing or decreasing), the options' expiry, and the options' moneyness. In Table 1, the step function is decreasing, while it is increasing in Table 2. In Tables 3 and 4, the options' expiry is extended. In Tables 5 and 6, the moneyness of the options is changed, from ATM (at-the-money) in Tables 1–4 to ITM (in-the-money) in Table 5 and OTM (out-of-the-money) in Table 6. All reported prices are computed using exact analytical formulae: the ones for the put and UOP options can be found in textbooks (see, e.g., Shreve 2010); those for step barrier options are given by Guillaume (2001, 2015) and those for multitouch barrier options are provided in this paper.

Table 1. Short-term, ATM, decreasing step barrier.

	Vol = 18%	**Vol = 36%**	**Vol = 64%**
Vanilla put	4.21028552	9.19640912	16.8915617
Standard UOP	3.87930345	6.11543647	7.41655712
3-step UOP	4.06327282	7.12573436	9.26066970
3-touch UOP	4.14387237	8.24464223	13.4221543

Table 2. Short-term, ATM, increasing step barrier.

	Vol = 18%	**Vol = 36%**	**Vol = 64%**
Vanilla put	4.21028552	9.19640912	16.8915617
Standard UOP	3.87930345	6.11543647	7.41655712
3-step barrier UOP	3.94774692	6.28171713	7.57564760
3-touch barrier UOP	4.12363140	8.08141477	13.0947085

Table 3. Longer term, ATM, decreasing step barrier.

	Vol = 18%	**Vol = 36%**	**Vol = 64%**
Vanilla put	6.77089322	16.2132539	30.4462253
Standard UOP	4.37918160	6.33005693	7.39619749
3-step barrier UOP	5.13331495	8.00612395	9.68798236
3-touch barrier UOP	5.96475363	12.3318524	21.0176847

Table 4. Longer term, ATM, increasing step barrier.

	Vol = 18%	**Vol = 36%**	**Vol = 64%**
Vanilla put	6.77089322	16.2132539	30.4462253
Standard UOP	4.37918160	6.33005693	7.39619749
3-step barrier UOP	4.51329511	6.46393309	7.49064685
3-touch barrier UOP	5.85533987	12.0464166	20.5970811

Table 5. Longer term, ITM, decreasing step barrier.

	Vol = 18%	Vol = 36%	Vol = 64%
Vanilla put	11.5899127	21.6573788	36.654626
Standard UOP	6.68560137	7.88644982	8.55349279
3-step barrier UOP	7.92208416	10.0057384	11.2146769
3-touch barrier UOP	9.65006219	16.0036889	24.9762416

Table 6. Longer term, OTM, decreasing step barrier.

	Vol = 18%	Vol = 36%	Vol = 64%
Vanilla put	3.37956019	11.4940697	24.6215650
Standard UOP	2.41880819	4.82385750	6.24758108
3-step barrier UOP	2.78825911	6.07420063	8.17314591
3-touch barrier UOP	3.11901323	9.02191302	17.2485329

In all tables, the following specifications hold:

- The underlying asset's value at the beginning of the option's life t_0 is $S(0) = 100$ and the riskless rate is equal to 3.5%.
- In the "short-term" setting, the option's expiry t_3 is equal to 6 months, while t_3 is 2 years in the "longer term" setting.
- The value of the constant knock-out barrier of the UOP option is equal to 110.
- The increasing up-and-out 3-step barrier is defined as the vector $[H_1 = 110, H_2 = 112, H_3 = 114]$, while the decreasing up-and-out 3-step barrier is defined as the vector $[H_1 = 114, H_2 = 112, H_3 = 110]$.
- The time intervals associated with each step are of equal size, i.e., $[t_0, t_1] = [t_1, t_2] = [t_2, t_3] = t_3/3$ (note, however, that unequal sizes of the time intervals are handled just as well by the analytical formula derived in Section 3).
- The weighting coefficients of the 3-touch UOP options are $\omega_0 = 1, \omega_1 = 0.75, \omega_2 = 0.5, \omega_3 = 0.25$.

Overall, the price differential observed between a standard UOP and a 3-touch UOP is substantial, reflecting the higher probability that the latter option will not expire worthless. The only setting in which the price differential is small is when volatility is low (18%) and expiry is short-term. However, this is the least significant setting inasmuch as all option prices are close to one another in it. When volatility is intermediate (36%) and the option is ATM, the price differential increases to 27% on a short-term expiry and it almost doubles on a longer time expiry. When volatility is high (64%) and the option is ATM, the price differential almost triples on a longer time expiry. The prices of ITM and OTM options display similar patterns.

Since a multitouch barrier option can be decomposed into a weighted sum of step barrier options, its value is sensitive to the price determinants specifically attached to step barrier options, such as the ordering of the steps (i.e., the distribution of the steps over time according to each step's distance to the origin S_0) and the relative sizes of the time intervals associated with each step. In this respect, one can notice that the prices of multitouch UOP options with decreasing steps in Tables 1 and 3 are higher than the prices of multitouch UOP options with increasing steps in Tables 2 and 4. For an explanation of this phenomenon and further insights into the specific price determinants of step barrier options, one can refer to Guillaume (2015).

Of course, the price differential between an UOP and a multitouch UOP is heavily dependent on the choice of the ω_i's, which is freely negotiated between the buyer and the seller of the option. If one decides to normalize the sum $\sum_{i=0}^{n} \omega_i$ to 1, then the prices of multitouch knock-out barrier options become lower than those of standard knock-out barrier options, which shows that multitouch barrier options can also be used to lower the cost of hedging, relative to standard barrier options. For instance, if we set $\omega_0 = 0.5, \omega_1 = 0.25, \omega_2 = 0.15, \omega_3 = 0.1$, then the prices of ATM, 2-year expiry, 3-touch UOP options with decreasing steps become 2.830891789, 5.391604028 and 8.504985192 when the volatility is 0.18%, 0.36% and 0.64%, respectively.

3. Analytical Valuation of Standard n-Touch Barrier Options

In this section, we show how to find an exact formula for the no-arbitrage value of a 3-touch up-and-out put, from which the values of other types of 3-touch barrier options can be inferred, as will be subsequently explained.

We begin by dealing with the computation of $E_Q(I' \times I_1)$ as defined in Section 2, which is the probability required to value a 3-step up-and-out put.

Let $\left\{X(t) = \ln\left(\frac{S(t)}{S(0)}\right), t \geq 0\right\}$. Then, by conditioning with respect to the absolutely continuous random variables $X(t_1)$, $X(t_2)$ and $X(t_3)$, and by using the Markov property of process X, the distribution under consideration can be written as the following multiple integral:

$$E_Q(I' \times I_1) = \int_{x_1=-\infty}^{h_1 \wedge h_2} \int_{x_2=-\infty}^{h_2 \wedge h_3} \int_{x_3=-\infty}^{k \wedge h_3} Q\left(\overline{X}_0^1 < h_1, X(t_1) \in dx_1\right) Q\left(\overline{X}_1^2 < h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right)$$
$$Q\left(\overline{X}_2^3 < h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \quad (13)$$

Since X is a Gaussian process, the random vector $[X(t_1), X(t_2), X(t_3)]$ follows a trivariate normal distribution. Under Q, each $X(t_i)$ has expectation μt_i, where $\mu = r - \sigma^2/2$, and variance $\sigma^2 t_i$, and the correlation coefficient between $X(t_i)$ and $X(t_j)$ is given by $\rho_{i.j} = \sqrt{\frac{t_i}{t_j}}$, $\forall i, j \in \mathbb{N}, i \leq j$. The first probability inside the integral in (13) is obtained by differentiating the classical formula for the joint cumulative distribution of the extremum of a Brownian motion with drift and its endpoint over a closed time interval (see, e.g., Shreve 2010). The next two probabilities can be obtained by using the following simple lemma.

Lemma 1. *Let $\{S(t), t \geq 0\}$ be a geometric Brownian motion whose instantaneous variations under a given probability measure P are driven by:*

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t) \quad (14)$$

where $B(t)$ is a standard Brownian motion, and $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$.

Let K and H be two positive real numbers such that $H > S(0)$ and $K \leq H$. Let T be a finite positive real number. Then, we have

$$P\left(\sup_{0<t\leq u \leq T} S(u) \leq H, S(T) \leq K | S(t) = S(0)e^x\right)$$
$$= N\left[\frac{k - x - \mu(T-t)}{\sigma\sqrt{T-t}}\right] - \exp\left(\frac{2\mu}{\sigma^2}(h-x)\right) N\left[\frac{k - 2h + x - \mu(T-t)}{\sigma\sqrt{T-t}}\right] \quad (15)$$

where $k = \ln\left(\frac{K}{S(0)}\right)$, $h = \ln\left(\frac{H}{S(0)}\right)$ and $\mu = \alpha - \sigma^2/2$ and $N[b]$, $\forall b \in \mathbb{R}$, is the univariate standard normal distribution function.

