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Amortizing Loans under Arbitrary Discount Functions

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Abstract: A general methodology for loan amortization under arbitrary discount functions is discussed. It is shown that it is always possible to uniquely define a scheme for constructing the loan amortization schedule with an arbitrary assigned discount function. It is also shown that, even if the loan amortization is carried out from the sequence of principal payments and the sequence of accrued interest, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. As a special case of the proposed approach, we derive the amortization method according to the law of simple interest.

Keywords: discount function; amortizing loans; outstanding balance; amortization schedule; compound interest; simple interest

JEL Classification: G31; G32; G33

1. Introduction

The past decade has seen a renewed interest in Italy in the valuation of amortizing loans, following an important debate on the consistency of the law of compound interest, also known as the law of exponential capitalization, with the principle, enshrined in Italian law, that interest produced in one period of time cannot produce interest in subsequent periods, a phenomenon called "anatocism".

In recent years, many Italian courts have produced conflicting rulings, in some cases accepting and in others denying the presence of anatocism in certain amortization schemes widespread in operating practices, such as, for example, the French amortization method, which is characterized by constant installments under the law of compound interest (for a review, see Annibali et al. (2017)). These conflicting rulings have animated an intense debate involving jurists, economists and mathematicians in an attempt to arrive at a shared solution that reconciles the financial mathematics of loan amortization with Italian law. Two main points are debated. The first is whether anatocism is present when amortizing loans are evaluated according to the law of compound interest (Fersini and Olivieri 2015). The second point concerns the possibility of exploring different amortization methods, with a focus on amortization methods consistent with the law of simple interest, also called the law of linear capitalization (Annibali et al. 2018; Mari and Aretusi 2018, 2019).

In Italy last year (2023), the issue landed in the Corte Suprema di Cassazione, the highest court in the judicial system established to ensure the correct application of the law, which will have to rule in the coming months on the compatibility with Italian law of the loan amortization techniques most widely used in operating practices (key documents are downloadable at www.cortedicassazione.it).

The problem also has international significance. Several international disputes have shown a general tendency not to accept compound interest (for a comprehensive review see Sinclair (2016)). This is motivated by the fact that the exponential nature of the law of compound interest has an explosive effect in the medium to long term, a factor that greatly affects the risk of default and, therefore, the ability to efficiently plan investments (Cerina 1993).

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In an attempt to guide the debate, some authors (Pressacco et al. 2022) proposed two different amortization schemes based on different inputs but sharing the same accrued interest calculation rule. In the first scheme, with no apparent reference to an underlying discount function, the input is the principal amount, the sequence of principal payments and the sequence of accrued interest is calculated by multiplying the interest rate by the outstanding balance. In the second scheme, the input is given by the principal amount, the sequence of installments calculated according to the law of compound interest and the sequence of accrued interest is calculated, as in the first scheme, by multiplying the interest rate by the outstanding balance. They claim that, in both of these amortization schemes, which are widespread in operating practices, there is no generation of interest on interest if and only if the principal payments are non-negative.

However, it is well known in the financial literature that the first amortization scheme proposed by Pressacco et al. (2022) includes the second as a special case of a single standard scheme under the law of compound interest (Ottaviani 1988). The authors probably do not realize that the presence of the law of compound interest is due to the assumption of the rule for calculating accrued interest and that, regardless of the positivity or negativity of principal payments, both proposed schemes involve the phenomenon of generating interest from interest (as we will show explicitly below).

The purpose of this paper is to provide a unified theoretical framework in which to find useful elements and insights for discussion. We focus on the possibility of establishing a general methodology for evaluating amortizing loans according to arbitrary financial laws and discuss a versatile methodology for loan amortization that allows for the unambiguous construction of a loan amortization schedule with any assigned discount function. Moreover, to monitor the interest generation process and understand the interest flow over time, an extended amortization schedule is introduced. Like a macro lens to uncover the intimate structure of the amortizing loan, the extended amortization schedule contains all the information needed to fully understand the loan repayment process. This level of customization is noteworthy because it can be adapted to various environments and financial scenarios. Different amortization methods can have varying effects on borrowers, including the total cost of borrowing, the distribution of interest payments over time and the pace of debt repayment. Research in this area can inform policymakers and consumer advocacy groups about the potential impacts on borrowers and help develop regulations that promote fair lending practices.

The approach we propose is fully consistent with the general Heath–Jarrow–Morton (HJM) methodology for pricing interest rate-sensitive contingent claims (Heath et al. 1992). Starting from the initial discount function, the HJM methodology provides a no-arbitrage-based pricing approach consistent with any assigned initial discount function. In our approach, the loan amortization methodology is developed starting from the initial discount function, i.e., the observed discount function at the evaluation time, with the support of some basic no-arbitrage arguments and without any reference to the decomposability property (Castellani et al. 2005), which plays no role in this context. In particular, it will be shown that the dynamic evolution of the outstanding balance during the lifetime of the loan is time-consistent and does not imply arbitrage.

As a consequence of the proposed methodology, two significant results are presented.

The first result allows us to design loan amortization using two different but equivalent schemes under any assigned discount function. In the first scheme, loan amortization is carried out starting from the knowledge of the discount function and the sequence of the loan installments; in the second scheme, loan amortization is performed starting from the sequence of principal payments and the sequence of accrued interest. It will be shown that, even if the second scheme is adopted, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. These findings will be presented more formally in Theorems 1 and 2.

As a second result, we derive the amortization method under the law of simple interest as a particular case of the proposed methodology. In this method, the generation of interest on interest is precluded. In fact, we will show that, under the law of simple interest, accrued interest is calculated on the present value of the outstanding balance and not on the outstanding balance itself as in the compound interest method of amortization. In this way, the interest component is removed from the outstanding balance and the interest compounding over time is avoided. The method of loan amortization according to the law of simple interest derived in this paper from the first principles of financial mathematics reproduces that obtained by Mari and Aretusi (2018, 2019). The inclusion of a loan amortization scheme under the law of simple interest, in which the interest-on-interest phenomenon is avoided, could be particularly useful for financial practitioners interested in alternative amortization methods that preclude compound interest.

