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Probability of Default and Default Correlations

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Abstract: We consider a system where the asset values of firms are correlated with the default thresholds. We first evaluate the probability of default of a single firm under the correlated assets assumptions. This extends Merton's probability of default of a single firm under the independent asset values assumption. At any time, the distance-to-default for a single firm is derived in the system, and this distance-to-default should provide a different measure for credit rating with the correlated asset values into consideration. Then we derive a closed formula for the joint default probability and a general closed formula for the default correlation via the correlated multivariate process of the first-passage-time default correlation model. Our structural model encodes the sensitivities of default correlations with respect to the underlying correlation among firms' asset values. We propose the disparate credit risk management from our result in contrast to the commonly used risk measurement methods considering default correlations into consideration.

Keywords: default correlation; probability of default; consistency; credit risk management; Kolmogorov forward equation; first-passage-time model; distance-to-default

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1. Introduction

In practice, one of the biggest challenges of credit risk models is modeling correlations between default events, between portfolios of correlated defaultable claims and between credit quality changes. Dependence among credit default events is indispensable for determining credit risk measures used to allocate capital for solvency and to investigate systemic risk. There is a huge market for asset securitization for default swaps (for example CDO, synthetic CDO etc.) and default swaps have a basket structure with many underlying references. It is clearly important to understand the prices and credit risk of tranches of CDOs (see [1]). Risk dependence of simultaneous defaults by financial institutions is also crucial in the study of the stability of the financial system. One of the most important measures of credit risk with dependence is the probability of default under the dependence in the portfolio or the financial system. It is also important to measure the joint probability of correlated defaults and probability of portfolio default.

In this paper, we study the credit risk with dependence to develop the first step in measuring default of single firm under the asset correlation assumption. We also approach the default of multiple firms with a structural model, initiated by [2–4]. We also provide an improved distance-to-default with dependence, with the hope of providing a better credit rating system. Our structural model can be used to bond portfolios and other structure products.

Default correlation is essential for managing credit risk, and the structure of default correlation is crucial to price structured credit derivatives as well as in the stability of financial institutions. Lucas [5] states that general conditions can produce non-zero default correlation and specifies that general economic conditions and more specific industry and regional factors produce systematic and idiosyncratic credit risk exposure analogous to systematic and unsystematic equity price risk.

Significantly biased perceptions of credit risk exposure can derive from omission (Das [6]) or underestimation (Jorion [7]) of default correlation. The most widely recognized examples are CreditMetrics; Crouhy, et al. [8] and CreditRisk$^+$ in Gundlach and Lehrbass [9] with incorrect dependence consideration due to the misunderstanding of the joint default probability. Giesecke [10] induces default correlation differently in a system of independent asset value changes through firm-specific default barriers that are themselves correlated random variables. Li [11] defines a random variable called "time-until-default".

Unlike the $n = 2$ case Rebholz [12], Zhou [13], Valužis [14] and Metzler [15], which use separable variable methods to deal with initial-boundary conditions by formal series, our formulas for evaluating probabilities of (multiple) defaults are precise and computational accessible. Solutions from previous literature in closed form are given by an infinite series of modified Bessel functions. Not only with computational intensity but also with inconsistency, The default correlations and probabilities of defaults from previous literature frequently attain absolute values greater than 1 as shown in Li and Krehbiel [16]. The evaluations of probability of default and default correlation from the previous literature are not only computationally intense but also inconsistent. Zhang and Melnik [17] discuss the first passage time for multivariate jump-diffusion processes without any explicit default correlation or default probability. Li and Krehbiel [16] indicate some numerical results to the inconsistency of their default rate and default correlation. Sacerdote et al. [18] recently expressed the joint density of the first passage time in terms of the solutions of a system of Volterra-Fredholm first kind integral equation, and the solution is again given by the same method of Zhou [13] in terms of the formal power series with modified Bessel function of the first kind. Erlenmaier and Gersbach [19] regard the optimal stopping time for the default as a non-random variable to deduce the default correlation and completely ignore the main difficulty in understanding the joint distribution of first-passage-times of two correlated asset firms. Erlenmaier and Gersbach [20] provide a connection between adverse macroeconomic shocks and loan default correlations and expresses the default correlation in integral form so that the partial derivatives of the default correlation can be traced analytically. But they do not discuss the default correlation itself and the relation with the default probability. They only discuss various properties of the default correlation with respect to the distances-to-default from implicit relation.

In this paper, we specify the linkage between companies as the correlation of logarithmic changes of underlying asset values. As mentioned in Li and Krehbiel [16], there is an inconsistency between the stochastic assumptions of Merton's firm-specific default probability model with the bivariate first passage time model of default correlation. We first evaluate the probability of single firm default with dependence by changing the driving Brownian motions among firms into a combination of driving Brownian motions through a linear algebra technique. Then we evaluate firm-specific default probabilities and default correlations under consistent stochastic assumptions in Propositions 1 and 2. To understand multiple defaults, we need to further extend the results from two joint default probability to one of n-joint default probability. By using the reflection principle in partial differential equations to solve the Kolmogorov forward equation with initial-boundary conditions, we resolve this problem to derive a closed form of the probability of multiple joint defaults in Theorem 1.

We further propose the probability of default algorithm for the dependent firms. The default algorithm in credit risk management follows from evaluating single firm default, two firms default, any k firms default simultaneously, and all firms default. For two firms case, Li and Krehbiel [16] show that the difference between KMV's and J. P. Morgan's approach and our probability of joint default is not trivial, among other things. Understanding default correlation and probabilities of the default state space is crucial for proper measurement of credit risk. Bond portfolio management is the quintessential example of default correlation application. Financial stability and the systemic risk are very important in the financial market. Our analysis provides a proper tool and technique to measure those risks. We propose a mixed pair default measure $MD_i(t)$ to incorporate the dependence by adding related default correlations. This measure should provide more information than the probability of default of a single firm from easy evaluations, and also give the credit risk estimate of the specific firm in the system.

The default probability and default correlation for more than 3 firms with constant drifts and constant diffusions can be consistently obtained in this paper. Our contribution concerns reconciliation of the assumptions of independent log asset changes with the correlated multivariate processes of the first-passage-time default correlation model. While Jarrow [21] attacks the use of Merton's structural credit risk model as a valid source of implied default probabilities due to the violation of the model's critical assumption that all the firm's assets trade in frictionless markets. The mixed default measure $MD_i(t)$ reflects all the market information and the default probability that is directly related to the i-th firm. Even if the probability of the specified firm default is smaller than one of other firms, the mixed default measure of the specified firm would be way bigger than one of the other firms due to the larger impact of default correlation. The default correlation matrix $\rho^D(t) = (\rho^D_{ij}(t))_{1\le i,j\le n}$ might be useful for regulators to access the possible default contagion from one financial institution to another. The mixed pair default $MD_i(t)$ measures all the possible default correlation from the market to this specified firm as well as its own probability of default under the dependence by adding the row of the default correlation matrix. On the other hand, if we add the column of the default correlation matrix, the sum of default correlations on the i-th column is the total effect of dependence from the specified i-th firm to rest of firms in the market. This quantity could be used to measure the significance of the firm in the market.

The remainder of this paper is organized as follows. Section 2 presents the first-passage-time model for probability of default with dependence. Section 3 provides risk management with default correlation. Section 4 gives numerical results of the paper and compares them with previous results. Section 5 concludes our findings. We give a complete proof of main results in Appendix.

2. First-Passage-Time Model for Probability of Default and Default Correlation

In the basic model for default correlations based on the first-passage-time model of Black and Cox [4], a firm defaults when its asset value breaches the default barrier. We assume that there are n credit entities with a collection of n firms (the *i*-th credit entity is referred to as the *i*-th firm). The *i*-th firm defaults whenever this firm's assets falls below the i-th firm specific threshold.

Assumption 1. *The dynamics of total asset values of both firms is given by*

$$d\ln V = \mu dt + \Omega dW(t) \tag{1}$$

where $\ln V^T = (\ln V_1(t), \cdots, \ln V_n(t))$, $\mu^T = (\mu_1, \cdots, \mu_n)$ *are constant drift terms for each firm and* $W^T(t)$ *is the independent standard n-dimensional Brownian motion. The log-asset covariance matrix is given by* $\Omega \cdot \Omega' = (\rho_{ij}\sigma_i\sigma_j)$ *with a constant correlation* $\rho_{ij} = Corr(d\ln V_i, d\ln V_j)$ *for* $1 \le i, j \le n$.

Assumption 2. *For each firm i, there exists a time-dependent value* $C_i(t)$ *such that the firm continues to operate and meet its contractual obligations if* $V_i(t) > C_i(t)$. *The firm defaults on all of its obligations immediately if* $V_i(t) \le C_i(t)$, *and some form of corporate restructuring takes place, for* $1 \le i \le n$.

For simplicity, we assume the drift vector and variance matrix of the total asset values' logarithmic change are constant. This is the same setup as Zhou [13] for $n = 2$. For simplicity, we focus on the the default threshold level $C_i(t) = e^{\lambda_i t} K_i$ for constant λ_i and K_i for $i = 1, 2, \cdots, n$ as proposed by Black and Cox [4]. Black and Cox [4] interpret $C_i(t)$ as the smallest possible value required by the safety covenant of a debt contract. One can extend to a random barrier and random interest rates as studied in Nielsen et al. [22], Leland [23] and Leland and Toft [24].

