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Article

Recognizing the Repeatable Configurations of Time-Reversible Generalized Langton's Ant is PSPACE-Hard

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Abstract: Chris Langton proposed a model of an artificial life that he named "ant": an agent- called ant- that is over a square of a grid moves by turning to the left (or right) accordingly to black (or white) color of the square where it is heading, and the square then reverses its color. Bunimovich and Troubetzkoy proved that an ant's trajectory is always unbounded, or equivalently, there exists no repeatable configuration of the ant's system. On the other hand, by introducing a new type of color where the ant goes straight ahead and the color never changes, repeatable configurations are known to exist. In this paper, we prove that determining whether a given finite configuration of generalized Langton's ant is repeatable or not is PSPACE-hard. We also prove the PSPACE-hardness of the ant's problem on a hexagonal grid.

Keywords: cellular automata; computational complexity; Langton's ant; Lorentz lattice gas; PSPACE-hard

1. Introduction

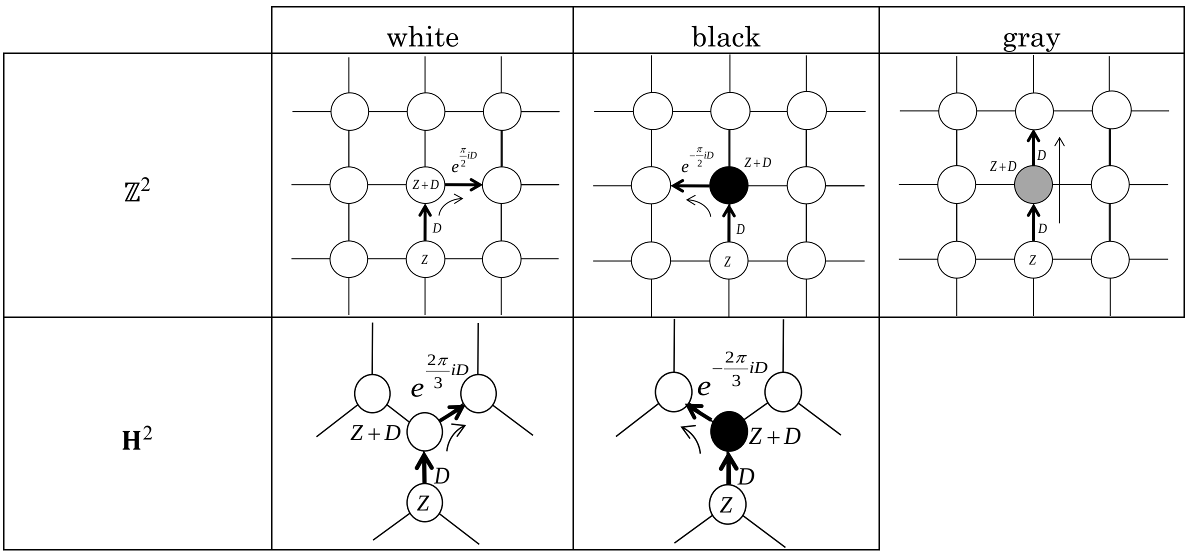
1.1. Generalized Langton's Ant

The virtual ant defined by Chris Langton [1–3] is the following cellular automaton. The "ant", represented by a pair (z, D) of a lattice point $z \in \mathbb{Z}^2 := \mathbb{Z} + i\mathbb{Z}$ and a direction $D \in \left\{e^0, e^{\frac{\pi}{2}i}, e^{\pi i}, e^{\frac{3\pi}{2}i}\right\}$, moves around on a two-dimensional square lattice $\mathbb{Z}^2$, where each lattice point, referred to as a "cell", is colored by white or black (later we will also introduce gray cells). Initially, the ant is sitting on a given cell with a given direction, say $z = 0$ and $D = 1$. It proceeds to travel from cell to cell according to the following rule (see Figure 1): the ant (z, D) moves to $\left(z + D, e^{\frac{\pi}{2}i}D\right)$ when $z + D$ is a white cell, and to $\left(z + D, e^{-\frac{\pi}{2}i}D\right)$ when $z + D$ is a black cell; the cell $z + D$ then reverses its color. In other words, the ant moves in the direction it is heading; when it lands on a white (or black) cell it rotates its direction 90° to the right (left); after that, the color of the cell changes to black (or white).

Langton's ant has been investigated independently as one model of Lorentz Lattice Gas Cellular Automata (LLGCA). Langton's ant corresponds to the Flipping Rotator (FR) model on $\mathbb{Z}^2$ [4]. In more general terms, the FR model was investigated on a triangular lattice $\mathbf{T}^2 = \mathbb{Z} + e^{\frac{\pi}{3}i}\mathbb{Z}$ [5] and on a hexagonal lattice $\mathbf{H}^2 = \mathbf{T}^2 - \sqrt{3}e^{\frac{\pi}{6}i}\mathbf{T}^2$ [6], too. On $\mathbf{H}^2$ the ant (z, D), where $z \in \mathbf{H}^2$ and $D \in \left\{ e^{\frac{2k\pi}{3}i} : k = 0, 1, 2\right\}$, moves to $\left(z + D, e^{\frac{2\pi}{3}i}D\right)$ and $\left(z + D, e^{-\frac{2\pi}{3}i}D\right)$ when $z + D$ is white and black, respectively (see Figure 1), and the cell $z + D$ then reverses its color. Another generalization of Langton's ant on $\mathbb{Z}^2$ is to introduce a third type of cell, called "gray" cell [7]. The ant (z, D) moves to $(z + D, D)$ if $z + D$ is a grey cell, and the cell $z + D$ never changes its color but remains gray forever (see Figure 1). In LLGCA models, gray cells were introduced naturally as empty lattice points. We remark that the $\mathbf{H}^2$ topology does not allow having such a gray cell.

We denote by R, L and S the ant's valid moves corresponding to the Right-turn, Left-turn and Straight-ahead respectively under these transaction rules; e.g., by R, L and S, the ant (z, D) on $\mathbb{Z}^2$ moves to $\left(z + D, e^{\frac{\pi}{2}i}D\right)$, $\left(z + D, e^{-\frac{\pi}{2}i}D\right)$ and $(z + D, D)$, respectively.

Figure 1. Transaction rules on each topology and each color of the cell that the ant is heading to.

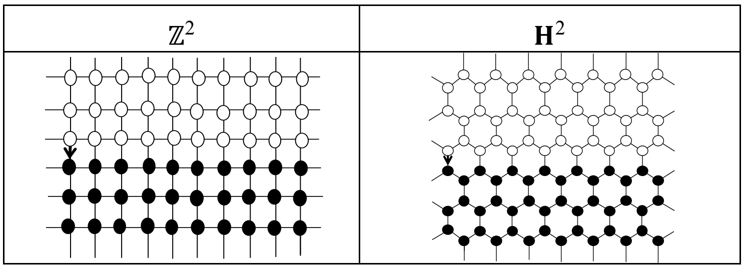


1.2. Recognizing the Repeatable Configurations of GLA

These transaction rules assure that Generalized Langton's Ant (GLA) is a time-reversible cellular automaton: the current configuration of GLA, consisting of a coloring of the cells and an ant's starting cell and direction, determines the past configurations as well as the future ones. As a consequence, the configurations of GLA are divided into the following two kinds: an ant's trajectory starting from a configuration of one kind is unbounded, never repeating the same configuration again; an ant's trajectory starting from a configuration of the other kind is bounded, repeating a finite series of configurations an infinitive number of times.

A "finite" configuration of GLA is defined by a finite coloring of the cells and an ant's starting cell and direction. Here, a coloring is finite if it has only a finite number of non-background-color cells. In this paper, we use the all white, all black, and half-and-half coloring as the background, where the half-and-half coloring gives white (black) color to the cells on the upper-half (lower-half) plane (see Figure 2Figure 2). We define size of a configuration by the minimum size of a closed square which contains the ant's starting cell and the all non-background-color cells. In this paper, we study the computational complexity of determining whether a given finite configuration of GLA is repeatable or not.

Figure 2. The half-and-half coloring.