This study provides a conceptual framework for evaluating amortization methods based on arbitrary financial laws, which appropriately extends the most common method of loan amortization, based on the law of compound interest, by including the latter as a special case. This paper could improve our understanding of loan amortization and its potential flexibility with the aim of providing a new perspective on traditional financial methods. We believe that one of the strengths of this paper is its ability to not only propose a unifying methodology, but also provide a detailed comparison with established practices that can be useful for both academic and professional audiences.

This paper is organized as follows. Section 2 outlines the general methodology for loan evaluation. Section 3 illustrates the standard amortization method. Section 4 presents the extended amortization schedule. In Section 5, Theorems 1 and 2 are stated and proved. Section 6 presents the loan amortization method under the law of compound interest as a particular case of our methodology. As a further special case, Section 7 provides the loan amortization method under the law of simple interest. The interest-on-interest question is discussed in both Sections 6 and 7. The interest-on-interest phenomenon under arbitrary discount functions is discussed in Section 8. Finally, Section 9 provides some conceptual insights into a different method of loan amortization under a linear capitalization scheme proposed in the literature (Annibali et al. 2018), showing that there is one and only one method of loan amortization under the law of simple interest, the one described in this study.

2. Methods

In this section, we provide a general methodology to design amortizing loans. The main goal is to show how to amortize a loan and properly construct amortization schedules under arbitrary discount functions.

2.1. Some Basic Results

Let us denote by $v(0,T)$ the discount function observed at the current time $t = 0$ (the present). It denotes the value at time $t = 0$ of one unit of money payable at a later time T and can incorporate credit risk (Duffie and Singleton 1999; Mari and Renò 2005). By standard no-arbitrage arguments, it follows that the discount function must be a strictly positive function (Duffie and Singleton 1999), i.e.,

$$v(0,T) > 0, \quad T \geq 0, \tag{1}$$

with

$$v(0,0) = 1. \tag{2}$$

Following the HJM methodology (Heath et al. 1992), the discount function is independent of the amount: if x_T denotes a monetary amount payable at time T ($T \geq 0$), its value at time $t = 0$, x_0, is given by

$$x_0 = x_T v(0,T). \tag{3}$$

The monetary amount x_0 is the present value of the amount x_T available at time T. The amount x_0 can be viewed as the spot price at time $t = 0$ of a contingent claim paying the amount x_T at time T (*spot evaluation*). Hence, the amounts x_0 and x_T can be transformed into each other and, therefore, are said to be financially equivalent.

The knowledge of the discount function allows us to define an equivalence relationship between monetary amounts due at different future times. Indeed, let us denote by x_{T_1} a sum of money due at time T_1 ($T_1 > 0$) and by x_{T_2} a sum of money due at time T_2 ($T_2 > 0$), which are financially equivalent if and only if they have the same present value,

$$x_{T_2} v(0, T_2) = x_{T_1} v(0, T_1). \tag{4}$$

In fact, to avoid arbitrage opportunities, the following forward pricing formula must hold (Brigo and Mercurio 2006; Heath et al. 1992):

$$x_{T_1} = x_{T_2} \frac{v(0, T_2)}{v(0, T_1)}. \tag{5}$$

In this regard, the amount x_{T_1} can be viewed as the forward price at time T_1 of a contingent claim paying the amount x_{T_2} at time T_2 (*forward evaluation*). Therefore, if the amounts x_{T_1} and x_{T_2} satisfy Equation (4) or, equivalently, Equation (5), they can be transformed into each other.

Stated in a different way, x_{T_1} and x_{T_2} are financially equivalent if and only if they differ only in the interest component. Indeed, Equation (5) shows that x_{T_2}, the financially equivalent amount of x_{T_1}, can be determined by first discounting x_{T_1} from time T_1 to current time $t = 0$, thus eliminating the interest component, and then imputing accrued interest in the time interval $[0, T_2]$ by capitalizing the obtained value from time $t = 0$ to time T_2.

The binary relation defined by Equation (5) is an equivalence relation. Indeed, it is trivially reflexive and symmetric. It is also a transitive relation because if $x_{T_1} v(0, T_1) = x_{T_2} v(0, T_2)$ and $x_{T_2} v(0, T_2) = x_{T_3} v(0, T_3)$, it follows that $x_{T_1} v(0, T_1) = x_{T_3} v(0, T_3)$, regardless of the temporal ordering of T_1, T_2 and T_3. The binary relation defined by Equation (5), being reflexive, symmetric and transitive, provides an equivalence relation between amounts of money due at different times.

The extension of the definition of financial equivalence to cash flows is straightforward. Indeed, let us consider the cash flow,

$$\mathbf{x} = \{x_{t_1}, x_{t_2}, \cdots, x_{t_n}\}, \tag{6}$$

where $0 < t_1 < t_2 < \cdots < t_n$. The amount S_T at time $T \geq 0$ is financially equivalent to the cash flow $\mathbf{x}$ if and only if the present value of S_T is equal to the present value of $\mathbf{x}$, that is, if and only if the following relationship holds:

$$S_T v(0, T) = \sum_{k=1}^{n} x_{t_k} v(0, t_k). \tag{7}$$

In fact, the following forward pricing formula must hold to avoid arbitrage opportunities (Brigo and Mercurio 2006; Heath et al. 1992):

$$S_T = \frac{1}{v(0, T)} \sum_{k=1}^{n} x_{t_k} v(0, t_k), \tag{8}$$

which extends by linearity to Equation (5). Again, the rationale is that, if Equation (8) is satisfied, the amount S_T can be transformed into the cash flow $\mathbf{x}$ and vice versa, because each term in the r.h.s. of Equation (8), i.e., $x_{t_k} v(0, t_k)/v(0, T)$, has only one financially equivalent amount x_{t_k} at time t_k.

Equation (8) has a very interesting financial interpretation: each term x_{t_k} is first discounted from time t_k to time $t = 0$ to eliminate the interest component; then, it is capitalized from time $t = 0$ to time T to include the interest accrued in the time interval $[0, T]$. In the case $T = 0$, Equation (8) becomes

$$S_0 = \sum_{k=1}^{n} x_{t_k} v(0, t_k). \tag{9}$$

Finally, we close this section by pointing out that the equivalence relationship is established at time $t = 0$ on the basis of the information contained in the discount function at time $t = 0$ and that it is not necessarily preserved over time. Due to the unpredictability of the time evolution of the discount function, monetary amounts that are financially equivalent at time $t = 0$ may no longer be financially equivalent at a later time.

