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A Modified Iterative Algorithm for Split Feasibility Problems of Right Bregman Strongly Quasi-Nonexpansive Mappings in Banach Spaces with Applications

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Abstract: In this paper, we present a new iterative scheme for finding a common element of the solution set $\mathcal{F}$ of the split feasibility problem and the fixed point set $F(T)$ of a right Bregman strongly quasi-nonexpansive mapping T in p-uniformly convex Banach spaces which are also uniformly smooth. We prove strong convergence theorem of the sequences generated by our scheme under some appropriate conditions in real p-uniformly convex and uniformly smooth Banach spaces. Furthermore, we give some examples and applications to illustrate our main results in this paper. Our results extend and improve the recent ones of some others in the literature.

Keywords: right Bregman strongly quasi-nonexpansive; split feasibility problem; fixed point

1. Introduction

Let E_1, E_2 be Banach spaces and C, Q be nonempty closed convex subsets of E_1 and E_2, respectively. Let $A\colon E_1 \to E_2$ be a bounded linear operator. The split feasibility problem (shortly, (SFP)) is as follows:

$$\text{Find } \ x \in C \ \text{ such that } \ Ax \in Q. \tag{1}$$

We denote the solution set of the problem (SFP) by $\mathcal{F} := \{x \in C : Ax \in Q\} = C \cap A^{-1}(Q)$. It is worth mentioning that (SFP) in finite-dimensional spaces was first introduced by Censor and Elfving [1] for modelling inverse problems which arise from phase retrievals and medical image reconstruction.

Note that, in finite dimensional Hilbert spaces, the strong convergence of a sequence is equivalent to the weak convergence and the boundedness of a sequence implies that there exists a strongly convergent subsequence. However, in infinite dimensional Hilbert spaces, the strong convergence of a sequence is not equivalent to the weak convergence and the boundedness of a sequence implies that there exists a weakly convergent subsequence. So, for some algorithms, we can prove only strong convergence theorems in finite dimensional Hilbert spaces, but we can prove weak and strong convergence theorems in infinite dimensional Hilbert spaces.

In [2], Byrne presented a new method $\{x_n\}$, which is called the CQ-algorithm for solving the problem (SFP) that does not involve matrix inverses, defined as follows:

For any $x_0 \in C$ and $n \geq 1$,

$$x_{n+1} = P_C(x_n + \gamma A^T(P_Q - I)x_n), \tag{2}$$

where P_C and P_Q is the orthogonal projections onto C and Q, respectively, $\gamma \in (0, \frac{2}{L})$, L is the largest eigenvalue of the matrix $A^T A$ and I is the identity matrix.

After that many authors [3–7] study extend some iterative algorithms from Hilbert spaces to Banach spaces by using Bregman's technic as follows:

In solving the problem (SFP) in p-uniformly convex real Banach spaces which are also uniformly smooth, Schopfer et al. [8] proposed the following algorithm $\{x_n\}$ defined as follows:

For any $x_1 \in E_1$ and $n \geq 1$,

$$x_{n+1} = \Pi_C J^*_{E_1}(J_{E_1}(x_n) - s_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))), \tag{3}$$

where Π_C denotes the Bregman projection and J the duality mapping.

Clearly, the algorithm (3) covers Byrne's CQ algorithm (2), which is a gradient-projection method (GPM) in convex minimization as a special case. The duality mapping of E_1 is sequentially weak-to-weak-continuous (see [8]) in Banach spaces such as the classical L_p $(2 < p < \infty)$ spaces.

In [9], Wang modified the algorithm (3) and proved strong convergence theorems for the following multiple-sets split feasibility problem (MSSFP):

$$\text{Find } \; x \in \bigcap_{i=1}^{r} C_i \; \text{ such that } \; Ax \in \bigcap_{j=1+r}^{r+s} Q_j, \tag{4}$$

where r, s are two given integers, C_i, $i = 1, 2, 3, \cdots, r$, is a closed convex subset in E_1 and Q_j, $j = r+1, \cdots, r+s$, is a closed convex subset in E_2. He defined the following: for each $n \in \mathbb{N}$,

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}}(x), & 1 \leq i(n) \leq r, \\ J^*_{E_1}[J_{E_1}(x) - s_n A^* J_{E_2}(Ax - P_{Q_{j(n)}}(Ax))], & r+1 \leq i(n) \leq r+s, \end{cases} \tag{5}$$

where $i : \mathbb{N} \to \mathcal{I}$ is the cyclic control mapping $i(n) = n \bmod (r+s) + 1$ and t_n satisfies

$$0 < s \leq s_n \leq \frac{q}{C_q \|A\|^q})^{\frac{1}{q-1}} \tag{6}$$

with a constant C_q and proposed the following algorithm $\{x_n\}$ defined as follows: For any $x_1 = \bar{x}$ and $n \geq 1$,

$$\begin{cases} y_n = T_n x_n, \\ D_n = \{w \in E_1 : d_p(y_n, w) \leq d_p(x_n, w)\}, \\ E_n = \{w \in E_1 : \langle x_n - w, J_p(\bar{x}) - J_p(x_n) \geq 0\}, \\ x_{n+1} = \Pi_{D_n \cap E_n}(\bar{x}). \end{cases} \tag{7}$$

Recently, Zegeye and Shahzad [10] proved a strong convergence theorem for a common fixed point of a finite family of right Bregman strongly nonexpansive mappings in the framework of real reflexive Banach spaces. Furthermore, they applied their method to approximate a common zero of a finite family of maximal monotone operators and a solution of a finite family of convex feasibility problems in reflexive real Banach spaces.

Let $f : E \to \mathbb{R}$ be a cofinite function which is bounded, uniformly Fŕechet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed convex subset of int(dom f) and let $T_i : C \to C$, for $i = 1, 2, \cdots, N$, be a finite family of right Bregman strongly nonexpansive mappings such that $F(T_i) = \widehat{F}(T_i)$ for each $i \in \{1, 2, \cdots, N\}$. Assume that $F := \widehat{F}(T_i)$ is nonempty. For any $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n$$

for each $n \geq 1$, where $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a point $\widehat{x}$.