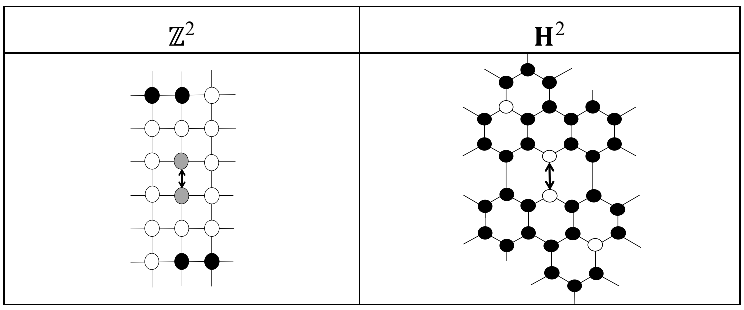


1.3. Previous Results

Bunimovch and Troubetskoy [4] proved the following: when there is no gray cell, an ant's trajectory on $\mathbb{Z}^2$ is always unbounded, or equivalently, there exists no repeatable configuration of the ant's system. As a matter of fact, the set of repeatable configurations of GLA on $\mathbb{Z}^2$ with no gray cell is empty, hence its recognition problem is trivial. On the other hand, repeatable configurations exist of GLA on $\mathbb{Z}^2$ with some gray cells (see Figure 3, see also [8]). The $\mathbf{H}^2$ model is also known to have repeatable configurations (see Figure 3, see also [6]).

The long-run behavior of Langton's ant on $\mathbb{Z}^2$ has been studied using both theories and experiments for more than two decades, yet it is still highly unpredictable. As a result, indicating hardness of the prediction, Gajardo, Moreira and Goles [9] proved that the following problem on $\mathbb{Z}^2$ with no gray cell is PTIME-hard: "Does the ant ever visit this given cell?".

Figure 3. Repeatable configurations of GLA.



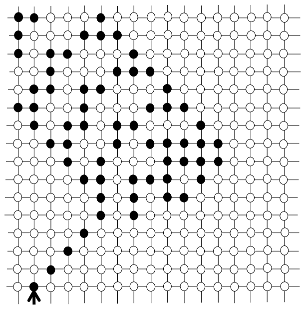
1.4. New Results

In this paper, we prove the following theorems:

Theorem 1. Recognizing the repeatable configurations of GLA on $\mathbb{Z}^2$ with gray cells is PSPACE-hard.
Theorem 2. Recognizing the repeatable configurations of GLA on $\mathbf{H}^2$ is PSPACE-hard.

To prove these theorems, we should have unbounded trajectories of the ant on each of the topologies. For the half-and-half background, Figure 2 shows such unbounded trajectories where the ant, starting from the arrow, walks from left to right by repeating the LLRR (LLLRRR) moves forever on $\mathbb{Z}^2$ ($\mathbf{H}^2$). On the other hand, for the monochromatic background, we do have the famous diagonal highway on $\mathbb{Z}^2$ (see Figure 4), but no provable unbounded trajectory is known on $\mathbf{H}^2$ [6]. For this reason, we can prove Theorem 1 for both of the monochromatic and half-and-half backgrounds, but Theorem 2 for only the half-and-half background.

Figure 4. The ant starting from the arrow proceeds as RLRLRLRLRL, and then starts repeating 104 steps forever, forming the famous diagonal highway going in a southeast direction.



We will prove Theorem 1 for "everywhere sparse" gray cells. For a given function $f\colon \mathbb{N} \to \mathbb{N}$, we say that a configuration is colored by f-sparse gray cells if the configuration size is n and its coloring has no more than one gray cell within any closed square of size no more than $\frac{n}{\sqrt{f(n)}}$. For example, a configuration colored by k-sparse gray cells, for any constant $k \in \mathbb{N}$, contains no more than k gray cells, and a configuration colored by n^2-sparse gray cells could have an arbitrary number of gray cells. We prove the following theorem:

Theorem 3. For any $\varepsilon > 0$ recognizing the repeatable configurations of GLA on $\mathbb{Z}^2$ colored by n^{ε}-sparse gray cells is PSPACE-complete.

1.5. Reduction

To prove Theorems 1–3, it is enough to reduce a known PSPACE-hard problem to the ant's problems on square and hexagonal grids. In this paper, we will reduce QBF (Quantified Boolean Formula) evaluation problem to the ant's problems. An instance of QBF is given by a closed CNF (Conjunctive Normal Form) formula, which is written as $Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \phi(x_1,\ldots,x_n)$ by an open CNF formula $\phi(x_1,\ldots,x_n)$ and arbitrary Boolean quantifiers $Q_i \in \{\exists,\forall\}$. Then, QBF evaluation problem asks the truth value of a given closed CNF formula. QBF evaluation problem is a well known PSPACE-complete problem [10].

1.6. Preparation

A (ant's walking) course is a sequence of ant's consecutive valid moves; it also represents a sequence of the induced coloring. When a coloring S of the ant's system turns to a coloring T by an ant's walking course w, we write as $S \xrightarrow{w} T$. When the ant, at a place I, moves to another place O by w, we write it as $I \xrightarrow{w} O$. The reverse of the ant (z, D) is $(z + D, -D)$. By time reversibility, we can define the reverse w^{-1} of w by the following walking course: reverse the order of the coloring in w, and reverse the ants therein. We write $S \underset{w^{-1}}{\overset{w}{\rightleftarrows}} T$, meaning that S turns to T by w, and T turns to S by w^{-1}.

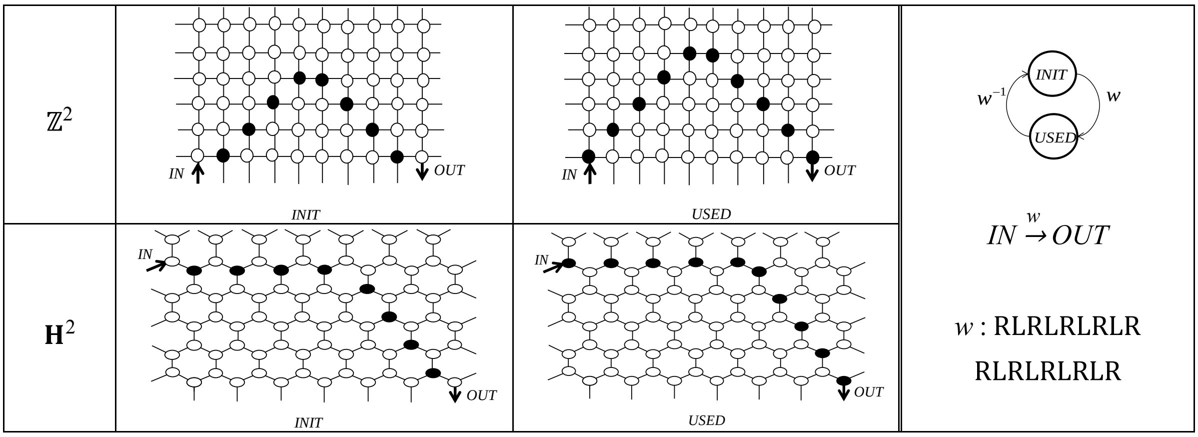
2. Gadgets

A gadget is a collection of GLA's coloring written on the all white background with an associated transition diagram and several input and output marked arrows. A polynomial number of gadgets are seamlessly connected to form an entire coloring of GLA, where some gadgets may be used after rotation or reflection. Note that the colors of reflected gadgets should be switched.

2.1. Path

We connect the rotations and reflections of PATH gadgets (see Figure 5) to form a long path along which the ant is guided. Its coloring is initially *INIT*, which turns to *USED* by a course $IN \xrightarrow{w} OUT$.