If the default status are described as Bernoulli random variables,

$$D_i(t) = \begin{cases} 1 & \text{if the firm i defaults by } t \\ 0 & \text{otherwise.} \end{cases} \qquad 1 \le i \le n$$

then we have, for $1 \le i \le n$,

$$D_i(t) = \{\tau_i \le t\}, \quad \tau_i = \min\{t \ge 0 : V_i(t) \le e^{\lambda_i t} K_i\}$$

as the first-passage-time that the i-th firm reaches its default threshold.

Define the default correlation $Corr(D_i(t), D_j(t)) = \rho_{ij}^D(t)$ between the firms i and j over the time interval $[0, t]$ as

$$Corr(D_i(t), D_j(t)) = \frac{Cov(D_i(t), D_j(t))}{\sqrt{Var(D_i(t)) \cdot Var(D_j(t))}} = \frac{E[D_i(t) \cdot D_j(t)] - E[D_i(t)] \cdot E[D_j(t)]}{\sqrt{Var(D_i(t)) \cdot Var(D_j(t))}} \tag{2}$$

Default correlation is central to determining the joint default probability that is the probability of multiple defaults. If $P(D_i(t) = 1) = P_i$, then

$$\begin{aligned} P(D_i(t) = 1 \text{ and } D_j(t) = 1) &= P_i P_j + \rho_{ij}^D(t)\sqrt{(1-P_i)(1-P_j)P_i P_j} \\ P(D_i(t) = 1 \text{ or } D_j(t) = 1) &= 1 - (1-P_i)(1-P_j) - \rho_{ij}^D(t)\sqrt{P_i P_j(1-P_i)(1-P_j)} \end{aligned}$$

We need to first derive the probability of i-th firm default under Assumptions 1 and 2.

Proposition 1. *Under Assumptions 1 and 2, the probability of i-th firm default is given by*

$$P(D_i(t) = 1) = P_i = N(-\frac{\ln(V_{i,0}/K_i)}{\Sigma_i\sqrt{t}} - \frac{\mu_i - \lambda_i}{\Sigma_i}\sqrt{t})$$

$$+(\frac{V_{i,0}}{K_i})^{\frac{2(\lambda_i - \mu_i)}{\sigma_i^2}} N(-\frac{\ln(V_{i,0}/K_i)}{\Sigma_i\sqrt{t}} + \frac{\mu_i - \lambda_i}{\Sigma_i}\sqrt{t})$$

and the distance-to-default at the time t for the i-th firm is given by

$$dd_i(t) = \frac{\ln(V_{i,0}/K_i) + (\mu_i - \lambda_i - \frac{1}{2}\Sigma_i^2)(T-t)}{\Sigma_i\sqrt{T-t}}$$

where Σ_i^2 is the i-th *eigenvalue of $\Omega \cdot \Omega^T$ and $N(\cdot)$ is the cumulative probability distribution function for a standard normal random variable.*

The proof of Proposition 1 is given in Appendix.

Then, the i-th firm-specific probability $P(D_i(t) = 1) = P(\tau_i \le t)$ of default given by Proposition 1 is derived under our Assumptions 1 and 2 with underlying asset correlations. The probability of i-th firm default is given by

$$p_i(t) = P(D_i(t) = 1) = N(-\frac{\ln(V_{i,0}/K_i)}{\sigma_i\sqrt{t}} - \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{t}) \tag{3}$$

$$+(\frac{V_{i,0}}{K_i})^{\frac{2(\lambda_i - \mu_i)}{\sigma_i^2}} N(-\frac{\ln(V_{i,0}/K_i)}{\sigma_i\sqrt{t}} + \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{t})$$

It is derived without the correlation Assumption 1 for each individual firm. We have $P_i(t) = p_i(t)$ if and only if $\rho_{ij} = 0$ for all $i \neq j$. The probability $p_i(t)$ of the i-th firm default is obtained assuming independence of the stochastic process in Merton (1974). Hence the firm-specific default probabilities (3) are not consistent with the default correlation of Assumptions 1 and 2, as shown in Li and Krehbiel [16]. The distance-to-default $dd_i(t)$ is also different from the distance-to-default with independent asset processes:

$$d_i(t) = \frac{\ln(V_{i,0}/K_i) + (\mu_i - \lambda_i - \frac{1}{2}\sigma_i^2)(T-t)}{\sigma_i\sqrt{T-t}}$$

Proposition 2. *The probability that either i-th firm or j-th firm defaults and the probability that both firms default under Assumptions 1 and 2 given $\{u_i, \lambda_i, \sigma_i, V_{i,0}, K_i, \rho\}$ are given by*

$$P(D_i(t) = 1 \text{ or } D_j(t) = 1) = 1 - F(b_i, b_j, t) = 1 - N_{2,\rho_{ij}}(\frac{b_i}{\sigma_i\sqrt{t}}, \frac{b_j}{\sigma_j\sqrt{t}}) \tag{4}$$

$$+N_{2,-\rho_{ij}}(-\frac{b_i}{\sigma_i\sqrt{t}}, \frac{b_j}{\sigma_j\sqrt{t}}) - N_{2,\rho_{ij}}(-\frac{b_i}{\sigma_i\sqrt{t}}, -\frac{b_j}{\sigma_j\sqrt{t}}) + N_{2,-\rho_{ij}}(\frac{b_i}{\sigma_i\sqrt{t}}, -\frac{b_j}{\sigma_j\sqrt{t}})$$

and the default correlation of the i-th firm and the j-th firm is

$$\rho_{ij}^D(t) = \frac{P_i + P_j - P_iP_j - P(D_i(t) = 1 \text{ or } D_j(t) = 1)}{\sqrt{P_i(1-P_i)P_j(1-P_j)}}$$

where $b_i = -\ln[\frac{K_i}{V_i(0)}] + (\mu_i - \lambda_i)t$, P_i is given in Proposition 1 for $1 \le i, j \le n$, $F(b_i, b_j, t) = P(\tau_{ij} > t)$ with $\tau_{ij} = \tau_i \wedge \tau_j = \min(\tau_i, \tau_j)$ and the probability of the standard bivariate normal distribution on $y_1 \le x_1$ and $y_2 \le x_2$ is denoted by

$$N_{2,\rho}(x_1, x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{y_1^2 - 2\rho y_1 y_2 + y_2^2}{2(1-\rho^2)}} dy_1 dy_2$$

and $n_{2,\rho}(y_1, y_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{y_1^2 - 2\rho y_1 y_2 + y_2^2}{2(1-\rho^2)}}$ is the standard bivariate normal distribution density function denoted by $N_2(0, 1, \rho)$ for random variables Y_i $(i = 1, 2)$ with mean zero and variance 1 and correlation $\rho = Cov(Y_1, Y_2)$.

By Proposition 2, the probability that both *i*-th firm and *j*-th firm default under Assumptions 1 and 2 is given by

$$P(D_i(t) = 1 \text{ and } D_j(t) = 1) = P_i + P_j - 1 + N_{2,\rho_{ij}}(\frac{b_i}{\sigma_i\sqrt{t}}, \frac{b_j}{\sigma_j\sqrt{t}}) \tag{5}$$

$$-N_{2,-\rho_{ij}}(-\frac{b_i}{\sigma_i\sqrt{t}}, \frac{b_j}{\sigma_j\sqrt{t}}) + N_{2,\rho_{ij}}(-\frac{b_i}{\sigma_i\sqrt{t}}, -\frac{b_j}{\sigma_j\sqrt{t}}) - N_{2,-\rho_{ij}}(\frac{b_i}{\sigma_i\sqrt{t}}, -\frac{b_j}{\sigma_j\sqrt{t}})$$

which is definitely different from

$$P(V_i(t) \le e^{\lambda_i t}K_i \text{ and } V_j(t) \le e^{\lambda_j t}K_j) = N_{2,\rho_{ij}}(\frac{b_i}{\sigma_i\sqrt{t}}, \frac{b_j}{\sigma_j\sqrt{t}}) \tag{6}$$

where (6) is used to evaluate the default correlation in Crouhy et al. [8] (also see Bielecki and Rutkowski [25], p. 117). The essential difference of these evaluations is discussed in [16], Section 2.1.

The proof of Proposition 2 follows from the same method in [16]. Formula (6) evaluates the probability of joint default on the horizon date, while (5) evaluates the joint default probability prior to and inclusive of the horizon date. Zhou's analogous default probability is achieved by solving the

Kolmogorov forward equation with initial-boundary conditions, with separable variable method and formal series of eigenfunctions of the boundary value problem. Zhou's closed-form solution is given by an infinite series involving the double integral of Bessel functions. The series solution is formal and not necessarily convergent because the domain of the boundary value problem is unbounded. Numerical evidence for those inconsistencies is illustrated in [16].