In this paper, we modify the Halpern-Mann iterative method for split feasibility problems and fixed point problems concerning right Bregman strongly quasi-nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. We prove strong convergence theorem of the sequences generated by our scheme under some appropriate conditions in real p-uniformly convex and uniformly smooth Banach spaces. Also, we give numerical examples of our result to study its efficiency and implementation. Our results extend and improve the recent ones of some others in the literature.

2. Preliminaries

Let E_1, E_2 be real Banach spaces and A: $E_1 \to E_2$ be a bounded linear operator. The dual (adjoint) operator of A, denoted by A^*, is a bounded linear operator defined by A^*: $E_2^* \to E_1^*$

$$\langle A^* \bar{y}, x \rangle := \langle \bar{y}, Ax \rangle \tag{8}$$

for all $x \in E_1$ and $\bar{y} \in E_2^*$ and the equalities $\|A^*\| = \|A\|$, $\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$ are valid, where

$$\mathcal{R}(A)^{\perp} := \{x^* \in E_2^* : \langle x^*, u \rangle = 0, \ \forall u \in \mathcal{R}(A)\}.$$

For more details on bounded linear operators and their duals, see [11,12].

Definition 1. *(1) The duality mapping $J_E^p : E \to E^*$ is defined by*

$$J_E^p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}.$$

(2) The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \implies \langle J_E^p x_n, y \rangle \to \langle J_E^p x, y \rangle$$

holds true for any $y \in E$.

We note here that l_p $(p > 1)$ spaces has such a property, but the L_p $(p > 2)$ space does not share this property. The domain of a convex function $f : E \to \mathbb{R}$ is defined by dom$f := \{x \in E : f(x) < +\infty\}$. When dom$f \neq \emptyset$, then we say that f is proper.

In the sequel, we adopt the following notations in this paper: $x_n \to x$ means that $x_n \to x$ strongly and $x_n \rightharpoonup x$ meansthat $x_n \to x$ weakly.

Definition 2 ([13])**.** *Let $f\colon E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. The Bregman distance with respect to f is defined by*

$$d_f(x, y) := f(y) - f(x) - \langle f'(x), y - x\rangle$$

for all $x, y \in E$.

The duality mapping J_E^p is actually the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. If $f = f_p$, then the Bregman distance with respect to f_p now becomes

$$\begin{aligned} d_p(x, y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, y\rangle. \end{aligned} \tag{9}$$

The Bregman distance is not symmetric and so it is not a metric, but it posses the following important properties: for all $w, x, y \in E$,

$$d_p(x, y) = d_p(x, w) + d_p(w, y) + \langle w - y, J_E^p x - J_E^p y\rangle \tag{10}$$

and

$$d_p(x, y) + d_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle. \tag{11}$$

Let $1 < q \le 2 \le p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \le 1, \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$$

and, for any $q > 1$, a Banach space E is said to be *q-uniformly smooth* if there exists $C_q > 0$ such that $\rho_E(t) \le C_q t^q$ for any $t > 0$.

Let $x, y \in E$ and $q > 1$. If a Banach space E is q-uniformly smooth, then there exists $C_q > 0$ such that

$$\|x - y\|^q \le \|x\|^q - q\langle J_E^p(x), y\rangle + C_q\|y\|^q. \tag{12}$$

Let $\dim(E) \ge 2$. The modulus of convexity of E is the function $\delta_E(\epsilon)$: $(0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1,\ \epsilon = \|x - y\|\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and, for any $p > 1$, a Banach space E is said to be p-uniformly convex if there is $C_p > 0$ such that $\delta_E(\epsilon) \ge C_p\epsilon^p$ for any $\epsilon \in (0, 2]$. More information concerning uniformly convex spaces can be found, for example, in the book by Goebel and Reich [14].

It is known that a Banach space E is p-uniformly convex and uniformly smooth if and only if its dual E^* is q-uniformly smooth and uniformly convex. It is also well known that the duality J_E^p is one-to-one, single valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^*.

For any p-uniformly convex Banach space E, the metric and the Bregman distance have the following relation :

$$\tau\|x-y\|^p \le d_p(x,y) \le \langle x-y, J_E^p x - J_E^p y\rangle, \tag{13}$$

where $\tau > 0$ is a fixed number.

Let C be a nonempty closed convex subset of E. The metric projection

$$P_C x := \arg\min_{y\in C} \|x-y\|$$

for all $x \in E$ is the unique minimizer of the norm distance, which can be characterized by a variational inequality

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \le 0 \tag{14}$$

for all $z \in C$.

Similarly, the Bregman projection is defined as follows:

$$\Pi_C x = \arg\min_{y\in C} d_p(x,y)$$

for all $x \in E$, which is the unique minimizer of the Bregman distance. In addition, the Bregman projection can also be characterized by a variational inequality

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \le 0 \tag{15}$$

for all $z \in C$, from which one has

$$d_p(\Pi_C x, z) \le d_p(x,z) - d_p(x, \Pi_C x) \tag{16}$$

for all $z \in C$.

Following [15,16], we will make use of the function V_p: $E^* \times E \to [0,+\infty)$ associated with f_p, which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p$$

for all $x \in E$ and $\bar{x} \in E^*$. Then V_p is nonnegative and

$$V_p(\bar{x}, x) = d_p(J_E^*(\bar{x}), x) = d_p(J_E^q(\bar{x}), x) \tag{17}$$

for all $x \in E$ and $\bar{x} \in E^*$. Moreover, by the subdifferential inequality, we have

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^*(\bar{x}) - x\rangle \le V_p(\bar{x} + \bar{y}, x) \tag{18}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see also [17,18]). In addition, V_p is convex in the first variable. Thus, for all $z \in E$,

$$d_p\Big(J_E^q\Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big), w\Big) = d_p\Big(J_E^*\Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big), w\Big) \le \sum_{i=1}^N t_i d_p(x_i, w), \tag{19}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum\limits_{i=1}^N = 1$. For more details, see [19,20].