Proof of Lemma 1. It is a corollary of a classical result given by Levy (1939) that:

$$P\left(\sup_{t \leq u \leq T} S(u) \leq H, S(T) \leq K | S(t)\right)$$
$$= N\left[\frac{\ln\left(\frac{K}{S(t)}\right) - \mu(T-t)}{\sigma\sqrt{T-t}}\right] - \left(\frac{H}{S(t)}\right)^{\frac{2\mu}{\sigma^2}} N\left[\frac{\ln\left(\frac{K}{S(t)}\right) - 2\ln\left(\frac{H}{S(t)}\right) - \mu(T-t)}{\sigma\sqrt{T-t}}\right] \quad (16)$$

which can be rewritten as:

$$P\left(\sup_{t\le u\le T} S(u)\le H, S(T)\le K|S(t)\right) \tag{17}$$

$$= N\left[\frac{\ln\left(\frac{K}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)-\mu(T-t)}{\sigma\sqrt{T-t}}\right]$$

$$-\exp\left(\frac{2\mu}{\sigma^2}\left(\ln\left(\frac{H}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)\right)\right)N\left[\frac{\ln\left(\frac{K}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)-2\left(\ln\left(\frac{H}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)\right)-\mu(T-t)}{\sigma\sqrt{T-t}}\right]$$

Therefore, by conditioning with respect to $\ln\left(\frac{S(t)}{S(0)}\right)$, we obtain:

$$E_P\left[\mathbf{1}_{\{\sup_{0<t\le u\le T} S(u)\le H, S(T)\le K\}}|S(0)\right] \tag{18}$$

$$=\int_{-\infty}^{h} P\left(\ln\left(\frac{S(t)}{S(0)}\right)\in dx\right)P\left(\begin{array}{l}\sup_{0<t\le u\le T}\ln\left(\frac{S(u)}{S(t)}\right)\le\ln\left(\frac{H}{S(0)}\right),\ln\left(\frac{S(T)}{S(t)}\right)\le\ln\left(\frac{K}{S(0)}\right)\\ \left|\ln\left(\frac{S(t)}{S(0)}\right)\in dx\right.\end{array}\right)$$

$$=\int_{-\infty}^{h}\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) \tag{19}$$

$$\left\{N\left[\frac{k-x-\mu(T-t)}{\sigma\sqrt{T-t}}\right]-\exp\left(\frac{2\mu}{\sigma^2}(h-x)\right)N\left[\frac{k-x-2(h-x)-\mu(T-t)}{\sigma\sqrt{T-t}}\right]\right\}dx$$

□

Let us now define the probability density functions φ_1, φ_2, ϕ_1 and ϕ_2 as follows:

$$\varphi_1(x_i)dx_i = Q(X(t_i)\in dx_i) = \frac{\exp\left(-\frac{1}{2}\left(\frac{x_i-\mu t_i}{\sigma\sqrt{t_i}}\right)^2\right)}{\sigma\sqrt{2\pi t_i}} \tag{20}$$

$$\varphi_2(x_i,x_j)dx_i dx_j = Q(X(t_j)\in dx_j|X(t_i)\in dx_i)$$

$$=\frac{\exp\left(-\frac{1}{2}\left(\frac{x_j-x_i-\mu(t_j-t_i)}{\sigma\sqrt{t_j-t_i}}\right)^2\right)}{\sigma\sqrt{2\pi(t_j-t_i)}}=\frac{\exp\left(-\frac{1}{2\left(1-\rho_{i,j}^2\right)}\left(\frac{x_j-\mu t_j}{\sigma\sqrt{t_j}}-\rho_{i.j}\frac{x_i-\mu t_i}{\sigma\sqrt{t_i}}\right)^2\right)}{\sigma\sqrt{2\pi t_j\left(1-\rho_{i,j}^2\right)}} \tag{21}$$

$$\phi_1(x_i)dx_i = Q\left(\overline{X}_0^i<h_i, X(t_i)\in dx_i\right)=\varphi_1(x_i)dx_i-\frac{\exp\left(\frac{2\mu h_i}{\sigma^2}-\frac{1}{2}\left(\frac{x_i-2h_i-\mu t_i}{\sigma\sqrt{t_i}}\right)^2\right)}{\sigma\sqrt{2\pi t_i}} \tag{22}$$

$$\phi_2(x_i,x_j)dx_i dx_j = Q\left(\overline{X}_i^j<h_j, X(t_j)\in dx_j|X(t_i)\in dx_i\right)$$

$$=\varphi_2(x_i,x_j)-\frac{\exp\left(\frac{2\mu}{\sigma^2}(h_j-x_i)-\frac{1}{2}\left(\frac{x_j+x_i-2h_j-\mu(t_j-t_i)}{\sigma\sqrt{t_j-t_i}}\right)^2\right)}{\sigma\sqrt{2\pi(t_j-t_i)}} \tag{23}$$

One can now express the valuation problem as the following explicit triple integral:

$$E_Q(I' \times I_1) = \int_{x_1=-\infty}^{h_1\wedge h_2} \int_{x_2=-\infty}^{h_2\wedge h_3} \int_{x_3=-\infty}^{k\wedge h_3} \phi_1(x_1)\phi_2(x_1,x_2)\phi_2(x_2,x_3)dx_3dx_2dx_1 \tag{24}$$

Let the function $\Phi_n[b_1,\ldots,b_n;\rho_{1.2},\ldots,\rho_{n-1.n}]$ be defined by the following convolution of Gaussian densities:

$$\Phi_n[b_1,\ldots,b_n;\rho_{1.2},\ldots,\rho_{n-1.n}] = \int_{D^n} \frac{\exp\left(-\frac{y_1^2}{2} - \sum_{i=2}^{n} \frac{(y_i-\rho_{i-1.i}y_{i-1})^2}{2\left(1-\rho_{i-1.i}^2\right)}\right)}{(2\pi)^{n/2}\prod_{i=2}^{n}\sqrt{1-\rho_{i-1.i}^2}} dy_n\ldots dy_1 \tag{25}$$

where $D^n =]-\infty,b_1] \times]-\infty,b_2] \ldots \times]-\infty,b_n]$, $b_i \in \mathbb{R}$, $\rho_{i-1.i} \in [0,1[$, $\forall i \in \{1,\ldots,n\}$.

One can notice that $\Phi_1[b_1] = N[b_1]$ and $\Phi_2[b_1,b_2;\rho_{1.2}] = N_2[b_1,b_2;\rho_{1.2}]$, where $N_2[b_1,b_2;\rho_{1.2}]$ is the bivariate standard normal distribution function. Then, performing the necessary calculations, one can obtain $E_Q(I' \times I_1)$ as given by (A1)–(A8) in Appendix A.

It is straightforward to show that the triple integral defining the function Φ_3 can be rewritten as the following single integral:

$$\Phi_3[b_1,b_2,b_3;\rho_{1.2},\rho_{2.3}] = \int_{x=-\infty}^{b_2} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N\left[\frac{b_1-\rho_{1.2}x}{\sqrt{1-\rho_{1.2}^2}}\right] N\left[\frac{b_3-\rho_{2.3}x}{\sqrt{1-\rho_{2.3}^2}}\right] dx \tag{26}$$

Since, on the one hand, the function $N[b]$ can be evaluated with adequate precision for all option valuation purposes, and, on the other hand, the exponential function is of class C^∞, the numerical evaluation of the integral in (35) does not raise any difficulty and can be implemented using classical quadrature methods (see, e.g., Davis and Rabinowitz 2007). The computational time using Gauss–Legendre quadrature is 0.005 s on an ordinary laptop personal computer, so that it takes approximately 0.01 s to compute the price of a 3-touch barrier option.

Alternatively, it is possible to obtain the probability under consideration as the solution of the following integration problem:

$$E_Q(I' \times I_1) = \int_{x_1=-\infty}^{h_1\wedge h_2} \int_{x_2=-\infty}^{h_2\wedge h_3} \int_{x_3=-\infty}^{k\wedge h_3} Q(X(t_1) \in dx_1, X(t_2) \in dx_2, X(t_3) \in dx_3) \tag{27}$$

$$Q\left(\overline{X}_0^1 < h_1 | X(t_1) \in dx_1\right) Q\left(\overline{X}_1^2 < h_2 | X(t_1) \in dx_1, X(t_2) \in dx_2\right)$$

$$Q\left(\overline{X}_2^3 < h_3 | X(t_2) \in dx_2, X(t_3) \in dx_3\right) dx_3dx_2dx_1$$

Substituting the four probabilities multiplied inside the integral in (27) yields:

$$E_Q(I' \times I_1) = \frac{1}{(2\pi)^{3/2}\sigma_{2|1}\sigma_{3|1.2}\sigma^3\sqrt{t_1t_2t_3}} \tag{28}$$

$$\times \int_{x_1=-\infty}^{h_1\wedge h_2} \int_{x_2=-\infty}^{h_2\wedge h_3} \int_{x_3=-\infty}^{k\wedge h_3} \exp\left(-\frac{1}{2}\left(\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}}\right)^2 - \frac{1}{2\sigma_{2|1}^2}\left(\frac{x_2-\mu t_2}{\sigma\sqrt{t_2}} - \rho_{1.2}\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}}\right)^2\right.$$

$$\left. - \frac{1}{2\sigma_{3|1.2}^2}\left(\frac{x_3-\mu t_3}{\sigma\sqrt{t_3}} - \rho_{1.3}\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}} - \frac{\rho_{2.3|1}}{\sigma_{3|1.2}}\left(\frac{x_2-\mu t_2}{\sigma\sqrt{t_2}} - \rho_{1.2}\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}}\right)\right)^2\right)\left(1-\exp\left(\frac{2h_1(x_1-h_1)}{\sigma^2 t_1}\right)\right)$$

$$\left(1-\exp\left(\frac{2(h_2-x_1)(x_2-h_2)}{\sigma^2(t_2-t_1)}\right)\right)\left(1-\exp\left(\frac{2(h_3-x_2)(x_3-h_3)}{\sigma^2(t_3-t_2)}\right)\right)dx_3dx_2dx_1$$

where:

$$\sigma_{2|1}=\sqrt{1-\rho_{12}^2},\ \rho_{2.3|1}=\frac{\rho_{2.3}-\rho_{1.2}\rho_{1.3}}{\sigma_{2|1}},\ \sigma_{3|1.2}=\sqrt{1-\rho_{1.3}^2-\rho_{2.3|1}^2}$$

This integral can be explicitly computed, yielding a linear combination of trivariate standard normal distribution functions $N_3[b_1,b_2,b_3;\rho_{1.2},\rho_{1.3},\rho_{2.3}]$, $(b_1,b_2,b_3)\in\mathbb{R}^3$. The result is not given because it is not easier to calculate or to evaluate numerically. In the remainder of this section, we will continue to use Φ_3 functions, but all results involving them could also be written in terms of N_3 functions.

Let us now proceed with $E_Q(I'\times I_4)$. We have:

$$E_Q(I_4)=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3\geq H_3\right) \tag{29}$$

$$=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2\right)-Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3\right)$$

The probability $Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3\right)=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3,\right.$ $\left.S(t_3)<H_3\right)$ has just been computed and the probability $Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2\right)$ can be obtained as follows:

$$\int_{x_1=-\infty}^{h_1\wedge h_2}\int_{x_2=-\infty}^{h_2} Q\left(\overline{X}_0^1<h_1,X(t_1)\in dx_1\right)Q\left(\overline{X}_1^2<h_2,X(t_2)\in dx_2|X(t_1)\in dx_1\right)dx_2dx_1 \tag{30}$$

$$=\int_{x_1=-\infty}^{h_1\wedge h_2}\int_{x_2=-\infty}^{h_2}\phi_1(x_1)\phi_2(x_1,x_2)dx_2dx_1$$

The solution to (30) is given by (A9)–(A11) in Appendix A.