2.2. Designing Amortizing Loans

The methodology outlined in the previous section can be employed to value amortizing loans under arbitrary discount functions. To show this, let us consider at time $t = 0$ a loan with a principal amount S_0 that will be repaid with a series of non-negative future installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{10}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. If we denote by $v(0, T)$ the discount function at time $t = 0$, the following relationship must hold, as a consequence of Equation (9):

$$S_0 = \sum_{k=1}^{n} R_k v(0, k). \tag{11}$$

Let us denote by M_k the outstanding balance after the payment of the k-th installment. By definition, M_k, $k = 1, 2, \cdots, n-1$, is the monetary amount due at time k that is financially equivalent to receiving the stream of future installments $\mathbf{r_k} = \{R_{k+1}, R_{k+2}, \cdots, R_n\}$. To avoid arbitrage opportunities, it can be computed from Equation (8), thus obtaining

$$M_k = \frac{1}{v(0, k)} \sum_{j=k+1}^{n} R_j v(0, j), \quad k = 1, 2, \cdots, n-1. \tag{12}$$

Equation (12) ensures that the dynamic evolution of the outstanding balance during the lifetime of the loan is time-consistent and does not imply arbitrage.

The values of the outstanding balance, M_k, $k = 1, 2, \cdots, n-1$, are strictly positive. Of course it must be $M_n = 0$ because, after the last payment at time n, the outstanding balance is zero. Moreover, since at time $t = 0$ the outstanding balance coincides with the principal amount, we pose $M_0 = S_0$. We note that each term R_j in Equation (12) is first discounted at time $t = 0$ to eliminate the interest component; then, it is capitalized from time 0 to time k to include the interest accrued in the time interval $[0, k]$. In this sense, the outstanding balance can be thought of as a mixture of principal and interest.

Equation (12) provides the so-called *prospective* method for computing the outstanding balance. In addition, since from Equation (11) we obtain

$$\sum_{j=k+1}^{n} R_j v(0, j) = S_0 - \sum_{j=1}^{k} R_j v(0, j), \quad k = 1, 2, \cdots, n-1, \tag{13}$$

we can recast Equation (12) in the following useful form:

$$M_k = \frac{1}{v(0, k)} \left(S_0 - \sum_{j=1}^{k} R_j v(0, j) \right), \quad k = 1, 2, \cdots, n-1, \tag{14}$$

that provides the so-called *retrospective* method for computing the outstanding balance.

The dynamics of the outstanding balance can also be determined recursively by comparing the outstanding balance at time $k-1$ with the outstanding balance at time k, thus obtaining

$$M_{k-1} = \frac{v(0,k)}{v(0,k-1)}(M_k + R_k), \quad k = 1,2,\cdots,n. \tag{15}$$

It should be noted that Equation (15) could have been obtained directly as a consequence of the financial equivalence between the outstanding balance M_{k-1} at time $k-1$ and the amount $M_k + R_k$ at time k, which is the sum of the outstanding balance at time k and the k-th installment. Finally, it is worth pointing out that Equation (12) can be recovered as the only solution of Equation (15) under the terminal condition $M_n = 0$, thus proving the equivalence of the above representations of the outstanding balance.

3. The Standard Amortization Schedule

Equation (15) can be cast in a more expressive form[1]:

$$M_k = M_{k-1} + i(0,k-1,k)M_{k-1} - R_k, \tag{16}$$

where

$$i(0,k-1,k) = \frac{v(0,k-1)}{v(0,k)} - 1, \tag{17}$$

is the one-period forward rate and quantifies the interest accrued in the time interval $[k-1,k]$ (Berk and De Marzo 2014). In this regard, we note that the dynamics of the outstanding balance has a simple structure driven by two components, namely accrued interest and loan repayments. If we recast Equation (16) in the following form:

$$R_k = M_{k-1} - M_k + i(0,k-1,k)M_{k-1}, \tag{18}$$

we can see that each installment R_k can be decomposed into two components, namely

$$R_k = C_k + I_k, \tag{19}$$

where

$$C_k = M_{k-1} - M_k, \tag{20}$$

and

$$I_k = i(0,k-1,k)M_{k-1}. \tag{21}$$

Equation (20) shows that C_k quantifies the change in the outstanding balance over the time interval $[k-1,k]$ and Equation (21) shows that I_k is the interest accrued over the same time interval. Finally, it is straightforward to show that the outstanding balance, M_k, can also be expressed as

$$M_k = S_0 - \sum_{j=1}^{k} C_j, \tag{22}$$

and that the following relationship holds:

$$\sum_{k=1}^{n} C_k = S_0. \tag{23}$$

For this reason, in the literature the numbers C_k ($k = 1,2,\cdots,n$) are called principal payments.

The standard amortization schedule is a table that shows all the financial information of the loan mentioned above (Broverman 2017; Pressacco et al. 2022). In particular, the amortization schedule exhibits for each k the vector

$$\phi_k = \{k, R_k, C_k, I_k, M_k\}, \tag{24}$$

starting from the initial vector $\phi_0 = \{0, 0, 0, 0, S_0\}$ which is reported in the first row of the table. All the financial quantities contained in ϕ_k can be easily computed in the proposed approach. For example (but this is not the only way), under an assigned discount function, the amortization schedule can be constructed iteratively as follows: starting from the principal amount $M_0 = S_0$ and the loan repayment plan, R_k, obtained as a solution of Equation (11) with $R_k \geq 0$ and $R_n > 0$, accrued interest I_k can be calculated by using Equation (21); then, C_k can be obtained from Equation (19) by taking the difference

$$C_k = R_k - I_k, \tag{25}$$

and, finally, M_k can be computed from Equation (20),

$$M_k = M_{k-1} - C_k. \tag{26}$$

A Numerical Example

To illustrate the standard amortization method, consider a loan with principal amount $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular intervals $k = 1, 2, \cdots, 5$. The values of the discount function at time $t = 0$ are reported in Table 1.