Let C be a nonempty, closed and convex subset of E. A mapping T: $C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x-y\|$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Let C be a convex subset of int (dom f_p), where $f_p(x) = (\frac{1}{p})\|x\|^p$, $2 \leq p < \infty$, and T be a self-mapping of C. A point $\widehat{x} \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to $\widehat{x}$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T is denoted by $\widehat{F}(T)$ (see [21]).

In general, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties.

Definition 3. *A nonlinear mapping T with a nonempty asymptotic fixed point set is said to be:*

(1) T is called right Bregman quasi-nonexpansive (shortly, R-BQNE) (see [22]) if $F(T) \neq \emptyset$ and

$$d_p(Tx, \bar{x}) \leq d_p(x, \bar{x})$$

for all $x \in C$ and $\bar{x} \in F(T)$.

(2) T is called right Bregman strongly quasi-nonexpansive (shortly, R-BSQNE) (see [23,24]) with respect to a nonempty $\widehat{F}(T)$ if

$$d_p(Tx, \widehat{x}) \leq d_p(x, \widehat{x})$$

for all $\widehat{x} \in \widehat{F}(T)$, $x \in C$, and if whenever $\{x_n\} \subset C$ is bounded, $\widehat{x} \in \widehat{F}(T)$ and $\lim_{n\to+\infty}(d_p(x_n, \widehat{x}) - d_p(Tx_n, \widehat{x})) = 0$, then it follows that $\lim_{n\to+\infty} d_p(x_n, Tx_n) = 0$.

(3) T is called right Bregman firmly nonexpansive (shortly, R-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

for all $x, y \in C$ or, equivalently,

$$d_p(Tx, Ty) + d_p(Ty, Tx) + d_p(x, Tx) + d_p(y, Ty) \leq d_p(x, Ty) + d_p(y, Tx)$$

for all $x, y \in C$.

Lemma 1 ([25]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$, that is, $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large number $k \in \mathbb{N}$): $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$. In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.*

Lemma 2 ([26]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n$$

for each $n \geq 0$, where

(i) $\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Results

Now, we give our main results in this paper.

Theorem 1. *Let E_1, E_2 be p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let A: $E_1 \to E_2$ be a bounded linear operator and A^*: $E_2^* \to E_1^*$ be the adjoint of A. Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T : C \to C$ be a right Bregman strongly quasi-nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$ and $F(T) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\alpha_n \leq b < 1$, $(1-\alpha_n)a < \gamma_n < \delta_n$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}(J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n), \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(u) + \gamma_n J^p_{E_1}(x_n) + \delta_n J^p_{E_1}(Tx_n))) \end{cases} \tag{20}$$

for each $n \geq 1$. Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < s \leq s_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $\Pi_{F(T)\cap\mathcal{F}}u$.

Proof. Firstly, we prove that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded. Setting $z_n := Au_n - P_Q(Au_n)$ for each $n \geq 1$. From (14), it follows that, for any $w \in F(T) \cap \mathcal{F}$,

$$\begin{aligned} \langle J^p_{E_2}(z_n), Au_n - Aw\rangle &= \|Au_n - P_Q(Au_n)\|^p + \langle J^p_{E_2}(z_n), P_Q(Au_n) - Aw\rangle \\ &\geq \|Au_n - P_Q(Au_n)\|^p \\ &= \|z_n\|^p. \end{aligned} \tag{21}$$

So, from (21) and (12), it follows that

$$\begin{aligned} & d_p(x_n, w) \\ & \leq d_p(J^q_{E_1^*}[J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n], w) \\ & = \frac{1}{q}\|J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(z_n)\|^q - \langle J^p_{E_1}(u_n), w\rangle + s_n\langle J^p_{E_2}(z_n), Aw\rangle + \frac{1}{p}\|w\|^p \\ & \leq \|J^p_{E_1}(u_n)\| - s_n\langle Au_n, J^p_{E_2}(z_n)\rangle + \frac{C_q(s_n\|A\|)^q}{q}\|J^p_{E_2}(z_n)\|^p \\ & \quad - \langle J^p_{E_1}(u_n), w\rangle + \frac{1}{p}\|w\|^p + s_n\langle Aw, J^p_{E_2}(z_n)\rangle \\ & = \frac{1}{q}\|u_n\|^p - \langle J^p_{E_1}(u_n), w\rangle + \frac{1}{q}\|w\|^p + s_n\langle J^p_{E_2}(z_n), Aw - Au_n\rangle \\ & \quad + \frac{C_q(s_n\|A\|)^q}{q}\|z_n\|^p \\ & = d_p(u_n, w) + s_n\langle J^p_{E_2}(z_n), Aw - Au_n\rangle + \frac{C_q(s_n\|A\|)^q}{q}\|z_n\|^p \\ & \leq d_p(u_n, w) - (s_n - \frac{C_q(s_n\|A\|)^q}{q})\|z_n\|^p. \end{aligned} \tag{22}$$

By using (c), we obtain

$$d_p(x_n, w) \leq d_p(u_n, w).$$

From (20), we have

$$
\begin{aligned}
&d_p(u_{n+1},w)\\
&\le d_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))],w)\\
&= d_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha_n)\beta_n J^p_{E_1}(u)\\
&\quad +(1-\alpha_n)\gamma_n J^p_{E_1}(x_n)+(1-\alpha_n)\delta_n J^p_{E_1}(Tx_n)],w)\\
&\le \alpha_n d_p(u,w)+(1-\alpha_n)\beta_n d_p(u,w)\\
&\quad +(1-\alpha_n)\gamma_n d_p(x_n,w)+(1-\alpha_n)\delta_n d_p(Tx_n,w)\\
&\le \alpha_n d_p(u,w)+(1-\alpha_n)\beta_n d_p(u,w)\\
&\quad +(1-\alpha_n)\gamma_n d_p(x_n,w)+(1-\alpha_n)\delta_n d_p(x_n,w)\\
&= (\alpha_n+(1-\alpha_n)\beta_n)d_p(u,w)+(1-\alpha_n)(\gamma_n+\delta_n)d_p(x_n,w)\\
&\le (\alpha_n+(1-\alpha_n)\beta_n)d_p(u,w)+(1-\alpha_n)(\gamma_n+\delta_n)d_p(u_n,w)\\
&\le \max\{d_p(u,w),d_p(u_n,w)\}\\
&\quad \cdots\\
&\le \max\{d_p(u,w),d_p(u_1,w)\}.
\end{aligned}
\tag{23}
$$