To tackle the terminal condition at expiry t_3, we use the following decomposition:

$$Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3\geq H_3,S(t_3)<K\right) \tag{31}$$

$$=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,S(t_3)<K\right)-Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3,S(t_3)<K\right)$$

where the term $Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,S(t_3)<K\right)$ can be handled as follows:

$$\int_{x_1=-\infty}^{h_1\wedge h_2}\int_{x_2=-\infty}^{h_2}\int_{x_3=-\infty}^{k} Q\left(\overline{X}_0^1<h_1,X(t_1)\in dx_1\right)Q\left(\overline{X}_1^2<h_2,X(t_2)\in dx_2|X(t_1)\in dx_1\right) \tag{32}$$

$$Q(X(t_3)\in dx_3|X(t_2)\in dx_2)dx_3dx_2dx_1$$

$$=\int_{x_1=-\infty}^{h_1\wedge h_2}\int_{x_2=-\infty}^{h_2}\int_{x_3=-\infty}^{k}\phi_1(x_1)\phi_2(x_1,x_2)\varphi_2(x_2,x_3)dx_3dx_2dx_1 \tag{33}$$

The solution to (33) is given by (A12)–(A15) in Appendix A.

Notice that $Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,S(t_3)<K\right)$ is the probability required to value an early-ending two-step up-and-out put option with step barrier $[H_1,H_2]$ on $[t_0,t_1]\cup[t_1,t_2]$.

Next, we deal with $E_Q(I' \times I_2)$:

$$E_Q(I' \times I_2) = Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right) \tag{34}$$

$$= Q\left(\overline{S}_0^1 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$$

$$-Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$$

where the probability $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ is already known and the probability $Q\left(\overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ is given by:

$$\int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_2 \wedge h_3} \int_{x_3=-\infty}^{k \wedge h_3} Q(X(t_1) \in dx_1) Q\left(\overline{X}_1^2 < h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right)$$
$$Q\left(\overline{X}_2^3 < h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \tag{35}$$

$$= \int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_2 \wedge h_3} \int_{x_3=-\infty}^{k \wedge h_3} \varphi_1(x_1)\phi_2(x_1, x_2)\phi_2(x_2, x_3) dx_3 dx_2 dx_1 \tag{36}$$

The solution is given by (A16)–(A19) in Appendix A.

Notice that $Q\left(\overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ is the probability required to value a forward-start 2-step up-and-out put option with step barrier $[H_2, H_3]$ on $[t_1, t_2] \cup [t_2, t_3]$.

We then proceed to $E_Q(I' \times I_7)$:

$$Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 \geq H_3, S(t_3) < K\right) \tag{37}$$

$$= Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right) - E_Q(I' \times I_2)$$

The term $Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right)$ can be obtained as follows:

$$Q\left(\overline{S}_1^2 < H_2, S(t_3) < K\right) - Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right) \tag{38}$$

where $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right)$ has already been calculated and:

$$Q\left(\overline{S}_1^2 < H_2, S(t_3) < K\right)$$

$$= \int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_2} \int_{x_3=-\infty}^{k} Q(X(t_1) \in dx_1) Q\left(\overline{X}_1^2 < h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right)$$
$$Q(X(t_3) \in dx_3 | X(t_2) \in dx_2) dx_3 dx_2 dx_1 \tag{39}$$

$$= \int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_2} \int_{x_3=-\infty}^{k} \varphi_1(x_1)\phi_2(x_1, x_2)\varphi_2(x_2, x_3) dx_3 dx_2 dx_1 \tag{40}$$

The solution to (40) is given by (A20)–(A23) in Appendix A.

Notice that $Q\left(\overline{S}_1^2 < H_2, S(t_3) < K\right)$ is the probability required to value a window up-and-out put option with barrier H_2.

The next case to handle is $E_Q(I' \times I_3)$:

$$E_Q\left(I' \times I_3\right) = Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right) \tag{41}$$

$$= Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 < H_3, S(t_3) < K\right) - E_Q\left(I' \times I_0\right)$$

The term $Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ can be computed as follows:

$$\int_{x_1=-\infty}^{h_1} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k\wedge h_3} Q\left(\overline{X}_0^1 < h_1, X(t_1) \in dx_1\right) Q(X(t_2) \in dx_2 | X(t_1) \in dx_1) \\ Q\left(\overline{X}_2^3 < H_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \tag{42}$$

$$= \int_{x_1=-\infty}^{h_1} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k\wedge h_3} \phi_1(x_1)\varphi_2(x_1,x_2)\phi_2(x_2,x_3) dx_3 dx_2 dx_1 \tag{43}$$

The solution to (43) is given by (A24)–(A27) in Appendix A.
Next, we deal with $E_Q(I' \times I_6)$:

$$E_Q\left(I' \times I_6\right) = Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 \geq H_3, S(t_3) < K\right) \tag{44}$$

$$= Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 \geq H_3, S(t_3) < K\right) - E_Q\left(I' \times I_4\right)$$

where:

$$Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 \geq H_3, S(t_3) < K\right) = Q\left(\overline{S}_0^1 < H_1, S(t_2) \geq H_3, S(t_3) < K\right) \\ + Q\left(\overline{S}_0^1 < H_1, S(t_2) < H_3, \overline{S}_2^3 \geq H_3, S(t_3) < K\right) \tag{45}$$

$$= \int_{x_1=-\infty}^{h_1} \int_{x_2=h_3}^{\infty} \int_{x_3=-\infty}^{k} Q\left(\overline{X}_0^1 < h_1, X(t_1) \in dx_1\right) Q(X(t_2) \in dx_2 | X(t_1) \in dx_1) \\ Q(X(t_3) \in dx_3 | X(t_2) \in dx_2) dx_3 dx_2 dx_1 + \int_{x_1=-\infty}^{h_1} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k} Q\left(\overline{X}_0^1 < h_1, X(t_1) \in dx_1\right) \\ Q(X(t_2) \in dx_2 | X(t_1) \in dx_1) Q\left(\overline{X}_2^3 \geq h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \tag{46}$$

$$= \int_{x_1=-\infty}^{h_1} \int_{x_2=h_3}^{\infty} \int_{x_3=-\infty}^{k} \phi_1(x_1)\varphi_2(x_1,x_2)\varphi_2(x_2,x_3) dx_3 dx_2 dx_1 \\ + \int_{x_1=-\infty}^{h_1} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k} \phi_1(x_1)\varphi_2(x_1,x_2)(\varphi_2(x_2,x_3) - \phi_2(x_2,x_3)) dx_3 dx_2 dx_1 \tag{47}$$

The solution to (47) is given by (A28)–(A31) in Appendix A.

$Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 \geq H_3, S(t_3) < K\right)$ is the probability required to value a partial-time 2-step barrier put with a knock-out barrier H_1 on $[t_0, t_1]$, a knock-in barrier H_2 on $[t_2, t_3]$ and no active barrier on $[t_1, t_2]$.

The penultimate case to tackle is $E_Q(I' \times I_5)$:

$$E_Q\left(I' \times I_5\right) = Q\left(\overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right) - E_Q\left(I' \times I_3\right) \tag{48}$$

where $Q\left(\overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ is computed as follows:

$$Q\left(S(t_1) < H_2, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right) + Q\left(S(t_1) \geq H_2, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$$

$$\begin{aligned} &= \int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k \wedge h_3} Q(X(t_1) \in dx_1) Q\left(\overline{X}_1^2 \geq h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right) \\ &Q\left(\overline{X}_2^3 < h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \end{aligned} \tag{49}$$

$$\begin{aligned} &+ \int_{x_1=h_2}^{\infty} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k \wedge h_3} Q(X(t_1) \in dx_1) Q\left(\overline{X}_1^2 \geq h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right) \\ &Q\left(\overline{X}_2^3 < h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) dx_3 dx_2 dx_1 \end{aligned} \tag{50}$$

$$\begin{aligned} &= \int_{x_1=-\infty}^{h_2} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k \wedge h_3} \varphi_1(x_1)(\varphi_2(x_1, x_2) - \phi_2(x_1, x_2)) \phi_2(x_2, x_3) dx_3 dx_2 dx_1 \\ &+ \int_{x_1=h_2}^{\infty} \int_{x_2=-\infty}^{h_3} \int_{x_3=-\infty}^{k \wedge h_3} \varphi_1(x_1)(\varphi_2(x_1, x_2) - \phi_2(x_1, x_2)) \phi_2(x_2, x_3) dx_3 dx_2 dx_1 \end{aligned} \tag{51}$$

The solution to (51) is given by (A32)–(A37) in Appendix A.

$Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$ is the probability required to value a 3- step in-and-in-and-out put option with knock-in steps H_1 and H_2, and knock-out step H_3.

Eventually, $E_Q(I' \times I_8)$ is dealt with:

$$E_Q(I' \times I_8) = Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, S(t_3) < K\right) - E_Q(I' \times I_5) \tag{52}$$

where $Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, S(t_3) < K\right)$ can be decomposed into:

$$Q\left(\overline{S}_0^1 \geq H_1, S(t_3) < K\right) - Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right) \tag{53}$$

$Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right)$ has already been calculated, and we have:

$$Q\left(\overline{S}_0^1 \geq H_1, S(t_3) < K\right) = Q(S(t_3) < K) - Q\left(\overline{S}_0^1 < H_1, S(t_3) < K\right) \tag{54}$$

where

$$Q\left(\overline{S}_0^1 < H_1, S(t_3) < K\right) = \int_{x_1=-\infty}^{h_1} \int_{x_3=-\infty}^{k} \phi_1(x_1) \varphi_2(x_1, x_3) dx_3 dx_1 \tag{55}$$

and is given by (A38) in Appendix A.

Retracing our steps, we see that we have completed the closed-form valuation of a 3-touch up-and-out put. As explained in Section 2, it suffices to take $\mu = r + \frac{\sigma^2}{2}$ instead of $\mu = r - \frac{\sigma^2}{2}$ in all the formulae obtained in this section to obtain the list of necessary probabilities under the $Q^{(S)}$ measure.

