Tolerance Modelling of Vibrations and Stability for Periodic Slender Visco-Elastic Beams on a Foundation with Damping. Revisiting

Jarosław Jędrysiak

Department of Structural Mechanics, Łódź University of Technology, al. Politechniki 6, 90-924 Łódź, Poland; jarek@p.lodz.pl; Tel.: +48-42-631-3564

Abstract: The mathematical modelling of certain problems of vibrations and stability for periodic slender visco-elastic beams is presented in this note. To consider these problems and take into account the effect of the microstructure, the tolerance modelling approach is proposed. Using this technique, the equation with non-continuous, periodic, highly oscillating coefficients is replaced by a system of differential equations with constant coefficients. Moreover, these governing equations describe the effect of the microstructure on the overall behavior of the beams under consideration. The tolerance modelling can lead to equations of two different tolerance models—the standard and the general, under weakened assumptions. This averaging tolerance method was assessed by comparison with the asymptotic homogenization, the governing equations of which omit this effect. My considerations were limited to proposing and presenting only mathematical models describing investigated beams. In a simple analytical example, the application of the presented average models is shown.

Keywords: periodic beam; microstructure; tolerance modelling; effect of microstructure

1. Introduction

Problems concerning the vibrations and stability of periodic, slender, visco-elastic beams were investigated in this contribution. Such beams can interact with a heterogeneous damping foundation. These beams have a periodic structure along their axis x and are made of many identical small repeatable elements, called periodic cells; cf. Figure 1 with fragments of the beams.
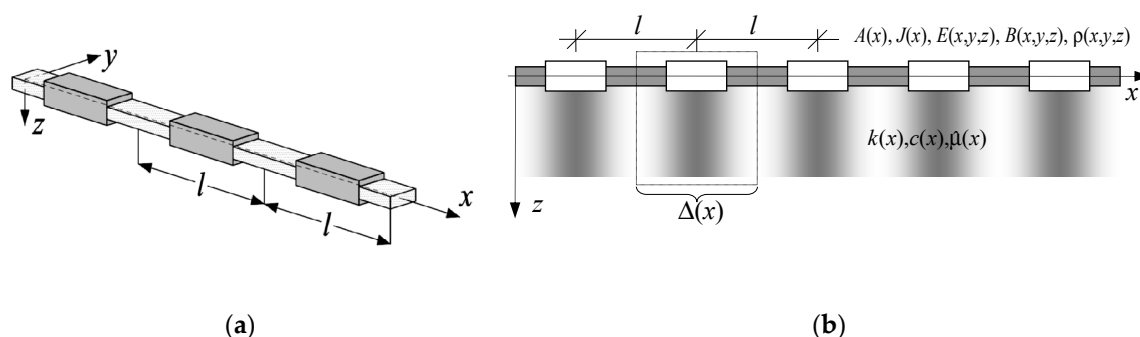

Figure 1. Fragments of periodic beams: (**a**) without foundation; (**b**) on a foundation.

Although beams are one of the simplest examples of periodical structures, they are often and widely used in different branches of engineering. Moreover, periodic objects can represent approximate models of some more complex systems.

The dynamics and stability of periodic, slender visco-elastic beams on a periodic damping foundation are described by partial differential equations with non-continuous, periodic, highly oscillating coefficients. The analysis of those problems using the aforementioned equations is rather difficult. Hence, various simplified models of such beams were proposed. Some of these models use the concept of the effective (averaged) stiffness (and other properties) of the beams. Among them, one can mention the models which are based on the theory of asymptotic homogenization (cf. Bensoussan et al. [1]), used for instance to analyze periodic plates; cf. Kohn and Vogelius [2]. This theory was used to consider equilibrium equations of microperiodic beams by Kolpakov [3–5]. Various other modelling methods were also proposed to investigate different composite media; e.g., Matysiak and Nagórko [6] showed a homogenization with microlocal parameters for periodic plates; a comprehensive study on composite beams vibrations was done by Hajianmaleki and Qatu [7]; Roque et al. [8] formulated a certain model of bending for laminated composite beams, capturing the effects of the microstructure, using a modified couple stress theory and a meshless method; Batra and Xiao [9] proposed a layer-wise third order shear and normal deformable plate/shell theory incorporating all geometric nonlinearities to study finite deformations of curved laminated beams; Grygorowicz et al. [10], Wittenbeck et al. [11] and Grygorowicz and Magnucka-Blandzi [12] analyzed dynamic stability of sandwich beams having variable mechanical properties of the core; a torsion of composite beams with a phase made of auxetic material was investigated by Stręk et al. [13] and Jopek and Stręk [14] using an analytical-numerical model based on analytical relations and finite element method; Pawlus [15] modelled the stability of three-layered annular plates made of fiber-reinforced composite facings and a foam core; some numerical investigations using a new approximation function in Carrera's unified formulation for composite layered beams were shown by Arruda et al. [16].

Other theoretical and numerical results were shown in many papers for composite functionally-graded structures. Bui et al. [17] applied meshless methods to analyze natural frequencies of sandwich beams with functionally graded composite core. Murin et al. [18] considered the effect of the shear correction function in modal analysis of functionally graded composite beams. A new theory with a generalization of layer-wise displacement approaches to dynamics of composite curved beams was proposed by Carpentieri et al. [19]. A free vibration optimization of composite functionally graded nano-beams was analyzed by Roque et al. [20]. Fantuzzi and Tornabene [21] proposed a strong formulation isogeometric analysis for laminated composite plates. A layer-wise theory and a differential quadrature method for composite plates were shown by Liu et al. [22]. Awrejcewicz et al. [23] presented a mathematical model of composite beams using the couple stress theory and size-dependent governing equations of the layers' motions. A semi-analytical method for stability problems of columns with closed cross-sections made of composite fiber-multilayered plate was proposed by Kołakowski and Mania [24] and Mania et al. [25] using the classical laminate plate theory. Eringen's nonlocal theory was applied to investigate the bending of composite functionally graded beams with internal porosity by Jouneghani et al. [26]. Ghayesh [27] and Ghayesh and Farokhi [28] proposed a vibration analysis of axially functionally-graded composite beams.

However, the effect of the microstructure size (cf. Brillouin [29]) is usually omitted in governing equations of these models. Hence, these models cannot be good tools to investigate this effect. However, this effect can play a role in the different behaviors of microheterogeneous structures; cf. [29,30]. That was observed and analyzed in a few papers; e.g., the differential quadrature method was applied to analyze vibration band gaps in periodic beams by Xiang and Shi [31] and to consider flexural wave band gaps in periodic composite plates by Cheng et al. [32]; the transfer matrix method was adapted to analysis of flexural wave propagation in a beam on an elastic foundation and for investigating natural frequencies of non-uniform periodic beams by Yu et al. [33] and Xu et al. [34], respectively; wave propagation in beams with periodically varying stiffness was considered by Chen [35] via applying the multireflection method; Zhi-Jing et al. [36] investigated vibration band gap properties of periodic Mindlin's plates using a spectral element method; a center finite difference method was used by Zhou et al. [37,38] to analyze band gaps of periodic thin plates with or without damping.

However, in governing equations of these models for various composite structures, the size of the microstructure parameter (the size of the periodicity cell) is often neglected. Therefore, non-asymptotic averaged models of the periodic composite structures are introduced, which are usually called the tolerance models. They are derived by applying the tolerance modelling technique; cf. Woźniak and Wierzbicki [39], Woźniak et al. [40] and Woźniak et al. [41].

The obtained tolerance model equations have constant coefficients, and in contrast to other averaged models, some of them depend on the size of the microstructure. The proposed method can be adopted to any differential equations with highly oscillating periodic or tolerance-periodic coefficients. This approach, in contrast to the asymptotic homogenization, allows one to analyze the effect of the microstructure size.

Various dynamical, stability and thermoelastic problems of periodic structures and composites were considered using this method. These applications can be found in a series of papers; e.g., the dynamics of microperiodic beams were analyzed by Mazur-Śniady [42]; periodic fluid-saturated grounds were considered by Dell'Isola et al. [43]; Wierzbicki and Woźniak [44] investigated the dynamics of plane periodic structures; Jędrysiak [45] studied the stability problems of periodic plates; an analysis of vibrations for periodic wavy-type plates was shown by Michalak [46]; Nagórko and Woźniak [47] considered the dynamics of thin plates reinforced periodically by stiffeners; Baron [48] analyzed vibrations of periodic medium-thickness plates; buckling and free vibrations of periodic thin plates on a foundation were investigated by Jędrysiak [49,50]; an application to dynamics of periodic thin plates with the microstructure size of an order of the plate thickness was presented by Mazur-Śniady et al. [51]; Tomczyk [52,53] analyzed the stability of periodic shells; the nonlinear dynamics of visco-elastic periodic plates were modelled by Jędrysiak [54]; Domagalski and Jędrysiak [55] studied the geometrically nonlinear vibrations of periodic beams; vibrations of periodic three-layered plates were shown by Marczak and Jędrysiak [56]; free vibrations of periodic thin plates with uncertain material properties were investigated by Jędrysiak and Ostrowski [57].