Thus $\{d_p(u_n,w)\}$ is bounded and, consequently, we have that $\{d_p(x_n,w)\}$ is bounded. Hence the sequence $\{x_n\}$ and $\{u_n\}$ are bounded. Setting

$$
y_n = J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))]
$$

for each $n\ge 1$. Then we have

$$
\begin{aligned}
&d_p(x_{n+1},w)\\
&\le d_p(u_{n+1},w)\\
&\le d_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))],w)\\
&= V_f(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n)),w)\\
&\le V_f(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))\\
&\quad -\alpha_n(J^p_{E_1}(u)-J^p_{E_1}(w)),w)-\langle -\alpha_n(J^p_{E_1}(u)-J^p_{E_1}(w)),\\
&\quad J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))]-w\rangle\\
&= V_f(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n)),w)\\
&\quad +\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(w),y_n-w\rangle\\
&= V_f(\alpha_n J^p_{E_1}(w)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n)),w)\\
&\quad +\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(w),y_n-w\rangle\\
&= d_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(w)+(1-\alpha_n)(\beta_n J^p_{E_1}(u)+\gamma_n J^p_{E_1}(x_n)+\delta_n J^p_{E_1}(Tx_n))],w)\\
&\quad +\alpha_n\langle_n J^p_{E_1}(u)-J^p_{E_1}(w),y_n-w\rangle\\
&\le \alpha_n d_p(w,w)+(1-\alpha_n)\beta_n d_p(w,w)+(1-\alpha_n)\gamma_n d_p(x_n,w)\\
&\quad +(1-\alpha_n)\delta_n d_p(Tx_n,w)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\gamma_n d_p(x_n,w) + (1-\alpha_n)\delta_n d_p(x_n,w) \\
&\quad + \alpha_n \langle J^p_{E_1}(u) - J^p_{E_1}(w), y_n - w\rangle \\
&= (1-\alpha_n)(\gamma_n+\delta_n) d_p(u_n,w) + \alpha_n \langle J^p_{E_1}(u) - J^p_{E_1}(w), y_n - w\rangle \\
&\leq (1-\alpha_n) d_p(u_n,w) + \alpha_n \langle J^p_{E_1}(u) - J^p_{E_1}(w), y_n - w\rangle.
\end{aligned} \tag{24}
$$

Now, we prove the strong convergence theorem by the two cases:

Case I. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d_p(x_n,w)\}$ is monotonically non-increasing for all $n \geq n_0$. Then $\{d_p(x_n,w)\}$ converges and, as $n \to \infty$,

$$
d_p(x_{n+1},w) - d_p(x_n,w) \to 0. \tag{25}
$$

Setting $t_n = J^q_{E_1^*}(\frac{\beta_n}{1-\alpha_n} J^p_{E_1}(x_n) + \frac{\gamma_n}{1-\alpha_n} J^p_{E_1}(x_n) + \frac{\delta_n}{1-\alpha_n} J^p_{E_1}(Tx_n))$. Then we have

$$
\begin{aligned}
d_p(t_n,w) &= d_p\Big(J^q_{E_1^*}\Big(\frac{\gamma_n}{1-\alpha_n} J^p_{E_1}(x_n) + \frac{\delta_n}{1-\alpha_n} J^p_{E_1}(Tx_n)\Big), w\Big) \\
&\leq \frac{\gamma_n}{1-\alpha_n} d_p(x_n,w) + \frac{\delta_n}{1-\alpha_n} d_p(Tx_n,w) \\
&\leq \frac{\gamma_n+\delta_n}{1-\alpha_n} d_p(x_n,w) \\
&\leq d_p(x_n,w).
\end{aligned} \tag{26}
$$

Therefore, we have

$$
\begin{aligned}
0 &\leq d_p(x_n,w) - d_p(t_n,w) \\
&= d_p(x_n,w) - d_p(x_{n+1},w) + d_p(x_{n+1},w) - d_p(t_n,w) \\
&\leq d_p(x_n,w) - d_p(x_{n+1},w) + d_p(u_{n+1},w) - d_p(t_n,w) \\
&\leq d_p(x_n,w) - d_p(x_{n+1},w) + \alpha_n d_p(u,w) + (1-\alpha_n) d_p(t_n,w) - d_p(t_n,w) \\
&= d_p(x_n,w) - d_p(x_{n+1},w) + \alpha_n d_p(u,w) - \alpha_n d_p(t_n,w) \to 0
\end{aligned} \tag{27}
$$

as $n \to \infty$. Again, we obtain

$$
\begin{aligned}
d_p(t_n,w) &\leq \frac{\gamma_n}{1-\alpha_n} d_p(x_n,w) + \frac{\delta_n}{1-\alpha_n} d_p(Tx_n,w) \\
&= \Big(1 - \frac{\beta_n+\delta_n}{1-\alpha_n}\Big) d_p(x_n,w) + \frac{\delta_n}{1-\alpha_n} d_p(Tx_n,w) \\
&= d_p(x_n,w) - \frac{\beta_n}{1-\alpha_n} d_p(x_n,w) + \frac{\delta_n}{1-\alpha_n}(d_p(Tx_n,w) - d_p(x_n,w)) \\
&\leq d_p(x_n,w) + \frac{\delta_n}{1-\alpha_n}(d_p(Tx_n,w) - d_p(x_n,w)).
\end{aligned} \tag{28}
$$