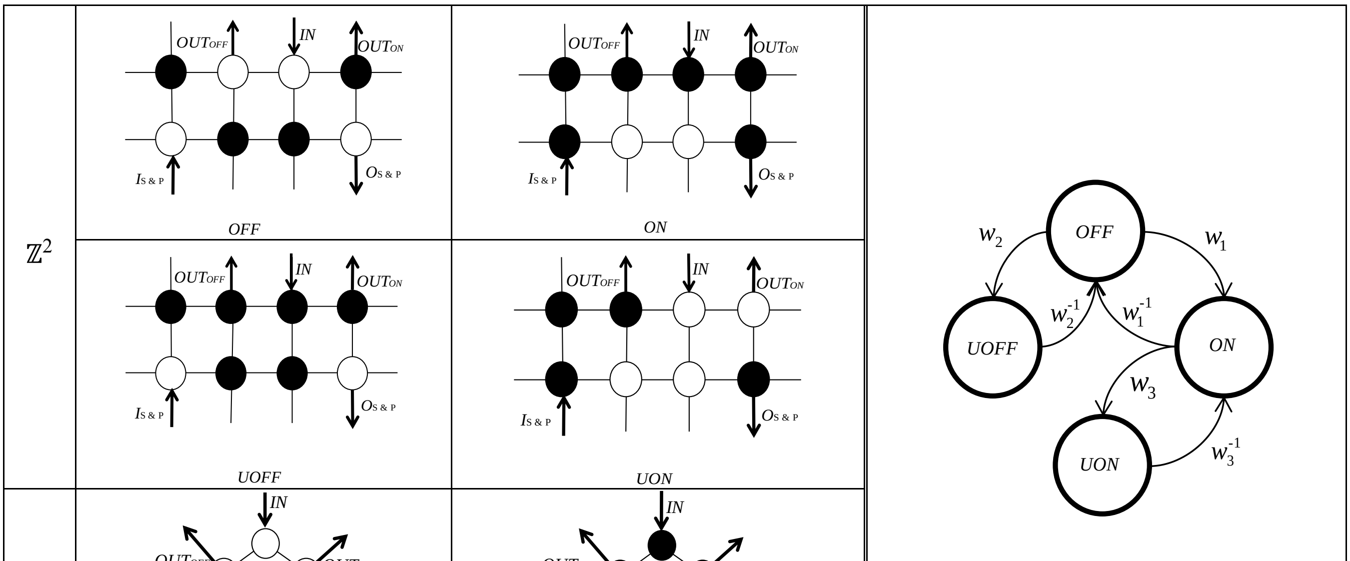
Figure 5. PATH.



2.2. Switch & Pass

A Switch & Pass (S&P) gadget (see Figure 6) can memorize 1 bit information by its coloring state. The coloring is initially *OFF*, which turns to *ON* by a walking course $I_{\mathrm{S\&P}} \xrightarrow{w_1} \mathrm{O}_{\mathrm{S\&P}}$; in other words, the ant is "Switching" the coloring state and "Passing" through it. When the coloring is *OFF*, the ant entering at *IN* walks along w_2 and exits at O_{OFF}, changing the coloring to *UOFF*. When the coloring is *ON*, the ant entering at *IN* walks along w_3 and exits at O_{ON}, changing the coloring to *UON*.

Figure 6. Switch & Pass.



	w_1	w_2	w_3
I/O	$I_{\text{S\&P}} \to O_{\text{S\&P}}$	$IN \to OUT_{OFF}$	$IN \to OUT_{ON}$
$\mathbb{Z}^2$	RLR RLR	RR	LL
$\mathbf{H}^2$	LRRRL	RR	LL

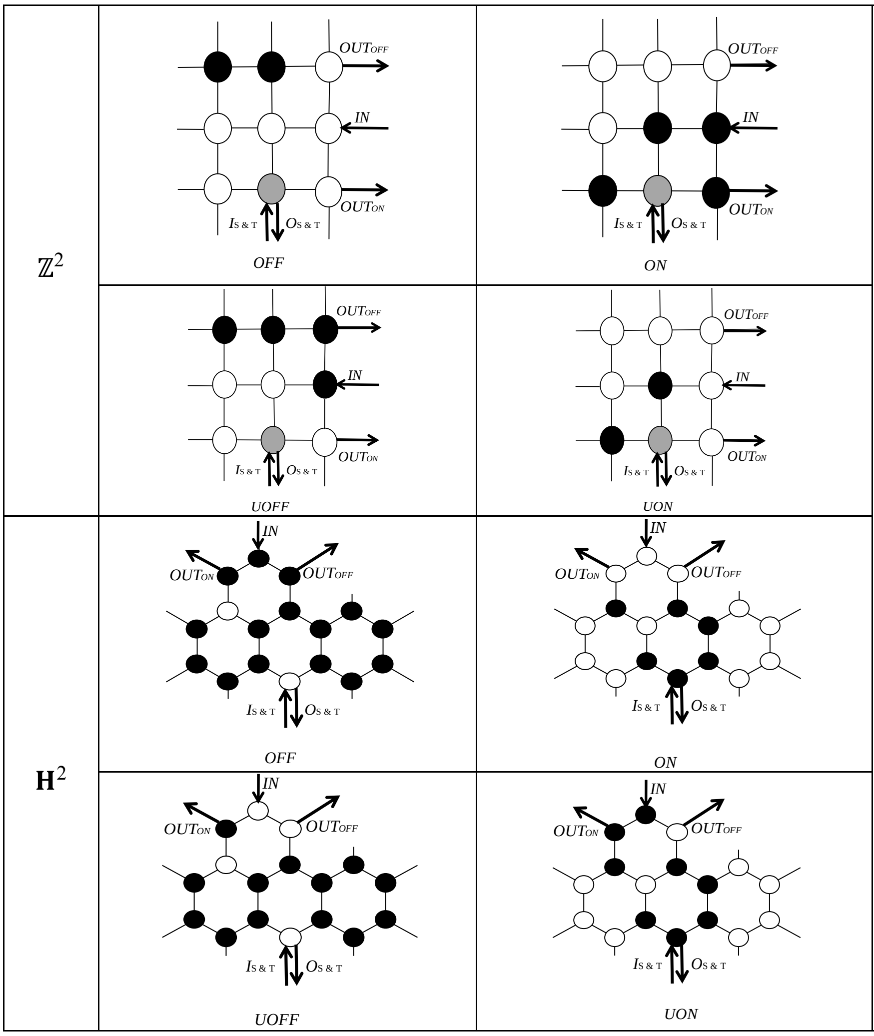
2.3. Switch & Turn

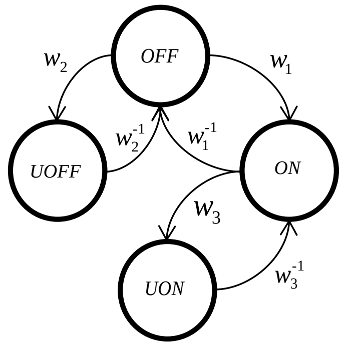
A Switch & Turn (S&T) gadget (see Figure 7) can memorize 1 bit information by its coloring state, too. The coloring is initially *OFF*, which turns to *ON* by an ant's walking course $I_{\text{S\&T}} \overset{w_1}{\to} O_{\text{S\&T}}$; in other words, since $O_{\text{S\&T}}$ is the reverse of $I_{\text{S\&T}}$, the ant is "Switching" the coloring state and "Turning" around. When the coloring is *OFF*, the ant entering at *IN* walks along w_2 and exits at OUT_{OFF}. When the coloring is *ON*, the ant entering at *IN* walks along w_3 and exits at OUT_{ON}.

2.4. Conjunction

The CONJunction (CONJ) gadget (see Figure 8, see also [9]) has two entrances and one exit. The coloring is initially *INIT*. The ant entering at IN_j walks along w_j and exits at *OUT*, for each $j = 1, 2$.

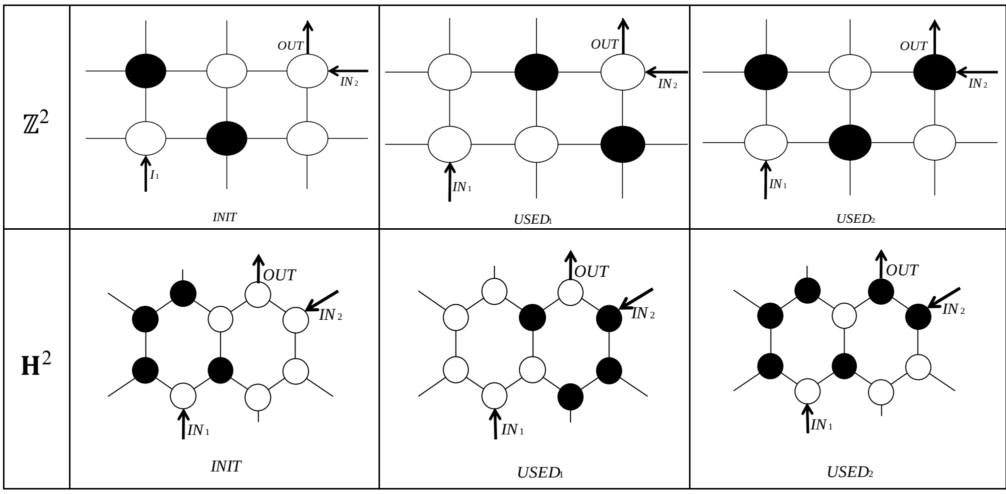
Figure 7. Switch & Turn.