We can now easily deduce other probabilities needed to recover the no-arbitrage prices of other types of 3-touch knock-out barrier options. Let us begin with a 3-touch up-and-out call. The value of this option is given by:

$$\exp(-rt_3)\left\{E_Q\left[S(t_3)\sum_{i=0}^{3}\omega_i \mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i, S(t_3)>K\}}\right] - KE_Q\left[\sum_{i=0}^{3}\omega_i \mathbf{1}_{\{\eta(S(t),t\in[0,t_3]), S(t_3)>K\}}\right]\right\} \tag{56}$$

From the Cameron–Martin–Girsanov theorem, all we need to compute is $E_Q\left[\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i,S(t_3)>K\}}\right]$. By the law of total probability and the continuity of paths of the process S, we have:

$$E_Q\left[\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i,S(t_3)>K\}}\right]=E_Q\left[\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i,S(t_3)\geq K\}}\right] \tag{57}$$

$$=E_Q\left[\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i\}}\right]-E_Q\left[I'\times\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i\}}\right] \tag{58}$$

Since the $\omega_i{}'$s are known and we have already obtained $E_Q\left[I'\times\sum_{i=0}^{3}\omega_i\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i\}}\right]$, we only have to calculate $E_Q\left[\sum_{i=0}^{3}\mathbf{1}_{\{\eta(S(t),t\in[0,t_3])=i\}}\right]$.

The term $E_Q(I_4)$ is given by (29). Moreover, since the event $\left\{\overline{S}_2^3<H_3\right\}$ includes the event $\{S(t_3)<H_3\}$, we have:

$$E_Q(I_1)=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3,S(t_3)<H_3\right) \tag{59}$$

$$E_Q(I_2)=Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3,S(t_3)<H_3\right)$$

$$E_Q(I_3)=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\geq H_2,\overline{S}_2^3<H_3,S(t_3)<H_3\right)$$

$$E_Q(I_5)=Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2\geq H_2,\overline{S}_2^3<H_3,S(t_3)<H_3\right)$$

Therefore, $E_Q(I_1)$, $E_Q(I_2)$, $E_Q(I_3)$ and $E_Q(I_5)$ are given by $E_Q(I'\times I_1)$, $E_Q(I'\times I_2)$, $E_Q(I'\times I_3)$ and $E_Q(I'\times I_5)$, respectively, along with the substitution $k=h_3$.

With regard to $E_Q(I_6)$, we have:

$$E_Q(I_6)=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\geq H_2\right)-Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\geq H_2,\overline{S}_2^3<H_3\right) \tag{60}$$

$$=Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\geq H_2\right)-E_Q(I_3)$$

$$=Q\left(\overline{S}_0^1<H_1\right)-Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2\right)-E_Q(I_3) \tag{61}$$

$Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2\right)$ has been dealt with in (30) and $Q\left(\overline{S}_0^1<H_1\right)$ is a textbook formula.

The probability $E_Q(I_7)$ can be derived as follows:

$$E_Q(I_7)=Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2<H_2\right)-Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3\right) \tag{62}$$

$$=Q\left(\overline{S}_1^2<H_2\right)-Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2\right)-Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3\right) \tag{63}$$

where $Q\left(\overline{S}_0^1\geq H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3\right)=E_Q(I_2)$, and has been dealt with in (59), and:

$$Q\left(\overline{S}_1^2<H_2\right)=\int_{x_1=-\infty}^{h_2}\int_{x_2=-\infty}^{h_2}\varphi_1(1)\phi_2(1,2)dx_2dx_1 \tag{64}$$

and is given by (A39) in Appendix A.

$Q\left(\overline{S}_1^2 \leq H_2\right)$ is the probability required to value a forward-start up-and-out put.

Finally, $E_Q(I_8)$ is dealt with as follows:

$$E_Q(I_8) = Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2\right) - Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3\right) \tag{65}$$

$$= Q\left(\overline{S}_0^1 \geq H_1\right) - Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2\right) - Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3\right) \tag{66}$$

$$= Q\left(\overline{S}_0^1 \geq H_1\right) - \left(Q\left(\overline{S}_1^2 < H_2\right) - Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2\right)\right) - E_Q(I_5) \tag{67}$$

where the probability $Q\left(\overline{S}_1^2 < H_2\right) - Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2\right)$ has been dealt with in (63), and the probability $Q\left(\overline{S}_0^1 \geq H_1\right) = 1 - Q\left(\overline{S}_0^1 < H_1\right)$ is a textbook formula.

Retracing our steps, we see that we have completed the closed-form valuation of a 3-touch up-and-out call.

Once formulae for 3-touch up-and-out calls and puts are known, formulae for 3-touch down-and-out calls and puts ensue as a corollary. Indeed, the symmetry of paths of Brownian motion entails:

$$Q\left(\overline{X}_i^j < h_j, X(t_j) < k\right) = Q\left(\underline{X}_i^j > -h_j, X(t_j) > -k\right), \forall H_j > S(t_i), \forall K > 0 \tag{68}$$

where we recall that $\underline{X}_i^j = \inf_{t_i \leq t \leq t_j} X(t)$.

The important practical consequence is that, in order to derive the formula for a 3-touch down-and-out call from the formula for a 3-touch up-and-out put, it suffices to multiply by -1 all the bounds (but not the correlation coefficients) of the cumulative distribution functions involved in the formula for a 3-touch up-and-out put. In other words, every function $\Phi_3[b_1, b_2, b_3; \pm\rho_{1.2}, \pm\rho_{2.3}]$ that appears in the formula for a 3-touch up-and-out put becomes $\Phi_3[-b_1, -b_2, -b_3; \pm\rho_{1.2}, \pm\rho_{2.3}]$ in the formula for a 3-touch down-and-out call. Obviously, the same transformation applies to cumulative distribution functions of a smaller order, i.e., functions $N[.]$ and $N_2[., .; .]$, which may appear in the formula for a 3-touch up-and-out put. For instance, from the probability $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2\right)$ given in Appendix A, one can immediately infer:

$$Q\left(\underline{S}_0^1 > H_1, \underline{S}_1^2 > H_2\right)$$

$$= N_2\left[\frac{-h_1 \wedge h_2 + \mu t_1}{\sigma\sqrt{t_1}}, \frac{-h_2 + \mu t_2}{\sigma\sqrt{t_2}}; \rho_{1.2}\right] - \exp\left(\frac{2\mu h_2}{\sigma^2}\right) N_2\left[\frac{-h_1 \wedge h_2 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{h_2 + \mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{69}$$

$$- \exp\left(\frac{2\mu h_1}{\sigma^2}\right) N_2\left[\frac{-h_1 \wedge h_2 + 2h_1 + \mu t_1}{\sigma\sqrt{t_1}}, \frac{-h_2 + 2h_1 + \mu t_2}{\sigma\sqrt{t_2}}; \rho_{1.2}\right] \tag{70}$$

$$+ \exp\left(\frac{2\mu(h_2 - h_1)}{\sigma^2}\right) N_2\left[\frac{-h_1 \wedge h_2 + 2h_1 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{h_2 - 2h_1 + \mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{71}$$

where $H_1 > S(0)$ in $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2\right)$ and $H_1 < S(0)$ in $Q\left(\underline{S}_0^1 > H_1, \underline{S}_1^2 > H_2\right)$.

Thus, there is no need to perform new analytical computations to obtain formulae for 3-touch down-and-out options.

To close Section 3, one can point out that, although we have not shown the details of the valuation of a 2-touch option, they are quite similar to those of the valuation of a 3-touch option, albeit simpler. Consequently, the formula for a 2-touch up-and-out put is given without proof in Appendix B.

4. Generalization to Other Types of Barriers, as Well as to Higher Dimension

In this section, we discuss extensions of the analytical method used in Section 3 to tackle a wider variety of barriers and a greater number of barrier crossings.

4.1. Outside Multitouch Payoff

An outside barrier option is a kind of multi-asset barrier option, the specificity of which is to define one asset, say S, with regard to which barrier monitoring is performed, and another asset, say V, with regard to which the moneyness of the option is tested at expiry. This allows, among other things, to take advantage of the volatility spread between S and V, as well as the correlation effects. When the barrier is knock-out, a classical strategy to optimize the expected payoff is to combine a low volatility on S, a high volatility on V and negative correlation between S and V. The payoff on a 3-touch up-and-out put with expiry t_3 and outside barrier is given by:

$$(K - V(t_3))^+ \sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t \in [0,t_3]) = i\}} \tag{72}$$

where $\eta(S(t), t \in [0, t_3])$, i.e., the number of predefined time intervals in which the barrier has been hit, is determined according to the variations of asset S.

The previous payoff can be slightly generalized by defining $t_4 > t_3$ as the expiry of the option and by considering an early-ending 3-touch up-and-out put with expiry t_4 and outside barrier as follows:

$$(K - V(t_4))^+ \sum_{i=0}^{3} \omega_i \mathbf{1}_{\{\eta(S(t), t \in [0,t_3]) = i\}} \tag{73}$$

where the step barrier is monitored on $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$ but not on $[t_3, t_4]$.

Under Q, the stochastic differentials of S and V are given by:

$$dS(t) = rS(t)dt + \sigma_S S(t) dB_1(t) \tag{74}$$

$$dV(t) = rV(t)dt + \sigma_V V(t) dB_2(t) \tag{75}$$

where $\sigma_S, \sigma_V > 0$, and the instantaneous correlation coefficient between $B_1(t)$ and $B_2(t)$ is denoted by $\theta_{1.2}$.

The $Q-$probability, denoted by p, that the maximum payout rate ω_0 is obtained at expiry, is the probability required to value an early-ending, outside 3-step up-and-out put option with expiry t_4, and is given by:

$$p = Q\left(\overline{S}_1(0,1) \le H_1, \overline{S}_1(1,2) \le H_2, \overline{S}_1(2,3) \le H_3, V(t_4) \le K\right) \tag{76}$$

$$= \int_{x_1=-\infty}^{h_1 \wedge h_2} \int_{x_2=-\infty}^{h_2 \wedge h_3} \int_{x_3=-\infty}^{h_3} \int_{x_4=-\infty}^{k} Q\left(\overline{X}(0,1) \le h_1, X(t_1) \in dx_1\right) Q\left(\overline{X}(1,2) \le h_2, X(t_2) \in dx_2 | X(t_1) \in dx_1\right)$$

$$Q\left(\overline{X}(2,3) \le h_3, X(t_3) \in dx_3 | X(t_2) \in dx_2\right) Q(Y(t_4) \in dx_4 | X(t_3) \in dx_3) dx_4 dx_3 dx_2 dx_1 \tag{77}$$

where $X(t) = \ln\left(\frac{S(t)}{S(0)}\right)$ and $Y(t) = \ln\left(\frac{V(t)}{V(0)}\right)$, and we use the equality in law between $Q\left(\overline{X}_i^j | X(t_i), X(t_j)\right)$ and $Q\left(\overline{X}_i^j | X(t_i), X(t_j), Y(t_j)\right)$.