Since $\alpha_n + \delta_n \leq 1$ and $\alpha_n \leq b < 1$, we have

$$
\begin{aligned}
a(d_p(x_n,w) - d_p(Tx_n,w)) &< \frac{\delta_n}{1-\alpha_n}(d_p(x_n,w) - d_p(Tx_n,w)) \\
&\leq d_p(x_n,w) - d_p(t_n,w) \to 0
\end{aligned} \tag{29}
$$

as $n \to \infty$. By using (c), we have

$$
d_p(x_n,w) - d_p(Tx_n,w) \to 0
$$

as $n \to \infty$. Since T is right Bregman strongly quasi-nonexpansive, we obtain

$$\lim_{n\to\infty} d_p(Tx_n, x_n) = 0,$$

which implies that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{30}$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\bar{x} \in C$. From (30), it follows that $\bar{x} \in F(T)$ since $F(T) = \widehat{F}(T)$.

Next, we prove that $A\bar{x} \in Q$, that is, $\bar{x} \in \mathcal{F}$. Setting

$$v_n = J^q_{E_1^*}[J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n].$$

From (16), (22) and (24), it follows that

$$\begin{aligned} d_p(v_n, x_n) &= d_p(v_n, \Pi_c v_n) \\ &\le d_p(v_n, w) - d_p(x_n, w) \\ &\le d_(u_n, w) - d_p(x_n, w) \\ &\le \alpha_n \mathcal{M} + d_p(x_{n-1}, w) - d_p(x_n, w) \to 0 \end{aligned} \tag{31}$$

as $n \to \infty$, where $\mathcal{M} > 0$ and $d_p(x_{n-1}, w) + \langle J^p_{E_1}(u) - J^p_{E_1}(w), y_{n-1} - w\rangle \le \mathcal{M}$. Hence we have

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{32}$$

From (22), it follows that, as $n \to \infty$,

$$\begin{aligned} \Big(s_n - \frac{C_q(s_n\|A\|)^q}{q}\Big)\|z_n\|^p &\le d(u_n, w) - d_p(x_n, w) \\ &\le \alpha_n \mathcal{M} + d_p(x_{n-1}, w) - d_p(x_n, w) \to 0. \end{aligned} \tag{33}$$

Since

$$s\Big(1 - \frac{C_q k^{q-1}(\|A\|)^q}{q}\Big) \le \Big(s_n - \frac{C_q(s_n\|A\|)^q}{q}\Big), \tag{34}$$

it follows that $\|z_n\|^p \to 0$ as $n \to \infty$, which implies that $\|Au_n - P_Q(u_n)\| \to 0$ as $n \to \infty$. By the definition of v_n, we have

$$\begin{aligned} 0 &\le \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n) \\ &\le s_n\|A^*\|\|J^p_{E_2}(Au_n - P_Q(Au_n))\| \\ &\le \Big(\frac{q}{C_q\|A\|^q}\Big)^{\frac{1}{q-1}} \|A^*\|\|Au_n - P_Q(Au_n)\| \to 0 \end{aligned} \tag{35}$$

as $n \to \infty$. Since $J^q_{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we obtain

$$\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \|J^q_{E_1^*}(J^p_{E_1}(v_n)) - J^q_{E_1^*}(J^p_{E_1}(u_n))\| \to 0 \tag{36}$$

as $n \to \infty$. From (3) and (36), we obtain

$$\|x_n - u_n\| \le \|x_n - y_n\| + \|y_n - u_n\| \to 0. \tag{37}$$

as $n \to \infty$. From (14), it follows that

$$\begin{aligned}
&\|(I - P_Q)A\bar{x}\|^p \\
&= \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), A\bar{x} - P_Q A\bar{x}\rangle \\
&= \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), A\bar{x} - Au_{n_i}\rangle + \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), Au_{n_i} - P_Q Au_{n_i}\rangle \\
&\quad + \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), P_Q Au_{n_i} - P_Q A\bar{x}\rangle \\
&\leq \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), A\bar{x} - Au_{n_i}\rangle + \langle J_{E_2}^p(A\bar{x} - P_Q A\bar{x}), Au_{n_i} - P_Q Au_{n_i}\rangle.
\end{aligned} \tag{38}$$

By the continuity of A and $\|x_n - u_n\| \to 0$ as $n \to \infty$, we have $Au_{n_i} \rightharpoonup A\bar{x}$ as $n \to \infty$. Thus, letting $i \to \infty$, we have

$$\|A\bar{x} - P_Q A\bar{x}\| = 0.$$

Hence $A\bar{x} = P_Q A\bar{x}$, that is, $A\bar{x} \in Q$. Therefore, we have that $\bar{x} \in F(T) \cap \mathcal{F}$.

Next, we prove that $\{x_n\}$ converges strongly to $\Pi_{P(T)\cap\mathcal{F}}u$. Now, we have

$$\begin{aligned}
&d_p(y_n, x_n) \\
&= d_p(J_{E_1^*}^q[\alpha_n J_{E_1}^p(u) + (1-\alpha_n)(\beta_n J_{E_1}^p(u) + \gamma_n J_{E_1}^p(x_n) + \delta_n J_{E_1}^p(Tx_n))], x_n) \\
&\leq \alpha_n d_p(u, x_n) + (1-\alpha_n)\beta_n d_p(u, x_n) \\
&\quad + (1-\alpha_n)\gamma_n d_p(x_n, x_n) + (1-\alpha)\delta_n d_p(Tx_n, x_n) \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Thus we have

$$\|y_n - x_n\| \to 0$$

as $n \to \infty$. Let $\widehat{x} = \Pi_{P(T)\cap\mathcal{F}}u$. From (24), we have

$$d_p(x_{n+1}, \widehat{x}) \leq (1-\alpha_n)d_p(x_n, \widehat{x}) + \alpha_n\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), y_n - \widehat{x}\rangle.$$

Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), x_n - \widehat{x}\rangle = \lim_{j\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), x_{n_j} - \widehat{x}\rangle$$

and $x_{n_j} \to \bar{x}$. Thus, from (15), it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), x_n - \widehat{x}\rangle &= \lim_{j\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), x_{n_j} - \widehat{x}\rangle \\
&= \langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), \bar{x} - \widehat{x}\rangle \\
&\leq 0.
\end{aligned}$$

Since $\|x_n - y_n\| \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(\widehat{x}), y_n - \widehat{x}\rangle \leq 0.$$

Hence, by Lemma 2, we conclude that $d_p(x_n, \widehat{x}) \to 0$ as $n \to \infty$. Therefore, $x_n \to \widehat{x}$ as $n \to \infty$ and, since $\|x_n - u_n\| \to 0$ as $n \to \infty$, we have $u_n \to \widehat{x}$ as $n \to \infty$.