Different thermomechanical problems of functionally graded media with microstructures were also considered using the tolerance modelling approach; e.g., Michalak and Wirowski [58], Perliński et al. [59] and Wirowski et al. [60] analyzed dynamics and stability of plates with longitudinally graded structures with or without an elastic foundations; free vibrations of transversally graded thin plates on an elastic foundation were presented by Jędrysiak and Kaźmierczak-Sobińska [61]; the dynamics of a thin-walled structure with a dense system of ribs was considered by Michalak [62]; Ostrowski and Michalak [63] showed an application for heat conduction in cylindrical composite conductors with non-uniform microstructure; the effect of microstructure in thermoelastic problems of transversally graded laminates was investigated by Pazera and Jędrysiak [64]; vibrations of thin plates with the microstructure size of an order of the plate thickness were analyzed by Jędrysiak [65]; Tomczyk and Szczerba [66] proposed a new asymptotic-tolerance model of dynamics for thin microstructured transversally graded shells; Jędrysiak [67] considered free vibrations of medium thickness functionally graded plates with a microstructure.

In this work, two new tolerance models of slender periodic beams are proposed—one (which can be called the "standard" or "classic" tolerance model) under the concept of slowly-varying functions and the second (it can be named "*general tolerance model*") under the concept of weakly-slowly-varying functions—suggested and shown also for thin periodic plates by Jędrysiak [68]. Governing equations of both the models describe the effect of the microstructure on the overall behavior of the periodic beams under consideration. The microstructural effect of periodic beams on a foundation plays a crucial role, for instance, in dynamical problems such as free vibrations or forced vibrations. Using the tolerance models one can investigate the higher order frequencies (and higher order vibrations; cf. Brillouin [29]) related to the microstructure of the beam, not only fundamental lower frequencies (and vibrations) related to the macrostructure of the beam. Moreover, these model equations are assessed by comparison with the governing equations obtained using the asymptotic homogenization, which omits this effect. Considerations in this note are restricted only to a theoretical analysis and

derivation of the problem. An application of the above-mentioned models is shown in a simple example for an infinite periodic beam.

2. Modelling Approach

2.1. Modelling Preliminaries

Denote by *Oxyz* the orthogonal Cartesian coordinate system and by *t* the time coordinate. Let the region of the undeformed beam be denoted by $\Omega \equiv \{(x, y, z) : -a/2 \leq y \leq a/2, -h/2 \leq z \leq h/2, x \in \Lambda\}$, where Λ is the beam axis; $\Lambda \equiv [0,L]$; $h(\cdot)$ is the beam height; $a(\cdot)$ is the beam width; *L* is the length of the beam. Derivatives of *x* are denoted by ∂. The "basic cell" on *Ox* is defined as $\Delta \equiv [-l/2, l/2]$. Hence, parameter *l* is the length of cell Δ. It is assumed that this parameter satisfies condition $h_{\max} <<l<< L$ and is called the microstructure parameter. Geometrical properties of the beam—height $h(\cdot)$ and width $a(\cdot)$; and foundation properties—Winkler's coefficient $k(\cdot)$, damping parameter $c(\cdot)$ and mass density of foundation $\hat{\mu}(\cdot)$ can be periodic functions in *x*, but material properties of the beam—modulus of elasticity $E = E(\cdot,y,z)$, modulus of visco-elasticity $B = B(\cdot,y,z)$ and mass density $\rho = \rho(\cdot,y,z)$ can be periodic functions in *x* and even functions in *y*, *z*. Let $u(x,z,t)$ ($x \in \overline{\Delta}$) be a vertical beam displacement (along *z*-axis) and *q* be total loadings in the *z*-axis direction.

In the modelling the assumptions and relations of the slender beam theory are applied:

- *The kinematic assumption of slender beams.*

$$u(x,z,t) = -z\partial w(x,t), \; w(x,z,t) = w(x,t), \tag{1}$$

where $w(\cdot,t)$ is the beam deflection.

The similar form to Equation (1) is introduced for virtual displacements:

$$\overline{u}(x,z) = -z\partial \overline{w}(x), \; \overline{w}(x,z) = \overline{w}(x), \tag{2}$$

where: $\overline{u}(\cdot)$—the virtual beam displacement; $\overline{w}(\cdot)$—the virtual beam deflection.

- *The strain–displacement relation.*

$$e = \partial u, \tag{3}$$

where: *e*—the beam strain.

- *The stress–strain relation.*

$$s = s_0 + Ee + B\dot{e}, \tag{4}$$

where: *s*—the stress; s_0—the initial stress; $E(\cdot)$, $B(\cdot)$—modulus of: elasticity, visco-elasticity, respectively (being periodic functions in *x*).

- *The virtual work equation*

$$\int_{\Lambda} \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} \rho \ddot{u}\overline{u}dzdydx + \int_{\Lambda} \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} \rho \ddot{w}\overline{w}dzdydx + \int_{\Lambda} \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} s\overline{e}dzdydx = \int_{\Lambda} q\overline{w}dx - \int_{\Lambda} (kw + \hat{\mu}\ddot{w} + c\dot{w})\overline{w}dx, \tag{5}$$

where: $\rho(\cdot)$—the mass density of the beam material; $k(\cdot)$—the Winkler's coefficient; *E*, *B*—modulus of: elasticity, visco-elasticity, respectively (being periodic functions in *x*); $\overline{e}(\cdot)$—the virtual beam strain.

Denoting stiffnesses of the plate: bending $d(\cdot)$, visco-elastic $b(\cdot)$; the mass density $\mu(\cdot)$; the rotational mass inertia $j(\cdot)$, being periodic functions in *x*, in the form:

$$\begin{aligned} d(x) &= \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} E(x,y,z)z^2 dzdy, \quad b(x) = \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} B(x,y,z)z^2 dzdy, \\ \mu(x) &= \int_{-a/2}^{b/2} \int_{-h/2}^{h/2} \rho(x,y,z)dzdy, \quad j(x) = \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} \rho(x,y,z)z^2 dzdy, \end{aligned} \tag{6}$$

and the axis force by:

$$n = \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} s_0 dz dy, \tag{7}$$

after some manipulations of Equation (5) combined with Equations (1)–(4), by applying the divergence theorem and the du Bois–Reymond lemma, the following governing equations can be derived:

- *The constitutive equation*

$$m = d\partial\partial w + b\partial\partial \dot{w}, \tag{8}$$

- *The equation of motion*

$$\partial\partial m - \partial(n\partial w) + \mu\ddot{w} - j\partial\partial\ddot{w} + kw + \hat{\mu}\ddot{w} + c\dot{w} = q. \tag{9}$$

Substituting Equation (8) into Equation (9), the governing equations of the periodic beams can be written as:

$$\partial\partial(d\partial\partial w) + \partial\partial(b\partial\partial\dot{w}) - \partial(n\partial w) + \mu\ddot{w} - j\partial\partial\ddot{w} + kw + \hat{\mu}\ddot{w} + c\dot{w} = q. \tag{10}$$

The above equation, similarly to Equations (8)–(9), has highly oscillating, non-continuous, periodic functional coefficients.

2.2. Introductory Concepts of the Tolerance Modelling

Concepts defined in books [1–3] are applied in modelling. They are also presented various papers, e.g., in [67], but some of them are reviewed below in the form adopted for beams.

Let $\Delta(x) \equiv x + \Delta$ be a cell at $x \in \Lambda_\Delta$, $\Lambda_\Delta = \{x \in \Lambda: \Delta(x) \subset \Lambda\}$. The averaging operator for an integrable function f is defined by

$$< f > (x) = l^{-1} \int_{\Delta(x)} f(\xi)d\xi, \quad x \in \Lambda, \ \xi \in \Delta(x). \tag{11}$$

For function f, a periodic one in x, the averaged value calculated from (11) is constant.