The covariance between $X(t_3)$ and $Y(t_4)$ can be written as follows:

$$\begin{aligned} \mathrm{cov}[X(t_3), Y(t_4)] &= \mathrm{cov}\left[\mu_S t_3 + \sigma_S \sqrt{t_3} Z_1, \mu_V t_4 + \sigma_V \sqrt{t_3}\left(\theta_{1.2} Z_1 + \sqrt{1 - \theta_{1.2}^2} Z_2\right) + \sigma_V \sqrt{t_4 - t_3} Z_3\right] \\ &= \sigma_1 \sigma_2 \theta_{1.2} t_3 \end{aligned} \tag{78}$$

where $\mu_S = r - \frac{\sigma_S^2}{2}$ and $\mu_V = r - \frac{\sigma_V^2}{2}$, and Z_1, Z_2, Z_3 are three independent, standard, normal random variables.

Hence, the correlation coefficient between $X(t_3)$ and $V(t_4)$ is equal to $\theta_{1.2}\sqrt{\frac{t_3}{t_4}}$ and we have:

$$p = \int_{x_1=-\infty}^{h_1 \wedge h_2} \int_{x_2=-\infty}^{h_2 \wedge h_3} \int_{x_3=-\infty}^{h_3} \int_{x_4=-\infty}^{k} \phi_1(x_1)\phi_2(x_1,x_2)\phi_2(x_2,x_3)\varphi_2(x_3,x_4)dx_4dx_3dx_2dx_1 \tag{79}$$

Performing the necessary calculations yields:

$$p = \Phi_4\left[\frac{h_1 \wedge h_2 - \mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2 \wedge h_3 - \mu_S t_2}{\sigma_S\sqrt{t_2}}, \frac{h_3 - \mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k - \mu_V t_4}{\sigma_V\sqrt{t_4}}; \rho_{1.2}, \rho_{2.3}, \rho_{3.4}\theta_{12}\right] \tag{80}$$

$$-\exp\left(\frac{2\mu_S h_1}{\sigma_S^2}\right)\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2-2h_1-\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2\wedge h_3-2h_1-\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{h_3-2h_1-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}h_1-\mu_V t_4}{\sigma_V\sqrt{t_4}}; \rho_{1.2}, \rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{81}$$

$$-\exp\left(\frac{2\mu_S h_2}{\sigma_S^2}\right)\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2+\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2\wedge h_3-2h_2-\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{h_3-2h_2-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}h_2-\mu_V t_4}{\sigma_V\sqrt{t_4}}; -\rho_{1.2}, \rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{82}$$

$$-\exp\left(\frac{2\mu_S h_3}{\sigma_S^2}\right)\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2+\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2\wedge h_3+\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{-h_3-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}h_3-\mu_V t_4}{\sigma_V\sqrt{t_4}}; \rho_{1.2}, -\rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{83}$$

$$+\exp\left(\frac{2\mu_S(h_2-h_1)}{\sigma_S^2}\right)$$

$$\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2-2h_1+\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{2h_1+h_2\wedge h_3-2h_2-\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{h_3-2h_2+2h_1-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}(h_2-h_1)-\mu_V t_4}{\sigma_V\sqrt{t_4}}; -\rho_{1.2}, \rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{84}$$

$$+\exp\left(\frac{2\mu_S(h_3-h_1)}{\sigma_S^2}\right)$$

$$\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2-2h_1+\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2\wedge h_3-2h_1+\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{-h_3+2h_1-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}(h_3-h_1)-\mu_V t_4}{\sigma_V\sqrt{t_4}}; \rho_{1.2}, -\rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{85}$$

$$+\exp\left(\frac{2\mu_S(h_3-h_2)}{\sigma_S^2}\right)$$

$$\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2-\mu_S t_1}{\sigma_S\sqrt{t_1}}, \frac{h_2\wedge h_3-2h_2+\mu_S t_2}{\sigma_S\sqrt{t_2}}, \\ \frac{-h_3+2h_2-\mu_S t_3}{\sigma_S\sqrt{t_3}}, \frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}(h_3-h_2)-\mu_V t_4}{\sigma_V\sqrt{t_4}}; -\rho_{1.2}, -\rho_{2.3}, \rho_{3.4}\theta_{1.2}\end{array}\right] \tag{86}$$

$$
-\exp\left(\frac{2\mu_S(h_3-h_2+h_1)}{\sigma_S^2}\right)
$$

$$
\Phi_4\left[\begin{array}{l}\frac{h_1\wedge h_2-2h_1-\mu_S t_1}{\sigma_S\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2+2h_1+\mu_S t_2}{\sigma_S\sqrt{t_2}},\\ \\ \frac{-h_3+2h_2-2h_1-\mu_S t_3}{\sigma_S\sqrt{t_3}},\frac{k-2\rho_{3.4}\theta_{1.2}\frac{\sigma_V}{\sigma_S}(h_3-h_2+h_1)-\mu_V t_4}{\sigma_V\sqrt{t_4}};-\rho_{1.2},-\rho_{2.3},\rho_{3.4}\theta_{1.2}\end{array}\right] \tag{87}
$$

where $h_i=\ln\left(\frac{H_i}{S(0)}\right), k=\ln\left(\frac{K}{V(0)}\right)$ and the function $\Phi_4[b_1,b_2,b_3,b_4;\rho_{1.2},\rho_{2.3},\rho_{3.4}]$ is defined by (25).

The quadruple integral defining the function Φ_4 can be rewritten as the following double integral:

$$
\Phi_4[b_1,b_2,b_3,b_4;\rho_{1.2},\rho_{2.3},\rho_{3.4}] \tag{88}
$$

$$
=\int\limits_{x_2=-\infty}^{b_2}\int\limits_{x_3=-\infty}^{\frac{b_3-\rho_{2.3}y_2}{\sqrt{1-\rho_{2.3}^2}}}\frac{1}{2\pi}\exp\left(-\frac{(x_2^2+x_3^2)}{2}\right)N\left[\frac{b_1-\rho_{1.2}x_2}{\sqrt{1-\rho_{1.2}^2}}\right]N\left[\frac{b_4-\rho_{3.4}\sqrt{1-\rho_{2.3}^2}x_3-\rho_{3.4}\rho_{2.3}x_2}{\sqrt{1-\rho_{3.4}^2}}\right]dx_2dx_3
$$

The numerical evaluation of (88) is just as easy as that of (26), for the same reasons as explained in Section 3. Extensive testing shows that a mere 16-point Gauss–Legendre double quadrature suffices to reach a minimum of 10^{-7} precision in less than one hundredth of a second, as long as $|\rho_{i.j}|<0.99$. If even more accuracy is needed, or if more extreme values of the correlation coefficients are encountered, a standard subregion adaptive algorithm will perform well, as explained by Berntsen et al. (1991), along with a Kronrod rule to reduce the number of required iterations (see, e.g., Davis and Rabinowitz 2007). These are widely used numerical integration techniques, and it is easy to find the available code or built-in functions in the usual scientific computing software.

Notice that the formula for an outside 3-step up-and-out put option without the early-ending feature, i.e., with expiry t_3, is immediately derived by substituting $\rho_{3.4}\theta_{1.2}$ with $\theta_{1.2}$ and by substituting t_4 with t_3 in (80)–(87).

It is possible to obtain a formula for the probability p in terms of quadrivariate standard normal distribution functions if one expresses the problem as the following integral:

$$
\begin{aligned}
p=\int\limits_{x_1=-\infty}^{h_1\wedge h_2}\int\limits_{x_2=-\infty}^{h_2\wedge h_3}\int\limits_{x_3=-\infty}^{h_3\wedge h_4}\int\limits_{x_4=-\infty}^{k}&Q(X(t_1)\in dx_1,X(t_2)\in dx_2,X(t_3)\in dx_3,Y(t_4)\in dx_4)\\
&Q\left(\overline{X}_0^1\le h_1|X(t_1)\in dx_1\right)Q\left(\overline{X}_1^2\le h_2|X(t_1)\in dx_1,X(t_2)\in dx_2\right)\\
&Q\left(\overline{X}_2^3\le h_3|X(t_2)\in dx_2,X(t_3)\in dx_3\right)dx_4dx_3dx_2dx_1
\end{aligned} \tag{89}
$$

The resulting formula is not given because it is more cumbersome and less easy to evaluate numerically than (80)–(87).

Once the probability p has been obtained, it is possible, using the same method, to explicitly calculate all the other probabilities involved in the valuation of an early-ending 3-touch up-and-out put with expiry t_4, i.e., the following set P of probabilities:

$$
\begin{aligned}
\mathrm{P}=\Big\{&Q\left(\overline{S}_0^1\ge H_1,\overline{S}_1^2\ge H_2,\overline{S}_2^3\ge H_3,V(t_4)<K\right),Q\left(\overline{S}_0^1\ge H_1,\overline{S}_1^2<H_2,\overline{S}_2^3<H_3,V(t_4)<K\right)\\
&Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\ge H_2,\overline{S}_2^3<H_3,V(t_4)<K\right),Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2<H_2,\overline{S}_2^3\ge H_3,V(t_4)<K\right)\\
&Q\left(\overline{S}_0^1\ge H_1,\overline{S}_1^2\ge H_2,\overline{S}_2^3<H_3,V(t_4)<K\right),Q\left(\overline{S}_0^1<H_1,\overline{S}_1^2\ge H_2,\overline{S}_2^3\ge H_3,V(t_4)<K\right)
\end{aligned}
$$

$$Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 \geq H_3, V(t_4) < K\right)\Big\} \tag{90}$$

Notice that, in contrast to a multitouch barrier option with a non-outside barrier, a new, elementary change of measure is required to obtain the option value, which is given by:

$$\sum_{i=1}^{3} \omega_i \left\{ \begin{array}{l} Ke^{-rt_4} Q(\eta(S(t), t \in [0, t_3]) = i, V(t_4) < K) \\ -V(0)Q^{(V)}(\eta(S(t), t \in [0, t_3]) = i, V(t_4) < K) \end{array} \right\} \tag{91}$$

The Radon–Nikodym derivative of the measure $Q^{(V)}$ w.r.t. Q is given by:

$$\frac{dQ^{(V)}}{dQ}\bigg|\mathcal{F}_t = \exp\left(\sigma_V \theta_{1.2} W_1(t) - \frac{\sigma_V^2 \theta_{1.2}^2}{2} t + \sigma_V \sqrt{1-\theta_{1.2}^2} W_2(t) - \frac{\sigma_V^2 (1-\theta_{1.2}^2)}{2} t\right) \tag{92}$$

where $W_1(t)$ and $W_2(t)$ are two independent, standard Brownian motions under $Q^{(V)}$, and $\mathcal{F}_t$ is the smallest filtration w.r.t. which both $W_1(t)$ and $W_2(t)$ are measurable; thus, under $Q^{(V)}$, we have:

$$E_{Q^{(V)}}[S(t)] = S(0) \exp\left(\left(r - \frac{\sigma_S^2}{2} + \sigma_S \sigma_V \theta_{1.2}\right) t\right),\ E_{Q^{(V)}}[V(t)] = V(0) \exp\left(\left(r + \frac{\sigma_V^2}{2}\right) t\right) \tag{93}$$

Table 7 reports numerical values for 3-touch OEEUOP (Outside Early-Ending Up and Out Put) option prices as functions of volatility and correlation. The option's expiry is $t_4 = 1$. All the other parameters that are not given inside Table 7 are identical to those in Table 1.