Case II. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$d_p(x_{n_i}, \widehat{x}) < d_p(x_{n_i+1}, \widehat{x})$$

for all $j \in \mathbb{N}$. Then, by Lemma 1, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$ as $n \to \infty$ such that

$$d_p(x_{m_k}, \widehat{x}) \leq d_p(x_{m_k+1}, \widehat{x}), \quad d_p(x_k, \widehat{x}) \leq d_p(x_{m_k+1}, \widehat{x})$$

for all $k \in \mathbb{N}$. Thus it follows from (27) and the same methods in the proof of Case I that

$$\|x_{m_k+1} - x_{m_k}\| \to 0, \quad \|Tx_{m_k} - x_{m_k}\| \to 0$$

as $k \to \infty$. Therefore, we have

$$\limsup_{k\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k+1} - \widehat{x}\rangle = \limsup_{k\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k} - \widehat{x}\rangle \leq 0. \tag{39}$$

From (24), also, we have

$$d_p(x_{m_k+1}, \widehat{x}) \leq (1 - \alpha_{m_k}) d_p(x_{m_k}, \widehat{x}) + \alpha_{m_k} \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k} - \widehat{x}\rangle. \tag{40}$$

Since $d_p(x_{m_k}, \widehat{x}) \leq d_p(x_{m_k+1}, \widehat{x})$, it follows from (40) that

$$\begin{aligned} \alpha_{m_k} d_p(x_{m_k}, \widehat{x}) &\leq d_p(x_{m_k}, \widehat{x}) - d_p(x_{m_k+1}, \widehat{x}) + \alpha_{m_k} \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k} - \widehat{x}\rangle \\ &\leq \alpha_{m_k} \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k} - \widehat{x}\rangle \end{aligned} \tag{41}$$

Since $\alpha_{m_k} > 0$, we obtain

$$d_p(x_{m_k}, \widehat{x}) \leq \langle J^p_{E_1}(u) - J^p_{E_1}(\widehat{x}), y_{m_k} - \widehat{x}\rangle. \tag{42}$$

Then, from (39), it follows that $d_p(x_{m_k}, \widehat{x}) \to 0$ as $k \to \infty$. This together with (40), we obtain $d_p(x_{m_k+1}, \widehat{x}) \to 0$ as $k \to \infty$. Since $d_p(x_k, \widehat{x}) \leq (x_{m_k+1}, \widehat{x})$ for all $k \in \mathbb{N}$, we have $x_k \to \widehat{x}$ as $k \to \infty$, which implies that $x_n \to \widehat{x}$ as $k \to \infty$.

Therefore, from the above two cases, we conclude that $\{x_n\}$ converges strongly to $\widehat{x} = \Pi_{P(T)\cap\mathcal{F}} u$. This completes the proof. □

Corollary 1 ([19]). *Let E_1, E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T : C \to C$ be a right Bregman strongly quasi-nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$ and $F(T) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\beta_n \leq b < 1$, $(1 - \beta_n)a < \gamma_n < \delta_n$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}(J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n), \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\beta_n J^p_{E_1}(u) + \gamma_n J^p_{E_1}(x_n) + \delta_n J^p_{E_1}(Tx_n)) \end{cases} \tag{43}$$

for each $n \geq 1$. Suppose the following condition is satisfied:

(a) $\lim_{n\to\infty} \beta_n = 0$;

(b) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(c) $0 < s \leq s_n \leq k < \left(\frac{q}{C_q \|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $\Pi_{F(T)\cap\mathcal{F}} u$.

Proof. If $\alpha_n = 0$ for all $n \geq 1$ in Theorem 1, then we obtain the desired conclusion. □

Corollary 2. *Let E_1, E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T : C \to C$ be a right Bregman strongly quasi-nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$ and $F(T) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$ is a sequences in $(0, 1)$ such that $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}(J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n), \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)) \end{cases} \tag{44}$$

for each $n \geq 1$. Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < s \leq s_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $\Pi_{F(T)\cap\mathcal{F}}u$.

Proof. If $\beta_n = \gamma_n = \delta_n = 0$ for all $n \geq 1$ in Theorem 1, then we obtain the desired conclusion. □

Next, we consider the mapping $T : C \to C$ defined by $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$, where T_i for each $i = 1, 2, \cdots, N$ is a right Bregman strongly quasi-nonexpansive mapping on E. Using the results in [10], we have the following:

Corollary 3. *Let E_1, E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$, where $T_i : C \to C$ for each $i = 1, 2, \cdots N$ be a finite family of right Bregman strongly quasi-nonexpansive mappings such that $F(T_i) = \widehat{F}(T_i) \neq \emptyset$ and $(\bigcap_{i=1}^{N} F(T_i)) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\alpha_n \leq b < 1$, $(1-\alpha_n)a < \gamma_n < \delta_n$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}(J^p_{E_1}(u_n) - s_n A^* J^p_{E_2}(I - P_Q)Au_n), \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(u) + \gamma_n J^p_{E_1}(x_n) + \delta_n J^p_{E_1}(Tx_n))) \end{cases} \tag{45}$$

for each $n \geq 1$. Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < s \leq s_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $\Pi_{F(T)\cap\mathcal{F}}u$.