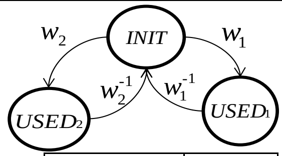




	w_1	w_2	w_3
I/O	$I_{S\&T} \rightarrow O_{S\&T}$	$IN \rightarrow OUT_{OFF}$	$IN \rightarrow OUT_{ON}$
$\mathbb{Z}^2$	ARRRA RRLLL LRA	RR	LL
$\mathbf{H}^2$	RLLLL LLRLL LLRRL LLRLL LR	LL	RR

Figure 8. CONJunction (CONJ).





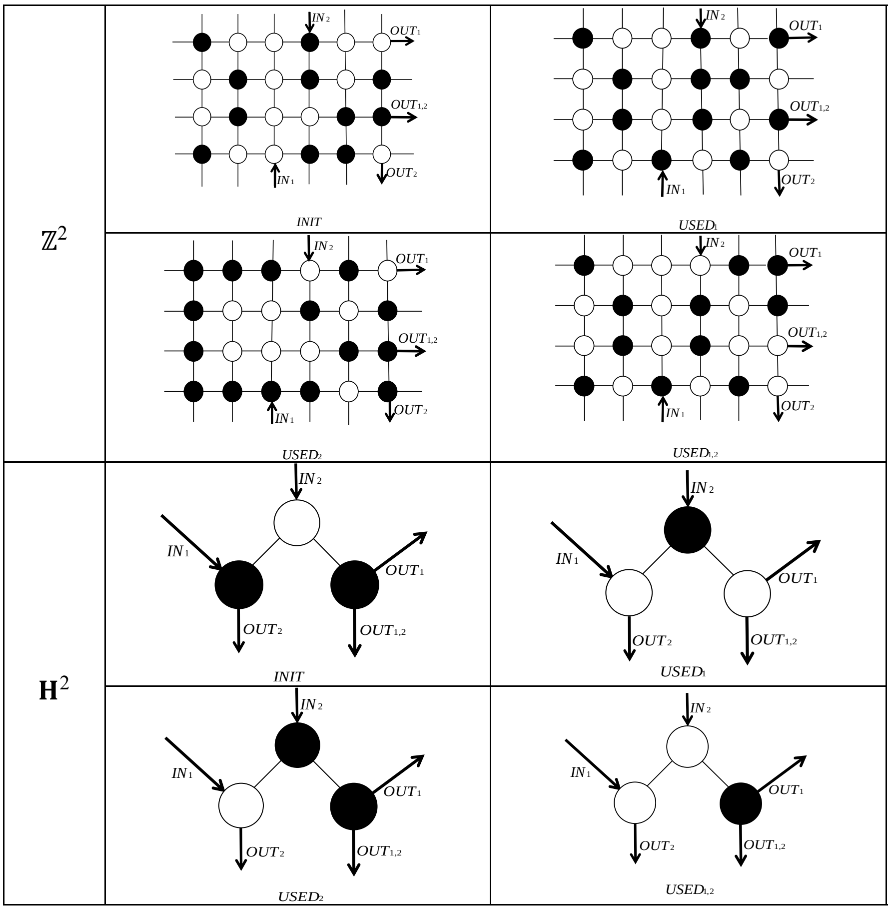
	w_1	w_2
I/O	$IN_1 \rightarrow OUT$	$IN_2 \rightarrow OUT$
$\mathbb{Z}^2$	RLRRRR LLLLRL	R
$\mathbf{H}^2$	RLRRRR RRLLLL LLRL	RR

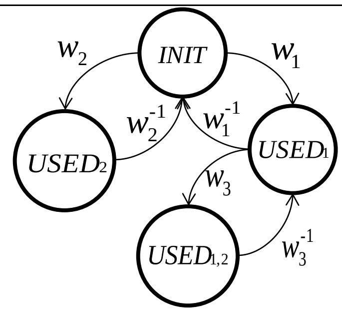
2.5. Pseudo-Crossing

A Pseudo-Crossing (PC) gadget (see Figure 9, see also [9]) has two entrances IN_1 and IN_2 and two exits OUT_1 and OUT_2 such that the ant entering at IN_j walks along w_j and exits at OUT_j for each

$j = 1, 2$. Since IN_1, IN_2, OUT_1, OUT_2 are placed clockwise in this order in 2D plane, these two walking courses w_1 and w_2 should be mutually crossing. Beginning from the initial coloring *INIT*, the ant can take mutually intersecting courses, first $IN_1 \xrightarrow{w_1} OUT_1$ and secondly $IN_1 \xrightarrow{w_3} OUT_{1,2}$ in order, changing the coloring to $INIT \xrightarrow{w_1} USED_1 \xrightarrow{w_3} USED_{1,2}$.

Figure 9. PC.