Table 7. 3-touch OEEUOP option prices with piecewise constant barrier as functions of volatility and correlation.

3-Touch OEEUOP	$\sigma_S = 20\%, \sigma_V = 50\%$	$\sigma_S = 50\%, \sigma_V = 20\%$	$\sigma_S = 35\%, \sigma_V = 35\%$
$\theta_{1.2} = -50\%$	12.6876269	3.19205121	6.93552897
$\theta_{1.2} = 50\%$	15.1581431	4.37818109	9.02213614
$\theta_{1.2} = 5\%$	14.0820718	3.84164681	8.09063875

4.2. Piecewise Exponential Affine Step Barrier

A more general and flexible form of barrier consists in replacing each constant H_i on each $[t_{i-1}, t_i]$ by a function of time. In general, only numerical approximations to the valuation problem can be attained in this new framework (Wang and Pötzelberger 1997; Novikov et al. 1999). However, as shown by Guillaume (2016), a remarkable exception is when the barrier is defined as a piecewise exponential affine function of time. Then, exact solutions can be found. This is all the more useful to notice as exponential functions display curvature, thus allowing for a wide variety of shapes. More precisely, let us define a barrier $g(t)$ for a standard geometric Brownian motion $S(t)$ as defined by (10) on a partition $\{[t_0 = 0, t_1], \ldots, [t_{n-1}, t_n = T]\}$ of $[0, T]$ as follows:

$$g(t) = \sum_{i=1}^{n} S(0) \exp(a_i + b_i(t - t_{i-1}))\, \mathrm{I}_{[t_{i-1}, t_i]}(t),\ a_i \in \mathbb{R},\ b_i \in \mathbb{R},\ i \in \{1, 2, \ldots, n\} \tag{94}$$

Then, the $Q-$probability, denoted by p, to receive a maximum payout rate ω_0 on a 3-touch up-and-out put option with expiry $T = t_3$, is defined by:

$$p = Q\left(\begin{array}{l} (S(t) < S(0)\exp(a_1 + b_1 t), \forall 0 \leq t \leq t_1) \\ \cap (S(t) < S(0)\exp(a_2 + b_2(t - t_1)), \forall t_1 \leq t \leq t_2) \\ \cap (S(t) < S(0)\exp(a_3 + b_3(t - t_2)), \forall t_2 \leq t \leq t_3) \cap S(t_3) < K \end{array} \right) \tag{95}$$

It can be shown that:

$$p = \Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-\mu_1t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-\mu_1t_1-\mu_2(t_2-t_1)}{\sigma\sqrt{t_2}}, \\ \frac{z_3-b_3t_3-\mu_1t_1-\mu_2(t_2-t_1)-\mu_3(t_3-t_2)}{\sigma\sqrt{t_3}}; \rho_{1.2}, \rho_{2.3}\end{array}\right] \tag{96}$$

$$-\exp\left(\frac{\lambda_1}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-2a_1-\mu_1t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-2a_1-\mu_1t_1-\mu_2(t_2-t_1)}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2a_1-\mu_1t_1-\mu_2(t_2-t_1)-\mu_3(t_3-t_2)}{\sigma\sqrt{t_3}}; \rho_{1.2}, \rho_{2.3}\end{array}\right] \tag{97}$$

$$-\exp\left(\frac{\lambda_2}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-\mu_1t_1+2\mu_2t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-2\alpha_2+\mu_1t_1-\mu_2(t_1+t_2)}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2\alpha_2+\mu_1t_1-\mu_2(t_1+t_2)-\mu_3(t_3-t_2)}{\sigma\sqrt{t_3}}; -\rho_{1.2}, \rho_{2.3}\end{array}\right] \tag{98}$$

$$+\exp\left(\frac{\lambda_3}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-2a_1-\mu_1t_1+2\mu_2t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-2\alpha_2+2a_1+\mu_1t_1-\mu_2(t_1+t_2)}{\sigma\sqrt{t_2}} \\ \\ \frac{z_3-b_3t_3-2\alpha_2+2a_1+\mu_1t_1-\mu_2(t_1+t_2)-\mu_3(t_3-t_2)}{\sigma\sqrt{t_3}}; -\rho_{1.2}, \rho_{2.3}\end{array}\right] \tag{99}$$

$$-\exp\left(\frac{\lambda_4}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-\mu_1t_1+2\mu_3t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-\mu_1t_1-\mu_2(t_2-t_1)+2\mu_3t_2}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2\alpha_3+\mu_1t_1+\mu_2(t_2-t_1)-\mu_3(t_2+t_3)}{\sigma\sqrt{t_3}}; \rho_{1.2}, -\rho_{2.3}\end{array}\right] \tag{100}$$

$$+\exp\left(\frac{\lambda_5}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-2a_1+2\mu_3t_1-\mu_1t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-2a_1-\mu_1t_1-\mu_2(t_2-t_1)+2\mu_3t_2}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2\alpha_3+2a_1+\mu_1t_1+\mu_2(t_2-t_1)-\mu_3(t_2+t_3)}{\sigma\sqrt{t_3}}; \rho_{1.2}, -\rho_{2.3}\end{array}\right] \tag{101}$$

$$+\exp\left(\frac{\lambda_6}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-2\mu_3t_1+2\mu_2t_1-\mu_1t_1}{\sigma\sqrt{t_1}}, \frac{z_2-b_2t_2-2\alpha_2+\mu_1t_1-\mu_2(t_1+t_2)+2\mu_3t_2}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2\alpha_3+2\alpha_2-\mu_1t_1+\mu_2(t_1+t_2)-\mu_3(t_2+t_3)}{\sigma\sqrt{t_3}}; -\rho_{1.2}, -\rho_{2.3}\end{array}\right] \tag{102}$$

$$-\exp\left(\frac{\lambda_7}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\frac{z_1-b_1t_1-2a_1-2\mu_3t_1+2\mu_2t_1-\mu_1t_1}{\sigma\sqrt{t_1}}, \\ \frac{z_2-b_2t_2-2\alpha_2+2a_1+\mu_1t_1-\mu_2(t_1+t_2)+2\mu_3t_2}{\sigma\sqrt{t_2}}, \\ \\ \frac{z_3-b_3t_3-2\alpha_3+2\alpha_2-2a_1-\mu_1t_1+\mu_2(t_1+t_2)-\mu_3(t_2+t_3)}{\sigma\sqrt{t_3}}; \\ -\rho_{1.2}, -\rho_{2.3}\end{array}\right] \tag{103}$$

where:

$$\alpha_2 = a_2 - b_2t_1,\ \alpha_3 = a_3 - b_3t_2 \tag{104}$$

$$\mu_i = \mu - \frac{\sigma^2}{2} - b_i, k = \ln(K/S(0))$$

$$\lambda_1 = 2\mu_1a_1, \lambda_2 = 2\mu_2\alpha_2 - 2\mu_1\mu_2t_1 + 2\mu_2^2t_1, \lambda_3 = 2\mu_1a_1 + 2\mu_2\alpha_2 - 4\mu_2a_1 - 2\mu_1\mu_2t_1 + 2\mu_2^2t_1$$

$$\lambda_4 = 2\mu_3\alpha_3 + 2\mu_3^2t_2 - 2\mu_1\mu_3t_1 - 2\mu_2\mu_3(t_2-t_1)$$

$$\lambda_5 = 2\mu_3\alpha_3 + 2\mu_1a_1 - 4\mu_3a_1 + 2\mu_3^2t_2 - 2\mu_1\mu_3t_1 - 2\mu_2\mu_3(t_2-t_1)$$

$$\lambda_6 = 2\mu_3\alpha_3 + 2\mu_2\alpha_2 - 4\mu_3\alpha_2 + 2(\mu_3 - \mu_2)^2 t_1 + 2\mu_1(\mu_3 - \mu_2)t_1 + 2\mu_3^2(t_2 - t_1) - 2\mu_2\mu_3(t_2 - t_1)$$

$$\lambda_7 = 2\mu_1 a_1 + 2\mu_2\alpha_2 - 4\mu_3\alpha_2 + 2\mu_3\alpha_3 + 2(\mu_3 - \mu_2)^2 t_1 + 2\mu_3^2(t_2 - t_1) - 2\mu_2\mu_3(t_2 - t_1) + 2(\mu_3 - \mu_2)(2a_1 + \mu_1 t_1)$$

$$z_1 = \min(a_1 + b_1 t_1, a_2), z_2 = \min(a_2 + b_2(t_2 - t_1), a_3), z_3 = \min(a_3 + b_3(t_3 - t_2), k)$$

Details on how this solution is obtained can be found in Guillaume (2016). Following the same method, it is possible to explicitly calculate all the other probabilities involved in the valuation of a 3-touch UOP with a barrier defined as a piecewise exponential affine function $g(t)$, as in (94). Table 8 reports a few numerical values of prices as functions of volatility and moneyness. Apart from the shape of the barrier, all the parameters in Table 8 are the same as those in Table 1. The function $g(t)$ is continuous at t_1 and t_2, but note that piecewise continuity on $[t_0, t_3]$ is sufficient for the formula in (96)–(103) to hold. The barrier $g(t)$ starts at 108.328707 on time t_0. Then, it takes values 109.965886 and 110.701441 at times t_1 and t_2, respectively, before ending at 111.441916 at expiry t_3.

Table 8. 3-touch UOP option prices with piecewise exponential affine step barrier.