Table 1. The discount function.

k	1	2	3	4	5
$v(0,k)$	0.9346	0.8573	0.7513	0.7084	0.6560

The amount of each payment can be computed by using Equation (11), thus obtaining

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}. \tag{27}$$

The standard amortization schedule, obtained by following the iterative procedure discussed above, is given in Table 2.

Table 2. The standard amortization schedule.

k	R_k	C_k	I_k	M_k
0	0	0	0	100
1	25.59	18.59	7.00	81.41
2	25.59	18.25	7.34	63.16
3	25.59	16.68	8.91	46.47
4	25.59	22.78	2.81	23.70
5	25.59	23.70	1.89	0

4. The Extended Amortization Schedule

Before proceeding further, it is necessary to explore one aspect that is definitely relevant to our analysis. Is it correct to identify accrued interest with paid interest? Looking at Equation (11), we can see that each term R_k is discounted at time $t = 0$. Discounting removes the interest component from R_k, thus providing the portion of the principal that is actually repaid with the k-th installment (in concordance also with the decomposition of a loan into single-payment loans). In this picture, the interest content of each installment is then given by the difference $R_k - R_k v(0,k)$. Let us pose, therefore,

$$S_{0,k} = R_k v(0,k), \tag{28}$$

and

$$J_k = R_k [1 - v(0,k)], \tag{29}$$

to indicate, respectively, the portion of principal and the portion of interest actually paid with the k-th installment. In addition to the representation provided by Equation (19), R_k also admits, therefore, the following decomposition:

$$R_k = S_{0,k} + J_k. \tag{30}$$

Of course, $S_{0,k} \neq C_k$ and $J_k \neq I_k$; however, the following equalities hold:

$$\sum_{k=1}^{n} S_{0,k} = \sum_{k=1}^{n} C_k = S_0, \tag{31}$$

$$\sum_{k=1}^{n} J_k = \sum_{k=1}^{n} I_k. \tag{32}$$

as a consequence of Equations (11) and (19). Since $S_{0,k}$ is the present value of R_k, it contains no interest and is, therefore, pure capital. For this reason, we will refer to the amounts $S_{0,k}$ ($k = 1, 2, \cdots, n$) as principal "bare" payments.

The financial quantities we have just introduced, namely $S_{0,k}$ and J_k, allow for a meaningful representation of outstanding balance. In fact, by substituting Equation (30) into Equation (16), we obtain

$$M_k = M_{k-1} - S_{0,k} + I_k - J_k. \tag{33}$$

Since I_k is the interest accrued in the time interval $[k-1, k]$ and J_k is the amount of interest actually paid with the k-th installment, it follows that whenever $J_k < I_k$, the interest component of M_k increases by the amount $I_k - J_k$; if $J_k > I_k$, the interest component of M_k decreases by the amount $J_k - I_k$. Furthermore, since $C_k = M_{k-1} - M_k$, Equation (33) also provides the relationship between C_k and $S_{0,k}$, namely

$$C_k = S_{0,k} + J_k - I_k, \tag{34}$$

showing that C_k, despite being called principal payment, contains a well-defined interest component. Moreover, let us denote by $D_{0,k}$ the value of the principal not yet actually repaid with the first k installments, i.e, the difference between S_0 and the sum of the first k principal bare payments,

$$D_{0,k} = S_0 - \sum_{j=1}^{k} S_{0,j} = \sum_{j=k+1}^{n} S_{0,j}, \quad k = 1, 2, \cdots, n-1. \tag{35}$$

By substituting Equation (28) into Equation (12) we obtain a very expressive relationship between M_k and $D_{0,k}$, namely

$$M_k = \frac{D_{0,k}}{v(0,k)}, \tag{36}$$

or, equivalently,

$$D_{0,k} = M_k v(0,k), \tag{37}$$

showing that $D_{0,k}$ is the present value of the outstanding balance M_k. Of course, it is $D_{0,n} = 0$ and $D_{0,0} = S_0$. Since $D_{0,k}$ is the present value of M_k, it contains no interest and is, therefore, pure capital[2]. As a consequence, the difference $M_k - D_{0,k}$ quantifies the interest component in the outstanding balance. It is given by

$$M_k - D_{0,k} = \sum_{j=1}^{k} (I_j - J_j), \tag{38}$$

as it is straightforward to prove by recursively applying Equation (33).

Finally, since $C_k = M_{k-1} - M_k$ and $S_{0,k} = D_{0,k-1} - D_{0,k}$, we also obtain the following interesting picture: C_k is given by the difference between the outstanding balance at time $k-1$ and the outstanding balance at time k; $S_{0,k}$ is given by the difference between the present value of the outstanding balance at time $k-1$ and the present value of the outstanding balance at time k.

In the extended amortization schedule, we will provide synoptically all relevant financial information about the loan, showing explicitly for each k the vector

$$\phi_k^{\text{ext}} = \{k, R_k, C_k, I_k, M_k, S_{0,k}, J_k, D_{0,k}\}, \tag{39}$$

starting from the initial vector $\phi_0^{\text{ext}} = \{0, 0, 0, 0, S_0, 0, 0, S_0\}$ reported in the first row of the table. In the extended amortization schedule, the traditional schedule is shown to the left of the vertical bar. On the right-hand side, some additional information is given concerning, for each epoch k, the financial quantities $S_{0,k}$, J_k and $D_{0,k}$. Like a macro lens to uncover the intimate structure of the amortizing loan, the part to the right of the vertical bar contains all the information needed to monitor the interest generation process and understand the interest flow over time.

A Numerical Example

Referring to the numerical example discussed in the previous section, the extended amortization schedule is shown in Table 3.