Let $\partial^k f$ denote the k–th gradient of function $f = f(x)$, $x \in \Lambda$, $k = 0, 1, \ldots, \alpha$, ($\alpha \geq 0$ and for considered problems of beams $\alpha = 2$); $\partial^0 f \equiv f$; and $\widetilde{f}^{(k)}(\cdot, \cdot)$ be a function defined in $\overline{\Lambda} \times R^m$.

Function $f \in H^2(\Lambda)$ is the tolerance-periodic function, $f \in TP_\delta^2(\Lambda, \Delta)$, if for $k = 0, 1, \ldots, 2$, the following conditions are satisfied:

- $(\forall x \in \Lambda) \ (\exists \widetilde{f}^{(k)}(x, \cdot) \in H^0(\Delta)) \ [\|\partial^k f|_{\Lambda_\Delta}(\cdot) - \widetilde{f}^{(k)}(x, \cdot)\|_{H^0(\Lambda, \Delta)} \leq \delta]$;
- $\int_{\Delta(\cdot)} \widetilde{f}^{(k)}(\cdot, \xi)d\xi \in C^0(\overline{\Lambda})$.

where function $\widetilde{f}^{(k)}(x, \cdot)$ is the *periodic approximation of $\partial^k f$ in $\Delta(x)$*, $x \in \Lambda$, $k = 0, 1, \ldots, 2$; and δ is the tolerance parameter, $\delta << 1$, related to the considered problems.

Function $F \in H^2(\Lambda)$ is the weakly-slowly-varying function, $F \in WSV_\delta^2(\Lambda, \Delta)$, if

- $F \in TP_\delta^2(\Lambda, \Delta)$;
- $(\forall(x; \xi) \in \Lambda) \ [(x; \xi) \Rightarrow F(x) \approx F(\xi) \wedge \partial F(x) \approx \partial F(\xi)]$.

Function $F \in H^2(\Lambda)$ is *the slowly-varying function*, $F \in SV_\delta^2(\Lambda, \Delta)$, if

- $F \in WSW_\delta^2(\Lambda, \Delta)$;
- $(\forall x \in \Lambda) \quad l\|\partial F(x)\|_{\Delta(x)} \approx 0$.

Function $\varphi \in H^2(\Lambda)$ is *the highly oscillating function*, $\varphi \in HO_\delta^2(\Lambda, \Delta)$, if

- $\varphi \in TP_\delta^2(\Lambda, \Delta)$;
- $(\forall x \in \Lambda) \ [\widetilde{\varphi}^{(k)}(x, \cdot)|_{\Delta(x)} = \partial^k \widetilde{\varphi}(x), \quad k = 0, 1, \ldots, 2]$.

- $\forall F \in SV_\delta^2(\Lambda, \Delta) \; \exists f \equiv \varphi F \in TP_\delta^2(\Lambda, \Delta) \quad \widetilde{f}^{(k)}(x, \cdot)|_{\Delta(x)} = F(x)\partial^k\widetilde{\varphi}(x)|_{\Delta(x)}, \quad k = 1, 2.$

For $k = 0$ let us denote $\widetilde{f} \equiv \widetilde{f}^{(0)}$.

Let $g(\cdot)$ be a highly oscillating function, defined on $\overline{\Lambda}$, $g \in HO_\delta^2(\Lambda, \Delta)$, continuous together with gradient $\partial^1 g$. Gradient $\partial^2 g$ is a piecewise continuous and bounded. Function $g(\cdot)$ is the fluctuation shape function of the second kind, $FS_\delta^2(\Lambda, \Delta)$, if it depends on l as a parameter and the following conditions hold:

- $\partial^k g \in O(l^{\alpha-k}) \quad$ for $k = 0, 1, \ldots, \alpha$, $\alpha = 2$, $\partial^0 g \equiv g$;
- $<g>(x) \approx 0 \quad$ for every $x \in \Lambda_\Delta$,

with the microstructure parameter l. The above second condition can be replaced by $<\mu g>(x) \approx 0$ for every $x \in \Lambda_\Delta$, where $\mu > 0$ is a certain periodic function.

2.3. Tolerance Modelling Assumptions

The introductory concepts are applied in the formulation of the modelling assumptions; cf. [1–3]. The first assumption is the micro-macro decomposition, introduced in the form:

$$w(x, t) = W(x, t) + g^A(x)V^A(x, t), \; A = 1, \ldots, P, \quad x \in \Lambda, \tag{12}$$

where $W(\cdot, t)$, $V^A(\cdot, t) \in WSV_\delta^2(\Lambda, \Delta)$ or $W(\cdot, t)$, $V^A(\cdot, t) \in SV_\delta^2(\Lambda, \Delta)$; i.e., they are weakly-slowly-varying functions of the second kind or slowly-varying functions of the second kind, respectively. Functions $W(\cdot, t)$ and $V^A(\cdot, t)$ are new kinematic unknowns, named the macrodeflection and the fluctuation amplitudes, respectively. Moreover, $g^A(\cdot)$, $A, B = 1, \ldots, P$, are the known fluctuation shape functions, which are usually postulated a priori in the considered problem and describe the unknown field (here: the deflection) oscillations caused by the beam inhomogeneity. These functions, apart from the periodicity condition, have to satisfy the following restrictions:

- $\partial^k g^A \in O(l^{2-k}) \quad$ for $k = 0, 1, \ldots, 2$;
- $<\mu g^A> = 0$;
- $<\mu g^A g^B> = 0 \quad$ for $A \neq B$, $A, B = 1, \ldots, P$.

The second assumption is the tolerance averaging approximation, in which it is assumed that terms $O(\delta)$ are negligibly small, e.g., for $f \in TP_\delta^2(\Lambda, \Delta)$, $F \in SV_\delta^2(\Lambda, \Delta)$ or $F \in WSV_\delta^2(\Lambda, \Delta)$ $g^A \in FS_\delta^2(\Lambda, \Delta)$, in:

$$\begin{aligned} <f>(x) &= <\overline{f}>(x) + O(\delta), \\ <fF>(x) &= <f>(x)F(x) + O(\delta), \\ <f\partial(g^A F)>(x) &= <f\partial g^A>(x)F(x) + O(\delta). \end{aligned} \tag{13}$$

The third assumption is the axis force restriction, in which it is assumed that terms involving fluctuating parts of the axis force can be neglected in comparing to terms with averaged parts; i.e.,:

$$n(x) = N(x) + \widetilde{n}(x),$$

$$N = <n>, \quad <\widetilde{n}> = 0, \tag{14}$$

where $N(\cdot) \in SV_\delta^2(\Lambda, \Delta)$ and $\widetilde{n}(\cdot) \in TP_\delta^2(\Lambda, \Delta)$ are averaged and fluctuating parts of axis force, respectively.

2.4. The Outline of the Tolerance Modelling Procedure

The modelling procedure can be outlined following [1], [2] or [3], but here it is similar to that modified one shown in [1]. The starting point of the modelling is the formulation of the virtual work equation in the form Equation (5). From the combining Equation (5) with Equations (1)–(4), using

the divergence theorem and the du Bois–Reymond lemma, after some manipulations the governing equations of the beam can be derived in the form of Equations (8)–(9) or (10).

Now, the first step of the tolerance modelling procedure is the substitution of the micro-macro decomposition (12) into Equation (10). In this case the governing dynamic Equation (10) does not hold; i.e., in the framework of macrodynamics there exists a residual field $r(\cdot)$ defined by:

$$
\begin{aligned}
r = {}& \partial\partial(d\partial\partial(W + g^A V^A) + b\partial\partial(\dot{W} + g^A \dot{V}^A)) - \partial(n\partial(W + g^A V^A)) + \\
& + \mu(\ddot{W} + g^A \ddot{V}^A) - j\partial\partial(\ddot{W} + g^A \ddot{V}^A) + \\
& + k(W + g^A V^A) + \hat{\mu}(\ddot{W} + g^A \ddot{V}^A) + c(\dot{W} + g^A \dot{V}^A) - q.
\end{aligned}
\tag{15}
$$

The next assumption is the residual orthogonality condition, in which the residual field $r(\cdot)$ has to satisfy the following conditions:

$$
< r > (x,t) = 0, \quad < rg^B > (x,t) = 0.
\tag{16}
$$

The above conditions (16), together with the assumptions—the tolerance averaging approximation (13) and the axis force restriction (14)—lead to a system of equations for the macrodeflection $W(\cdot,t)$ and the fluctuation amplitudes: $V^A(\cdot,t)$, $A = 1,\ldots,P$. The form of these governing equations depends on the specification of the class of slowly-varying functions $W(\cdot,t)$, $V^A(\cdot,t)$ (weakly-slowly-varying or slowly-varying function).

