Public Capital and the Labor Income Share:
Technical Appendix

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1 Introduction

The main text refers to this technical appendix in several occasions. It describes the macroeconomic model used in the paper and reports the relevant mathematical derivations, including first-order conditions, log-linearization, and transitional dynamics. This appendix complements the main text by providing additional technical derivations.

2 The Model

This section describes the dynamic general equilibrium model. It subsequently discusses the behavior firms, individual households, aggregate households, and the government. The model is summarized in Table A1.

2.1 Firms

The goods market is perfectly competitive. Adjustment costs to private capital formation are introduced because of the exogenously given rate of interest.

2.1.1 Production Function

The production function transforms private capital, $K(t)$, and labor, $L(t)$, into homogeneous output, $Y(t)$, according to the following Constant Elasticity of Substitution (CES) specification:

$$Y(t) = \left\{ [A_K(t)K(t)]^{\frac{\sigma-1}{\sigma}} + [A_L(t)L(t)]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

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where $\sigma \equiv d\ln(L/K)/d\ln(Y_K/Y_L) > 0$ is the substitution elasticity between labor and private capital. The terms $A_K(t)$ and $A_L(t)$ capture capital-augmenting and labor-augmenting technical change, respectively. The CES production function embeds the Leontief technology for $\sigma = 0$, the Cobb-Douglas technology for $\sigma = 1$, and the linear technology for $\sigma \to \infty$. Private capital and labor are said to be gross complements if $\sigma < 1$, and gross substitutes if $\sigma > 1$.

The stock of public capital, $K_G(t)$, enters the production function through the factor-augmentation terms, A_K and A_L:

$$A_i(t) \equiv \rho_i K_G(t)^{\eta_i}, \qquad i = \{K, L\}, \tag{2}$$

where η_i represents the elasticity of the factor-augmentation term and $\rho_i > 0$ is a scaling factor. If $\eta_K = \eta_L = \eta$, public capital augments private inputs in the same proportion and is therefore said to be factor-neutral. Private capital-augmenting public capital is captured by $\eta_K > \eta_L = 0$, whereas labor-augmenting public capital is described by $\eta_L > \eta_K = 0$.

Using equation (1), one finds the following marginal productivity conditions for private inputs:

$$Y_K(t) \equiv \frac{\partial Y(t)}{\partial K(t)} = A_K(t)^{(\sigma-1)/\sigma} \left(\frac{Y(t)}{K(t)}\right)^{1/\sigma}, \tag{3}$$

$$Y_L(t) \equiv \frac{\partial Y(t)}{\partial L(t)} = A_L(t)^{(\sigma-1)/\sigma} \left(\frac{Y(t)}{L(t)}\right)^{1/\sigma}, \tag{4}$$

where subscripts denote partial derivatives. The marginal productivity of public capital is

$$Y_G(t) \equiv \frac{\partial Y(t)}{\partial K_G(t)} = Y(t) \left(\frac{Y_K(t)K(t)}{Y(t)} \frac{E'_K(t)}{E_K(t)} + \frac{Y_L(t)L(t)}{Y(t)} \frac{E'_L(t)}{E_L(t)}\right), \tag{5}$$

which can be rewritten as:

$$\theta_G(t) \equiv \frac{Y_G(t)K_G(t)}{Y(t)} = \theta_K(t)\eta_K + \theta_L(t)\eta_L > 0, \tag{6}$$

where

$$\theta_K(t) \equiv \frac{Y_K(t)K(t)}{Y(t)} > 0, \qquad \theta_L(t) \equiv \frac{Y_L(t)L(t)}{Y(t)} > 0, \qquad \theta_K(t) + \theta_L(t) = 1.$$

Note that public capital is assumed to have a positive effect on private factor productivity. From the above, we know that $\theta_K + \theta_L + \theta_G > \theta_K + \theta_L = 1$. To ensure diminishing returns with respect to *broad* capital—thus excluding endogenous growth—I impose the following conditions:

- Factor-neutral case: $\eta + \theta_K < 1$, so that $\eta < 1 - \theta_K = \theta_L$
- Capital-augmenting case: $\theta_K(1 + \eta_K) < 1$

No conditions are required for the labor-augmenting case, since $\eta_K = 0$ and $\theta_K < 1$.