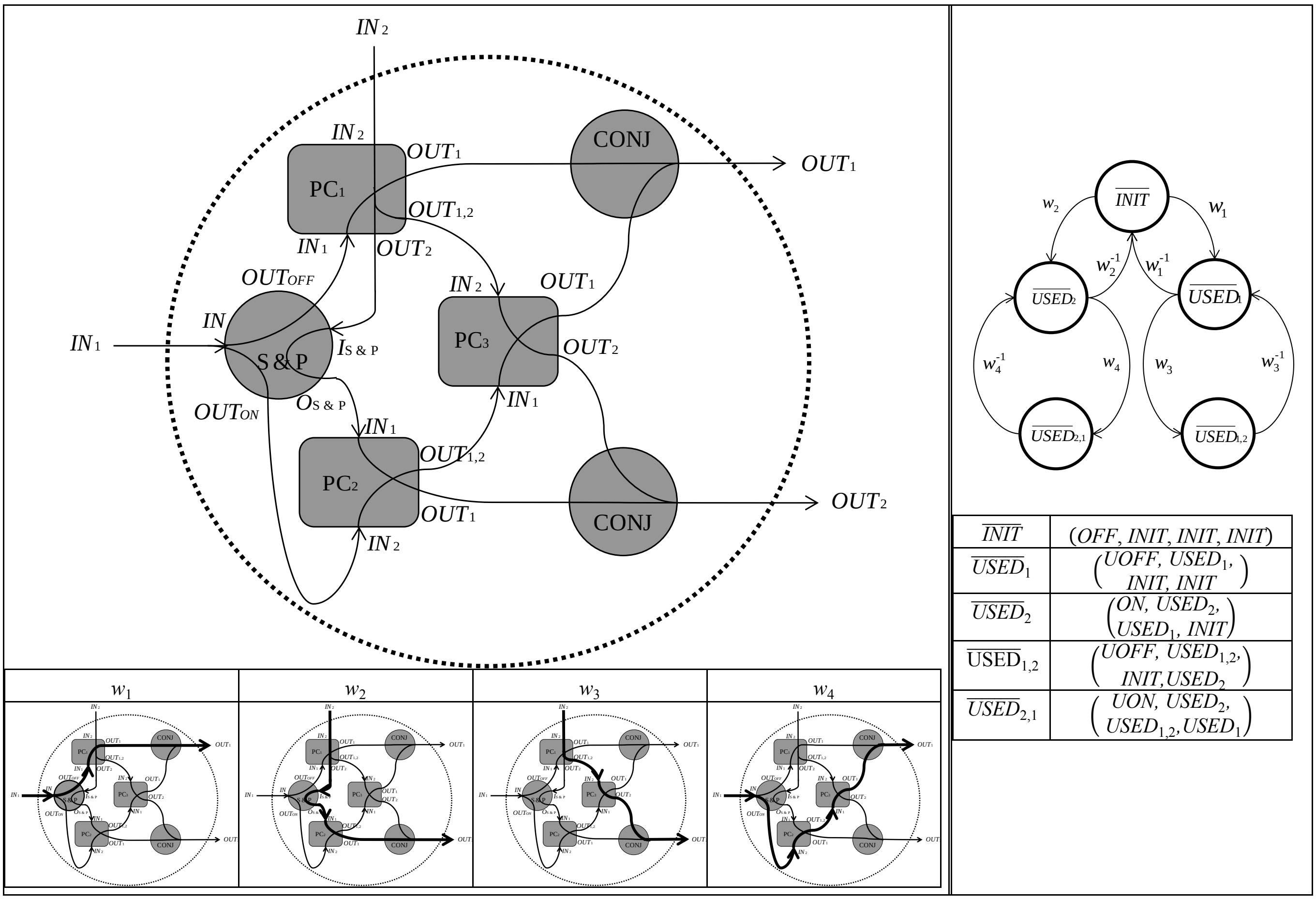


	w_1	w_2	w_3
I/O	IN_1 →OUT$_1$	IN_2 →OUT_2	IN_2 →$OUT_{1,2}$
$\mathbb{Z}^2$	RLRL RLR	LRRL RRRR LLLL RLRR LRRL RRLR LRLR	LRL RL
$\mathbf{H}^2$	LRL	R	LR

2.6. Crossing

A CROSSing (CROSS) gadget (see Figure 10) is placed at each crossing point of two intersecting paths on 2D plane. It is built by one S&P gadget and three PC gadgets, named PC_1, PC_2 and PC_3. The coloring of a CROSS gadget can be represented by the coloring of all gadgets composing it, that are S&P, PC_1, PC_2, PC_3, two CONJ gadgets and many PATH gadgets connecting them. We indicate the coloring of a CROSS gadget only by those of (S&P, PC_1, PC_2, PC_3). The coloring of CONJ (PATH) gadgets and PATH gadgets are initially *INIT*, that turn to USED$_j$ (*USED*) when the ant has passed through them. The ant can take mutually intersecting courses by first $IN_1 \xrightarrow{w_1} OUT_1$ and secondly $IN_2 \xrightarrow{w_3} OUT_2$, as well as by first $IN_2 \xrightarrow{w_2} OUT_2$ and secondly $IN_1 \xrightarrow{w_4} OUT_1$. In Figure 10, each of these paths are depicted by thick lines, where those gadgets that passed through are shown by the thick lines that have turned to used coloring, while the other gadgets remain in their initial coloring.

Figure 10. CROSS.



$\overline{INIT}$	$(OFF, INIT, INIT, INIT)$
$\overline{USED}_1$	$(UOFF, USED_1, INIT, INIT)$
$\overline{USED}_2$	$(ON, USED_2, USED_1, INIT)$
$\overline{\mathrm{USED}}_{1,2}$	$(UOFF, USED_{1,2}, INIT, USED_2)$
$\overline{USED}_{2,1}$	$(UON, USED_2, USED_{1,2}, USED_1)$

3. CNF Formulae Evaluation

In this section, we construct an EVAL_ϕ gadget for evaluating the truth value of a CNF formula ϕ by inputting a given truth-value assignment a. It has one entrance IN and two exits OUT_{FALSE} and OUT_{TURE}. The ant entering at IN is routed to exit at OUT_{FALSE} (OUT_{TRUE}) if $\phi(a)$ is FALSE (TRUE).

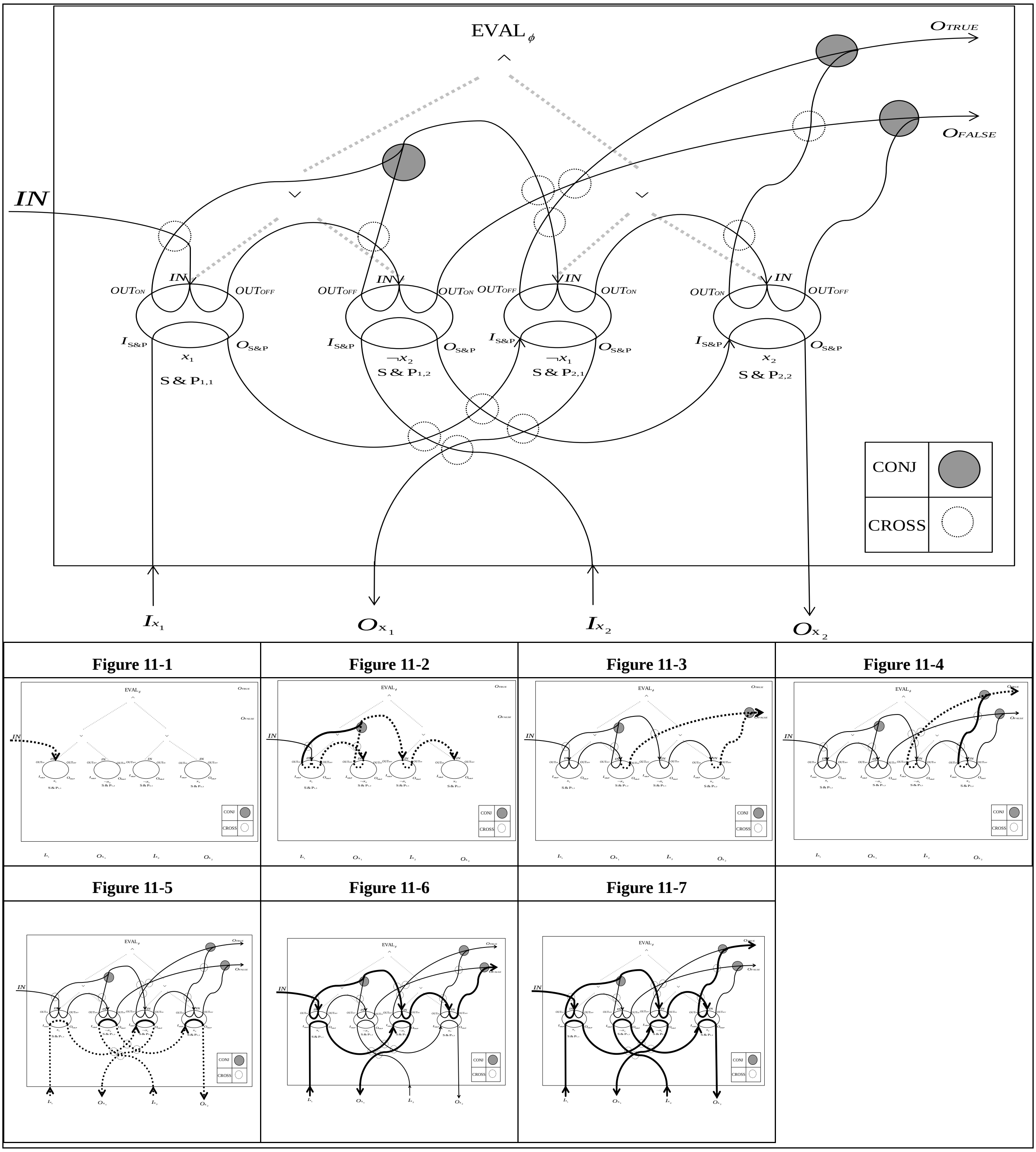
Let $\mathrm{POS} = \{x_1,\ldots,x_n\}$ and $\mathrm{NEG} = \{-x_1,\ldots, -x_n\}$. Let $y_{i,j} \in \mathrm{POS} \cup \mathrm{NEG}$ be the jth literal in the ith term of $\phi(x_1,\ldots,x_n)$, where $1 \le i \le m$ and $1 \le j \le k_i$. These literals $y_{i,j}$ are implemented in EVAL_ϕ by S&P gadgets, named $\mathrm{S\&P}_{i,j}$. Then, draw PATHs between $\mathrm{S\&P}_{i,j}$ in the following manner to define an evaluator EVAL_ϕ of ϕ: $IN \to IN$ of $\mathrm{S\&P}_{1,1}$ (see Figure 11-1). When $y_{i,j} \in \mathrm{POS}$ (NEG), OUT_{OFF} (OUT_{ON}) of $\mathrm{S\&P}_{i,j} \to IN$ of $\mathrm{S\&P}_{i,j+1}$ (see Figure 11-2). When $y_{i,k_i} \in \mathrm{POS}$(NEG), OUT_{OFF} (OUT_{ON}) of $\mathrm{S\&P}_{i,k_i} \to OUT_{\mathrm{FALSE}}$ (see Figure 11-3). When $y_{m,j} \in \mathrm{POS}$ (NEG), OUT_{ON} (OUT_{OFF}) of $\mathrm{S\&P}_{m,j} \to OUT_{\mathrm{TRUE}}$ (see Figure 11-4).