Proof. If $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$ in Theorem 1, then we obtain the desired conclusion. □

Corollary 4. *Let H_1, H_2 be two real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A.*

Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T : C \to C$ be a right Bregman strongly quasi-nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$, $I - T$ is demiclosed at zero and $F(T) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\alpha_n \leq b < 1$, $(1 - \alpha_n)a < \gamma_n < \delta_n$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and

$$\begin{cases} x_n = P_C\big(u_n - s_n A^*(I - P_Q)Au_n\big), \\ u_{n+1} = P_C\big(\alpha_n u + (1 - \alpha_n)(\beta_n u + \gamma_n x_n + \delta_n T x_n)\big) \end{cases} \tag{46}$$

for each $n \geq 1$. Suppose the following conditions are satisfied:

(a) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(b) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < s \leq s_n \leq k < \dfrac{2}{\|A\|^2}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $\Pi_{F(T)\cap\mathcal{F}} u$.

Proof. Let $E = H$ in Theorem 1. Since the duality mappings $J^q_{E_1^*}$, $J^p_{E_1}$ and $J^p_{E_2}$ are the identity mapping in a Hilbert space H, from Theorem 1, we obtain the desired conclusion. □

Corollary 5. *Let H_1, H_2 be two real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Suppose that the problem (SFP) has a nonempty solution set $\mathcal{F}$. Let $T : C \to C$ be a right Bregman strongly quasi-nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$, $I - T$ is demiclosed at zero and $F(T) \cap \mathcal{F} \neq \emptyset$. Suppose that $\{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\beta_n \leq b < 1$, $(1 - \beta_n)a < \gamma_n < \delta_n$, $a \in (0, \frac{1}{2})$. For any fixed $u \in C$, let the sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$ and*

$$\begin{cases} x_n = P_C\big(u_n - s_n A^*(I - P_Q)Au_n\big), \\ u_{n+1} = P_C\big(\beta_n u + \gamma_n x_n + \delta_n T x_n\big) \end{cases} \tag{47}$$

for each $n \geq 1$. Suppose the following conditions are satisfied:

(a) $\lim\limits_{n\to\infty} \beta_n = 0$;

(b) $\sum\limits_{n=1}^{\infty} \beta_n = \infty$;

(c) $0 < s \leq s_n \leq k < \dfrac{2}{\|A\|^2}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to a point $\Pi_{F(T)\cap\mathcal{F}} u$.

Proof. Let $E = H$ and $\alpha_n = 0$ for each $n \geq 1$. Since the duality mappings $J^q_{E_1^*}$, $J^p_{E_1}$ and $J^p_{E_2}$ are the identity mapping in a Hilbert space H, from Theorem 1, we obtain the desired conclusion. □

Remark 1. *A prototype for the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}, \{\delta_n\}$ and $\{s_n\}$ in Theorem 1 are as follows:*

$$\alpha_n = \frac{1}{n+1}, \quad \beta_n = 1 - \frac{2an+1}{n+1}, \quad \gamma_n = \frac{\frac{1}{2}an}{n+1}, \quad \delta_n = \frac{\frac{3}{2}an}{n+1}, \quad a \in \left(0, \frac{1}{2}\right)$$

and

$$s_n = \left(\frac{n}{n+1}\right)\left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$$

for each $n \geq 0$.

4. Some Numerical Examples

In this section, we present some preliminary numerical results to illustrate the main result, Theorem 1. All codes were written in Matlab 2013b and run on Sumsung i-3 Core laptop.

Example 1. *We find a numerical example in* $(\mathbb{R}^3, \|\cdot\|_2)$ *of the problem considered in Theorem 1 of the previous section. Now, take*

$$C := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle \geq b\},$$

where $a = (2, -6, 1)$ *and* $b = -4$. *Then we have*

$$P_C(x) = \frac{b - \langle a, x\rangle}{\|a\|_2^2} a + x.$$

Let $Q := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle c, x\rangle = d\}$, *where* $c = (3, 5, 7)$ *and* $d = 2$. *Then we have*

$$\Pi_Q(x) = P_Q(x) = \max\left\{0, \frac{d - \langle c, x\rangle}{\|c\|_2^2}\right\} c + x.$$

Suppose that the mapping T *in Theorem 1 is defined as* $T := P_C$, *the metric projection on* C. *Then the problem considered in Theorem 1 reduces to the problem:*

$$\text{Find } \ x \in F(T) \cap C(= C) \ \text{ such that } \ Ax \in Q. \tag{48}$$

Let $\mathcal{F}$ *denote the set of solutions of the problem (48) with* $\mathcal{F} \neq \emptyset$. *Furthermore, let*

$$\alpha_n = \frac{1}{n+1}, \ \beta_n = \frac{n}{8(n+1)}, \ \gamma_n = \frac{n}{8(n+1)}, \ \delta_n = \frac{3n}{8(n+1)}, \ A = \begin{pmatrix} 4 & -6 & -8 \\ -5 & 1 & -5 \\ -8 & -5 & 4 \end{pmatrix}.$$

Then our iterative processes (20) becomes

$$\begin{cases} x_n = P_C\big(u_n - s_n A^T(I - P_Q)Au_n\big), \\ u_{n+1} = P_C\Big[\dfrac{u}{n+1} + \Big(1 - \dfrac{1}{n+1}\Big)\Big(\dfrac{n}{2(n+1)}u + \dfrac{n}{8(n+1)}x_n + \dfrac{3n}{8(n+1)}P_C x_n\Big)\Big] \end{cases} \tag{49}$$

for each $n \geq 1$. *Now, we make different choices of* u, u_1, s_n *and take* $\dfrac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-2}$ *as our stopping criterion.*

Case I. Take $u = (1, 1, 1)$, $u_1 = (3, 0, 4)$ and $s_n = 0.0137$. Then we have the numerical analysis tabulated in Table 1 and show in Figure 1.

