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Article

Algorithmic Solution of Stochastic Differential Equations

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Abstract: This brief note presents an algorithm to solve ordinary stochastic differential equations (SDEs). The algorithm is based on the joint solution of a system of two partial differential equations and provides strong solutions for finite-dimensional systems of SDEs driven by standard Wiener processes and with adapted initial data. Several examples illustrate its use.

Keywords: stochastic differential equations; strong solution; PDE-based algorithm

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1. Introduction

In general, it is hard to find explicit expression for closed-form solutions for stochastic differential equations (SDEs), see [1–5,8–12,16] to name just a few monographs. In none of the aforementioned monographs one can find a general algorithm to find such solutions. Nowadays, sophisticated computer packages provide powerful tools to seek for solutions, e.g., MAPLE, MATHEMATICA, MATLAB, *etc*. Our aim is to provide a general algorithm to find explicit expression for the strong solution of initial value problems related to systems of finite-dimensional ordinary SDEs

$$\begin{aligned} dX(t) &= f(t,X(t))dt + \sum_{j=1}^{m} g^j(t,X(t))dW_j(t), \qquad (1)\\ X(0) &= X_0 \in \mathbb{R}^d, \quad t \geq 0, \end{aligned}$$

driven by independent Wiener processes W_j on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which is completed with respect to all $\mathbb{P}$-null sets. The drift $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and diffusion coefficients $g^j : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are supposed to be Borel-measurable functions such that the strong solution of the IVP (1) exists (a.s.). The initial value X_0 in (1) is completely independent of the increments of all Wiener processes W_j for $t \geq 0$.

The algorithm is applicable to both Itô- and Stratonovich-interpreted SDEs (1). It is mainly based on the fundamental theorem of stochastic calculus which originates from Itô's works [6,7]. The algorithm can be extended to SDEs integrated in the sense of α-calculus, see [13,14]. For the readers, the readership is supposed to be familiar with the basic concepts of stochastic calculus and real analysis (e.g., for an excellent survey, see [15]).

The paper is organized in 3 sections as follows. After this short introduction, Section 2 presents the algorithm. In Section 3 we discuss its application to solve numerous examples where the solution is and is not known from the literature. An appendix states the fundamental theorem of stochastic calculus, also known as Itô formula, and a theorem on the existence of unique strong solutions of linear SDEs.

2. The Linear PDE-Based Algorithm

Suppose the SDE with sufficiently smooth coefficients f, g_j is solvable (in the strong sense). Define the partial differential operators $\mathcal{L}^j$ by

$$\mathcal{L}^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{m}\sum_{i,k=1}^{d} g_i^j(t,x)g_k^j(t,x)\frac{\partial^2}{\partial x_i \partial x_k} \tag{2}$$

$$\mathcal{L}^j = \sum_{k=1}^{d} g_k^j(t,x)\frac{\partial}{\partial x_k} \tag{3}$$

where $j = 1, 2, ..., m$, $0 \leq t \leq T$ and $x \in \mathbb{R}^d$. In passing, note that $\mathcal{L}^0$ is called the infinitesimal generator of SDE (1) and occurs in several fundamental theorems of stochastic analysis (*cf.* Itô formula in appendix).

2.1. The algorithm for Itô SDEs

[1] Find the general solution $F = F(t,x)$ of the PDE-problem

$$\mathcal{L}^0 F = a_0 + a_1 F \tag{4}$$

with appropriate constants $a_0, a_1 \in \mathbb{R}$.

[2] Find the general solution $F = F(t,x)$ of the PDE-problems

$$\mathcal{L}^j F = b_0^j + b_1^j F \tag{5}$$

with appropriate constants $b_0^j, b_1^j \in \mathbb{R}$.

[3] Consider all common solutions $F = F(t,x)$ of steps [1] and [2]. Check whether there are invertible solutions F with respect to x. If there are no common, invertible solutions then the algorithm stops and one knows that there are no invertible transformations F to linear SDEs (6) which form common solutions of linear PDE-problems (4) and (5), otherwise one continues with step [4].

[4] Solve the linear SDE (or look up its solution in literature)

$$dU(t) \;=\; [a_0 + a_1 U(t)]dt + \sum_{j=1}^{m} [b_0^j + b_1^j U(t)] dW_j(t) \tag{6}$$

with $U(0) = F(0, X_0)$ and coefficients a_i and b_i^j from steps [1] and [2].