For each $x_k \in \mathrm{POS}$, EVAL_ϕ has an entrance I_{x_k} and an exit O_{x_k} such that the ant entering at I_{x_i} switches the coloring states of all S&P gadgets corresponding to the literals $y_{i,j} \in \{x_k, -x_k\}$ from OFF to ON, and exits at O_{x_i}. Figure 11-5 is drawing such PATHs, where the ports $I_{x_1}, O_{x_1}, I_{x_2}, O_{x_2},\ldots, I_{x_n}, O_{x_n}$ should be placed in this order for further development in the next section.

Suppose that the coloring of the composing gadgets of EVAL_ϕ are all initialized. For every $a \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^n$, we denote by $INIT_a$ the coloring of EVAL_ϕ after taking the walking courses

$I_{x_k} \to O_{x_k}$ for all x_k such that a_{x_k} = TRUE in the descending order of k. Consequently, $INIT_a$ sets the coloring of every S&P$_{i,j}$ with $y_{i,j} \in \{x_k, \neg x_k\}$ $OFF(ON)$ if a_k = FALSE (TRUE).

Figure 11. EVAL$_\phi$.



Lemma 1. If the coloring of $\mathrm{EVAL}_{\boldsymbol{\phi}}$ is $\mathrm{INIT}_{\mathrm{a}}$ and $\boldsymbol{\phi}(\mathrm{a}) = \mathrm{FALSE}$ (TRUE), then the ant entering at *IN* exits at OUT_{FALSE} (OUT_{TRUE}).

Proof. First, suppose that $\phi(a) = \mathrm{FALSE}$. We can assume that the ℓth term of ϕ is FALSE and the ith term of ϕ is TRUE for every $1 \le i < l$. So, we can assume that for every $1 \le j \le k_\ell$, if $y_{\ell,j} \in \mathrm{POS}$ (NEG) then $\mathrm{S\&P}_{\ell,j}$ is *OFF* (*ON*), and for every $1 \le i < l$ there exits $1 \le j_i \le k_i$ such that if $y_{i,j_i} \in \mathrm{POS}$ (NEG) then $\mathrm{S\&P}_{i,j_i}$ is *ON*(*OFF*); in addition, for every $1 \le j < j_i$, if $y_{i,j} \in \mathrm{POS}$ (NEG) then $\mathrm{S\&P}_{i,j}$ is *OFF* (*ON*). Given these colorings of the S&Ps, the ant walks from *IN* to OUT_{FALSE} in the following way: $IN \to IN$ of $\mathrm{S\&P}_{1,1}$; *IN* of $\mathrm{S\&P}_{i,j} \to OUT_{OFF}$ (OUT_{ON}) of $\mathrm{S\&P}_{i,j} \to IN$ of $\mathrm{S\&P}_{i,j+1}$; *IN* of $\mathrm{S\&P}_{i,j_i} \to OUT_{ON}$ (OUT_{OFF}) of $\mathrm{S\&P}_{i,j_i} \to IN$ of $\mathrm{S\&P}_{i+1,1}$; *IN* of $\mathrm{S\&P}_{\ell,j} \to OUT_{OFF}$ (OUT_{ON}) of $\mathrm{S\&P}_{\ell,j} \to IN$ of $\mathrm{S\&P}_{\ell,j+1}$; *IN* of $\mathrm{S\&P}_{\ell,k_\ell} \to OUT_{OFF}$ (OUT_{ON}) of $\mathrm{S\&P}_{\ell,k_\ell} \to OUT_{OFF}$. Figure 11-6 illustrates this walking course by a thick line when $\phi = (x_1 \vee -x_2) \wedge (-x_1 \vee x_2)$ and $(a_1, a_2) = (\mathrm{TRUE}, \mathrm{FALSE})$, where the gadgets have passed through, shown by the thick line, have turned to used coloring states, while the other gadgets still remain in their initial coloring.

Secondly, suppose that $\phi(a) = \mathrm{TRUE}$. Then, all terms of ϕ are TRUE. So, we can assume that for every $1 \le i \le m$ there exits $1 \le j_i \le k_i$ such that if $y_{i,j_i} \in \mathrm{POS}$ (NEG) then $\mathrm{S\&P}_{i,j_i}$ is *ON*(*OFF*); in addition, for every $1 \le j < j_i$, if $y_{i,j} \in \mathrm{POS}$ (NEG) then $\mathrm{S\&P}_{i,j}$ is *OFF* (*ON*). Given these colorings of the S&Ps, the ant walks from *IN* to OUT_{TRUE} in the following way: $IN \to IN$ of $\mathrm{S\&P}_{1,1}$; *IN* of $\mathrm{S\&P}_{i,j} \to OUT_{OFF}$ (OUT_{ON}) of $\mathrm{S\&P}_{i,j} \to IN$ of $\mathrm{S\&P}_{i,j+1}$; *IN* of $\mathrm{S\&P}_{i,j_i} \to OUT_{ON}$ (OUT_{OFF}) of $\mathrm{S\&P}_{i,j_i} \to IN$ of $\mathrm{S\&P}_{i+1,1}$; *IN* of $\mathrm{S\&P}_{m,j} \to OUT_{OFF}$ (OUT_{ON}) of $\mathrm{S\&P}_{m,j} \to IN$ of $\mathrm{S\&P}_{m,j+1}$; *IN* of $\mathrm{S\&P}_{m,j_m} \to OUT_{ON}$ (OUT_{OFF}) of $\mathrm{S\&P}_{m,j_m} \to OUT_{ON}$. Figure 11-7 illustrate this walking course when $\phi = (x_1 \vee -x_2) \wedge (-x_1 \vee x_2)$ and $(a_1, a_2) = (\mathrm{TRUE}, \mathrm{TRUE})$.

Let $USED_a$ be the coloring of an EVAL_ϕ gadget derived from $INIT_a$ by the ant's walking course from the entrance to an exit of EVAL_ϕ.

4. Boolean Quantifiers Evaluation

Let $\phi(x_1,\ldots,x_n)$ be an arbitrary CNF formula of the variables $x_1,\ldots,x_n$. For arbitrary taken quantifiers $\mathrm{Q}_i \in \{\exists, \forall\}$, let $\phi_i(x_1,\ldots,x_i) := \mathrm{Q}_{i+1}x_{i+1}\mathrm{Q}_{i+2}x_{i+2}\ldots\mathrm{Q}_n x_n \phi(x_1,\ldots,x_n)$. In particular, $\phi_n(x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n)$. Let $\mathrm{EVAL}_n \coloneqq \mathrm{EVAL}_\phi$ built in the previous section. We have already defined $INIT_{n,a} := INIT_a$ as a coloring of EVAL_n for every $a \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^n$. In the following, for each $i = n, n-1,\ldots,0$, we define inductively an EVAL_i gadget and its coloring $INIT_{i,a}$ for every $a \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^i$. Figure 12 depicts our construction of an EVAL_i gadget when $\mathrm{Q}_{i+1} = \forall$, containing an already constructed EVAL_{i+1} gadget. We describe the coloring of EVAL_i by those of the (S&T, S&P, CONJ, EVAL_{i+1}) gadgets therein. For $a \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^i$ and $b \in \{\mathrm{FALSE}, \mathrm{TRUE}\}$, let $(a,b) = (a_1,\ldots,a_i,b) \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^{i+1}$. Let $INIT_{i,a}$ be the coloring of EVAL_i that sets the coloring of the (S&T, S&P, CONJ, EVAL_{i+1}) gadgets as $\left(UOFF, OFF, INIT, INIT_{i+1,(a,\mathrm{FALSE})}\right)$; the coloring of the other gadgets are set to be initialized. Then, we denote by $USED_{i,a}$ the coloring derived from $INIT_{i,a}$ by the ant's walking course from the entrance to an exit of EVAL_i. We have already defined $USED_{n,a} := USED_a$. In the following lemma, $USED_{i,a}$ are inductively defined, too.