Table 3. The extended amortization schedule.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	25.59	18.59	7.00	81.41	23.92	1.67	76.08
2	25.59	18.25	7.34	63.16	21.94	3.65	54.14
3	25.59	16.68	8.91	46.47	19.23	6.36	34.92
4	25.59	22.78	2.81	23.70	18.13	7.46	16.79
5	25.59	23.70	1.89	0	16.79	8.80	0

5. Uncovering the Financial Law behind an Amortizing Loan

In this section, we discuss a loan amortization technique that can be configured as a second well-defined amortization scheme (Pressacco et al. 2022). With no apparent reference to an underlying discount function, in this scheme the input is given by the principal amount, S_0, the sequence of principal payments, C_k, and the sequence of accrued interest, I_k. To simplify the notation, let us pose

$$B_k = S_0 - \sum_{j=1}^{k} C_j, \quad B_0 = S_0. \tag{40}$$

We assume that the sequences of numbers C_k and I_k satisfy the following conditions:

(G1) $B_n = 0$;

(G2) $I_k = f(k)B_{k-1} \ (f(k) > -1), \quad k = 1, 2, \cdots, n$;

(G3) $C_k + I_k \geq 0, \quad k = 1, 2, \cdots, n-1, \quad C_n + I_n > 0$.

From this figure, the loan installments and outstanding balance are calculated as follows:

$$R_k = C_k + I_k, \tag{41}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0. \tag{42}$$

Condition *(G1)* ensures that $M_n = 0$, i.e.,

$$\sum_{k=1}^{n} C_k = S_0, \tag{43}$$

and that $M_k = B_k$; condition *(G2)* also allows for negative rates to be taken into account; condition *(G3)* ensures that the installments, R_k, are non-negative with $R_n > 0$. Moreover, conditions *(G1)–(G3)* imply that

$$M_k > 0, \quad k = 1, 2, \cdots, n-1. \tag{44}$$

Indeed, if there is $\bar{k}$ such that $M_{\bar{k}} \leq 0$, $\bar{k} = 1, 2, \cdots, n-1$, it follows that

$$M_{\bar{k}+1} = (1 + f(\bar{k}+1))M_{\bar{k}} - R_{\bar{k}} \leq 0$$

and so on until time n where $M_n < 0$ since $R_n > 0$.

We will show that, even if this second scheme is adopted, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. In addition, we will show that this second amortization scheme is equivalent to the scheme discussed in Section 3. These results are more formally described by the following Theorems 1 and 2. In particular, Theorem 1 summarizes the findings obtained in Section 3.

Theorem 1. *Let S_0 a strictly positive number and consider for $k = 1, 2, \cdots, n$: (i) a sequence of strictly positive numbers $v(0,k)$; (ii) a sequence of non-negative numbers R_k, with $R_n > 0$, such that*

$$S_0 = \sum_{k=1}^{n} R_k v(0,k). \tag{45}$$

If M_k is computed according to

$$M_k = \frac{1}{v(0,k)} \sum_{j=k+1}^{n} R_j v(0,j), \quad k = 1, 2, \cdots, n-1, \tag{46}$$

and $M_n = 0$, then there exist a unique sequence of numbers C_k and a unique sequence of numbers I_k, $k = 1, 2, \cdots, n$, satisfying conditions (G1)–(G3), such that the amortizing schedule can be computed according to the following rules:

$$R_k = C_k + I_k, \tag{47}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0. \tag{48}$$

Proof of Theorem 1. Under the assumptions of Theorem 1, the sequences of numbers C_k and I_k are given by Equations (20) and (21), respectively. Then, it is straightforward to verify that conditions *(G1)–(G3)* hold with $f(k) = i(0, k-1, k)$. □

The converse is also true. Indeed, we will prove that the following proposition holds.

Theorem 2. *Let S_0 a strictly positive number and consider for $k = 1, 2, \cdots, n$: (i) a sequence of numbers C_k and (ii) a sequence of numbers I_k satisfying conditions (G1)–(G3). If the amortizing schedule is computed according to the following rules:*

$$R_k = C_k + I_k, \tag{49}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0, \tag{50}$$

there exists a unique sequence of numbers,

$$v(0,k) = \prod_{j=1}^{k} \frac{1}{1+f(j)} \quad k = 1,2,\cdots,n, \tag{51}$$

such that the following relationships hold:

$$S_0 = \sum_{k=1}^{n} R_k v(0,k), \tag{52}$$

$$M_k = \frac{1}{v(0,k)} \sum_{j=k+1}^{n} R_j v(0,j), \quad k = 1,2,\cdots,n-1. \tag{53}$$

Moreover, the numbers $v(0,k)$, $k = 1,2,\cdots,n$, are strictly positive.

Proof of Theorem 2. Preliminarily, we note that, from condition *(G2)*, the numbers $v(0,k)$ defined by Equation (51) are strictly positive since $f(k) > -1$. By substituting Equation (49) into Equation (50), we obtain

$$M_k = M_{k-1} + f(k)M_{k-1} - R_k, \tag{54}$$

where condition *(G2)* has been used. Solving with respect to M_{k-1}, we obtain

$$M_{k-1} = \frac{R_k + M_k}{1+f(k)}. \tag{55}$$

By using Equation (51), we can rewrite Equation (55) in the following recursive form:

$$M_{k-1} = \frac{v(0,k)}{v(0,k-1)}(R_k + M_k), \quad k = 1,2,\cdots,n, \tag{56}$$

with $v(0,0) = 1$. Equations (52) and (53) can be then recovered by backward induction starting from $M_n = 0$ and recalling that $M_0 = S_0$. To prove the uniqueness, we observe that the system of n linear equations in the n unknowns $v(0,k)$, $k = 1,2,\cdots,n$, described by Equation (56), admits one and only one solution. □

As Theorem 2 clearly shows, the rule for calculating interest, expressed by condition *(G2)*, plays a crucial role in identifying the discount function, allowing it to be uniquely determined. Moreover, we note that Equation (51) can be cast in the following recursive form:

$$v(0,k) = \frac{v(0,k-1)}{1+f(k)}, \quad k = 1,2,\cdots,n, \tag{57}$$

with $v(0,0) = 1$.

As an example, it is easy to verify that the discount function represented in Table 1 can be easily discovered from the amortization schedule shown in Table 2 by using Equation (51) or, equivalently, Equation (57).