3-Touch UOP	$\sigma = 18\%$	$\sigma = 36\%$	$\sigma = 64\%$
$K = 100$	5.02931082	8.09776189	12.2562049
$K = 110$	9.36383182	11.652507	15.5356909
$K = 90$	1.59173168	4.82830602	9.09761268

It is a quite straightforward extension to value a 3-touch OEEUOP with a piecewise exponential affine barrier, in the same way as we moved on from a 3-touch UOP to a 3-touch OEEUOP when the barrier was piecewise constant. Table 9 provides a few numerical results when the process $S(t)$ and the barrier $g(t)$ have the same specifications as in Table 8, and when the process $V(t)$ has the same specifications as in Table 7.

Table 9. 3-touch OEEUOP option prices with piecewise exponential affine step barrier.

3-Touch OEEUOP	$\sigma_S = 20\%, \sigma_V = 50\%$	$\sigma_S = 50\%, \sigma_V = 20\%$	$\sigma_S = 35\%, \sigma_V = 35\%$
$\theta_{1.2} = -50\%$	4.04980406	2.65336826	5.40003822
$\theta_{1.2} = 50\%$	8.53096357	4.95979365	9.36116401
$\theta_{1.2} = 5\%$	6.45391124	3.89816913	7.55304833

4.3. Higher Dimension

So far, exact results have been provided only for 3-touch barrier options. It is important to know if an $n-$touch barrier option remains analytically tractable for $n > 3$. Let us refer to the maximum number of crossings n as the dimension of the $n-$touch barrier option. Such terminology is justified by the fact that n is the dimension of the integral problem associated with an $n-$touch barrier option. Clearly, as n increases, the analytical calculations become more and more time-consuming, and the resulting formulae more and more cumbersome. Among all possible $n-$touch barrier option valuation formulae, the $n-$touch up-and-in put and the $n-$touch down-and-in call are the ones that should require the fewest multidimensional integrals to compute, and thus the ones that should result in the most compact formulae. Indeed, consider the following probability p, where the $K_i{}'$s are fixed positive real numbers and the $H_i{}'$s are all greater than zero, with $H_1 > S(0)$:

$$p = Q\left(\begin{array}{l} \overline{S}_0^1 \geq H_1, S(t_1) < K_1, \overline{S}_1^2 \geq H_2, S(t_2) < K_2, \overline{S}_2^3 \geq H_3, S(t_3) < K_3, \\ \overline{S}_3^4 \geq H_4, S(t_4) < K_4, \overline{S}_4^5 \geq H_5, S(t_5) \leq K_5 \end{array}\right)$$

$$= \exp\left(\frac{2\mu}{\sigma^2}(h_5 - h_4 + h_3 - h_2 + h_1)\right) \tag{105}$$

$$\Phi_5\left[\begin{array}{l}\frac{k_1-2h_1-\mu t_1}{\sigma\sqrt{t_1}}, \frac{k_2-2h_2+2h_1+\mu t_2}{\sigma\sqrt{t_2}}, \frac{k_3-2h_3+2h_2-2h_1-\mu t_3}{\sigma\sqrt{t_3}}, \frac{k_4-2h_4+2h_3-2h_2+2h_1+\mu t_4}{\sigma\sqrt{t_4}} \\ \frac{k_5-2h_5+2h_4-2h_3+2h_2-2h_1-\mu t_5}{\sigma\sqrt{t_5}}; -\rho_{1.2}, -\rho_{2.3}, -\rho_{3.4}, -\rho_{4.5}\end{array}\right]$$

where the function $\Phi_5[b_1, b_2, b_3, b_4, b_5; \rho_{12}, \rho_{23}, \rho_{34}, \rho_{45}]$ is defined by (25).

In theory, the probability p in (105) could be used to value a 5-step up-and-in put by taking K_1, K_2, K_3, K_4 high enough for the probability $Q(S(t_1) < K_1, S(t_2) < K_2, S(t_3) < K_3, S(t_4) < K_4)$ to become "very" close to 1. One should beware, though, of the numerical errors entailed by taking the appropriate limits w.r.t. the $K_i{}'$s in (105). A little testing shows that they can be big, so that one cannot get around the analytical derivation of the following probability:

$$Q\left(\overline{S}_0^1 \geq H_1, \overline{S}_1^2 \geq H_2, \overline{S}_2^3 \geq H_3, \overline{S}_3^4 \geq H_4, \overline{S}_4^5 \geq H_5, S(t_5) \leq K_5\right) \tag{106}$$

which involves many more 5-dimensional Gaussian integrals than p.

To evaluate the function Φ_5, it can be shown that the 5-dimensional integral defining Φ_5 can be rewritten as the following triple integral, which is significantly faster to evaluate numerically:

$$\begin{aligned}\Phi_5[b_1,\ldots,b_5;\rho_{1.2},\ldots,\rho_{4.5}] = \frac{1}{\sqrt{8\pi^3}} \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{\frac{b_3-\rho_{2.3}x_2}{\sqrt{1-\rho_{2.3}^2}}} \int_{x_4=-\infty}^{\frac{b_4-\rho_{3.4}(x_3\sqrt{1-\rho_{2.3}^2}+\rho_{2.3}x_2)}{\sqrt{1-\rho_{3.4}^2}}} \exp\left(-\frac{x_2^2+x_3^2+x_4^2}{2}\right) N\left[\frac{b_1-\rho_{1.2}x_2}{\sqrt{1-\rho_{1.2}^2}}\right] \\ N\left[\frac{b_5-\rho_{4.5}\left(x_4\sqrt{1-\rho_{3.4}^2}+\rho_{3.4}\left(x_3\sqrt{1-\rho_{2.3}^2}+\rho_{2.3}x_2\right)\right)}{\sqrt{1-\rho_{4.5}^2}}\right] dx_4 dx_3 dx_2\end{aligned} \tag{107}$$

The probability p can also be expressed in terms of the pentavariate standard normal cumulative distribution function N_5, as follows:

$$p = \exp\left(\frac{2\mu}{\sigma^2}(h_5-h_4+h_3-h_2+h_1)\right) \tag{108}$$

$$N_5\left[\begin{array}{l}\frac{k_1-2h_1-\nu t_1}{\sigma\sqrt{t_1}}, \frac{k_2-2h_2+2h_1+\nu t_2}{\sigma\sqrt{t_2}}, \frac{k_3-2h_3+2h_2-2h_1-\nu t_3}{\sigma\sqrt{t_3}}, \frac{k_4-2h_4+2h_3-2h_2+2h_1+\nu t_4}{\sigma\sqrt{t_4}} \\ \frac{k_5-2h_5+2h_4-2h_3+2h_2-2h_1-\nu t_5}{\sigma\sqrt{t_5}}; -\rho_{1.2}, \rho_{1.3}, -\rho_{1.4}, \rho_{1.5}-\rho_{2.3}, \rho_{2.4}, -\rho_{2.5}, -\rho_{3.4}, \rho_{3.5}, -\rho_{4.5}\end{array}\right]$$

An exact three-dimensional quadrature rule for the numerical evaluation of the five-dimensional function N_5 can be found in Guillaume (2018). However, this approach is not faster than implementing the formula in (107).

Whatever the multitouch payoff considered, it is clear that, as n increases, so does the size of the obtained formulae. Thus, although it is possible to calculate closed-form formulae for multitouch barrier options for $n > 3$, the number of terms involved makes it a quite tedious task. To give an idea of the required effort, one can examine the number of multivariate normal distribution functions involved in the calculation of the probability needed to value a mere 5-step knock-out put in Guillaume (2015). For an arbitrary dimension n, the only realistic solution would be to find a way to automate the calculation.

However, even if a computer program were able to write down the exact solution of the $n-$touch barrier option valuation problem for any $n \in \mathbb{N}$, with all the necessary details for its immediate implementation, there would still be a more fundamental issue to be addressed than the size of the resulting formula, namely, the numerical evaluation of the multivariate Gaussian integral of order $n \in \mathbb{N}$. It is well known by specialists of numerical integration that it is impossible to evaluate such an integral with arbitrary accuracy and efficiency for any $n \in \mathbb{N}$, due to the notorious "curse of dimensionality". For more background on this topic, the reader may refer to Genz and Bretz (2009). In high dimension, the conditional Monte Carlo simulation method pioneered by Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001) remains extensively used by practitioners, due to its flexibility and its good mix of speed (computational time grows linearly in dimension) and precision (no discretization of each $[t_{i-1}, t_i]$ is required). Notice that the closed-form solutions that one can derive in low dimension are still a useful way to increase the speed of convergence of the Monte Carlo simulation in high dimension, either as accurate benchmarks that can be used as control variates, or as a way to extend the Brownian bridge over time intervals larger than $[t_{i-1}, t_i]$. Another possible solution in high dimension is to notice that the function Φ_n, as a convolution of Gaussian densities, can be numerically evaluated by means of the fast Gauss transform algorithm pioneered by Greengard and Strain (1991). Examples of applications of this numerical method to quantitative problems in finance can be found in Broadie and Yamamoto (2005).

5. Conclusions

This paper has shown how to compute exact prices, at least in moderate dimension, of valuable kinds of barrier options known as multitouch options. The latter capture the flexibility of step barrier options without exposing investors to the risk of 'sudden death' (nullification of the claim at the first passage to the barrier before expiry), thus making multitouch options significantly less risky than barrier options that do not gradually knock out. Hedging parameters can be derived by mere differentiation of the obtained formulae. Implementing dynamic hedging with multitouch options is easier than with other barrier options, as gammas do not 'explode' (i.e., do not go to infinity) near the barrier, thanks to the multiple chances of staying alive entailed by the payoff.

Based on the valuation method presented here, it would be a straightforward extension to derive exact formulae for partial-time multitouch options. An interesting but more challenging extension would consist in letting the rates of participation in the payoff at expiry be defined as functions of the highest or lowest values reached by the underlying asset during the option's lifetime. What seems much less easy to find is a way to analytically price a multitouch option in arbitrary dimension, i.e., with an arbitrary number of crossings, because it rapidly becomes impossible to evaluate the resulting multidimensional integrals with sufficient precision in a reasonable amount of time. This does not mean that closed-form formulae in low dimension are not valuable though, because most traded contracts only incorporate a few possibilities of breaching the barrier. In particular, the most popular multitouch option in the markets, known as the 'baseball option', is merely a 3-touch option, and this case is precisely the one we have focused on in this paper. Moreover, exact values in low dimension can serve as useful benchmarks to speed up approximate valuation by simulation in high dimension in the form of control variates.