Theorem 1. *The evaluation of the probability that either firm defaults under Assumptions 1 and 2 given* $\{\mu_i, \lambda_i, \sigma_i, V_{i,0}, K_i, \rho_{ij}\}_{1\le i,j\le n}$ *is given by*

$$\begin{aligned} P(\tau \le t) &= P(D_1(t) = 1 \, or \; D_2(t) = 1 \cdots, or \, D_n(t) = 1) \\ &= 1 - F(b_1, b_2, \cdots, b_n, t) \end{aligned}$$

$$F(b_1, b_2, \cdots, b_n, t) = \sum_{i=1}^{2^n} (-1)^{signQ_i} N_{\Omega_{Q_i}\Omega_{Q_i}^T}(sign(z_1)b_1, sign(z_2)b_2, \cdots, sign(z_n)b_n) \tag{7}$$

where $\tau = \min\{\tau_1, \cdots, \tau_n\}$, $(sign(z_1), \cdots, sign(z_n) \in Q_i$ *with* $sign(z_i) = \pm 1$, $signQ_i$ *is the sum of negative 1 in* $(sign(z_1), \cdots, sign(z_n))$, $\Omega_{Q_i} = diag(sign(z_1), \cdots, sign(z_n)) \cdot \Omega$, *and* $N_A(x_1, \cdots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \frac{1}{(2\pi t)^{n/2}\sqrt{\det A}} \exp(-\frac{1}{2t} z^T A z) dz_1 \cdots dz_n$.

We solve the Kolmogorov forward equation with initial-boundary conditions using the reflection principle in partial differential equations. This is an extension of $n = 2$ case in [16]. We first solve the Kolmogorov forward equation with initial condition from the constant coefficients in the partial differential operator. The solution gives the n-dimensional Gaussian distribution function that satisfies all three boundary conditions automatically, except the boundary on the finite regions $x_i = b_i$ for $1 \le i \le n$. This is the main difficulty that led Rebholz [12], Zhou [13] and Valužis [14] to use the separable variable methods to obtain a formal series solution, even in $n = 2$ case. In contrast, we use the reflection principle first to shift the boundary region as the boundary of the first quadrant, and then extend the solution to the whole plane $\mathbb{R}^n$ by odd extension. After identifying the involution through the odd extension and using the Dirac function properties, the probability of at-least one default among all n firms is obtained in Theorem 1. For the completeness of the result, we present the detailed proof in Appendix. The probability of at-least one default in Theorem 1 is given by 2^n terms with standard multivariate normal distribution function.

For all the firms under Assumptions 1 and 2, we can use Propositions 1 and 2 and Theorem 1 to determine any possible combination of default events. We outline the basic scheme and hope this will be useful in determining systemic risk.

Probability of single firm default We first use Proposition 1 to evaluate the probability $P(D_i(t) = 1) = P_i(t)$ of i-th firm default at the time t. The probability P_i of the i-th firm default incorporates with other firms through Assumptions 1 and 2, and this is essentially different from the probability p_i of the i-th firms default in (3) where the i-th firm is independent of the rest firms.

Probability of two firms default For any pair of two firms, we first evaluate the probability $P(D_i(t) = 1 \, \text{or} \, D_j(t) = 1)$ that either i-th firm or j-th firm defaults by Proposition 2. There are $\begin{pmatrix} n \\ 2 \end{pmatrix}$ many pairs of two firms from those n firms. Similarly for k firms, there are $\begin{pmatrix} n \\ k \end{pmatrix}$ many k firms from those n firms.

Hence, the probability $P(D_i(t) = 1 \, \text{and} \, D_j(t) = 1) = P_{ij}(t)$ that both i-th firm and j-th firm default at the time t is given by

$$P(D_i(t) = 1 \, \text{and} \, D_j(t) = 1) = P_i + P_j - P(D_i(t) = 1 \, \text{or} \, D_j(t) = 1)$$

Probability of three firms default For any triple firms (i, j, k), we apply the sub-matrix taking the i, j, k rows and columns and the probability $P(D_i(t) = 1 \text{ or } D_j(t) = 1 \text{ or } D_k(t) = 1)$ that either i-th firm or j-th firm or k-th firm defaults by Theorem 1 for $n = 3$. Hence, the probability $P(D_i(t) = 1 \text{ and } D_j(t) = 1 \text{ and } D_k(t) = 1) = P_{ijk}(t)$ that both i-th firm and j-th firm and k-th firm default at the time t is given by

$$P_{ijk}(t) = -P_i(t) - P_j(t) - P_k(t) + (P_{ij}(t) + P_{ik}(t) + P_{jk}(t))$$
$$+P(D_i(t) = 1 \text{ or } D_j(t) = 1 \text{ or } D_k(t) = 1)$$

Probability of k firms default For $k \geq 2$, we apply Theorem 1 for all $k \times k$ sub-matrix from $\Omega\Omega^T$ to evaluate the probability $P(\cup_{j=1}^k D_{i_j}(t) = 1)$, and then evaluate the probability $P_{i_1 i_2 \cdots i_k}(t)$ that k firms (from i_1-th to i_k-th firms) both default at the time t from the following identity and previous steps,

$$P(\cup_{j=1}^k D_{i_j}(t) = 1) = \sum_{j=1}^k P_{i_j}(t) - \sum_{j_1, j_2} P_{i_{j_1} i_{j_2}}(t) + \cdots + (-1)^{k-1} P_{i_1 i_2 \cdots i_k}(t)$$

Probability of all n firms default Repeat the previous step for $2 \leq k \leq n - 1$. Evaluate the probability $P(\tau \leq t)$ that either firm defaults among n firms by Theorem 1. Therefore, the probability $P_{12\cdots n}(t)$ that all n firms default at the time t is derived from the following identity.

$$P(\tau \leq t) = \sum_{i=1}^n P_i(t) - \sum_{i,j} P_{ij}(t) + \cdots + \sum (-1)^{n-2} P_{i_1 i_2 \cdots i_{n-1}}(t) + (-1)^{n-1} P_{12\cdots n}(t)$$

3. Risk Management with the Default Correlation

A portfolio manager would not only be concerned with the default of a single firm but also be worried with the probability of multiple defaults in the portfolio. Similarly, a regulator would have to estimate the probability of multiple defaults of a number of financial institutions (banks) if there is no bailout to let a default of a single financial institution (the too big to fail puzzle). Modeling credit risk from a portfolio perspective is indispensable if derivatives on credit portfolios are priced and risk measures of credit portfolios are to be computed. New regulatory rules make it mandatory for financial institutions to extend their risk measures from a single-contract setting to a portfolio perspective.

An effective credit risk measurement in a portfolio consists of three important components: (1) the probability of default for each individual firm over various investment horizons; (2) the joint probability of default between any two firms in the portfolio during various investment horizons; and (3) the magnitude of financial loss in the case of default. As noted in the literature, the precise quantity of the default correlations is the most crucial and the most challenging part of credit risk analysis in a portfolio.

The structural credit risk models assume that all of the firms assets trade and their values are observable. The major drawback of the structure credit risk models is that both firms' assets are not tradable and their volatilities are unobservable. See Jarrow [26] for more debates between structural models and reduced form models. In practice, one can calibrate the observable stock prices and stock price's volatility to estimate the firms' asset values and volatilities, and further minimize the error between the market observed stock price and volatility and the model's asset value and volatility. In industry practice, the structural models are still used incorrectly as a theoretical model for estimating the firm's default probability. Reduced form model is adapted to estimate default probabilities from historical default data. Jarrow [21] shows that the implied default probability form a structural credit risk model and the default correlations obtained from credit risk copula models lead to mis-specified estimates.

3.1. CreditRisk$^+$ Incorporating Default Correlations

Credit Suisse Financial Products (CSFP) introduced the CreditRisk$^+$ in October 1997. CreditRisk$^+$ applies techniques from actuarial mathematics in an enhanced way to compute the probabilities for portfolio loss levels. Among other fundamental ideas in CreditRisk$^+$, the linear relationship between the systemic risk factors and the probabilities of default is assumed; the defaults of obligators are assumed independent; correlations among obligators are assumed implicitly due to common risk factors which drive the probability of defaults.

The original CreditRisk$^+$ does not allow correlations in modelling default events. Akkaya et al. [27] extend the CreditRisk$^+$ that is able to model default correlations among segments while preserving the analytical solution for the loss distribution. The portfolio loss X is given by $\sum_{A \in S} \mathbf{1}_A \mu_A$, where $\mathbf{1}_A$ is the default indicator for obligor A with $P[\mathbf{1}_A = 1] = p_A$ and μ_A is the exposure net of recovery for a single-segment portfolio S. For the N different and correlated segments $S_1, \cdots, S_N$, the total loss is given by

$$X = \sum_{k=1}^{N} X_k = \sum_{k=1}^{N} \sum_{A \in S_k} \mathbf{1}_A \mu_A$$

The expected loss and variance of X are given by

$$\begin{aligned} EL(X) &= \sum_{k=1}^{N} \sum_{A \in S_k} p_A \mu_A, \\ Var(X) &= \sum_{k=1}^{N} Var(X_k) \\ &= \sum_{k=1}^{N} \sigma_{S_k}^2 EL^2(X_k) + \sum_{k \neq l} Corr(S_k, S_l) \sigma_{S_k} \sigma_{S_l} EL(X_k) EL(X_l) + \sum_{k=1}^{N} \sum_{A \in S_k} p_A \mu_A^2 \end{aligned}$$

The variance $Var(X)$ contains a term $\sum_{k \neq l} Corr(S_k, S_l) \sigma_{S_k} \sigma_{S_l} EL(X_k) EL(X_l)$ which is the default correlation measured in the extended CreditRisk$^+$ in [27]. The analytic solution for the default correlation depends on the assumption on the portfolio loss distributions. The multivariate structure with nontrivial correlation leads a pragmatic simplification to reduce into a single-segment model.