3. Results—The Governing Equations

3.1. The General Tolerance Model Equations

By using the residual orthogonality condition Equation (16) with the modelling assumptions (12)–(14), introducing following denotations of averaged coefficients:

$$
\begin{aligned}
& D \equiv < d >, && D^A \equiv < d\partial\partial g^A >, && D^{AB} \equiv < d\partial\partial g^A \partial\partial g^B >, \\
& \widetilde{B} \equiv < b >, && \widetilde{B}^A \equiv < b\partial\partial g^A >, && \widetilde{B}^{AB} \equiv < b\partial\partial g^A \partial\partial g^B >, \\
& \widetilde{m} \equiv < \mu >, && \widetilde{m}^A \equiv l^{-2} < \mu g^A >, && \widetilde{m}^{AB} \equiv l^{-4} < \mu g^A g^B >, \\
& \vartheta \equiv < j >, && \vartheta^A \equiv l^{-1} < j\partial g^A >, && \vartheta^{AB} \equiv l^{-2} < j\partial g^A \partial g^B >, \\
& C \equiv < c >, && C^A \equiv l^{-2} < c g^A >, && C^{AB} \equiv l^{-4} < c g^A g^B >, \\
& K \equiv < k >, && K^A \equiv l^{-2} < k g^A >, && K^{AB} \equiv l^{-4} < k g^A g^B >, \\
& \hat{m} \equiv < \hat{\mu} >, && \hat{m}^A \equiv l^{-2} < \hat{\mu} g^A >, && \hat{m}^{AB} \equiv l^{-4} < \hat{\mu} g^A g^B >, \\[4pt]
& N \equiv < n >, && N^{AB} \equiv l^{-2} < n\partial\partial g^A g^B >, && \overset{\smile AB}{N} \equiv -l^{-2} < n\partial g^A \partial g^B >= -N^{AB}, \\
& Q \equiv < q >, && Q^A \equiv l^{-2} < q g^A >;
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
& \overline{N}^A \equiv l^{-2} < n g^A >, && \overline{N}^{AB} \equiv l^{-4} < n g^A g^B >, \\
& \overline{D}^A \equiv l^{-2} < d g^A >, && \overline{D}^{AB} \equiv l^{-4} < d g^A g^B >, \\
& \hat{D}^{AB} \equiv l^{-2} < d\partial g^A \partial g^B >, && \overset{\smile AB}{D} \equiv l^{-2} < d\partial\partial g^A g^B >, \\
& \overline{B}^A \equiv l^{-2} < b g^A >, && \overline{B}^{AB} \equiv l^{-4} < b g^A g^B >, \\
& \hat{B}^{AB} \equiv l^{-2} < b\partial g^A \partial g^B >, && \overset{\smile AB}{B} \equiv l^{-2} < b\partial\partial g^A g^B >, \\
& \overline{\vartheta}^A \equiv l^{-2} < j g^A >, && \overline{\vartheta}^{AB} \equiv l^{-4} < j g^A g^B >, && \widetilde{\vartheta}^{AB} \equiv l^{-3} < j g^A \partial g^B >;
\end{aligned}
$$

the equations for $W(\cdot,t)$ and $V^A(\cdot,t)$ are derived:

- The general tolerance constitutive equations.

$$M = D\partial\partial W + D^A V^A + \underline{l^2 \overline{D}^A \partial\partial V^A} + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A + \underline{l^2 \overline{B}^A \partial\partial\dot{V}^A},$$
$$M^A = D^A \partial\partial W + D^{AB} V^B + \underline{l^2 \overset{\smile}{D}{}^{AB} \partial\partial V^B} + \widetilde{B}^A \partial\partial\dot{W} + \widetilde{B}^{AB} \dot{V}^B + \underline{l^2 \overset{\smile}{B}{}^{AB} \partial\partial\dot{V}^B};$$

(18)

- The general tolerance equations of motion.

$$\partial\partial M - \partial(N\partial W) + (\widetilde{m} + \hat{m})\ddot{W} + l^2(\widetilde{m}^A + \hat{m}^A)\ddot{V}^A - \partial(\vartheta\partial\ddot{W}) - l\partial(\vartheta^A \ddot{V}^A) +$$
$$+KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A - \underline{l^2 \partial(\overline{N}^A \partial V^A + \overline{\vartheta}^A \partial\ddot{V}^A)} = Q,$$

$$M^A + l^2(\widetilde{m}^A + \hat{m}^A)\ddot{W} + l\vartheta^A \partial\ddot{W} + l^2(l^2 m^{AB} + l^2 \hat{m}^{AB} + \vartheta^{AB})\ddot{V}^B -$$
$$-l^2 N^{AB} V^B + l^2 K^A W + l^4 K^{AB} V^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B + \underline{l^3 \widetilde{\vartheta}^{AB} \partial\ddot{V}^B} +$$
$$+l^2 \partial\partial(\overline{D}^A \partial\partial W + l^2 \overline{D}^{AB} \partial\partial V^B) + \underline{l^2[(\overset{\smile}{D}{}^{AB} - 4\hat{D}^{AB}) - l^2 \overline{N}^{AB}]\partial\partial V^B} -$$
$$-\underline{l^2 \partial(\overline{\vartheta}^A \partial\ddot{W} + l^2 \overline{\vartheta}^{AB} \partial\ddot{V}^B + l\widetilde{\vartheta}^{AB} \ddot{V}^B)} = l^2 Q^A.$$

(19)

By substituting Equation (18) into Equation (19) the averaged governing equations of the periodic beams can be written as:

$$\partial\partial[D\partial\partial W + D^A V^A + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A + l^2 \partial\partial(\overline{D}^A V^A + \overline{B}^A \dot{V}^A)] -$$
$$-\partial(N\partial W) + (\widetilde{m} + \hat{m})\ddot{W} + l^2(\widetilde{m}^A + \hat{m}^A)\ddot{V}^A - \partial(\vartheta\partial\ddot{W}) - l\partial(\vartheta^A \ddot{V}^A) +$$
$$+KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A - \underline{l^2 \partial(\overline{N}^A \partial V^A + \overline{\vartheta}^A \partial\ddot{V}^A)} = Q,$$

$$D^A \partial\partial W + D^{AB} V^B + \widetilde{B}^A \partial\partial\dot{W} + \widetilde{B}^{AB} \dot{V}^B + \underline{l^2 \overset{\smile}{B}{}^{AB} \partial\partial\dot{V}^B} +$$
$$+l^2(\widetilde{m}^A + \hat{m}^A)\ddot{W} + l\vartheta^A \partial\ddot{W} + l^2(l^2 m^{AB} + l^2 \hat{m}^{AB} + \vartheta^{AB})\ddot{V}^B -$$
$$-l^2 N^{AB} V^B + l^2 K^A W + l^4 K^{AB} V^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B + \underline{l^3 \widetilde{\vartheta}^{AB} \partial\ddot{V}^B} +$$
$$+l^2 \partial\partial(\overline{D}^A \partial\partial W + l^2 \overline{D}^{AB} \partial\partial V^B) + \underline{l^2[2(\overset{\smile}{D}{}^{AB} - 2\hat{D}^{AB}) - l^2 \overline{N}^{AB}]\partial\partial V^B} -$$
$$-\underline{l^2 \partial(\overline{\vartheta}^A \partial\ddot{W} + l^2 \overline{\vartheta}^{AB} \partial\ddot{V}^B + l\widetilde{\vartheta}^{AB} \ddot{V}^B)} = l^2 Q^A.$$

(20)

Equation (20) stands the system of the governing equations of the general tolerance model of slender visco-elastic periodic beams on a periodic foundation with damping. They are the system for the macrodeflection W and the fluctuation amplitudes V^A, A = 1, ... , P, and they have a physical sense only if all these unknowns are weakly-slowly-varying functions in x. For every basic unknown: the macrodeflection W and the fluctuation amplitudes V^A, A = 1, ... , P, two boundary conditions should be formulated at both ends of the beam (four conditions together for every unknown). Characteristic features of Equation (20), similarly to Equations (18)–(19), are that they have constant coefficients and describe the effect of the microstructure size on the overall dynamic and stability behavior of the beams under consideration by terms involving the microstructure parameter l. Moreover, the governing equations of the general tolerance model include additional terms, which are underlined.