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Appendix A. Solutions to the Integration Problems in Section 3

(1) $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$

$$= \Phi_3\left[\frac{h_1 \wedge h_2 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{h_2 \wedge h_3 - \mu t_2}{\sigma\sqrt{t_2}}, \frac{k \wedge h_3 - \mu t_3}{\sigma\sqrt{t_3}}; \rho_{1.2}, \rho_{2.3}\right] \tag{A1}$$

$$- \exp\left(\frac{2\mu h_1}{\sigma^2}\right)\Phi_3\left[\frac{h_1 \wedge h_2 - 2h_1 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{h_2 \wedge h_3 - 2h_1 - \mu t_2}{\sigma\sqrt{t_2}}, \frac{k \wedge h_3 - 2h_1 - \mu t_3}{\sigma\sqrt{t_3}}; \rho_{1.2}, \rho_{2.3}\right] \tag{A2}$$

$$- \exp\left(\frac{2\mu h_2}{\sigma^2}\right)\Phi_3\left[\frac{h_1 \wedge h_2 + \mu t_1}{\sigma\sqrt{t_1}}, \frac{h_2 \wedge h_3 - 2h_2 - \mu t_2}{\sigma\sqrt{t_2}}, \frac{k \wedge h_3 - 2h_2 - \mu t_3}{\sigma\sqrt{t_3}}; -\rho_{1.2}, \rho_{2.3}\right] \tag{A3}$$

$$-\exp\left(\frac{2\mu h_3}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \quad (A4)$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_2+2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \quad (A5)$$

$$+\exp\left(\frac{2\mu(h_3-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \quad (A6)$$

$$+\exp\left(\frac{2\mu(h_3-h_2)}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right] \quad (A7)$$

$$-\exp\left(\frac{2\mu(h_3-h_2+h_1)}{\sigma^2}\right) \quad (A8)$$

$$\Phi_3\left[\frac{h_1\wedge h_2-2h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2+2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right]$$

(2) $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2\right)$

$$=N_2\left[\frac{h_1\wedge h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-\mu t_2}{\sigma\sqrt{t_2}};\rho_{1.2}\right]-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2-\mu t_2}{\sigma\sqrt{t_2}};-\rho_{1.2}\right] \quad (A9)$$

$$-\exp\left(\frac{2\mu h_1}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-2h_1-\mu t_2}{\sigma\sqrt{t_2}};\rho_{1.2}\right] \quad (A10)$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}};-\rho_{1.2}\right] \quad (A11)$$

(3) $Q\left(\overline{S}_0^1 < H_1, \overline{S}_1^2 < H_2, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{h_1\wedge h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \quad (A12)$$

$$-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \quad (A13)$$

$$-\exp\left(\frac{2\mu h_1}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2-2h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \quad (A14)$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_2+2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \quad (A15)$$

(4) $Q\left(\overline{S}_1^2 < H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \quad (A16)$$

$$-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\Phi_3\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \quad (A17)$$

$$-\exp\left(\frac{2\mu h_3}{\sigma^2}\right)\Phi_3\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \tag{A18}$$

$$+\exp\left(\frac{2\mu(h_3-h_2)}{\sigma^2}\right)\Phi_3\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2\wedge h_3-2h_2+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right] \tag{A19}$$

(5) $Q\left(\overline{S}_1^2 < H_2, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right]-\Phi_3\left[\frac{h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \tag{A20}$$

$$-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\left\{\begin{array}{l}\Phi_3\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right]\\ -\Phi_3\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right]\end{array}\right\} \tag{A21}$$

$$+\exp\left(\frac{2\mu h_1}{\sigma^2}\right)\Phi_3\left[\frac{-h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \tag{A22}$$

$$-\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{-h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_2+2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \tag{A23}$$

(6) $Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 < H_3, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \tag{A24}$$

$$-\exp\left(\frac{2\mu h_1}{\sigma^2}\right)\Phi_3\left[\frac{-h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},\rho_{2.3}\right] \tag{A25}$$

$$-\exp\left(\frac{2\mu h_3}{\sigma^2}\right)\Phi_3\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \tag{A26}$$

$$+\exp\left(\frac{2\mu(h_3-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{-h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \tag{A27}$$

(7) $Q\left(\overline{S}_0^1 < H_1, \overline{S}_2^3 \geq H_3, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right] \tag{A28}$$

$$-\exp\left(\frac{2\mu h_1}{\sigma^2}\right)\Phi_3\left[\frac{-h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_3+2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right] \tag{A29}$$

$$+\exp\left(\frac{2\mu h_3}{\sigma^2}\right)\Phi_3\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_3-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \tag{A30}$$

$$-\exp\left(\frac{2\mu(h_3-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{-h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k-2h_3+2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.2},-\rho_{2.3}\right] \tag{A31}$$

(8) $Q\left(\overline{S}_1^2 \geq H_2, \overline{S}_2^3 < H_3, S(t_3) < K\right)$

$$=\Phi_3\left[\frac{-h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \tag{A32}$$

$$+\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\left\{\begin{array}{c}\Phi_3\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right]\\ \Phi_3\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right]\end{array}\right\} \tag{A33}$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)\Phi_3\left[\frac{-h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_2+2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},\rho_{2.3}\right] \tag{A34}$$

$$-\exp\left(\frac{2\mu h_3}{\sigma^2}\right)\Phi_3\left[\frac{-h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right] \tag{A35}$$

$$-\exp\left(\frac{2\mu(h_3-h_2)}{\sigma^2}\right)\left\{\begin{array}{c}\Phi_3\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right]\\ -\Phi_3\left[\frac{h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right]\end{array}\right\} \tag{A36}$$

$$\begin{array}{l}-\exp\left(\frac{2\mu(h_3-h_2+h_1)}{\sigma^2}\right)\\ \Phi_3\left[\frac{-h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_3-2h_2+2h_1+\mu t_2}{\sigma\sqrt{t_2}},\frac{k\wedge h_3-2h_3+2h_2-2h_1-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.2},-\rho_{2.3}\right]\end{array} \tag{A37}$$

(9) $Q\left(\overline{S}_0^1 < H_1, S(t_3) < K\right)$

$$= N_2\left[\frac{-h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{k-\mu t_3}{\sigma\sqrt{t_3}};-\rho_{1.3}\right]+\exp\left(\frac{2\mu h_1}{\sigma^2}\right)N_2\left[\frac{-h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{k-2h_1-\mu t_3}{\sigma\sqrt{t_3}};\rho_{1.3}\right] \tag{A38}$$

(10) $Q\left(\overline{S}_1^2 < H_2\right)$

$$= N_2\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{h_2-\mu t_2}{\sigma\sqrt{t_2}};\rho_{1.2}\right]-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)N_2\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{-h_2-\mu t_2}{\sigma\sqrt{t_2}};-\rho_{1.2}\right] \tag{A39}$$

Appendix B. Closed-Form Formula for the Value of a 2-Touch Up-and-Out Put

Under the modeling assumptions and the notations defined in Section 2, the no-arbitrage value of a 2-touch up-and-out put with strike K, expiry t_2, constant knock-out levels H_1 and H_2 monitored on $[t_0, t_1]$ and $[t_1, t_2]$ respectively, such that $H_1 < H_2$, is given by:

$$Ke^{-rt_2}\left\{\omega_0\psi_1\left(\mu^{(Q)}\right)+\omega_1\left(\psi_2\left(\mu^{(Q)}\right)+\psi_3\left(\mu^{(Q)}\right)\right)+\omega_2\psi_4\left(\mu^{(Q)}\right)\right\} \tag{A40}$$

$$-S(0)\left\{\omega_0\psi_1\left(\mu^{(Q^{(S)})}\right)+\omega_1\left(\psi_2\left(\mu^{(Q^{(S)})}\right)+\psi_3\left(\mu^{(Q^{(S)})}\right)\right)+\omega_2\psi_4\left(\mu^{(Q^{(S)})}\right)\right\}$$

where:

$$-\ \mu^{(Q)} = r-\frac{\sigma^2}{2},\ \mu^{(Q^{(S)})} = r+\frac{\sigma^2}{2} \tag{A41}$$

$$-\ \psi_1(\mu) = N_2\left[\frac{h_1\wedge h_2-\mu t_1}{\sigma\sqrt{t_1}},\frac{k\wedge h_2-\mu t_2}{\sigma\sqrt{t_2}};\rho_{1.2}\right] \tag{A42}$$

$$-\exp\left(\frac{2\mu h_1}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1-\mu t_1}{\sigma\sqrt{t_1}},\frac{k\wedge h_2-2h_1-\mu t_2}{\sigma\sqrt{t_2}};\rho_{1.2}\right] \tag{A43}$$

$$-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2+\mu t_1}{\sigma\sqrt{t_1}},\frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}};-\rho_{1.2}\right] \tag{A44}$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}},\frac{k\wedge h_2-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}};-\rho_{1.2}\right] \tag{A45}$$

$$-\psi_2(\mu) = N_2\left[\frac{h_2-\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-\mu t_2}{\sigma\sqrt{t_2}}; \rho_{1.2}\right] - N_2\left[\frac{h_1-\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-\mu t_2}{\sigma\sqrt{t_2}}; \rho_{1.2}\right] \tag{A46}$$

$$+\exp\left(\frac{2\mu h_1}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1-\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_1-\mu t_2}{\sigma\sqrt{t_2}}; \rho_{1.2}\right] \tag{A47}$$

$$-\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\left(\begin{array}{c} N_2\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \\ -N_2\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \end{array}\right) \tag{A48}$$

$$-\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{A49}$$

$$-\psi_3(\mu) = \exp\left(\frac{2\mu h_2}{\sigma^2}\right)N_2\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{A50}$$

$$-\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{A51}$$

$$-\psi_4(\mu) = N_2\left[\frac{-h_2+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{A52}$$

$$+\exp\left(\frac{2\mu h_2}{\sigma^2}\right)\left(\begin{array}{c} N_2\left[\frac{h_2+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \\ -N_2\left[\frac{h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \end{array}\right) \tag{A53}$$

$$+\exp\left(\frac{2\mu(h_2-h_1)}{\sigma^2}\right)N_2\left[\frac{h_1\wedge h_2-2h_1+\mu t_1}{\sigma\sqrt{t_1}}, \frac{k\wedge h_2-2h_2+2h_1-\mu t_2}{\sigma\sqrt{t_2}}; -\rho_{1.2}\right] \tag{A54}$$

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