Due to the non-uniqueness in the parameter estimation, one has to assess the impact of the choice of parameters on the overall loss distribution. This reduced-form approach to default correlation does not reflect any underlying structure of the assets. The conditionally independent defaults (CID) approach creates default dependence among firms through the dependence of the firm's intensity processes on a common set of state variables. Contagion models extend the CID model to account for the empirical information of default clustering and to include the existence of credit risk contagion mechanisms between firms.

3.2. CreditMetrics Incorporating Default Correlations

From the CreditMetrics^{TM} Technical Document, we examined several rating changes and defaults in order to establish that such correlations indeed exist. The direct method to estimate joint rating change likelihoods is to test credit rating time series across many firms which are synchronized in time as a statistical approach. The advantage of this method is that it is independent of the underlying process and the joint distribution shape and the underlying correlation. The disadvantage is that it treats all firms with a same credit rating identical. To improve this method, the CreditMetrics^{TM} estimates credit quality correlation through bond spreads. Whenever bond price histories are available, it is reasonable to estimate a certain type of credit correlation by extracting credit spreads from the bond prices and then estimating the correlation in the movements of credit spreads. This adapts a reduced-form approach. The biggest drawback of this approach is that the bond spread data is notoriously scarce, especially for low credit quality issues.

For the structural model, CreditMetrics uses the correlation between equity returns as a proxy for the correlation of asset returns. The drawback of this replacement is overlooking discrepency between equity and asset correlation. It is better than using a fixed correlation as well as much more readily availability than credit spreads or actual joint rating changes. Due to the scarcity of data for many obligors and the impossibility of storing a huge correlation matrix, one has to rely on correlations within a set of indices and mapping scheme to build the obligor-by-obligor correlations from the index correlations.

Copula methods have been extensively used in both industry and academia to assess the joint default probability of groups of obligors. However, the choice of the copula and its calibration is still an issue to debate. The copula reduced form model separates the modelling and estimation of individual default probabilities from the modelling and calibration of the estimation of credit risk dependence (the copula). The inconsistency among the reduced form methods is also unavoidable. The copula model for pricing collateralized debt obligations (CDO's) and implied default correlations is misused. Jarrow [21] points out that the default correlation obtained from credit risk copula models for computing VaR measures leads to misspecified estimates.

3.3. Proposed Mixed Pair Defaults Incorporating Default Correlations

Default dependence, mostly represented by default correlation, is one of most important features in credit derivatives pricing, hedging and risk management. Despite the inconsistency of the individual default probability and the default correlation, the default correlation measured in Theorem 1 gives the proper indication of the default dependence under Assumptions 1 and 2. We define a mixed pair for the defaults incorporating default correlations.

Under Assumptions 1 and 2, we define a mixed pair of defaults for each firm by $(P_i(t), \rho_{ij}^D(t))$ for $1 \leq i, j \leq n$. The mixed pair of defaults indicates the measure of the default probability and the default correlation under the realistic correlated asset assumption. Both measures should play an important role in credit risk management. We can simply use the sum of the default probability and the default correlation for the default measure under Assumptions 1 and 2:

$$MD_i(t) = P_i(t) + \sum_{j \neq i} \rho_{ij}^D(t), 1 \leq i \leq n$$

where $MD_i(t)$ is the mixed default measure for the i-th firm and the default correlation $\rho_{ij}^D(t)$ between the i-th firm and the j-th firm is given in Proposition 2. A single default correlation between two firms can be extended to a default correlation matrix. Theoretically the mixed default measure can be larger even the individual firm specific default probability is quite lower due to the larger default correlation from the underlying asset correlation between this firm and the other highly "toxic" firm. The mixed default measure $MD_i(t)$ reflects all the market information and the default probability that is directly related to the i-th firm. Even if $P_i(t) < P_k(t)$, the mixed default measure $MD_i(t) > MD_k(t)$ due to the impact of default correlations.

The default correlation matrix $\rho^D(t) = (\rho_{ij}^D(t))_{1 \leq i,j \leq n}$ might be useful for regulators to access the possible default mitigation from one financial institution to another. The mixed pair default $MD_i(t)$ measures all the possible default correlation from the market to this specified firm as well as its own probability of default under the dependence by adding the row of the default correlation matrix. On the other hand, if we add the column of the default correlation matrix, the quantity $\sum_{j \neq i} \rho_{ji}^D(t)$ is the total effect of dependence from the specified i-th firm to the rest of firms in the market.

The individual asset risk is usually characterized by its return variance in the modern investment theory. The covariance matrix for a portfolio can be a measure of a risk for the portfolio. Similarly, the default correlation matrix (or default covariance) is the key to managing a bond portfolio or other portfolios. One of main risk management tasks is to keep the default risk and the default correlation under control. The ideal situation would spread possible defaults out rather than cluster, if the default cannot be avoided in the risk management. The mixed default measure for each firm or a single defaultable bond would provide an important measure in risk management for fixed income portfolios, bank management, insurance industry and capital management, etc. More empirical analysis on the mixed default measure will be given in a future study.

4. Numerical Analysis

In this section, we show the numerical analysis of our main results. For computation simplification, we assume $\lambda_i = \mu_i (i = 1, 2, \cdots, n)$ throughout this section.

Credit quality parameters, $\frac{V}{K}$, for the numerical analysis of Propositions 1 and 2 and Theorem 1 are selected from implied standardized distances to default estimated from the Standard and Poor's 2010 annual cumulative default study, Credit Week: 20 April 2011. Implied standardized distance to default, Z_i based on $p_i(t)$ in (3), can be estimated from the study's cumulative default rates $P(Z, t)$ to the historical default rate $\tilde{A}_i(t)$, under the maintained hypothesis, $\lambda_i = \mu_i$ by minimizing the sum of squared errors. We have $Z_i = \frac{ln(V_{i,0}/K_i)}{\sigma_i}$ and

$$Z = arg \quad min_Z \sum_t \left(\frac{P(Z,t)}{t} - \frac{\tilde{A}(t)}{t}\right)^2$$

Empirical analysis of default correlations can be done by using firms' data $(V_{i,0}, K_i, \rho, \mu_i, \lambda_i)$ to compute the default correlation by Proposition 2. We use the Standard and Poor's 2010 annual cumulative default study to first estimate the standard distance to default from default data.

Table 1 is borrowed from Li and Krehbiel [16], and presents the standardized distance to default for S & P credit ratings implied by the cumulative default probabilities reported in the 2010 cumulative default study.

Table 1. Implied values Z from 2010 Cumulative Default Study.

	AAA	AA	A	BBB	BB	B	C	INV	SPEC	ALL
Z	8.916	8.769	7.569	5.980	3.902	2.515	1.138	7.024	2.782	4.457
V/K	35	33	21	11	5	3	2	17	3	6

Assuming the annualized standard deviation of logarithmic change of asset market value is $\sigma_1 = 0.4$, the S & P BBB rated bonds with a ratio of initial asset value to default threshold $\frac{V_{1,0}}{K_1} = 11$, and $\sigma_2 = 0.3$ for bonds rated BB by $S\&P$ with $\frac{V_{2,0}}{K_2} = 5$, and $\sigma_3 = 0.2$ for bonds rated AA $\frac{V_{3,0}}{K_3} = 33$. The constant correlation matrix of the logarithms of asset values is given by parameters $(\rho_{12}, \rho_{13}, \rho_{23})$:

$$\rho = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 0.16 & 0.12\rho_{12} & 0.08\rho_{13} \\ 0.12\rho_{12} & 0.09 & 0.06\rho_{23} \\ 0.08\rho_{13} & 0.06\rho_{23} & 0.04 \end{pmatrix}$$

Table 2 illustrates the probability of default for each asset under the correlation changes. We evaluate the probability of default for each individual firm under Assumptions 1 and 2 in Proposition 1, and compare the probability of default from Merton (1974) under the isolated stochastic process for $t = 1$. For simplicity, we have $\mu_i = \lambda_i$ for $i = 1, 2, 3$. The ratio $\frac{V_{1,0}}{K_1} = 11$ for the BBB rated bonds with $\sigma_1 = 0.4$, $\frac{V_{2,0}}{K_2} = 5$ for the BB rated bonds with $\sigma_2 = 0.3$ and $\frac{V_{3,0}}{K_3} = 33$ for the AA rated bonds with $\sigma_3 = 0.2$.

Table 2. Eigenvalues and Probability of Default with underlying asset correlations.