[5] Let F^{-1} be the inverse of invertible common solution with respect to x from step [3]. Define the final solution of the algorithm by

$$X(t) \;:=\; F^{-1}(t, U(t)) \tag{7}$$

for $0 \leq t \leq T$.

Note that the linear system (6) has always a strong solution (as stated by Theorem 2 in Section 3.3). The easiest choices are when all $b_0^j = 0$ or all $b_1^j = 0$. One is also tempted to seek for solutions with $a_0 = b_0^j = 0$ or $a_1 = b_1^j = 0$. These are the cases of pure homogeneous or non-state-dependent equations for U, respectively. However, there is no guarantee that one of this special choices works in a closed form (series solutions can be observed). If the algorithm stops with no found solution then one might also try to find common solutions to nonlinear PDE-problems — which is a much harder task to solve (however, not impossible in certain special cases).

2.2. *The algorithm for Stratonovich SDEs*

The linear PDE-based algorithm for Stratonovich based SDEs (1) is very similar to that of Itô SDEs. One only has to exchange the generator $\mathcal{L}^0$ by the 1st order partial differential operator

$$\bar{\mathcal{L}}^0 \;=\; \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t,x) \frac{\partial}{\partial x_i} \tag{8}$$

in step [1].

3. Examples

With a series of fairly simple examples, we shall demonstrate the systematic applicability of our algorithm.

3.1. *Linear equations: geometric Brownian motion*

Most commonly cited example in mathematical finance is that of geometric Brownian motion satisfying the Itô equation

$$dX(t) \;=\; aX(t)dt + \sigma X(t) dW(t) \tag{9}$$

with $\sigma \neq 0$. Let us go through the steps $[1] - [5]$ of the algorithm. The simplest choice is to transform to the Wiener process $dX(t) = dW(t)$ itself. For this purpose, in step $[1]$ we solve

$$\mathcal{L}^0 F(t,x) = \left(\frac{\partial}{\partial t} + ax\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) F(t,x) = 0$$

One finds that its solution is given by

$$F(t,x) = \frac{1}{\sigma}\left(\ln[\frac{x}{F_0}] - (a - \frac{\sigma^2}{2})t\right)$$

with real constant $F_0 \neq 0$, since

$$\mathcal{L}^0 F(t,x) = \frac{1}{\sigma}\left(-a + \frac{\sigma^2}{2} + ax\frac{1}{x} + \frac{1}{2}\sigma^2 x^2(-\frac{1}{x^2})\right) = 0$$

In step [2], one needs to solve

$$\mathcal{L}^1 F(t,x) = \sigma x \frac{\partial}{\partial x} F(t,x) = 1$$

One easily checks that $F(t,x)$ defined as above represents a common solution of steps [1] and [2]. Moreover, $F(t,x)$ is invertible with respect to x and its inverse is

$$F^{-1}(t,x) = \exp\left((a - \frac{\sigma^2}{2})t + \sigma x\right) F_0$$

where F_0 is any real constant. Thus, we can successfully proceed through step [3]. Step [4] brings up the trivial solution $X(t) = W(t)$ of the transformed equation $dX(t) = dW(t)$. Finally, the final answer is provided by step [5] and is given by

$$X(t) = F^{-1}(t, W(t)) = \exp\left((a - \frac{\sigma^2}{2})t + \sigma W(t)\right) X(0)$$

with the choice $F_0 = X(0)$ to match the initial data correctly.

3.2. Bernoulli-type and logistic equations

Consider stochastic Bernoulli equations

$$dX(t) \quad = \quad rX(t)(K - [X(t)]^{n-1})dt + \sigma X(t) dW(t) \tag{10}$$

which are interpreted in Itô sense. The most popular representative of these equations is the case with $n = 2$, also called the logistic growth model with growth rate $r > 0$ and environmental capacity $K > 0$. These equations have numerous applications in mathematical biology (e.g., to model the growth of populations). For simplicity, we suppose that $X(0) > 0$ (*i.e.*, we seek only for physically meaningful solutions and additionally avoid technical difficulties of mathematical stopping procedures). To carry out the procedure of the algorithm, we begin with step [1] solving

$$\mathcal{L}^0 F(x) = \left(rx(K - x^{n-1})\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) F(x) = a_0 + a_1 F(x)$$