6. Amortizing Loans under the Law of Compound Interest

As a special case of the general approach proposed in this paper, we derive the amortization method according to the law of compound interest, which is the most common way of amortizing loans. In such a case, the discount function at time $t = 0$ is expressed as follows:

$$v(0,T) = \frac{1}{(1+i)^T}, \tag{58}$$

where i denotes the interest rate level at time $t = 0$. Within this framework, one-period forward rates are constant, namely

$$i(0, k-1, k) = i. \tag{59}$$

6.1. The Amortization Method

Let us consider at time $t = 0$ a loan with a principal amount S_0 which will be repaid with a series of future non-negative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{60}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. From Equation (11), the following relationship must hold:

$$S_0 = \sum_{k=1}^{n} \frac{R_k}{(1+i)^k}. \tag{61}$$

According to the law of compound interest, the dynamics of the outstanding balance, described by Equation (16), becomes

$$M_k = M_{k-1} + iM_{k-1} - R_k, \tag{62}$$

so that each installment can be decomposed in the following form:

$$R_k = C_k + I_k, \tag{63}$$

where

$$C_k = M_{k-1} - M_k, \tag{64}$$

quantifies the change in the outstanding balance in the time interval $[k-1, k]$, and

$$I_k = iM_{k-1}, \tag{65}$$

is the interest accrued over the same time interval. Then, the amortization method is uniquely defined according to the schemes provided by Theorem 1 or Theorem 2.

Finally, we emphasize that, under the law of compound interest, the outstanding balance has a very appealing representation. In fact, applying Equation (12) we obtain,

$$M_k = \sum_{j=k+1}^{n} \frac{R_j}{(1+i)^{j-k}}. \tag{66}$$

However, attention should be paid to financial interpretations of this formula. It does not represent the present value at time k of the installments not yet repaid at that time. Since k is a future time instant, the discount function at that epoch is unknown (it is a random variable).

Numerical Examples

To illustrate the amortization method, consider a loan with a principal amount of $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular intervals $k = 1, 2, \cdots, 5$. We assume that the interest rate level is $i = 10\%$. The amount of each installment is computed according to Equation (61),

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}, \tag{67}$$

where

$$v(0,k) = \frac{1}{(1+i)^k}. \tag{68}$$

The extended amortization schedule is depicted in Table 4.

Table 4. Constant installments.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	26.38	16.38	10.00	83.62	23.98	2.40	76.02
2	26.38	18.02	8.36	65.60	21.80	4.58	54.22
3	26.38	19.82	6.56	45.78	19.82	6.56	34.40
4	26.38	21.80	4.58	23.98	18.02	8.36	16.38
5	26.38	23.98	2.40	0	16.38	10.00	0

Looking at Table 4, we note the correspondence $I_k = J_{n-k+1}$ (and $C_k = S_{0,n-k+1}$). However, such a relationship is accidental. In fact, if we consider the loan described in the previous example but with constant principal payments, $C_k = S_0/n$, this correspondence disappears, as the amortization schedule presented in Table 5 clearly shows.

Table 5. Constant principal payments.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	30.00	20.00	10.00	80.00	27.27	2.73	72.73
2	28.00	20.00	8.00	60.00	23.14	4.86	49.59
3	26.00	20.00	6.00	40.00	19.53	6.47	30.05
4	24.00	20.00	4.00	20.00	16.39	7.61	13.66
5	22.00	20.00	2.00	0	13.66	8.34	0

6.2. The Interest-on-Interest Phenomenon

Under the law of compound interest, the interest accrued in the time interval $[k-1,k]$ is computed on the outstanding balance at time $k-1$ according to

$$I_k = iM_{k-1}. \tag{69}$$

We recall that M_{k-1} is related to $D_{0,k-1}$ by Equation (36) which, in the law of compound interest, becomes

$$M_{k-1} = (1+i)^{k-1} D_{0,k-1}. \tag{70}$$

The interest accrued in the time interval $[k-1,k]$ can be, therefore, expressed as

$$I_k = i(1+i)^{k-1} D_{0,k-1}. \tag{71}$$

Since $D_{0,k-1}$ is pure capital, Equation (71) shows that the phenomenon of generating interest on interest is implicit in the law of compound interest and arises as a consequence of calculating accrued interest according to Equation (69), i.e., by multiplying the outstanding balance at each epoch $k-1$ by the interest rate i.

Equation (71) allows us to quantify the amount of the interest-on-interest component at each epoch. It is given by

$$A_k = i(1+i)^{k-1} D_{0,k-1} - iD_{0,k-1} = iD_{0,k-1}\left[(1+i)^{k-1} - 1\right], \tag{72}$$

or, in terms of the outstanding balance, by

$$A_k = iM_{k-1}\left[1 - (1+i)^{-(k-1)}\right]. \tag{73}$$

Table 6 shows, in the first and second rows, respectively, the interest-on-interest component of the loan amortization examples depicted in Tables 4 and 5.

Table 6. The interest-on-interest component.

k	1	2	3	4	5
A_k	0	0.76	1.14	1.14	0.76
A_k	0	0.73	1.04	0.99	0.63

As discussed below, amortizing loans designed under the law of simple interest are not affected by this mechanism of interest compounding over time.

Finally, we show that the law of compound interest is the only financial law characterized by the property that accrued interest in each time interval $[k-1, k]$ is calculated as a given percentage, say i, of the outstanding balance at time $k-1$, as described by Equation (69). This result is a consequence of Theorem 2, with $f(k) = i$. In fact, from Equation (51) we obtain

$$v(0,k) = \frac{1}{(1+i)^k}. \tag{74}$$

Therefore, adopting Equation (69) as the rule for calculating accrued interest (Pressacco et al. 2022) uniquely identifies the law of compound interest with its interest-on-interest component.