The ratio of marginal products of private factors of production is given by:

$$\frac{Y_K(t)}{Y_L(t)} = \left(\frac{\rho_K}{\rho_L}\right)^{\frac{\sigma-1}{\sigma}} K_G(t)^{\frac{(\eta_K - \eta_L)(\sigma-1)}{\sigma}} \left(\frac{K(t)}{L(t)}\right)^{-\frac{1}{\sigma}}, \tag{7}$$

If $\sigma > 1$ and $\eta_K - \eta_L > 0$, an increase in $K_G(t)$ increases the relative marginal product of $K(t)$. Thus, public capital is biased toward private capital. In the more empirically-plausible case of $\sigma < 1$ and $\eta_K - \eta_L > 0$, however, an increase in $K_G(t)$ increases the marginal product of $L(t)$. Public capital is then biased toward labor.

2.1.2 First-Order Conditions

Following Uzawa (1969), we postulate a concave accumulation function, $\Phi(\cdot)$, which links net capital accumulation to gross investment ($I(t)$):

$$\dot{K}(t) = \left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] K(t), \tag{8}$$

where δ is the rate of depreciation of private capital.

Firms maximize the net present value of their cash flow subject to the capital accumulation constraint and the stock of public capital (which firms take as given). The current-value Hamiltonian is:

$$\mathcal{H}_F(t) = Y(t) - w(t)L(t) - I(t) + q(t)\left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] K(t), \tag{9}$$

where the production function is given in (1) and $q(t)$ is the (current-value) co-state variable (which is also known as Tobin's q). The prices of investment goods $P_I(t)$ and output $P_Y(t)$ are normalized to unity. The first-order conditions are:

$$w(t) = A_L(t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y(t)}{L(t)}\right)^{\frac{1}{\sigma}}, \tag{10}$$

$$1 = q(t)\Phi'\left(\frac{I(t)}{K(t)}\right), \tag{11}$$

$$\dot{q}(t) = -q(t)\left[\Phi\left(\frac{I(t)}{K(t)}\right) - \frac{I(t)}{K(t)}\Phi'\left(\frac{I(t)}{K(t)}\right) - (r+\delta)\right] - A_K(t)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y(t)}{K(t)}\right)^{\frac{1}{\sigma}}. \tag{12}$$

Equation (10) represents a standard labor demand function. Equation (11) pins down the optimal investment level conditional on the existing stock of capital and its market value (i.e., Tobin's q). Finally, equation (12) describes the evolution of Tobin's q. Note that the first term on its right-hand side is the marginal product of capital.

2.2 Households

2.2.1 Individual Households

Lifetime utility at time t of a representative household born at time $v \leq t$ is given by:

$$\Lambda(v,t) = \int_t^{\infty} \ln U(v,\tau) e^{(\alpha+\beta)(t-\tau)} d\tau, \tag{13}$$

subject to:

$$\dot{A}(v,t) = (r+\beta)A(v,t) + w(t)L(v,t) - T(t) - C(v,t), \tag{14}$$

$$U(v,t) \equiv C(v,t)^{\varepsilon}\left[1 - L(v,t)\right]^{1-\varepsilon}, \qquad 0 < \varepsilon < 1, \tag{15}$$

where $w(t)$ is the (age-independent) real wage, $C(v,t)$ is private consumption, $L(v,t)$ is labor supply,[1] $T(t)$ are lump-sum taxes, α is the pure rate of time preference, and β is the instantaneous probability of death. Private consumption is used as the numeraire and its price is set to unity.

Full consumption $X(v,t)$ is defined as

$$X(v,t) \equiv P(t)U(v,t) \equiv w(t)\left[1 - L(v,t)\right] + C(v,t) \tag{16}$$

where $P(t)$ is a 'true' price index (to be derived below) and $U(v,t)$ is the subutility index given in

[1] Total time is normalized to unity. Hence, leisure is defined as $1 - L(v,t)$.

(15). The household budget identity can now be written as

$$\dot{A}(v,t) = (r+\beta)A(v,t) + w(t) - T(t) - X(v,t). \tag{17}$$

The household's problem is solved by means of two-stage budgeting. In the first stage, the household decides on consumption and savings. The household's current-value Hamiltonian is defined as

$$\mathcal{H}^{H}(v,t) \equiv \ln U(v,t) + \lambda(v,t)\left[(r+\beta)\,A(v,t) + w(t) - T(t) - P(t)U(v,t)\right], \tag{18}$$

where $U(t)$ is the control variable, $A(t)$ denotes the state variable, and λ is the co-state variable. The first-order conditions are:

$$1/U\,(v,t) = \lambda\,(v,t)\,P\,(t)\,, \tag{19}$$

$$\frac{\dot{\lambda}\,(v,t)}{\lambda\,(v,t)} = \alpha - r. \tag{20}$$

Combining (19)–(20) and noting (16) gives the household's Euler equation for full consumption:

$$\frac{\dot{X}\,(v,t)}{X\,(v,t)} = r - \alpha, \tag{21}$$

which can be used in (21) to obtain the Euler equation for felicity:

$$\frac{\dot{U}\,(v,t)}{U\,(v,t)} + \frac{\dot{P}\,(t)}{P\,(t)} = r - \alpha. \tag{22}$$