Suppose that $n \neq 1$. By choosing the parameters

$$a_0 = -r(1-n), \quad a_1 = (1-n)(rK - n\frac{\sigma^2}{2})$$

one recognizes that the (deterministically known) Leibniz transform

$$F(x) = x^{1-n}$$

represents a solution of step [1]. Step [2] asks for the calculation of $\mathcal{L}^1 F = \sigma x \frac{\partial}{\partial x} F$. The Leibniz transform gives

$$\mathcal{L}^1 F(x) = \sigma(1-n)x^{1-n} = \sigma(1-n)F(x)$$

hence we may choose the constants $b_0^1 = 0$ and $b_1^1 = \sigma(1-n)$. Clearly, $F(x) = x^{1-n}$ is a common invertible solution of both step [1] and [2]. Thus, step [3] provides the solution $F(x) = x^{1-n}$ with its inverse

$$F^{-1}(x) = \sqrt[1-n]{x}$$

Step [4] takes into consideration the linear Itô equation

$$dU(t) = (1-n)[-r + (rK - n\frac{\sigma^2}{2})U(t)]dt + (1-n)\sigma U(t)dW(t)$$

which can be solved by Theorem 2 from the appendix. Its solution is

$$U(t) = \varphi(t)\left([X(0)]^{1-n} - r(1-n)\int_0^t \varphi^{-1}(s)ds\right)$$

where

$$\varphi(t) = \exp\left((1-n)(rK - \frac{\sigma^2}{2})t + \sigma(1-n)W(t)\right)$$

with $t \geq 0$. Finally, step [5] leads to the solution of (10)

$$X(t) = F^{-1}(U(t)) = \exp\left((rK - \frac{\sigma^2}{2})t + \sigma W(t)\right)\left([X(0)]^{1-n} + r(n-1)\int_0^t \varphi^{-1}(s)ds\right)^{1/(1-n)}$$

3.3. *A nonlinear example (pure Stratonovich diffusions)*

Pure Stratonovich diffusions are governed by Stratonovich SDEs

$$dX(t) = \sigma(X(t)) \circ dW(t) \tag{11}$$

with differentiable coefficients $\sigma(x) \geq 0$. For our analysis, we suggest to transfer this equation to its equivalent Itô SDE (see [2])

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dW(t) \tag{12}$$

Now, we apply our algorithm. For this purpose, it is interesting to observe that the transform

$$F(x) = \int^x \frac{dz}{\sigma(z)} \tag{13}$$

has vanishing operator image for $\mathcal{L}^0$. By chain rule, we find that

$$\mathcal{L}^0 F(x) = \frac{1}{2}\sigma(x)\sigma'(x)\frac{dF(x)}{dx} + \frac{1}{2}[\sigma(x)]^2\frac{d^2F(x)}{dx^2} = \frac{1}{2}\sigma'(x) - \frac{1}{2}\sigma'(x) = 0$$

Moreover, one encounters

$$\mathcal{L}^1 F(x) = \sigma(x) \frac{dF(x)}{dx} = \sigma(x) \frac{1}{\sigma(x)} = 1$$

Therefore, steps [1], [2] and [3] with constants $a_0 = a_1 = b_1^1 = 0$ and $b_0^1 = 1$ can be successfully conducted, and its common solution is of the form (13) with existing inverse F^{-1} (invertible since $\sigma \geq 0$). The trivial solution of step [4] is $U(t) = W(t)$ with $U(0) = F(X(0))$. Eventually, step [5] yields the solution

$$X(t) = F^{-1}(U(t))$$

As a fairly simple sub-example, one can treat

$$dX(t) = \frac{m^2}{4} dt + m\sqrt{X(t)} dW(t) = m\sqrt{X(t)} \circ dW(t)$$

with $X(0) \geq 0$, which is also known as Bessel-type diffusion with dimension parameter $m > 0$. Here,

$$F(x) = \frac{1}{m} \int^x \frac{dz}{\sqrt{z}} = \frac{2}{m}\sqrt{x}$$

Hence, its inverse is $F^{-1}(x) = \left(\frac{m}{2}x\right)^2$. Therefore, the algorithm gives the positive solution