Table 1. Example 1, case I.

s_n	Time Taken	Number of Iterations	$\|x_{n+1} - x_n\|_2$	$\|u_{n+1} - u_n\|_2$
0.0137	0.0400	2	0.6758	4.2169
		3	0.0122	0.0453
		4	0.0177	0.0497
		5	0.0158	0.0265
		6	0.0146	0.0194
		7	0.0130	0.0154
		8	0.0113	0.0127
		9	0.0098	0.0107
		10	0.0085	0.0091
		11	0.0074	0.0079

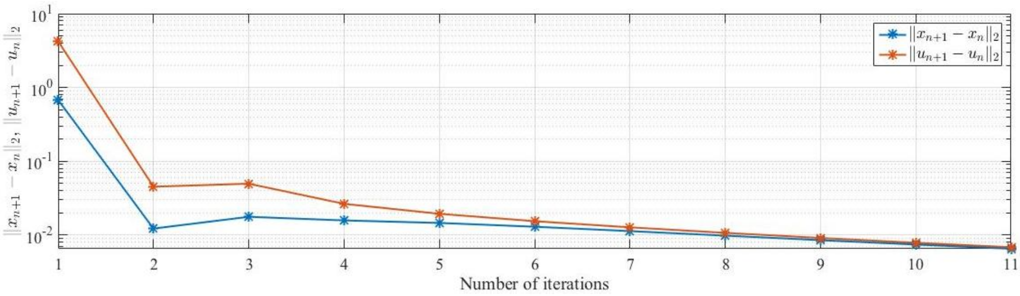

Figure 1. Example 1, case I.

Case II. Take $u = (1,1,1)$, $u_1 = (3,0,4)$ and $s_n = 0.0001$. Then we have the numerical analysis tabulated in Table 2 and show in Figure 2.

Table 2. Example 1, case II.

s_n	Time Taken	Number of Iterations	$\|x_{n+1}-x_n\|_2$	$\|u_{n+1}-u_n\|_2$
0.0001	0.0400	2	1.7588	3.9690
		3	0.1950	0.1968
		4	0.0181	0.0182
		5	0.0265	0.0266
		6	0.0318	0.0320
		7	0.0303	0.0305
		8	0.0271	0.0273
		9	0.0237	0.0239
		10	0.0207	0.0209
		11	0.0181	0.0182

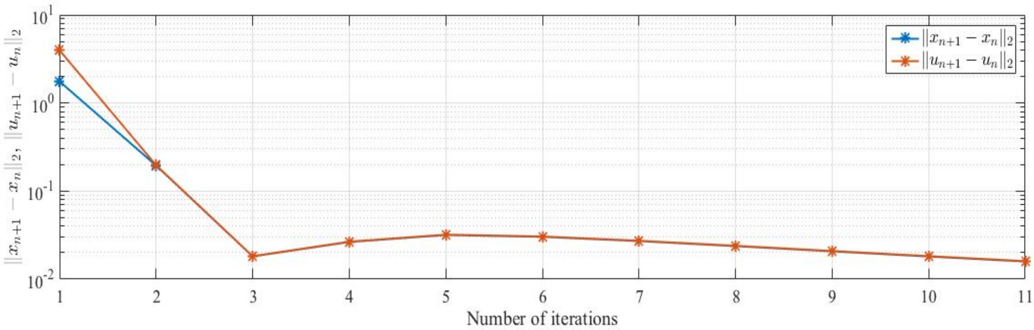

Figure 2. Example 1, case II.

Case III. Take $u = (1,1,1)$, $u_1 = (3,0,4)$ and $s_n = 0.0000001$. Then we have the numerical analysis tabulated in Table 3 and show in Figure 3.

Table 3. Example 1, case III.

s_n	Time Taken	Number of Iterations	$\|x_{n+1}-x_n\|_2$	$\|u_{n+1}-u_n\|_2$
0.0000001	0.0400	2	1.7720	3.9674
		3	0.1969	0.1969
		4	0.0179	0.0179
		5	0.0272	0.0272
		6	0.0326	0.0326
		7	0.0311	0.0311
		8	0.0278	0.0278
		9	0.0244	0.0244
		10	0.0213	0.0213
		11	0.0186	0.0186

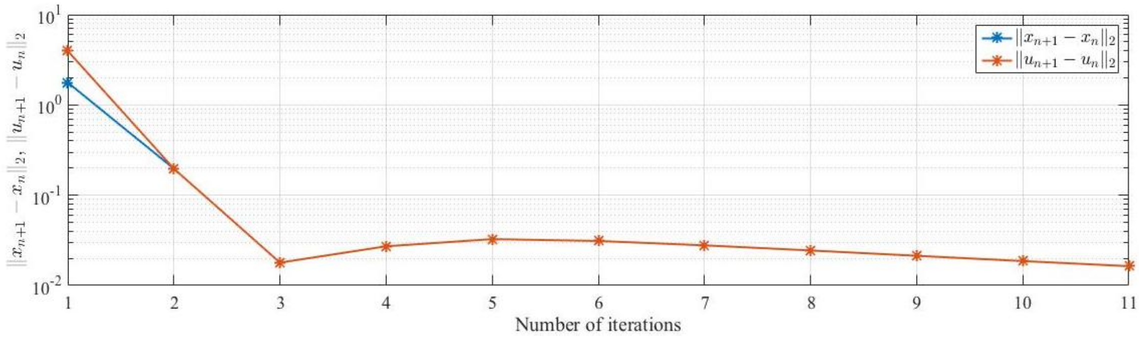

Figure 3. Example 1, case III.

Remark 2. *We make the following comments from Example 1. We observe that different choices of s_n has no effect in terms of number of iterations obtained and the time taken for the convergence of our algorithm.*

5. Conclusions

Our iterative processes can be used for finding a common element of the solution set $\mathcal{F}$ of the split feasibility problem and the fixed point set $F(T)$ of a right Bregman strongly quasi-nonexpansive mapping T in p-uniformly convex Banach spaces, which are also uniformly smooth.

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