Integrating the household budget identity (17) gives rise to the lifetime budget constraint for the household:

$$A\,(v,t) + H(t) = \int_t^{\infty} X(v,\tau)e^{(r+\beta)(t-\tau)}d\tau, \tag{23}$$

$$H\,(t) \equiv \int_t^{\infty} \left[w(\tau) - T(\tau)\right] e^{(r+\beta)(t-\tau)}d\tau, \tag{24}$$

where the following no-Ponzi game (NPG) condition has been imposed:

$$\lim_{\tau\to\infty} A(v,\tau)e^{(r+\beta)(t-\tau)} = 0. \tag{25}$$

By substituting (16) and (19) into (23) and simplifying one obtains:

$$
\begin{aligned}
A(v,t)+H(t) &= \int_t^{\infty} P(\tau)U(v,\tau)e^{(r+\beta)(t-\tau)}d\tau \\
&= \int_t^{\infty} [\lambda(v,\tau)]^{-1}\, e^{(r+\beta)(t-\tau)}d\tau. \qquad (26)
\end{aligned}
$$

Using (20) allows us to write:

$$
\lambda(v,\tau) = \lambda(v,t)e^{(r-\alpha)(t-\tau)}. \qquad (27)
$$

Substituting (27) into (26) gives

$$
\begin{aligned}
A(v,t)+H(t) &= \int_t^{\infty} \left[\lambda(v,t)e^{(r-\alpha)(t-\tau)}\right]^{-1} e^{(r+\beta)(t-\tau)}d\tau \\
&= 1/[\lambda(v,t)(\alpha+\beta)] \\
&= X(v,t)/(\alpha+\beta), \qquad (28)
\end{aligned}
$$

where (19) has been used. Obviously, in view of (16) and (28) it also holds that

$$
P(t)U(v,t) = (\alpha+\beta)\left[A(v,t)+H(t)\right]. \qquad (29)
$$

In the second stage, $C(t)$ and $1-L(t)$ are allocated so that $U(\cdot)$ is maximized subject to (16). The first-order condition is

$$
\frac{C(v,t)}{1-L(v,t)} = \frac{\varepsilon}{1-\varepsilon}w(t), \qquad (30)
$$

which is substituted into (16) to arrive at:

$$
\begin{aligned}
C(v,t) &= \varepsilon X(v,t), \qquad (31) \\
w(t)\left[1-L(v,t)\right] &= (1-\varepsilon)X(v,t), \qquad (32)
\end{aligned}
$$

By substituting (31) and (32) into the definition of $U(t)$ (given in (15) above) we obtain the expression of the price index:

$$
P(t) \equiv \left(\frac{1}{\varepsilon}\right)^{\varepsilon}\left(\frac{w(t)}{1-\varepsilon}\right)^{1-\varepsilon} \qquad (33)
$$

2.2.2 Aggregation of the Financial Wealth Equation

The size of each cohort of age v at time t is a fraction $\beta e^{\beta(v-t)}$ of the total population.[2] The relation between total or aggregate financial wealth and the wealth of each individual households is therefore:

$$A(t) = \int_{-\infty}^{t} A(v,t)\beta e^{\beta(v-t)} dv. \tag{34}$$

To derive an equation for the growth of financial wealth and take therefore the first derivative of this equation with respect to time t (using Leibnitz's rule):

$$\begin{aligned} \dot{A}(t) &= \int_{-\infty}^{t} [-\beta A(v,t) + \dot{A}(v,t)]\beta e^{\beta(v-t)} dv + \beta A(t,t) \\ &= -\beta A(t) + \int_{-\infty}^{t} \dot{A}(v,t)\beta e^{\beta(v-t)} dv, \end{aligned} \tag{35}$$

where the fact that households are born without financial wealth (so that $A(t,t) = 0$) has been used in going from the first to the second expression. Substituting (17) into (35) gives

$$\begin{aligned} \dot{A}(t) &= -\beta A(t) + \int_{-\infty}^{t} [(r+\beta)A(v,\tau) + w(\tau) - T(\tau) - X(v,\tau)]\, \beta e^{\beta(v-t)} dv \\ &= rA(t) + w(t) - T(t) - X(t). \end{aligned} \tag{36}$$

2.2.3 Aggregation of Felicity

Aggregate felicity is defined as:

$$U(t) = \int_{-\infty}^{t} U(v,t)\beta e^{\beta(v-t)} dv. \tag{37}$$

Differentiating with respect to time gives:

$$\dot{U}(t) = \int_{-\infty}^{t} [-\beta U(v,t) + \dot{U}(v,t)]\beta e^{\beta(v-t)} dv + \beta U(t,t). \tag{38}$$

[2] We assume large cohorts, so that frequencies and probabilities coincide by the law of large numbers.