ρ_{12}	ρ_{13}	ρ_{23}	Σ_1	Σ_2	Σ_3	P_1	p_1	P_2	p_2	P_3	p_3
−0.75	−0.75	−0.75	0.0625	0.0625	6.25	8.67E-22	2.04E-09	0.000%	5.73E-05	16.19%	2.87E-05
−0.75	−0.75	−0.5	0.0658	1.9187	3.5781	8.93E-21	2.04E-09	24.527%	5.73E-05	6.45%	2.87E-05
−0.75	−0.75	−0.25	0.0008	1.0856	4.2886	0.00E+00	2.04E-09	12.242%	5.73E-05	9.13%	2.87E-05
−0.75	−0.75	0	0.0037	1	4.2463	0.00E+00	2.04E-09	10.752%	5.73E-05	8.97%	2.87E-05
−0.75	−0.75	0.25	0.0032	0.5625	4.8093	0.00E+00	2.04E-09	3.188%	5.73E-05	11.08%	2.87E-05
−0.75	−0.75	0.5	0.1603	0.5	2.3397	2.11E-09	2.04E-09	2.284%	5.73E-05	2.23%	2.87E-05
−0.75	−0.75	0.75	0.0625	0.0625	6.25	8.67E-22	2.04E-09	0.000%	5.73E-05	16.19%	2.87E-05
0	0.25	0.75	0.0439	1	3.2061	2.50E-30	2.04E-09	10.752%	5.73E-05	5.08%	2.87E-05
0	0.5	0.75	0.009722	1	3.615322	1.22E-130	2.04E-09	10.752%	5.73E-05	6.59%	2.87E-05
0	0.75	0.75	0.003684	1	4.246484	0.00E+00	2.04E-09	10.752%	5.73E-05	8.97%	2.87E-05
0	−0.25	0.75	0.0439	1	3.2061	2.50E-30	2.04E-09	10.752%	5.73E-05	5.08%	2.87E-05
0	−0.5	0.75	0.009722	1	3.615322	1.22E-130	2.04E-09	10.752%	5.73E-05	6.59%	2.87E-05
0	−0.75	0.75	0.003684	1	4.246484	0.00E+00	2.04E-09	10.752%	5.73E-05	8.97%	2.87E-05
0.5	0.25	0.75	0.0398	0.596293	4.338889	2.81E-33	2.04E-09	3.714%	5.73E-05	9.32%	2.87E-05
0.5	0.5	0.75	0.0625	0.330165	4.732365	8.67E-22	2.04E-09	0.509%	5.73E-05	10.80%	2.87E-05
0.5	0.75	0.75	0.025696	0.25	5.474196	1.36E-50	2.04E-09	0.129%	5.73E-05	13.51%	2.87E-05
0.5	0.25	0.5	0.165568	0.5625	3.397018	3.79E-09	2.04E-09	3.188%	5.73E-05	5.78%	2.87E-05
0.5	0.5	0.5	0.25	0.25	4	1.62E-06	2.04E-09	0.129%	5.73E-05	8.04%	2.87E-05
0.5	0.75	0.5	0.0625	0.330165	4.732365	8.67E-22	2.04E-09	0.509%	5.73E-05	10.80%	2.87E-05

Under the asset correlation changes, we see how the eigenvalues of the covariance matrix varying. These simple numerical results illustrate the important role of the underlying asset correlations, and the probability of default P_1 behaves more complex than the probability of p_1. Note that p_i in Merton [3] does not vary at all for $i = 1, 2, 3$. But the probability of default P_i is very sensitive to the correlations $\rho_{12}, \rho_{13}, \rho_{23}$.

We compute the distance-to-default for each individual firm under Assumptions 1 and 2, and compare the distance-to-default from [3] under the isolated stochastic process for $t = 1$. For simplicity, we have $\mu_i = \lambda_i$ for $i = 1, 2, 3$. The ratio $\frac{V_{1,0}}{K_1} = 11$ for the BBB rated bonds with $\sigma_1 = 0.4$, $\frac{V_{2,0}}{K_2} = 5$ for the BB rated bonds with $\sigma_2 = 0.3$ and $\frac{V_{3,0}}{K_3} = 33$ for the AA rated bonds with $\sigma_3 = 0.2$.

Table 3 illustrates the distance-to-default in Proposition 1 under Assumptions 1 and 2. The distance-to-default is the key to the rating of the underlying firm or bonds. We see the changes of dd_i through the variations of ρ_{12}, ρ_{13} and ρ_{23}. The distance-to-default dd_1 increases dramatically when the correlation ρ_{23} of other two firms increases from -0.5 to -0.25, and decreases rapidly when ρ_{23} increases from 0.25 to 0.5. This is the essence of correlation. As the correlation of other firms varies, the distance-to-default as well as the credit rating of the first firm may change dramatically. We see the big difference between dd_3 and d_3. Without considering the underlying asset correlations, the third firm or bonds has the distance-to-default $d_3 = 17.383$. Its actual distance-to-default dd_3 with the underlying asset correlation into consideration is much smaller than d_3. Hence, the third firm (AA rated bonds) is overrated if we have the correlation input.

We evaluate the time series of the probability of default for each individual firm under Assumptions 1 and 2 in Proposition 1 as well as the time series of the distance-to-default, and compare the probability of default and the distance-to-default from [3] under the isolated stochastic process for $t = 0.5, 1, 1.5, 2, 2.5, 3$ and $\rho_{12} = 0.5, \rho_{13} = 0.25$ and $\rho_{23} = 0.75$. For simplicity, we have $\mu_i = \lambda_i$ for $i = 1, 2, 3$. The ratio $\frac{V_{1,0}}{K_1} = 11$ for the BBB rated bonds with $\sigma_1 = 0.4$, $\frac{V_{2,0}}{K_2} = 5$ for the BB rated bonds with $\sigma_2 = 0.3$ and $\frac{V_{3,0}}{K_3} = 33$ for the AA rated bonds with $\sigma_3 = 0.2$.

Table 4 illustrates that three firms or bonds positively correlated have their probabilities of default and distances-to-default changing as t varies. The probability of default P_3 for the third AA rated bonds increases clearly, and P_1 stays stagnantly. The distances-to-default for both three assets decrease as t increases, as well as the difference $dd_i(t) - d_i(t)$ gets smaller as t varies.

Table 3. Comparing distances-to-default.

ρ_{12}	ρ_{13}	ρ_{23}	dd_1	d_1	dd_2	d_2	dd_3	d_3
−0.75	−0.75	−0.75	9.467	5.795	6.313	5.215	0.149	17.383
−0.75	−0.75	−0.5	9.220	5.795	0.469	5.215	0.903	17.383
−0.75	−0.75	−0.25	84.764	5.795	1.024	5.215	0.653	17.383
−0.75	−0.75	0	39.391	5.795	1.109	5.215	0.666	17.383
−0.75	−0.75	0.25	42.361	5.795	1.771	5.215	0.498	17.383
−0.75	−0.75	0.5	5.789	5.795	1.923	5.215	1.521	17.383
−0.75	−0.75	0.75	9.467	5.795	6.313	5.215	0.149	17.383
0	0.25	0.75	11.340	5.795	1.109	5.215	1.057	17.383
0	0.5	0.75	24.270	5.795	1.109	5.215	0.888	17.383
0	0.75	0.75	39.474	5.795	1.109	5.215	0.666	17.383
0	−0.25	0.75	11.340	5.795	1.109	5.215	1.057	17.383
0	−0.5	0.75	24.270	5.795	1.109	5.215	0.888	17.383
0	−0.75	0.75	39.474	5.795	1.109	5.215	0.666	17.383
0.5	0.25	0.75	11.920	5.795	1.698	5.215	0.637	17.383
0.5	0.5	0.75	9.467	5.795	2.514	5.215	0.520	17.383
0.5	0.75	0.75	14.879	5.795	2.969	5.215	0.325	17.383
0.5	0.25	0.5	5.690	5.795	1.771	5.215	0.976	17.383
0.5	0.5	0.5	4.546	5.795	2.969	5.215	0.748	17.383
0.5	0.75	0.5	9.467	5.795	2.514	5.215	0.520	17.383

Table 4. Time-Series Default Probability and Distance-to-Default.

t	P_1	p_1	P_2	p_2	P_3	p_3	dd_1	d_1	dd_2	d_2	dd_3	d_3
0.5	0	1.15E-17	0.000135	6.34E-09	0.254433	2.1E-35	16.928	8.336	10.352	5.549	1.637	12.221
1	0	1.02E-09	0.006953	2.87E-05	0.420328	1.15E-18	12.020	5.795	6.573	3.824	0.637	8.541
1.5	0	4.92E-07	0.02754	0.000509	0.510553	4.76E-13	9.814	4.650	4.757	3.040	0.095	6.892
2	0	1.12E-05	0.056322	0.00222	0.568797	3.18E-10	8.499	3.956	3.591	2.562	−0.286	5.898
2.5	0	7.49E-05	0.087814	0.005468	0.610285	1.62E-08	7.602	3.475	2.739	2.229	−0.585	5.212
3	4.2E-265	0.000269	0.11916	0.010089	0.641746	2.25E-07	6.939	3.115	2.069	1.977	−0.835	4.700

Compute $F(b_1, b_2, b_3, t)$ by Theorem 1 for $n = 3$. We have $(sign(z_1), sign(z_2), sign(z_3)) = (\pm 1, \pm 1, \pm 1) \in Q_i$, and

$$\Omega_{Q_i}\Omega_{Q_i}^T = \begin{pmatrix} \sigma_1^2 & sign(z_1)sign(z_2)\rho_{12}\sigma_1\sigma_2 & sign(z_1)sign(z_3)\rho_{13}\sigma_1\sigma_3 \\ sign(z_1)sign(z_2)\rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & sign(z_2)sign(z_3)\rho_{23}\sigma_2\sigma_3 \\ sign(z_1)sign(z_3)\rho_{13}\sigma_1\sigma_3 & sign(z_2)sign(z_3)\rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}$$