7. Amortizing Loans under the Law of Simple Interest

As a special case of the general approach proposed in this paper, we derive the amortization method under the law of simple interest. In this case, the discount function is given by

$$v(0,T) = \frac{1}{1+iT}, \tag{75}$$

where i denotes the interest rate level at time $t = 0$. Within this framework, one-period forward rates are not constant and are given by

$$i(0,k-1,k) = \frac{i}{1+i(k-1)}. \tag{76}$$

7.1. The Amortization Method

Let us consider at time $t = 0$ a loan with a principal amount S_0 which will be repaid with a series of future non-negative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{77}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. From Equation (11), the following relationship must hold:

$$S_0 = \sum_{k=1}^{n} \frac{R_k}{1+ik}. \tag{78}$$

Under the law of simple interest, the dynamics of the outstanding balance, described by Equation (16), becomes

$$M_k = M_{k-1} + \frac{iM_{k-1}}{1+i(k-1)} - R_k, \tag{79}$$

so that each installment can be decomposed in the following form:

$$R_k = C_k + I_k, \tag{80}$$

where

$$C_k = M_{k-1} - M_k, \tag{81}$$

quantifies the change in the outstanding balance in the time interval $[k-1, k]$, and

$$I_k = \frac{iM_{k-1}}{1 + i(k-1)}, \tag{82}$$

is the interest accrued over the same time interval. Then, the amortization method is uniquely defined according to the schemes provided by Theorem 1 or Theorem 2. It is worth noting a very important difference from the amortization method based on the law of compound interest. Indeed, looking at Equation (82), we observe that, under the law of simple interest, the interest accrued in the time interval $[k-1, k]$ is calculated multiplying by i the present value of the outstanding balance at time $k-1$. In this way, the interest component of the outstanding balance is removed, thus preventing interest compounding over time.

As a final remark, consider a single-payment loan, i.e., a loan with a principal amount of S_0 repaid with a single strictly positive installment $R_n = S_0(1 + in)$ at time n. In this case, the dynamics of the outstanding balance can be determined by applying Equation (12), thus obtaining

$$M_k = (1 + ik)S_0, \quad k = 1, 2, \cdots, n-1, \tag{83}$$

and $M_n = 0$. Therefore, the outstanding balance grows linearly over time until time n and then equals 0 due to the payment of the n-th installment. By applying Equation (82), we see that accrued interest is constant over each time interval, namely

$$I_k = iS_0, \tag{84}$$

just as required by the law of simple interest. The significant implications of Equation (82) will be further discussed below.

Numerical Examples

To illustrate the amortization method with simple interest, consider a loan with a principal amount of $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular time intervals $k = 1, 2, \cdots, 5$. We assume that the interest rate level is $i = 10\%$. The amount of each installment is computed by using Equation (78), thus obtaining

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}, \tag{85}$$

with

$$v(0,k) = \frac{1}{1 + ik}. \tag{86}$$

The extended amortization schedule is depicted in Table 7.

Table 7. Constant installments.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	25.69	15.69	10.00	84.31	23.35	2.34	76.65
2	25.69	18.03	7.66	66.29	21.41	4.28	55.24
3	25.69	20.17	5.52	46.12	19.76	5.93	35.48
4	25.69	22.14	3.55	23.98	18.35	7.34	17.13
5	25.69	23.98	1.71	0	17.13	8.56	0

In the case of constant principal payments, the amortization schedule is shown in Table 8.

Table 8. Constant principal payments.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	30.00	20.00	10.00	80.00	27.27	2.73	72.73
2	27.27	20.00	7.27	60.00	22.73	4.55	50.00
3	25.00	20.00	5.00	40.00	19.23	5.77	30.77
4	23.08	20.00	3.08	20.00	16.48	6.59	14.29
5	21.43	20.00	1.43	0	14.29	7.14	0

7.2. The Absence of the Interest-on-Interest Phenomenon

Under the law of simple interest, the interest accrued in the time interval $[k-1, k]$ is computed on the present value of the outstanding balance M_{k-1}, as expressed by Equation (82), namely

$$I_k = \frac{iM_{k-1}}{1+i(k-1)}, \tag{87}$$

and not on M_{k-1} as required by the law of compound interest, i.e., $I_k = iM_{k-1}$. In this way, the interest compounding over time, i.e., the generation of interest on interest, is precluded. Indeed, we recall that M_{k-1} is related to $D_{0,k-1}$ by Equation (36) which, in the law of simple interest, becomes

$$M_{k-1} = (1+i(k-1))D_{0,k-1}. \tag{88}$$

The accrued interest in the time interval $[k-1, k]$ is, therefore, given by

$$I_k = iD_{0,k-1}. \tag{89}$$

Since $D_{0,k-1}$ is pure capital and, therefore, contains no interest, capitalization of interest over time is avoided.

Moreover, we show that the law of simple interest is the only financial law in which interest accrued in each time interval $[k-1, k]$ is calculated as a given percentage, say i, of the present value of the outstanding balance at time $k-1$, namely

$$I_k = iv(0, k-1)M_{k-1}. \tag{90}$$

This result is a consequence of Theorem 2, with

$$f(k) = iv(0, k-1). \tag{91}$$

In fact, by substituting Equation (91) into Equation (51), we obtain

$$v(0, k) = \frac{1}{1+ik}. \tag{92}$$

Finally, we emphasize that an appealing formula similar to Equation (66) also applies to the outstanding balance computed according to the law of simple interest. In fact, from Equation (12) we obtain

$$M_k = \sum_{j=k+1}^{n} \frac{R_j}{1+i_k(j-k)}, \tag{93}$$

where

$$i_k = i(0, k, k+1) = \frac{i}{1+ik}. \tag{94}$$

Again, attention must be paid to financial interpretations of this formula. It does not represent the present value at time k of the installments not yet repaid at that time. Since k is a future time instant, the discount function at that epoch is unknown (it is a random variable). However, Equation (93) has its own relevance in that the parameters i_k can play the role of a strategic variables in early repayment decisions.

8. The Interest-on-Interest Phenomenon under Arbitrary Discount Functions

What about the phenomenon of interest on interest when a loan is designed according to an arbitrary discount function? The answer depends on the parameterization adopted to represent the discount function. Consider the following generalized compound interest representation:

$$v(0,k) = \frac{1}{\prod_{j=1}^{k}(1+i_j^c)}. \tag{95}$$

We note that, given an arbitrary discount function, the sequence of rates $\{i_k^c\}$ is uniquely determined. In this representation, the interest accrued in the time interval $[k-1,k]$, computed using Equation (21), is given by

$$I_k = i_k^c M_{k-1}. \tag{96}$$

To describe the same discount function, we could have used a different parameterization such as, for example, the following generalized simple interest representation:

$$v(0,k) = \frac{1}{1+\sum_{j=1}^{k} i_j^s}. \tag{97}$$

Again, the sequence of rates $\{i_k^s\}$ is uniquely determined. In this new representation, the interest accrued in the time interval $[k-1,k]$, computed using Equation (21), is given by

$$I_k = i_k^s \frac{M_{k-1}}{1+\sum_{j=1}^{k-1} i_j^s}. \tag{98}$$

Of course, both representations produce the same amortization schedule, but with different sequences of rates $\{i_k^c\}$ and $\{i_k^s\}$. However, in the generalized compound interest representation, the phenomenon of interest on interest occurs, which results from multiplying the outstanding balance M_{k-1} by the interest rate i_k^c, as shown in Equation (96), the outstanding balance being a mixture of principal and interest. On the other hand, in the generalized simple interest representation, Equation (98) shows that the generation of interest on interest is avoided, since in this case accrued interest is computed multiplying the present value of the outstanding balance M_{k-1} by the interest rate i_k^s.