Lemma 2. For every $\mathrm{a} \in \{\mathrm{FALSE}, \mathrm{TRUE}\}^i$, if the coloring of EVAL_i is $INIT_{i,a}$ and $\boldsymbol{\phi}_i(\mathrm{a}_1,\ldots,\mathrm{a}_i) = \mathrm{FALSE}$ (TRUE) then the ant entering to the EVAL_i gadget at IN_i exits at $OUT_{i,\mathrm{FALSE}}$ ($OUT_{i,\mathrm{TRUE}}$).

Proof. Lemma 1 proves the $i = n$ case of Lemma 2. Let $a \in \{\text{FALSE}, \text{TRUE}\}^i$. By the backward induction hypothesis, Lemma 2 is assumed to hold for both $\phi_{i+1}(a, \text{FALSE})$ and $\phi_{i+1}(a, \text{TRUE})$. In the following proof, we prove Lemma 2 for $\phi_i(a)$ when $\text{Q}_i = \text{Q}_\forall$.

Case 1. Suppose that $\phi_{i+1}(a, \text{FALSE}) = \text{FALSE}$. The ant walks as follows (Figure 12-1): $IN_i \to \text{S\&T} \to \text{CONJ} \to IN_{i+1}$, changing the coloring of $(\text{S\&T}, \text{CONJ})$ from $(UOFF, INIT)$ to $(OFF, USED_1)$; next, since the coloring of EVAL_{i+1} is $INIT_{i+1,(a,\,\text{FALSE})}$, by Lemma 2 for $\phi_{i+1}(a, \text{FALSE})$, the ant starting from IN_{i+1} reaches to $OUT_{i+1,\text{FALSE}}$, which changes the coloring of EVAL_{i+1} from $INIT_{i+1,\,(a,\,\text{FALSE})}$ to $USED_{i+1,\,(a,\,\text{FALSE})}$; finally, the ant walks from $OUT_{i+1,\,\text{FALSE}}$ to $OUT_{i,\,\text{FALSE}}$.

Case 2. Suppose $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$ and $\phi_{i+1}(a, \text{TRUE}) = \text{FALSE}$. First, the ant walks as follows (see Figure 12-2): $IN_i \to \text{S\&T} \to \text{CONJ} \to IN_{i+1}$; next, since $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$, the ant entering EVAL_{i+1} at IN_{i+1} reaches to $OUT_{i+1,\text{TRUE}}$, changing the coloring of EVAL_{i+1} from $INIT_{i+1,\,(a,\,\text{FALSE})}$ to $USED_{i+1,\,(a,\,\text{FALSE})}$; after that, the ant walks as $OUT_{i+1,\text{TRUE}} \to IN$ of $\text{S\&P} \to OUT_{OFF}$ of $\text{S\&P} \to I_{\text{S\&T}}$, which change the coloring of the S&P gadget from *OFF* to *ON*, and the coloring of the PATHs taken from *INIT* to *USED*; then, getting into the S&T gadget, the ant turns around and exits at $O_{\text{S\&T}}$, switching the coloring of the S&T gadget from *OFF* to *ON*.

Secondly, the ant takes the following reversed walking course (see Figure 12-3): $O_{\text{S\&T}} \to \text{S\&P} \to \text{EVAL}_{i+1} \to \text{CONJ} \to IN$ of S&T, changing the colorings of $(\text{S\&P}, \text{EVAL}_{i+1}, \text{CONJ})$ from $(UOFF,\ USED_{i+1,(a,\,\text{FALSE})},\ USED_1)$ to $\left(ON, INIT_{(a,\,FALSE)}, INIT\right)$; the coloring of the taken PATH taken and CROSS gadgets are initialized, too.

We remark that the gadgets that passed through, shown by the dotted line in Figure 12-3, are those gadgets that changed to the used color by the ant's forward walking, but then initialized by the ant's backward walking. These gadgets can be used again without violating the rules of the transition diagram given in Section 2; in other words, the already completely used gadgets, e.g., *USED* PATH gadgets or $USED_{1,2}$ CROSS gadgets, will never be used without being initialized in this manner.

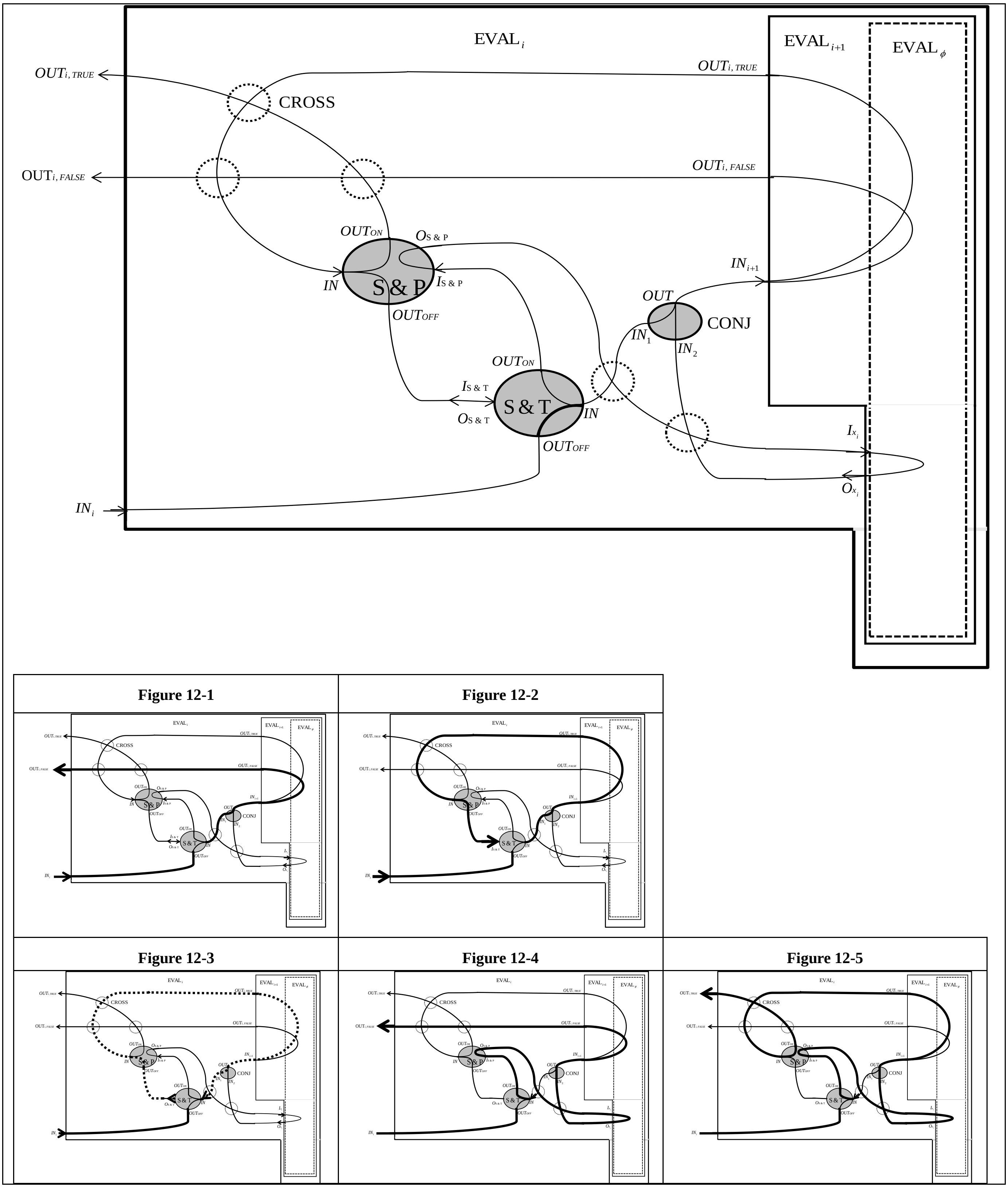
Finally, the ant walks as follows (see Figure 12-4): IN of $\text{S\&T} \to OUT_{ON}$ of $\text{S\&T} \to I_{\text{S\&P}} \to O_{\text{S\&P}} \to I_{x_i}$ of $\text{EVAL}_\phi \to O_{x_i}$ of $\text{EVAL}_\phi \to \text{CONJ} \to IN_{i+1}$, changing the coloring of $(\text{S\&T}, \text{S\&P}, \text{EVAL}_{i+1}, \text{CONJ})$ from $(ON, OFF, INIT_{(a,\,\text{FALSE})}, INIT)$ to $(UON, ON, INIT_{(a,\,\text{TRUE})}, USED_2)$; we remark that, as shown in Figure 11, the walking course $I_{x_i} \to O_{x_i}$ going through the EVAL_ϕ gadget have polynomially many crossing points with other such courses $I_{x_j} \to O_{x_j}$ inside the EVAL_ϕ gadget; next, since $\phi_{i+1}(a, \text{TRUE}) = \text{FALSE}$, the ant entering to EVAL_{i+1} at IN_{i+1} reaches $OUT_{i+1,\text{FALSE}}$; finally, the ant walks as $OUT_{i+1,\,\text{FALSE}} \to OUT_{i,\,\text{FALSE}}$.