For instance with the quadrant $(1, -1, -1) \in Q_i$, we have $signQ_i = 2$ and

$$\Omega_{Q_i}\Omega_{Q_i}^T = \begin{pmatrix} \sigma_1^2 & -\rho_{12}\sigma_1\sigma_2 & -\rho_{13}\sigma_1\sigma_3 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ -\rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}$$

Hence $(-1)^{signQ_i} N_{\Omega_{Q_i}\Omega_{Q_i}^T}(sign(z_1)b_1, sign(z_2)b_2, sign(z_3)b_3)$

$$= (-1)^2 N_{\Omega_{Q_i}\Omega_{Q_i}^T}(b_1, -b_2, -b_3) = N_{n,A_i}\left(\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, -\frac{b_3}{\sigma_3\sqrt{t}}\right)$$

where A_i is the matrix $\Omega_{Q_i}\Omega_{Q_i}^T$ with $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and N_{n,A_i} is the standard multivariate normal distribution with A_i. Hence, we have $F(b_1, b_2, b_3, t)$ given by the following.

$$N_{\Omega_{Q_1}\Omega_{Q_1}^T}(b_1, b_2, b_3) - N_{\Omega_{Q_2}\Omega_{Q_2}^T}(-b_1, b_2, b_3) + N_{\Omega_{Q_3}\Omega_{Q_3}^T}(-b_1, -b_2, b_3) - N_{\Omega_{Q_4}\Omega_{Q_4}^T}(b_1, -b_2, b_3)$$

$$+N_{\Omega_{Q_5}\Omega_{Q_5}^T}(-b_1, b_2, -b_3) - N_{\Omega_{Q_6}\Omega_{Q_6}^T}(-b_1, -b_2, -b_3) + N_{\Omega_{Q_7}\Omega_{Q_7}^T}(b_1, -b_2, -b_3) - N_{\Omega_{Q_8}\Omega_{Q_8}^T}(b_1, b_2, -b_3)$$

$$= N_{n,A_1}(\frac{b_1}{\sigma_1\sqrt{t}}, \frac{b_2}{\sigma_2\sqrt{t}}, \frac{b_3}{\sigma_3\sqrt{t}}) - N_{n,A_2}(-\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, \frac{b_3}{\sigma_3\sqrt{t}}) + N_{n,A_3}(-\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, \frac{b_3}{\sigma_3\sqrt{t}})$$
$$-N_{n,A_4}(\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, \frac{b_3}{\sigma_3\sqrt{t}}) + N_{n,A_5}(-\frac{b_1}{\sigma_1\sqrt{t}}, \frac{b_2}{\sigma_2\sqrt{t}}, -\frac{b_3}{\sigma_3\sqrt{t}}) - N_{n,A_6}(-\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, -\frac{b_3}{\sigma_3\sqrt{t}})$$
$$+N_{n,A_7}(\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}, -\frac{b_3}{\sigma_3\sqrt{t}}) - N_{n,A_8}(\frac{b_1}{\sigma_1\sqrt{t}}, \frac{b_2}{\sigma_2\sqrt{t}}, -\frac{b_3}{\sigma_3\sqrt{t}})$$

where we give a complete formula for $n = 3$ that the probability of either three firms default in the above.

The following is the algorithm for the numerical evaluation in practice for $n = 3$ case.

Probability of Default with underlying correlations With the explicit numbers, we first compute $P_1(t), P_2(t)$ and $P_3(t)$ by Proposition 1 as in Table 2, and evaluate the distance-to-default $dd_1(t), dd_2(t)$ and $dd_3(t)$ for each three firms as in Table 3. (One can compare these values with $p_i(t)$ and $d_i(t)$ from previous evaluation to adjust their credit ratings with the consideration of the underlying correlations under Assumptions 1 and 2.)

Probabilities of either two firms default at the time *t* From the three firms with BBB, BB and AA rated by the S & P, we choose any pair (there are three choices) and evaluate the chosen two firms either one will be default at the time t by Proposition 2. Hence we obtain

$$P(D_1(t) = 1 \,\text{or}\, D_2(t) = 1),\;\; P(D_1(t) = 1 \,\text{or}\, D_3(t) = 1),\;\; P(D_2(t) = 1 \,\text{or}\, D_3(t) = 1)$$

Probability of two firms both default at the time *t* By the previous two steps, we can evaluate both i-th firm and j-th firm default at the time t,

$$P_{12}(t) = P_1(t) + P_2(t) - P(D_1(t) = 1 \,\text{or}\, D_2(t) = 1)$$
$$P_{13}(t) = P_1(t) + P_3(t) - P(D_1(t) = 1 \,\text{or}\, D_3(t) = 1)$$
$$P_{23}(t) = P_2(t) + P_3(t) - P(D_2(t) = 1 \,\text{or}\, D_3(t) = 1)$$

Probability of both three firms default at the time *t* This is to analyze the systemic risk by the probability of both firms in the considered system to default at the time t. We first evaluate either firm from the three firms default by Theorem 1,

$$P(D_1(t) = 1 \,\text{or}\, D_2(t) = 1 \,\text{or}\, D_3(t) = 1)$$

and hence the probability $P_{123}(t)$ is given by

$$P_{123}(t) = P(D_1(t) = 1 \,\text{or}\, D_2(t) = 1 \,\text{or}\, D_3(t) = 1) - P_1(t) - P_2(t) - P_3(t) + P_{12}(t) + P_{13}(t) + P_{23}(t)$$

Default Correlations and Mixed defaults We can now get the default correlation matrix $(\rho_{ij}^D(t))$ by Proposition 2. Now we can formulate the mixed pair matrix in the following,

$$\begin{pmatrix} P_1(t) & \rho_{12}^D(t) & \rho_{13}^D(t) \\ \rho_{12}^D(t) & P_2(t) & \rho_{23}^D(t) \\ \rho_{13}^D(t) & \rho_{23}^D(t) & P_3(t) \end{pmatrix}$$

By adding the rows of the mixed pair matrix, we obtain the mixed default measures

$$MD_1(t) = P_1(t) + \rho_{12}^D(t) + \rho_{13}^D(t)$$

and $MD_2(t), MD_3(t)$ to access the other firms effected on the firm. The quantity $\rho_{12}^D(t) + \rho_{13}^D(t)$ is the total effect of dependence from the specified first firm to the market of rest firms. The quantity $\rho_{12}^D(t) + \rho_{23}^D(t)$ is the total effect of dependence from the specified second firm to the market of rest firms. The quantity $\rho_{13}^D(t) + \rho_{23}^D(t)$ is the total effect of dependence from the specified third firm to the market of rest firms.

5. Conclusions

We give the answer to this issue by providing explicit formulas of the probability of multiple defaults and default correlations in this paper under the dependence. Merton [3] derives the probability of a single firm default under the isolated asset firm value hypothesis. By using a linear algebra technique, we change the driving Brownian motions into uncorrelated ones and apply the formula of Merton [3] to derive the probability of single firm default under the correlated asset values. In order to solve the probabilities of multiple defaults, we use the reflection principle in PDE to solve the Kolmogorov forward equation with initial-boundary conditions to evaluate the probability that any collection of some firms from the choice of all firms default in our main result Theorem 1. Then we propose an algorithm to determine probabilities of any multiple choice of firms defaults.

We further propose the mixed pair defaults to understand all the possible default correlations from the market to effect the specific firm. The sum of the *i*-th column of the default correlation matrix provide the quantity of the *i*-th firm to effect the market from the default correlation perspective. The mixed default pair would provide an important measure in credit risk management for fixed income portfolio, financial stability and insurance industry.

We also point out that the difference between the probability of single firm default under dependence in our result and the one used in structural models presently, as well as the difference between the joint default probability in this paper and the one used by KMV's and J. P. Morgan's. By fully considering the underlying asset correlations, the probability that multiple firms default can be measured precisely with parameters with respect to the time horizon. Our results can be applied in broad pricing and risk management related to credit risk.

The probability of multiple defaults and the default correlations are important factors in evaluating the credit risk of portfolios or correlated firms in credit markets. The financial industry and regulators recognize the importance of controlling default risks and default correlations for the stability of financial markets. A closed-form formula for evaluating default correlations in credit risk analysis is absolutely necessary for both practical and theoretical purposes. The importance and significance of the probability of multiple defaults and default correlations has been addressed extensively.