3.2. The (Standard) Tolerance Model Equations

By using the residual orthogonality condition Equation (16) with the modelling assumptions Equations (12)–(14) and the concept of the slowly-varying function, while introducing denotations of averaged coefficients similar to Equation (17):

$$
\begin{aligned}
&D \equiv\; <d>, & D^A \equiv\; &<d\partial\partial g^A>, & D^{AB} \equiv\; &<d\partial\partial g^A\partial\partial g^B>, \\
&\widetilde{B} \equiv\; <b>, & \widetilde{B}^A \equiv\; &<b\partial\partial g^A>, & \widetilde{B}^{AB} \equiv\; &<b\partial\partial g^A\partial\partial g^B>, \\
&\widetilde{m} \equiv\; <\mu>, & \widetilde{m}^A \equiv\; l^{-2}&<\mu g^A>, & \widetilde{m}^{AB} \equiv\; l^{-4}&<\mu g^A g^B>, \\
&\vartheta \equiv\; <j>, & \vartheta^A \equiv\; l^{-1}&<j\partial g^A>, & \vartheta^{AB} \equiv\; l^{-2}&<j\partial g^A\partial g^B>, \\
&C \equiv\; <c>, & C^A \equiv\; l^{-2}&<cg^A>, & C^{AB} \equiv\; l^{-4}&<cg^A g^B>, \\
&K \equiv\; <k>, & K^A \equiv\; l^{-2}&<kg^A>, & K^{AB} \equiv\; l^{-4}&<kg^A g^B>, \\
&\hat{m} \equiv\; <\hat{\mu}>, & \hat{m}^A \equiv\; l^{-2}&<\hat{\mu} g^A>, & \hat{m}^{AB} \equiv\; l^{-4}&<\hat{\mu} g^A g^B>, \\
&N \equiv\; <n>, & N^{AB} \equiv\; l^{-2}&<n\partial\partial g^A g^B>, & \overset{\smile}{N}{}^{AB} \equiv\; -l^{-2}&<n\partial g^A\partial g^B> = -N^{AB}, \\
&Q \equiv\; <q>, & Q^A \equiv\; l^{-2}&<qg^A>;
\end{aligned}
\tag{21}
$$

the following equations for unknowns $W(\cdot,t)$ and $V^A(\cdot,t)$ are derived:

- The standard tolerance constitutive equations.

$$
\begin{aligned}
M &= D\partial\partial W + D^A V^A + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A, \\
M^A &= D^A\partial\partial W + D^{AB} V^B + \widetilde{B}^A\partial\partial\dot{W} + \widetilde{B}^{AB}\dot{V}^B;
\end{aligned}
\tag{22}
$$

- The standard tolerance equations of motion.

$$
\begin{aligned}
\partial\partial M - \partial(N\partial W) + (\widetilde{m}+\hat{m})\ddot{W} + l^2(\widetilde{m}^A+\hat{m}^A)\ddot{V}^A - \partial(\vartheta\partial\ddot{W}) - l\partial(\vartheta^A\ddot{V}^A) + \\
+KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A = Q,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
M^A + l^2(\widetilde{m}^A+\hat{m}^A)\ddot{W} + l\vartheta^A\partial\ddot{W} + l^2(l^2 m^{AB} + l^2\hat{m}^{AB} + \vartheta^{AB})\ddot{V}^B - \\
-l^2 N^{AB} V^B + l^2 K^A W + l^4 K^{AB} V^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B = l^2 Q^A.
\end{aligned}
$$

Equation (22) can be substituted into Equation (23); this leads to the averaged governing equations of the periodic beams in the following form:

$$
\begin{aligned}
\partial\partial[D\partial\partial W + D^A V^A + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A] - \\
-\partial(N\partial W) + (\widetilde{m}+\hat{m})\ddot{W} + l^2(\widetilde{m}^A+\hat{m}^A)\ddot{V}^A - \partial(\vartheta\partial\ddot{W}) - l\partial(\vartheta^A\ddot{V}^A) + \\
+KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A = Q,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
D^A\partial\partial W + D^{AB} V^B + \widetilde{B}^A\partial\partial\dot{W} + \widetilde{B}^{AB}\dot{V}^B + \\
+l^2(\widetilde{m}^A+\hat{m}^A)\ddot{W} + l\vartheta^A\partial\ddot{W} + l^2(l^2 m^{AB} + l^2\hat{m}^{AB} + \vartheta^{AB})\ddot{V}^B - \\
-l^2 N^{AB} V^B + l^2 K^A W + l^4 K^{AB} V^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B = l^2 Q^A.
\end{aligned}
$$

Equation (24) stands the governing equations of the standard tolerance model of slender visco-elastic periodic beams on a periodic foundation with damping. Similarly to Equation (20), they are the system for the macrodeflection W and the fluctuation amplitudes V^A, $A = 1, \ldots, P$. Two boundary conditions at both ends of the beam should be formulated only for the macrodeflection W (four conditions together) but not for the fluctuation amplitudes V^A, $A = 1, \ldots, P$. Characteristic features of these equations are that they: have constant coefficients; describe the effect of the microstructure size on the overall dynamic and stability behavior of the beams under consideration by terms involving the microstructure parameter l; have a physical sense only if all the basic unknowns W and V^A are slowly-varying functions in x.

3.3. The Asymptotic Model Equations

In order to evaluate the above theoretical results (models) obtained in the framework of the tolerance modelling technique, the approximate model will be presented, the governing equations of which neglect the effect of the microstructure size. These equations can be derived applying the proper asymptotic modelling procedure (cf. [2,3] or directly from Equations (22)–(23)) after vanishing terms with the microstructure parameter *l*. Using some of the denotations in Equation (21) these equations can be written in the form:

- The asymptotic constitutive equations.

$$
\begin{aligned}
M &= D\partial\partial W + D^A V^A + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A, \\
M^A &= D^A \partial\partial W + D^{AB} V^B + \widetilde{B}^A \partial\partial\dot{W} + \widetilde{B}^{AB} \dot{V}^B;
\end{aligned}
\tag{25}
$$

- The asymptotic equations of motion.

$$
\begin{aligned}
\partial\partial M - \partial(N\partial W) + (\widetilde{m} + \hat{m})\ddot{W} - \partial(\vartheta\partial\ddot{W}) + KW + C\dot{W} &= Q, \\
M^A &= 0.
\end{aligned}
\tag{26}
$$

After combining Equation (25) with Equation (26), the averaged governing equations of the periodic beams take the following form:

$$
\begin{aligned}
\partial\partial[D\partial\partial W + D^A V^A + \widetilde{B}\partial\partial\dot{W} + \widetilde{B}^A \dot{V}^A] - \partial(N\partial W) + \\
+ (\widetilde{m} + \hat{m})\ddot{W} - \partial(\vartheta\partial\ddot{W}) + KW + C\dot{W} = Q,
\end{aligned}
\tag{27}
$$

$$
D^A \partial\partial W + D^{AB} V^B + \widetilde{B}^A \partial\partial\dot{W} + \widetilde{B}^{AB} \dot{V}^B = 0.
$$

Equation (27), similarly to Equations (25)–(26), represents the asymptotic model of slender visco-elastic periodic plates on a periodic foundation with damping, which neglects the effect of the microstructure size on the overall behavior of the beams. Two boundary conditions at both ends of the beam should be formulated only for the macrodeflection *W* (four conditions together). The above equations have constant coefficients, similarly to Equation (20) or (24) for the tolerance models, in the contrast to Equations (8)–(9) or (10) with non-continuous, highly oscillating, periodic, functional coefficients.

4. Results—A Simple Theoretical Example

4.1. Introduction

As an example, a homogeneous elastic unbounded beam without the effects of the rotational inertia, the axis force and the foundation is considered. Its periodicity is caused by the periodic distribution of the system of two concentrated masses M_1 and M_2 (Figure 2).
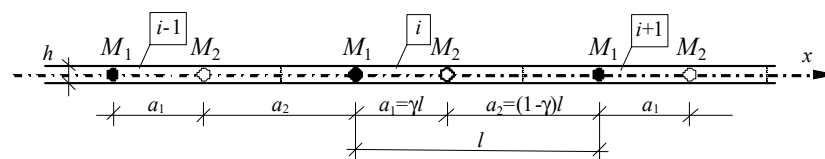
Figure 2. Fragment of periodic beam with a periodically distributed system of two concentrated masses, M_1 and M_2.