Case 3. Suppose $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$ and $\phi_{i+1}(a, \text{TRUE}) = \text{TRUE}$. First, the ant walks as in Case 2: $IN_i \to \text{S\&T} \to \text{CONJ} \to IN_{i+1} \to OUT_{i+1,\,\text{TRUE}} \to IN$ of $\text{S\&P} \to OUT_{OFF}$ of $\text{S\&P} \to I_{\text{S\&T}} \to O_{\text{S\&T}} \to \text{S\&P} \to \text{EVAL}_{i+1} \to \text{CONJ} \to IN$ of $\text{S\&T} \to OUT_{ON}$ of $\text{S\&T} \to I_{\text{S\&P}} \to O_{\text{S\&P}} \to I_{x_i}$ of $\text{EVAL}_\phi \to O_{x_i}$ of $\text{EVAL}_\phi \to \text{CONJ} \to IN_{i+1}$.

Secondly, the ant walks as follows (see Figure 12-5): since the coloring of EVAL_{i+1} has become $INIT_{(a,\,\text{TRUE})}$ and $\phi_{i+1}(a, \text{TRUE}) = \text{TRUE}$, the ant reaches $OUT_{i+1,\text{TRUE}}$; after that, since the coloring of the S&P gadget has become *ON*, the ant walks as $OUT_{i+1,\text{TRUE}} \to IN$ of $\text{S\&P} \to OUT_{ON}$ of $\text{S\&P} \to OUT_{i,\text{TRUE}}$.

By these three Cases, Lemma 2 has shown to hold for $\phi_i(a)$ when $\mathrm{Q}_i = \forall$. Since $\exists$ is a logical dual of $\forall$, a gadget of the $\mathrm{Q}_i = \exists$ case is obtained from Figure 12 by switching the labels $OUT_{i',\mathrm{TRUE}}$ and $OUT_{i',\mathrm{FALSE}}$ for $i' = i$ and i+1. Accordingly, rewriting the above proof attains that of the $\mathrm{Q}_i = \exists$ case.

Figure 12. EVAL_i when $\mathrm{Q}_i = \forall$.

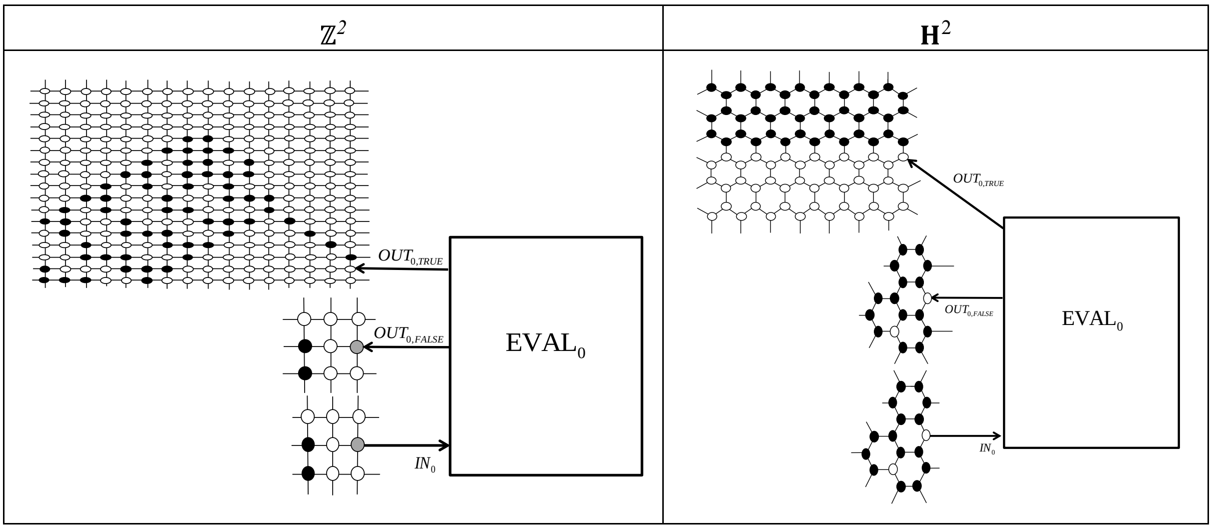


5. Polynomial-Time Reduction

For a given closed QBF formula ϕ_0, Lemma 2 and its proof gives a polynomial-time construction of an EVAL_0 gadget such that if $\phi_0 = \mathrm{FALSE}$ (TRUE) then the ant placed at IN_0 of $INIT_{0,\,\mathrm{FALSE}^n}$ coloring of EVAL_0 finally reaches $OUT_{0,\,\mathrm{FALSE}}$ ($OUT_{0,\mathrm{TRUE}}$). So, as illustrated in Figure 13, plugging reflectors (see Figure 3) in both IN_0 and $INIT_{0,\,\mathrm{FALSE}^n}$, and a diagonal highway (see Figure 4) in $OUT_{0,\,\mathrm{TRUE}}$ gives an initial configuration of GLA such that $\phi_0 = \mathrm{FALSE}$ (TRUE) if, and only if, the ant stays in a bounded area (goes out of any bounded area). This establishes an efficient reduction from the QBF evaluation problem to the recognition problem of the repeatable coloring of GLA on $\mathbb{Z}^2$ with gray cells, proving Theorem 1. For the $\mathbf{H}^2$ model, plugging a highway along the horizon of the half-and-half background (see Figure 1) to $OUT_{0,\mathrm{TRUE}}$ gives an efficient reduction, too, proving Theorem 2.

Among our gadgets on $\mathbb{Z}^2$ given in Section 2, only Switch & Turn gadget uses gray cell. In addition, the Switch & Turn gadget contains only one gray cell. So, putting these Switch & Turn gadgets mutually away from each other makes a size-n configuration of GLA colored by n^{ε}-sparse gray cells for the reduction, proving Theorem 3.

Figure 13. Reduction from QBF evaluation problem to the recognition problem of the repeatable coloring of GLA.



6. Open Questions

We can construct all gadgets shown in Section 2 on the $\mathbf{T}^2$ (triangular lattice) model with gray cells, excepting the Switch & Turn gadget. Although we are lacking the Switch & Turn gadget, we believe that the recognition problem of the repeatable configurations of GLA on $\mathbf{T}^2$ with gray cells is PSPACE-hard. The experimental results by Wang and Cohen [6] showed that randomly generated configurations of GLA on $\mathbf{H}^2$ for the monochromatic background, fall into the repeatable configurations with high probability. As far as we know, it is challenging to find even one provably unrepeatable configuration of GLA on $\mathbf{H}^2$ for the monochromatic background. Perhaps it is more challenging to prove the following: "an ant's trajectory starting from a repeatable size- n configuration of GLA is always at most a polynomial of *n*". If this were true, then the recognition problem of the repeatable configurations of GLA would belong to PSPACE.

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