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Appendix

Proof of Proposition 1. We use the symmetric matrix property in $\Omega \cdot \Omega^T$ to incorporate with Assumption 1 by standard Linear Algebra diagonalization. Under Assumptions 1 and 2, we first derive the default probability for each individual firm then, we use the first-passage-time to get the default probability and new distance-to-default corresponding to eigenvalues of the covariance matrix. Note that $\Omega \cdot \Omega^T$ is a real symmetric matrix if and only if there is a spectral decomposition

$$\Omega \cdot \Omega^T = \sum_{i=1}^{n} \Sigma_i^2 q_i q_i^T \tag{A1}$$

where $\Omega \cdot \Omega^T q_i = \Sigma_i^2 q_i$ due to the covariance matrix's positivity and $\{q_1, \cdots, q_n\}$ is an orthnormal basis such that the orthnormal basis can form an orthogonal matrix Q such that

$Q^T\Omega \cdot \Omega^T Q = \begin{pmatrix} \Sigma_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Sigma_n^2 \end{pmatrix}$. Hence, the new covariance matrix can be treated as uncorrelated diffusions as in [3]. The eigenvalues of the covariance matrix $\Omega \cdot \Omega^T = (\rho_{ij}\sigma_i\sigma_j)_{1\le i,j\le n}$ are given by

$$\Sigma_1^2, \Sigma_2^2, \cdots, \Sigma_n^2$$

where $\Sigma_i = \sigma_i$ if and only if $\rho_{ij} = 0$ for all $i \neq j$. If $n = 2$, we have

$$\Sigma_1^2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\rho\sigma_1\sigma_2}\right), \Sigma_2^2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\rho\sigma_1\sigma_2}\right)$$

Then, the i-th firm specific probability of default $P(D_i(t) = 1) = P(\tau_i \le t)$ is given by

$$P_i(t) = P(D_i(t) = 1) = N\left(-\frac{\ln(V_{i,0}/K_i)}{\Sigma_i\sqrt{t}} - \frac{\mu_i - \lambda_i}{\Sigma_i}\sqrt{t}\right) \tag{A2}$$

$$+\left(\frac{V_{i,0}}{K_i}\right)^{\frac{2(\lambda_i-\mu_i)}{\sigma_i^2}} N\left(-\frac{\ln(V_{i,0}/K_i)}{\Sigma_i\sqrt{t}} + \frac{\mu_i - \lambda_i}{\Sigma_i}\sqrt{t}\right)$$

The underlying asset values of these firms are now uncorrelated through the new driving Brownian motions with respect to the orthnormal basis. Hence, one can adapt Merton's structural model for the isolated firm asset value to find the probability of default in (A2) for the i-th firm. □

Proof of Theorem 1. The evaluation of the probability that either firm defaults with default correlation is reduced to the following probability.

$$P(D_1(t) = 1 \text{ or } D_2(t) = 1 \cdots, \text{or } D_n(t) = 1)$$

$$= P(\tau_1 \le t \text{ or } \tau_2 \le t \cdots, \text{or } \tau_n \le t) = P(\tau \le t)$$

where $\tau = \min\{\tau_1, \tau_2, \cdots, \tau_n\}$ and $\tau_i = \min\{t \ge 0 : V_i(t) \le e^{\lambda_i t}K_i\}$ for $i = 1, 2, \cdots, n$.

If $\lambda_i = \mu_i$, then we can set $X_i(t) = -\ln[e^{-\lambda_i t}V_i(t)/V_i(0)]$ and $b_i = -\ln[K_i/V_i(0)]$. The default condition $V_i(t) \le e^{\lambda_i t}K_i$ is equivalent to $X_i(t) \ge b_i$.

If $\lambda_i \neq \mu_i$, then set $X_i(t) = -(\lambda_i - \mu_i)t - \ln[e^{-\lambda_i t}V_i(t)/V_i(0)]$ and $b_i(t) = -\ln[\frac{e^{\lambda_i t}K_i}{e^{\mu_i t}V_i(0)}]$, the default is equivalent to $X_i(t) \ge b_i(t)$.

For both cases, we have $dX(t) = -\Omega dW(t)$ with $X(t)^T = (X_1(t), \cdots, X_n(t))$ and n-dimensional independent Brownian motion $W(t)^T = (W_1(t), \cdots W_n(t))$.

Let $f(x_1, x_2, \cdots, x_n, t)$ be the transition probability density of a particle in the region $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i < b_i \, 1 \le i \le n$ before the time t. Therefore, we have

$$\begin{aligned} F(b_1, b_2, \cdots, b_n, t) &= \int_{-\infty}^{b_1}\int_{-\infty}^{b_2}\cdots\int_{-\infty}^{b_n} f(x_1, x_2, \cdots, x_n, t)dx_1dx_2\cdots dx_n = P(\tau > t) \\ &= 1 - P(\tau \le t) = 1 - P(D_1(t) = 1 \text{ or } D_2(t) = 1 \cdots, \text{or } D_n(t) = 1) \end{aligned}$$

Following the standard Kolmogorov forward equation for the transition probability density $f(x_1, x_2, \cdots, x_n, t)$, we have the classical problem to determine the density from the PDE, i.e., the transition probability density is the solution of the Kolmogorov forward equation

$$\frac{1}{2}\left(\sum_{i=1}^{n}\sigma_i^2\frac{\partial^2 f}{\partial x_i^2} + 2\sum_{1\le i\neq j\le n}\rho_{ij}\sigma_i\sigma_j\frac{\partial^2 f}{\partial x_i\partial x_j}\right) = \frac{\partial f}{\partial t}, \quad x_i < b_i, 1 \le i \le n$$

subject to the following boundary conditions:

$$\begin{aligned}
&f(-\infty, x_2, \cdots, x_n, t) = f(x_1, \cdots, -\infty, \cdots, x_n, t) = f(x_1, \cdots, x_{n-1}, -\infty, t) = 0\\
&f(x_1, x_2, \cdots, x_n, 0) = \delta(x_1)\delta(x_2)\cdots\delta(x_n)\\
&F(b_1, b_2, \cdots, b_n, t) \leq 1, \;\; t > 0\\
&f(b_1, x_2, \cdots, x_n, t) = \cdots = f(x_1, \cdots, x_{n-1}, b_n, t) = 0
\end{aligned}$$

where $\delta(x)$ is the Dirac's delta function with $\int_{-\infty}^{\infty} \delta(x)dx = 1$. The equation has a solution

$$f(x_1, x_2, \cdots, x_n, t) = \frac{1}{(2\pi t)^{n/2} \det(\Omega\Omega^T)^{1/2}} \exp(-\frac{1}{2t} x^T (\Omega\Omega^T)^{-1} x) \tag{A3}$$

as a time-varying Gaussian distribution with the symmetric matrix $\Omega\Omega^T$, where $x^T = (x_1, \cdots, x_n)$. It is straightforward to verify that the multivariate normal distribution function $f(x_1, x_2, \cdots, x_n, t)$ satisfies the Kolmogorov forward equation and the first three boundary conditions, except $f(b_1, x_2, \cdots, x_n, t) = \cdots = f(x_1, \cdots, x_{n-1}, b_n, t) = 0$.

For completeness, we present the complete proof of the simple reflection principle of the partial differential equation for this problem. This settles the correlated default probability from the first-passage-time approach.

Let $Pu = (\frac{\partial}{\partial t} - \frac{1}{2}(\sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + 2\sum_{1\leq i\neq j\leq n} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2}{\partial x_i \partial x_j}))u = 0$ be the partial differential equation with $u(x, 0) = 0$, $u(-\infty, x_2, t) = \cdots = f(x_1, \cdots, -\infty, t) = 0$, $u(b_1, x_2, \cdots, x_n, t) = u(x_1, \cdots, x_{n-1}, b_n, t) = 0$. Let $X_i = -x_i + b_i (1 \leq i \leq n)$ be a coordinate shift, and $g(X_1, X_2, \cdots, X_n, t) = u(x_1, x_2, \cdots, x_n, t)$ be the function in terms of the new variables X_i for $1 \leq i \leq n$. Thus, we have

$$Pg(X_1, X_2, \cdots, X_n, t) = 0, \quad 0 \leq X_i < \infty, 1 \leq i \leq n$$

$$g(X_1, X_2, \cdots, X_n, 0) = \delta(X_1)\delta(X_2)\cdots\delta(X_n)$$

$$g(+\infty, X_2, t) = \cdots = g(X_1, \cdots, +\infty, t) = 0$$

$$g(X_1, \cdots, 0|_{X_i}, \cdots, X_n, t) = 0, 0 \leq X_i < \infty, \; 1 \leq i \leq n$$

Now, we use the reflection principle to solve the above boundary valued PDE. Define $h(X_1, \cdots, X_n, t)$ be the odd extension of $g(X_1, \cdots, X_n, t)$ as the following:

$$h(y_1, y_2, \cdots, y_n, t) = (-1)^{\sum_{i=1}^{n} sign^{-} y_i} g(sign(y_1)y_1, \cdots, sign(y_n)y_n, t)$$

where $sign^{-}(y_i) = -1$ if $sign(y_i) = -1$ and $sign^{-}(y_i) = 0$ if $sign(y_i) = 1$, and the odd extension of g runs over the 2^n quadrants in the Euclidean plane $\mathbb{R}^n$ with respect to the quadrant where $(sign(y_1), \cdots, sign(y_n))$ belongs to. For example when $n = 2$, the odd extension of g in first quadrant Q_1 can be expressed by

$$h(y_1, y_2, t) = \begin{cases} g(y_1, y_2, t) & \text{if } (y_1, y_2) \in Q_1 \\ -g(-y_1, y_2, t) & \text{if } (y_1, y_2) \in Q_2 \\ g(-y_1, -y_2, t) & \text{if } (y_1, y_2) \in Q_3 \\ -g(y_1, -y_2, t) & \text{if } (y_1, y_2) \in Q_4 \end{cases}$$

where Q_i is the i-th quadrant in the Euclidean plane $\mathbb{R}^2$. Similarly, we define an odd extension for the initial value for $\delta(y_1, y_2, \cdots, y_n)$ accordingly:

$$\delta(y_1, y_2, \cdots, y_n) = (-1)^{\sum_{i=1}^{n} sign^{-} y_i} \delta(sign(y_1)y_1) \cdots \delta(sign(y_n)y_n)$$