Let us denote the distance between mass M_1 and M_2 by a_1 and between M_2 and M_1 as a_2 (hence $a_2 = l - a_1$); cf. Figure 3.
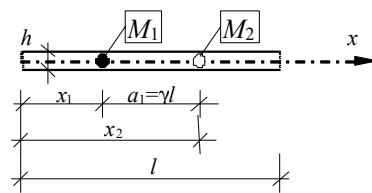
Figure 3. The periodicity cell of the considered beam.

Young's modulus E, mass density μ, height h and width b of the beam are assumed to be constant. Load q is neglected. In considerations only one fluctuation shape function $g = g^1$, $A = P = 1$, is assumed in the form:

$$g(x) = l^2[\sin(2\pi x/l) + c], \tag{28}$$

where the constant c is calculated from the condition $<\mu g> = 0$. The form of the fluctuation shape function g (Equation (28)) is related to the form of the periodicity cell shown in Figure 3. Denote also $V \equiv V^1$.

Hence, the averaged coefficients (17) other than zero can be written as:

$$
\begin{aligned}
d &= \tfrac{1}{12}Ebh^3, \\
D &\equiv d, \qquad D^{11} \equiv d < \partial\partial g\partial\partial g >, \qquad \widetilde{m} \equiv < \mu >, \qquad \widetilde{m}^{11} \equiv l^{-4} < \mu gg >, \\
\overline{D}^1 &\equiv l^{-2}d < g >, \quad \overline{D}^{11} \equiv l^{-4}d < gg >, \quad \hat{D}^{11} \equiv l^{-2}d < \partial g\partial g >, \quad \breve{D}^{11} \equiv l^{-2}d < \partial\partial gg > .
\end{aligned} \tag{29}
$$

4.2. Example—Frequencies of a Travelling Wave

The aim of the example is to obtain formulas of frequencies of a travelling wave for the unbounded beam with the periodic distribution of the system of two concentrated masses M_1 and M_2 introduced above.

4.2.1. The General Tolerance Model

Using the averaged coefficients Equation (29) the governing equations of the general tolerance model Equation (20) for the considered beam can be written as:

$$D\partial\partial\partial\partial W + \underline{l^2\partial\partial\partial\partial\overline{D}^1 V} + \widetilde{m}\ddot{W} = 0,$$

$$D^{11}V + l^4m^{11}\ddot{V} + \underline{l^2\overline{D}^1\partial\partial\partial\partial W} + \underline{l^4\overline{D}^{11}\partial\partial\partial\partial V} + \underline{2l^2(\breve{D}^{11} - 2\hat{D}^{11})\partial\partial V} = 0. \tag{30}$$

Introduce the wave number k (e.g., $k = 2\pi/L$). Solutions to Equation (30) will be assumed as:

$$W(x,t) = A_W \exp i(kx - \omega t), \quad V(x,t) = A_V \exp i(kx - \omega t), \tag{31}$$

with A_W, A_V as amplitudes, ω as a frequency. By substituting Equation (31) into Equation (30) the following system of algebraic equations is obtained:

$$(Dk^4 - \widetilde{m}\omega^2)A_W + \underline{l^2\overline{D}^1 k^4 A_V} = 0,$$

$$\underline{l^2\overline{D}^1 k^4 A_W} + [D^{11} + \underline{l^4\overline{D}^{11}k^4} - \underline{2l^2(\breve{D}^{11} - 2\hat{D}^{11})k^2} - l^4m^{11}\omega^2]A_V = 0. \tag{32}$$

While comparing the main determinant of the above system of equations to zero, the characteristic equation can be written as:

$$l^4 \widetilde{m} m^{11} \omega^4 - \left\{ \widetilde{m}[D^{11} + \underline{l^4 \overline{D}^{11} k^4} - \underline{2l^2 (\breve{D}^{11} - 2\hat{D}^{11})k^2}] + l^4 m^{11} D k^4 \right\} \omega^2 +$$
$$+ [D^{11} + \underline{l^4 \overline{D}^{11} k^4} - \underline{2l^2 (\breve{D}^{11} - 2\hat{D}^{11})k^2}] D k^4 - (\underline{l^2 \overline{D}^1 k^4})^2 = 0, \tag{33}$$

from which the following formulas of the travelling wave frequencies in the framework of the general tolerance model can be obtained as

$$\omega_{-,+} = \sqrt{ \frac{\widetilde{m}[D^{11} + l^4 k^4 \overline{D}^{11} - 2l^2 k^2 (\breve{D}^{11} - 2\hat{D}^{11})] + l^4 k^4 m^{11} D}{2 l^4 \widetilde{m} m^{11}} \mp \frac{\sqrt{ \left\{ l^4 k^4 m^{11} D - \widetilde{m}[D^{11} - 2l^2 k^2 (\breve{D}^{11} - 2\hat{D}^{11}) + l^4 k^4 \overline{D}^{11}] \right\}^2 + (2 l^4 k^4 \overline{D}^1)^2 \widetilde{m} m^{11}}}{2 l^4 \widetilde{m} m^{11}} }, \tag{34}$$

where ω_- is the fundamental lower frequency related to the averaged macrostructure of the beam with the periodic distribution of the system of two concentrated masses, but ω_+ is the higher frequency related to the microstructure of the beam.

4.2.2. The Standard Tolerance Model

By applying the averaged coefficients Equation (29) the governing equations of the standard tolerance model Equation (24) for the considered beam can be written as:

$$D \partial \partial \partial \partial W + \widetilde{m} \ddot{W} = 0,$$
$$D^{11} V + l^4 m^{11} \ddot{V} = 0. \tag{35}$$

It should be noted that, in contrast to the general tolerance model, two independent differential equations of the beam under consideration are obtained. The first of them describes vibrations of beam for a "macro" scale, and the second equation determines "microvibrations" related to a periodic system of two concentrated masses. By substituting solutions of Equation (31) into Equation (35) the uncoupled system of algebraic equations can be written in the form:

$$(D k^4 - \widetilde{m} \omega^2) A_W = 0,$$
$$(D^{11} - l^4 m^{11} \omega^2) A_V = 0. \tag{36}$$

The characteristic equation in this case takes the form:

$$l^4 \widetilde{m} m^{11} \omega^4 - (\widetilde{m} D^{11} + l^4 m^{11} D k^4) \omega^2 + D^{11} D k^4 = 0, \tag{37}$$

The formulas of the travelling wave frequencies in the framework of the standard tolerance model have the form:

$$\widetilde{\omega}_- = \sqrt{\frac{D k^4}{\widetilde{m}}}, \quad \widetilde{\omega}_+ = \sqrt{\frac{D^{11}}{l^4 m^{11}}}, \tag{38}$$

where $\widetilde{\omega}_-$ is the fundamental lower frequency related to the averaged macrostructure of the beam with the periodic distribution of the system of two concentrated masses, but $\widetilde{\omega}_+$ is the higher frequency related to the microstructure of the beam.

4.2.3. The Asymptotic Model

Using the averaged coefficients Equation (29) the governing equations of the asymptotic model Equation (27) for the considered beam can be written as:

$$D \partial \partial \partial \partial W + \widetilde{m} \ddot{W} = 0, \quad D^{11} V = 0. \tag{39}$$

It should be noted that, in contrast to both the tolerance models—general and standard, only one differential equation of the beam under consideration is obtained. This Equation $(39)_1$ describing vibrations of beam for a "macro" scale and the effect of the microstructure of the beam, in the form of the higher order vibrations, is neglected. By substituting Solutions (31) to Equation $(39)_1$ the algebraic equation is obtained:

$$(Dk^4 - \widetilde{m}\omega^2)A_W = 0, \tag{40}$$

which leads to the characteristic equation in the form:

$$\widetilde{m}\omega^2 - Dk^4 = 0, \tag{41}$$

but the formula of the travelling wave frequency in the framework of the asymptotic model is:

$$\omega_0 = \sqrt{\frac{Dk^4}{\widetilde{m}}}, \tag{42}$$

where ω_0 is the fundamental lower frequency related to the averaged macrostructure of the beam with the periodic distribution of the system of two concentrated masses The effect of the microstructure of the beam in from of the higher frequency is omitted.