$$X(t) = \left(\frac{m}{2} W(t) + \sqrt{X(0)}\right)^2$$

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References

1. Allen, E. *Modeling with Itô Stochastic Differential Equations*; Springer-Verlag: New York, NY, USA, 2008.
2. Arnold, L. *Stochastic Differential Equations*; John Wiley & Sons: New York, NY, USA, 1974.
3. Friedman, A. *Stochastic Differential Equations and Applications 1 & 2*; Academic Press: New York, NY, USA, 1975/1976.
4. Gard, T.C. *Introduction to Stochastic Differential Equations*; Marcel Dekker: Basel, Swizherland, 1988.
5. Gikhman, I.I.; Skorochod, A.V. *Stochastische Differentialgleichungen*; Akademie-Verlag: Berlin, Germany, 1971.
6. Itô, K. Stochastic integral. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 519–524.
7. Itô, K. On a formula concerning stochastic differential equations. *Nagoya Math. J.* **1951**, *3*, 55–65.
8. Karatzas, I.; Shreve, S. *Brownian Motion and Stochastic Calculus*; Springer-Verlag: New York, NY, USA, 1988.
9. Krylov, N.V. *Introduction to the Theory of Diffusion Processes*; AMS: Providence, RI, USA, 1995.
10. Øksendal, B. *Stochastic Differential Equations*; Springer-Verlag: New York, NY, USA, 1985.

11. Protter, P. *Stochastic Integration and Differential Equations*; Springer-Verlag: New York, NY, USA, 1990.
12. Revuz, D.; Yor, M. *Continuous Martingales and Brownian Motion*, 2nd ed.; Springer-Verlag: New York, NY, USA, 1994.
13. Schurz, H. Stochastic α-calculus, a fundamental theorem and Burkholder-Davis-Gundy-type estimates. *Dynam. Syst. Applic.* **2006**, *15*, 241–268.
14. Schurz, H. New stochastic integrals, oscillation theorems and energy identities. *Commun. Appl. Anal.* **2009**, *13*, 181–194.
15. Shiryaev, A.N. *Probability*, 2nd ed.; Springer-Verlag: Berlin, Germany, 1996.
16. Stroock, D.W.; Varadhan, S.R.S. *Multidimensional Diffusion Processes*; Springer-Verlag: New York, NY, USA, 1982.

Appendix: Itô Formula, Linear Solutions of SDEs and a Remark on Nonlinear Versions

The above-mentioned algorithm is based on the fundamental theorem of stochastic calculus, also known as Itô Lemma or Itô Formula. Its proof can be found in e.g. [2,8] and many other basic texts, and it traces back to its original [7].

Theorem 1. *Assume that $F \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and X satisfies the Itô SDE (1). Then $Y = F(t, X(t))$ satisfies (a.s.) the Itô SDE*

$$dY \quad = \quad \mathcal{L}^0 F(t, X(t))dt + \sum_{j=1}^{m} \mathcal{L}^j F(t, X(t))dW_j(t) \tag{14}$$

on $0 \leq t \leq T$, where the differential operators $\mathcal{L}^0$ and $\mathcal{L}^j$ are defined as in (2) and (3), respectively.

As seen in previous sections, the presented algorithm is based on transformations to linear systems of SDEs. The solutions of linear equations of SDEs

$$dX(t) \quad = \quad (a_0 + a_1 X(t))dt + \sum_{j=1}^{m} (b_0^j + b_1^j X(t))dW_j(t) \tag{15}$$

are well-known (see [2]).

Theorem 2. *The unique strong solution $X \in \mathbb{R}^d$ satisfying the linear Itô SDE (15) with real constants a_0, a_1, b_0^j and $b_1^j \in \mathbb{R}^1$ is given by*

$$X(t) \quad = \quad \varphi(t) \left(X(0) + (a_0 - \sum_{j=1}^{m} b_0^j b_1^j) \int_0^t \varphi^{-1}(s)ds + \sum_{j=1}^{m} b_0^j \int_0^t \varphi^{-1}(s)dW_j(s) \right) \tag{16}$$

with fundamental solution

$$\varphi(t) = \exp \left([a_1 - \frac{1}{2} \sum_{j=1}^{m} (b_1^j)^2]t + \sum_{j=1}^{m} b_1^j W_j(t) \right), \quad t \geq 0$$

Remark 1. *An extension of the algorithm to a nonlinear-PDE-based version is conceivable and subject to future research. For example, Equation (4) in step [1] and (5) in step [2] can be substituted by nonlinear systems of PDEs*

$$\mathcal{L}^0 F = G_0(t, x, F), \quad \mathcal{L}^j F = G_j(t, x, F) \tag{17}$$

with appropriate nonlinear functions G_0 and G_j, respectively. However, to find explicit expressions for common solutions F of $m+1$-dimensional systems of nonlinear PDEs such as (17) (i.e. G_0 or G_j nonlinear in F, $j = 1, 2, ..., m$) seems to be an extremely difficult field of research.