To provide some numerical examples, let us first consider the discount function given in Table 1. The rate sequences $\{i_k^c\}$ and $\{i_k^s\}$ are depicted in Table 9.

Table 9. The discount function and the rates $\{i_k^c\}$, $\{i_k^s\}$.

k	1	2	3	4	5
$v(0,k)$	0.9346	0.8573	0.7513	0.7084	0.6560
i_k^c	7.00%	9.02%	14.11%	6.06%	7.99%
i_k^s	7.00%	9.65%	16.46%	8.06%	11.28%

The same amortization schedule given in Table 3 can be obtained by using both the generalized compound interest representation provided by Equation (95) and the generalized simple interest representation provided by Equation (97) with the sequence of rates $\{i_k^c\}$ and $\{i_k^s\}$, respectively, shown in Table 9.

Table 10 gives the sequences of the rates $\{i_k^c\}$ and $\{i_k^s\}$ in the case of the numerical examples discussed in Section 6, where the law of compound interest was used (with constant interest rate $i = 10\%$). The same amortization schedules depicted in Tables 4 and 5 can be obtained by using the generalized simple interest representation provided by Equation (97) with the sequence of rates $\{i_k^s\}$ shown in Table 10.

Table 10. The discount function and the rates $\{i_k^c\}$, $\{i_k^s\}$.

k	1	2	3	4	5
$v(0,k)$	0.909091	0.826446	0.751315	0.683013	0.620921
i_k^c	10%	10%	10%	10%	10%
i_k^s	10.00%	11.00%	12.10%	13.31%	14.64%

Table 11 gives the sequences of the rates $\{i_k^c\}$ and $\{i_k^s\}$ in the case of the numerical examples discussed in Section 7 where the law of simple interest was used (with constant interest rate $i = 10\%$). The same amortization schedules depicted in Tables 7 and 8 can be obtained by using the generalized compound interest representation provided by Equation (95) with the sequence of rates $\{i_k^c\}$ shown in Table 11.

Table 11. The discount function and the rates $\{i_k^c\}$, $\{i_k^s\}$.

k	1	2	3	4	5
$v(0,k)$	0.909091	0.833333	0.769231	0.714286	0.666667
i_k^c	10.00%	9.09%	8.33%	7.69%	7.14%
i_k^s	10%	10%	10%	10%	10%

The ability to design amortizing loans with arbitrary discount functions increases the level of customization and can have varying effects on borrowers, including the total cost of borrowing, the distribution of interest payments over time and the pace of debt repayment. However, great care should be taken to use financial representations that are consistent with the law and provide consumers with accurate information about the representation of the financial scheme and the rates used, so as to avoid the phenomenon of interest-on-interest generation where this is not permitted. Research in this area can inform policymakers and consumer advocacy groups about the potential impacts on borrowers and help develop regulations that promote fair lending practices.

9. Discussion and Concluding Remarks

In this paper, we have provided a general methodology for valuing amortizing loans with arbitrary discount functions. The proposed approach is fully consistent with the general Heath–Jarrow–Morton methodology (Heath et al. 1992) for pricing interest rate-sensitive contingent loans. The entire methodology was developed from the knowledge of the initial discount function and using some basic no-arbitrage arguments. It is valid whatever stochastic model is used to describe the evolution of the discount function. No reference is made to the decomposability property (Castellani et al. 2005), which plays no role in this context. This approach is perfectly consistent with the fundamental theorems of asset pricing (Delbaen and Schachermayer 1994).

Although we have discussed loans with installment payments at regular time intervals, the extension to the case of time intervals of variable amplitude is straightforward.

As a special case of the proposed methodology, we have illustrated the amortization method based on the law of simple interest and shown that, in this case, the phenomenon of generating interest on interest is precluded. Some authors proposed a different method for valuing amortizing loans under a linear capitalization scheme (Annibali et al. 2018). To illustrate their procedure, let us consider at time $t = 0$ a loan with a principal amount S_0 that will be repaid with a series of future non-negative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{99}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. The starting point of their analysis is that the loan principal and each installment are linearly capitalized at loan maturity n, using the interest rate i observed at time $t = 0$, thus obtaining

$$S_0(1 + in) = \sum_{k=1}^{n} R_k \big[1 + i(n - k)\big]. \tag{100}$$

We point out that this approach can be considered as a special case of the methodology proposed in this study with the following discount function:

$$v(0, k) = \frac{1 + i(n - k)}{1 + in}. \tag{101}$$

However, it should be noted that this procedure produces spurious results that are not consistent with the law of simple interest. Consider, for example, a loan with a principal amount S_0 at time $t = 0$ that will be repaid with a single strictly positive installment $R_n = S_0(1 + in)$ at time n. Following this approach, the dynamics of the outstanding balance is given by

$$M_k = \frac{1 + in}{1 + i(n - k)} S_0, \tag{102}$$

showing that the outstanding balance does not follow a linear behavior, as it should be according to the law of simple interest and as obtained from Equation (83). In a different but equivalent way, interest does not accrue linearly over time. For these reasons, we believe that there is one and only one method of loan amortization under the law of simple interest, the one described in this study.

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Notes

1 Unless otherwise stated, the index k takes values from 1 to n.

2 We remark that, in the case of early repayment at time k, M_k (and not $D_{0,k}$, which is its present value) is the amount the borrower is required to repay to the lender.

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