For example when $n = 2$,

$$\delta(y_1, y_2) = \begin{cases} \delta(y_1)\delta(y_2) & \text{if } (y_1, y_2) \in Q_1 \\ -\delta(-y_1)\delta(y_2) & \text{if } (y_1, y_2) \in Q_2 \\ \delta(-y_1)\delta(-y_2) & \text{if } (y_1, y_2) \in Q_3 \\ -\delta(y_1)\delta(-y_2) & \text{if } (y_1, y_2) \in Q_4 \end{cases}$$

Then, we solve the initial value problem on the whole plane $\mathbb{R}^n$:

$$Ph(y_1, y_2, \cdots, y_n, t) = 0, \quad t > 0, (y_1, \cdots, y_n) \in \mathbb{R}^n; \quad h(y_1, \cdots, y_n, 0) = \delta(y_1, y_2, y_2, \cdots, y_n)$$

The problem can be solved through the standard heat kernel on $\mathbb{R}^2$,

$$h(y_1, \cdots, y_n, t) = \int_{\mathbb{R}^n} f(y_1 - z_1, y_2 - z_2, \cdots, y_n - z_n, t)\delta(z_1, z_2, \cdots, z_n) dz_1 dz_2 \cdots dz_n$$

where $f(z_1, z_2, \cdots, z_n, t) = \frac{1}{(2\pi t)^{n/2}\det(\Omega\Omega^T)^{1/2}} \exp(-\frac{1}{2t} z^T (\Omega\Omega^T)^{-1} z)$. Due to the odd extension, $h(y_1, \cdots, y_n, t) = 0$ whenever $y_i = 0$ for some i. Therefore $g(y_1, y_2, \cdots, y_n, t)$ is simply the restriction of $h(y_1, y_2, \cdots, y_n, t)$ on the first quadrant Q_1. For $(y_1, y_2, \cdots, y_n) \in Q_1$,

$$\begin{aligned} g(y_1, y_2, \cdots, y_n, t) &= h(y_1, y_2, \cdots, y_n, t) \\ &= \int_{\mathbb{R}^n} f(y_1 - z_1, y_2 - z_2, \cdots, y_n - z_n, t)\delta(z_1, z_2, \cdots, z_n) dz_1 dz_2 \cdots dz_n \end{aligned}$$

By using the definition of $\delta(z_1, z_2, \cdots, z_n)$ on different regions to decompose the integral into 2^n terms corresponding to each quandrant, we obtain $g(y_1, y_2, \cdots, y_n, t)$

$$\begin{aligned} &= \int_{\mathbb{R}^n} f(y_1 - z_1, y_2 - z_2, \cdots, y_n - z_n, t)\delta(z_1, z_2, \cdots, z_n) dz_1 dz_2 \cdots dz_n \\ &= \sum \int f(y_1 - z_1, y_2 - z_2, \cdots, y_n - z_n, t)(-1)^{\sum_{i=1}^n sign^- z_i} \delta(sign(z_1)z_1) \cdots \delta(sign(z_n)z_n) dz_1 dz_2 \cdots dz_n \\ &= \sum \int_{Q_1} (-1)^{\sum_{i=1}^n sign^- z_i} f(y_1 - sign(z_1)z_1, y_2 - sign(z_2)z_2, \cdots, y_n - sign(z_n)z_n, t)\delta(z_1)\delta(z_2) \cdots \delta(z_n) dz_1 dz_2 \cdots dz_n \end{aligned}$$

where the second identity follows from the odd extension of $\delta(z_1, z_2, \cdots, z_n)$ from the first quadrant to all the plane $\mathbb{R}^n$, the third identity from the change of variables to switch all different quadrants into the first quadrant Q_1. Therefore the solution for the boundary conditions is given by $u(x_1, x_2, \cdots, x_n, t)$

$$\begin{aligned} &= g(X_1, X_2, \cdots, X_n, t) \\ &= g(-x_1 + b_1, -x_2 + b_2, \cdots, -x_n + b_n, t) \\ &= \sum \int_{Q_1} (-1)^{\sum_{i=1}^n sign^- z_i} f(-x_1 + b_1 - sign(z_1)z_1, \cdots, -x_n + b_n - sign(z_n)z_n, t)\delta(b_1 - z_1) \cdots \delta(b_n - z_n) dz_1 \cdots dz_n \\ &= \sum_{i=1}^{2^n} (-1)^{signQ_i} \int_{-\infty}^{b_1} \cdots \int_{-\infty}^{b_n} f(-x_i + sign(z_i)y_i + b_i(1 - sign(z_i)), t)\delta(y_1) \cdots \delta(y_n) dy_1 \cdots dy_n \\ &= \sum_{i=1}^{2^n} (-1)^{signQ_i} \int_{-\infty}^{b_1} \cdots \int_{-\infty}^{b_n} f_{\Omega_{Q_i}}(sign(z_i)x_i + b_i(1 - sign(z_i)) - y_i, t)\delta(y_1) \cdots \delta(y_n) dy_1 \cdots dy_n \\ &= \sum_{i=1}^{2^n} (-1)^{signQ_i} f_{\Omega_{Q_i}}(sign(z_1)x_1 + b_1(1 - sign(z_1)), \cdots, sign(z_n)x_n + b_n(1 - sign(z_n)), t) \end{aligned}$$

where the first two equalities follow from the coordinate shift, the third from the previous identity with the substitution $y_i = -x_i + b_i$ $(1 \le i \le n)$, sign Q_i is the number of negative 1 for the representative $(\pm 1, \pm 1, \cdots, \pm 1)$ in Q_i, the fourth identity from the variable change $y_i = b_i - z_i$ $(1 \le i \le n)$ and $(sign(z_1), \cdots, sign(z_n))$ is the representative of Q_i, the fifth identity from the even function property of f, where $f_{\Omega_{Q_i}}$ is the function $f(sign(z_1)z_1, \cdots, sign(z_n)z_n, t)$ which is same as replacing Ω by

$$\operatorname{diag}\left(sign(z_1), \cdots, sign(z_n)\right)\Omega = \Omega_{Q_i}$$

and the last from the Dirac delta function property.

Thus we have $F(b_1, b_2, \cdots, b_n, t)$

$$\begin{aligned}
&= \int_{-\infty}^{b_n} \cdots \int_{-\infty}^{b_1} u(x_1, x_2, \cdots, x_n, t) dx_1 \cdots dx_n \\
&= \sum_{i=1}^{2^n} (-1)^{signQ_i} \int_{-\infty}^{b_n} \cdots \int_{-\infty}^{b_1} f_{\Omega_{Q_i}}(sign(z_1)x_1 + b_1(1 - sign(z_1)), \cdots, sign(z_n)x_n + b_n(1 - sign(z_n))) dx_1 \cdots dx_n \\
&= \sum_{i=1}^{2^n} (-1)^{signQ_i} N_{\Omega_{Q_i}\Omega_{Q_i}^T}(sign(z_1)b_1, sign(z_2)b_2, \cdots, sign(z_n)b_n)
\end{aligned}$$

where the last equality follows from various changing variables and $N_A(x_1, \cdots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \frac{1}{(2\pi t)^{n/2}\sqrt{\det A}} \exp(-\frac{1}{2t} z^T A z) dz_1 \cdots dz_n$.

For example $n = 2$, the representative $(sign(z_1), sign(z_2)) = (\pm 1, \pm 1)$ for the quadrants with $sign(z_1) = \pm 1$ and $sign(z_2) = \pm 1$ has

$$(1,1) \in Q_1, (-1,1) \in Q_2, (-1,-1) \in Q_3, (1,-1) \in Q_4$$

$$signQ_1 = 0, signQ_2 = 1, signQ_3 = 2, signQ_4 = 1$$

$$\Omega_{Q_1} = \Omega, \Omega_{Q_2} = diag(-1,1)\Omega, \Omega_{Q_3} = diag(-1,-1)\Omega, \Omega_{Q_4} = diag(1,-1)\Omega$$

Hence $F(b_1, b_2, t) = \sum_{i=1}^{2^2} (-1)^{signQ_i} N_{\Omega_{Q_i}}(sign(z_1)b_1, sign(z_2)b_2)$. Thus, we have

$$F(b_1, b_2, t) = N_{2,\rho}\left(\frac{b_1}{\sigma_1\sqrt{t}}, \frac{b_2}{\sigma_2\sqrt{t}}\right) - N_{2,-\rho}\left(-\frac{b_1}{\sigma_1\sqrt{t}}, \frac{b_2}{\sigma_2\sqrt{t}}\right) + N_{2,\rho}\left(-\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}\right) - N_{2,-\rho}\left(\frac{b_1}{\sigma_1\sqrt{t}}, -\frac{b_2}{\sigma_2\sqrt{t}}\right) \tag{A4}$$

where $N_{2,\rho}(x_1, x_2) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{y_1^2 - 2\rho y_1 y_2 + y_2^2}{2(1-\rho^2)}} dy_1 dy_2$ is the probability of the standard bivariate normal distribution on $y_1 \leq x_1$ and $y_2 \leq x_2$, and $\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{y_1^2 - 2\rho y_1 y_2 + y_2^2}{2(1-\rho^2)}}$ is the standard bivariate normal distribution density function $N_2(0, 1, \rho)$ for random variable Y_i with mean zero and variance 1 and correlation $\rho = Cov(Y_1, Y_2)$. □

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