4.3. Comments on the Example

By analyzing results obtained in this simple example it can be observed that both the tolerance models—general and standard, allow one to investigate the effect of the microstructure in dynamical problems, which occurs in the possibility of considering higher order vibrations related to the microstructure. In addition, it seems that the general tolerance model better describes the lower order fundamental vibrations, taking into account the effect of microstructure. Compared with both the tolerance models, the asymptotic model only allows one to study lower order fundamental vibrations.

5. Remarks

By summing up the above theoretical considerations, some remarks can be formulated.

- The tolerance modelling method allows one to derive averaged equations of models of slender visco-elastic beams on a periodic damping foundation with constant coefficients, which replace the classic equations with non-continuous, periodic coefficients.
- Using various definitions of concepts of the weakly-slowly-varying function and the slowly-varying function. two different tolerance models can be derived in the framework of the tolerance approach—the general tolerance model and the standard tolerance model.
- The governing equations of both the tolerance models involve terms which describe the effect of the microstructure size on overall dynamic and stability behavior of these beams; hence both models allow one to analyze dynamic and stability problems of the beams under consideration on the macro-and the micro-level.
- The form of the model equations depends on the class of slowly-varying, basic, unknown functions. The weakly-slowly-varying functions lead to the general tolerance model equations, which involve additional terms depending on the microstructure parameter; but the slowly-varying functions lead to the standard tolerance model equations.
- In contrast to the governing equations of the standard tolerance model for slender visco-elastic beams on a periodic damping foundation, the equations of the general tolerance model for these beams include additional terms describing the effect of the microstructure size also in stationary problems.
- In contrast, the asymptotic model allows one to analyze dynamic and stability problems of the beams under consideration only at the macro-level, without the effect of the microstructure.

- Based on the presented simple example, it can be observed that the proposed general tolerance model also allows one in such a simple case not only to analyze the effect of microstructure in the form of higher order vibrations (microvibrations), but also to consider and relate this effect with the fundamental lower order vibrations (macrovibrations).

Some applications of the general tolerance model equations of the slender visco-elastic beams on a periodic damping foundation will be presented in forthcoming papers. It also seems that the proposed tolerance models can be used in this or suitably adapted form in the analysis of problems related to nanostructures, e.g., nano-beams, but it is an open problem.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bensoussan, A.; Lions, J.L.; Papanicolaou, G. *Asymptotic Analysis for Periodic Structures*; North-Holland: Amsterdam, The Netherlands, 1978.
2. Kohn, R.V.; Vogelius, M. A new model for thin plates with rapidly varying thickness. *Int. J. Solids Struct.* **1984**, *20*, 333–350. [CrossRef]
3. Kolpakov, A. Calculation of the characteristics of thin elastic rods with a periodic structure. *J. Appl. Math. Mech.* **1991**, *55*, 358–365. [CrossRef]
4. Kolpakov, A. Application of homogenization method to justification of 1-D model for beam of periodic structure having initial stresses. *Int. J. Solids Struct.* **1998**, *35*, 2847–2859. [CrossRef]
5. Kolpakov, A. The governing equations of a thin elastic stressed beam with a periodic structure. *J. Appl. Math. Mech.* **1999**, *63*, 495–504. [CrossRef]
6. Matysiak, S.J.; Nagórko, W. Microlocal parameters in the modelling of microperiodic plates. *Ing. Arch.* **1989**, *59*, 434–444. [CrossRef]
7. Hajianmaleki, M.; Qatu, M.S. Vibrations of straight and curved composite beams: A review. *Compos. Struct.* **2013**, *100*, 218–232. [CrossRef]
8. Roque, C.M.C.; Fidalgo, D.S.; Ferreira, A.J.M.; Reddy, J.N. A study of a microstructure-dependent composite laminated Timoshenko beam using a modified couple stress theory and a meshless method. *Compos. Struct.* **2013**, *96*, 532–537. [CrossRef]
9. Batra, R.C.; Xiao, J. Finite deformations of curved laminated St. Venant–Kirchhoff beam using layer-wise third order shear and normal deformable beam theory (TSNDT). *Compos. Struct.* **2013**, *97*, 147–164. [CrossRef]
10. Grygorowicz, M.; Magnucki, K.; Malinowski, M. Elastic buckling of a sandwich beam with variable mechanical properties of the core. *Thin Walled Struct.* **2015**, *87*, 127–132. [CrossRef]
11. Wittenbeck, L.; Grygorowicz, M.; Paczos, P. Numerical analysis of sandwich beam with corrugated core under three-point bending. *AIP Conf. Proc.* **2015**, *1648*, 800007.
12. Grygorowicz, M.; Magnucka-Blandzi, E. Mathematical modeling for dynamic stability of sandwich beam with variable mechanical properties of core. *Appl. Math. Mech.* **2016**, *37*, 1361–1374. [CrossRef]
13. Strek, T.; Jopek, H.; Fraska, A. Torsion of elliptical composite beams with auxetic phase. *Phys. Status Solidi Basic Res.* **2016**, *253*, 1359–1368. [CrossRef]
14. Jopek, H.; Strek, T. Torsion of a two-phased composite bar with helical distribution of constituents. *Phys. Status Solidi* **2017**, *254*, 1700050. [CrossRef]
15. Pawlus, D. Stability of three-layered annular plate with composite facings. *Appl. Compos. Mater.* **2017**, *24*, 141–158. [CrossRef]
16. Arruda, M.R.T.; Castro, L.M.S.; Ferreira, A.J.M.; Garrido, M.; Gonilha, J.; Correia, J.R. Analysis of composite layered beams using Carrera unified formulation with Legendre approximation. *Compos. Part B Eng.* **2018**, *137*, 39–50. [CrossRef]
17. Bui, T.Q.; Khosravifard, A.; Zhang, C.; Hematiyan, M.R.; Golu, M.V. Dynamic analysis of sandwich beams with functionally graded core using a truly meshfree radial point interpolation method. *Eng. Struct.* **2013**, *47*, 90–104. [CrossRef]

18. Murin, J.; Aminbaghai, M.; Hrabovsky, J.; Kutiš, V.; Kugler, S. Modal analysis of the FGM beams with effect of the shear correction function. *Compos. Part B* **2013**, *45*, 1575–1582. [CrossRef]

19. Carpentieri, G.; Tornabene, F.; Ascione, A.; Fraternali, F. An accurate one-dimensional theory for the dynamics of laminated composite curved beams. *J. Sound Vibration* **2015**, *336*, 96–105. [CrossRef]

20. Roque, C.M.C.; Martins, P.A.L.S.; Ferreira, A.J.M.; Jorge, R.M.N. Differential evolution for free vibration optimization of functionally graded nano beams. *Compos. Struct.* **2016**, *156*, 29–34. [CrossRef]

21. Fantuzzi, N.; Tornabene, F. Strong formulation isogeometric analysis (SFIGA) for laminated composite arbitrarily shaped plates. *Compos. Part B Eng.* **2016**, *96*, 173–203. [CrossRef]

22. Liu, B.; Ferreira, A.J.M.; Xing, Y.F.; Neves, A.M.A. Analysis of composite plates using a layerwise theory and a differential quadrature finite element method. *Compos. Struct.* **2016**, *156*, 393–398. [CrossRef]

23. Awrejcewicz, J.; Krysko, V.A.; Pavlov, S.P.; Zhigalov, M.V.; Krysko, A.V. On natural frequencies of non-uniform beams modulated by finite periodic cells. *Int. J. Solids Struct.* **2017**, *117*, 39–50. [CrossRef]

24. Kołakowski, Z.; Mania, R.J. Influence of the coupling matrix B on the interactive buckling of FML-FGM columns with closed cross-sections under axial compression. *Compos. Struct.* **2017**, *173*, 70–77. [CrossRef]

25. Mania, R.J.; Madeo, A.; Zucco, G.; Kubiak, T. Imperfection sensitivity of post-buckling of FML channel section column. *Thin Walled Struct.* **2017**, *114*, 32–38. [CrossRef]

26. Jouneghani, F.Z.; Dimitri, R.; Tornabene, F. Structural response of porous FG nanobeams under hygro-thermo-mechanical loadings. *Compos. Part B Eng.* **2018**, *152*, 71–78. [CrossRef]

27. Ghayesh, M.H.; Farokhi, H. Bending and vibration analyses of coupled axially functionally graded tapered beams. *Nonlinear Dyn.* **2018**, *91*, 17–28. [CrossRef]

28. Ghayesh, M.H. Vibration analysis of shear-deformable AFG imperfect beams. *Compos. Struct.* **2018**, *200*, 910–920. [CrossRef]

29. Brillouin, L. *Wave Propagation in Periodic Structures*; Dover Pub. Inc.: Dover, UK, 1953.

30. Robinson, C.W.; Leppelmeier, G.W. Experimental verification of dispersion relations for layered composites. *J. Appl. Mech.* **1974**, *41*, 89–91. [CrossRef]

31. Xiang, H.J.; Shi, Z.F. Analysis of flexural vibration band gaps in periodic beams using differential quadrature method. *Comput. Struct.* **2009**, *87*, 1559–1566. [CrossRef]

32. Cheng, Z.B.; Xu, Y.G.; Zhang, L.L. Analysis of flexural wave bandgaps in periodic plate structures using differential quadrature element method. *Int. J. Mech. Sci.* **2015**, *100*, 112–125. [CrossRef]

33. Yu, D.; Wen, J.; Shen, H.; Xiao, Y.; Wen, X. Propagation of flexural wave in periodic beam on elastic foundations. *Phys. Lett. A* **2012**, *376*, 626–630. [CrossRef]

34. Xu, Y.; Zhou, X.; Wang, W.; Wang, L.; Peng, F.; Li, B. On natural frequencies of non-uniform beams modulated by finite periodic cells. *Phys. Lett. A* **2016**, *380*, 3278–3283. [CrossRef]

35. Chen, T. Investigations on flexural wave propagation of a periodic beam using multi-reflection method. *Arch. Appl. Mech.* **2013**, *83*, 315–329. [CrossRef]

36. Wu, Z.J.; Li, F.M.; Wang, Y.Z. Vibration band gap properties of periodic Mindlin plate structure using the spectral element method. *Meccanica* **2014**, *49*, 725–737. [CrossRef]

37. Zhou, X.Q.; Yu, D.Y.; Shao, X.; Wang, S.; Tian, Y.H. Band gap characteristics of periodically stiffened-thin-plate based on center-finite-difference-method. *Thin Walled Struct.* **2014**, *82*, 115–123. [CrossRef]

38. Zhou, X.Q.; Yu, D.Y.; Shao, X.; Wang, S.; Zhang, S.Q. Simplified-super-element-method for analyzing free flexural vibration characteristics of periodically stiffened-thin-plate filled with viscoelastic damping material. *Thin Walled Struct.* **2015**, *94*, 234–252. [CrossRef]

39. Woźniak, C.; Wierzbicki, E. *Averaging Techniques in Thermomechanics of Composite Solids. Tolerance Averaging Versus Homogenization*; Publishing House of Częstochowa University of Technology: Częstochowa, Poland, 2000.

40. *Thermomechanics of Heterogeneous Solids and Structures*; Woźniak, C.; Michalak, B.; Jędrysiak, J. (Eds.) Publishing House of Łódź University of Technology: Łódź, Poland, 2008.

41. Woźniak, C. *Mathematical Modelling and Analysis in Continuum Mechanics of Microstructured Media*; Publishing House of Łódź University of Technology: Gliwice, Poland, 2010.

42. Mazur-Śniady, K. Macro-dynamic of micro-periodic elastic beams. *J. Theor. Appl. Mech.* **1993**, *31*, 781–793.

43. Dell'Isola, F.; Rosa, L.; Woźniak, C. A micro-structural continuum modelling compacting fluid-saturated grounds. *Acta Mech.* **1998**, *127*, 165–182. [CrossRef]

44. Wierzbicki, E.; Woźniak, C. On the dynamics of combined plane periodic structures. *Arch. Appl. Mech.* **2000**, *70*, 387–398. [CrossRef]

45. Jędrysiak, J. On the stability of thin periodic plates. *Eur. J. Mech. A Solids* **2000**, *19*, 487–502. [CrossRef]

46. Michalak, B. The meso-shape functions for the meso-structural models of wavy-plates. *ZAMM* **2001**, *81*, 639–641. [CrossRef]

47. Nagórko, W.; Woźniak, C. Nonasymptotic modelling of thin plates reinforced by a system of stiffeners. *Electr. J. Polish Agric. Univ. Civil Eng.* **2002**, *5*, 8.

48. Baron, E. On dynamic behaviour of medium-thickness plates with uniperiodic structure. *Arch. Appl. Mech.* **2003**, *73*, 505–516. [CrossRef]

49. Jędrysiak, J. The length-scale effect in the buckling of thin periodic plates resting on a periodic Winkler foundation. *Meccanica* **2003**, *38*, 435–451. [CrossRef]

50. Jędrysiak, J. Free vibrations of thin periodic plates interacting with an elastic periodic foundation. *Int. J. Mech. Sci.* **2003**, *45*, 1411–1428. [CrossRef]

51. Mazur-Śniady, K.; Woźniak, C.; Wierzbicki, E. On the modelling of dynamic problems for plates with a periodic structure. *Arch. Appl. Mech.* **2004**, *74*, 179–190. [CrossRef]

52. Tomczyk, B. A non-asymptotic model for the stability analysis of thin biperiodic cylindrical shells. *Thin Walled Struct.* **2007**, *45*, 941–944. [CrossRef]

53. Tomczyk, B. Dynamic stability of micro-periodic cylindrical shells. *Mech. Mech. Eng.* **2010**, *14*, 137–150.

54. Jędrysiak, J. Geometrically nonlinear vibrations of thin visco-elastic periodic plates on a foundation with damping: Non-asymptotic modelling. *J. Theor. Appl. Mech.* **2016**, *54*, 945–961. [CrossRef]

55. Domagalski, Ł.; Jędrysiak, J. Nonlinear vibrations of periodic beams. *J. Theor. Appl. Mech.* **2016**, *54*, 1095–1108. [CrossRef]

56. Marczak, J.; Jędrysiak, J. Some remarks on modelling of vibrations of periodic sandwich structures with inert core. *Compos. Struct.* **2018**, *202*, 752–758. [CrossRef]

57. Jędrysiak, J.; Ostrowski, P. The effect of uncertain material properties on free vibrations of thin periodic plates. *Meccanica* **2017**, *52*, 3713–3730. [CrossRef]

58. Michalak, B.; Wirowski, A. Dynamic modelling of thin plate made of certain functionally graded materials. *Meccanica* **2012**, *47*, 1487–1498. [CrossRef]

59. Perliński, W.; Gajdzicki, M.; Michalak, B. Modelling of annular plates stability with functionally graded structure interacting with elastic heterogeneous subsoil. *J. Theor. Appl. Mech.* **2014**, *52*, 485–498.

60. Wirowski, A.; Michalak, B.; Gajdzicki, M. Dynamic modelling of annular plates of functionally graded structure resting on elastic heterogeneous foundation with two modules. *J. Mech.* **2015**, *31*, 493–504. [CrossRef]

61. Jędrysiak, J.; Kaźmierczak-Sobińska, M. On free vibration of thin functionally graded plate bands resting on an elastic foundation. *J. Theor. Appl. Mech.* **2015**, *53*, 629–642. [CrossRef]

62. Michalak, B. 2D tolerance and asymptotic models in elastodynamics of a thin-walled structure with dense system of ribs. *Arch. Civil Mech. Eng.* **2015**, *15*, 449–455. [CrossRef]

63. Ostrowski, P.; Michalak, B. The combined asymptotic-tolerance model of heat conduction in a skeletal micro-heterogeneous hollow cylinder. *Compos. Struct.* **2015**, *134*, 343–352. [CrossRef]

64. Pazera, E.; Jędrysiak, J. Effect of microstructure in thermoelasticity problems of functionally graded laminates. *Compos. Struct.* **2018**, *202*, 296–303. [CrossRef]

65. Jędrysiak, J. Tolerance modelling of free vibration frequencies of thin functionally graded plates with one-directional microstructure. *Compos. Struct.* **2017**, *161*, 453–468. [CrossRef]

66. Tomczyk, B.; Szczerba, P. A new asymptotic-tolerance model of dynamic and stability problems for longitudinally graded cylindrical shells. *Compos. Struct.* **2018**, *202*, 473–481. [CrossRef]

67. Jędrysiak, J. Tolerance modelling of free vibrations of medium thickness functionally graded plates. *Compos. Struct.* **2018**, *202*, 1253–1262. [CrossRef]

68. Jędrysiak, J. General and standard tolerance models of thin two-directional periodic plates. In *Shell Structures: Theory and Applications*; Pietraszkiewicz, W., Witkowski, W., Eds.; Taylor & Francis: London, UK, 2018; Volume 4, pp. 101–104.