Using (22) and (29) in (38) gives

$$\begin{aligned}\dot{U}(t) &= -\beta U(t) + \int_{-\infty}^{t}\left[r-\alpha-\frac{\dot{P}(t)}{P(t)}\right]U(v,t)\beta e^{\beta(v-t)}dv + \beta U(t,t) \\ &= \left[r-\alpha-\frac{\dot{P}(t)}{P(t)}\right]U(t) - \beta\left[U(t)-U(t,t)\right] \\ &= \left[r-\alpha-\frac{\dot{P}(t)}{P(t)}\right]U(t) - \frac{\beta(\alpha+\beta)A(t)}{P(t)}. \qquad (39)\end{aligned}$$

Since $X(t) \equiv P(t)U(t)$ (39) can also be written as

$$\begin{aligned}\frac{\dot{U}(t)}{U(t)} &= \frac{\dot{X}(t)}{X(t)} - \frac{\dot{P}(t)}{P(t)} = \left[r-\alpha-\frac{\dot{P}_U(t)}{P(t)}\right] - \frac{\beta(\alpha+\beta)A(t)}{P(t)U(t)} \\ \frac{\dot{X}(t)}{X(t)} &= r-\alpha-\frac{\beta(\alpha+\beta)A(t)}{X(t)}. \qquad (40)\end{aligned}$$

This is equation (TA1.3) in Table A.1.

2.3 The Government

The government invests in public capital $I_G(t)$ and consumes goods $C_G(t)$. Total public spending is financed by: (i) lump-sum taxes, $T(t)$; and/or (ii) public debt, $B(t)$. The government's budget constraint is:

$$\dot{B}(t) = rB(t) + I_G(t) + C_G(t) - T(t). \qquad (41)$$

Imposing the no-Ponzi game condition $\lim_{\tau\to\infty} B(\tau)e^{-r(\tau-t)} = 0$ gives the government's intertemporal budget constraint

$$B(t) = \int_{t}^{\infty}[T(\tau) - I_G(\tau) - C_G(\tau)]e^{-r(\tau-t)}d\tau. \qquad (42)$$

In order to remain solvent, the government adjusts lump-sum taxes in reaction to developments in spending. This adjustment does not have to be instantaneous. Instead, the government is allowed to use debt-financing so as to delay the tax change by k periods, after which taxes are permanently raised to their new level.

Government capital accumulates according to a concave function similar to that for private capital accumulation:

$$\dot{K}_G(t) = \left[\Phi_G\left(\frac{I_G(t)}{K_G(t)}\right) - \delta_G\right] K_G(t), \tag{43}$$

where δ_G is the rate of depreciation of public capital ($\delta_G \leq \delta$).

3 Solving the Log-Linearized Model

This section derives the impact effect and transitional dynamics of a permanent public investment shock using the log-linearized model.

3.1 Log-Linearization

The model is log-linearized around the initial steady state. The results are reported in Table A.2. Notational conventions are:

$$\tilde{x}(t) \equiv \frac{dx(t)}{x}, \qquad \dot{\tilde{x}}(t) \equiv \frac{d\dot{x}(t)}{x} = \frac{\dot{x}(t)}{x}, \tag{44}$$

where x is the steady-state value of $x(t)$. For a number of variables a slightly different notation is used; asset-like variables (e.g., H, F, B, and A) are defined as:

$$\tilde{x}(t) \equiv \frac{r dx(t)}{Y}, \qquad \dot{\tilde{x}}(t) \equiv \frac{r d\dot{x}(t)}{Y}, \tag{45}$$

and lump-sum taxes are defined as

$$\tilde{T}(t) \equiv \frac{dT(t)}{Y}. \tag{46}$$

We will make use of the Laplace transform technique (Judd, 1982), which allows us to analyze time-varying fiscal shocks.[3] The Laplace transformation of $x(t)$ evaluated at s is given by

$$\mathcal{L}\{x, s\} \equiv \int_0^\infty x(t)e^{-st}\mathrm{d}t. \tag{47}$$

Intuitively, $\mathcal{L}\{x, s\}$ represents the present value of $x(t)$ using s as the discount rate.

[3] See Kreyszig (1993) for a good introduction on the Laplace transform technique.

3.2 The Public Investment Shock

We consider a *permanent* and *unanticipated* increase in public investment occurring at time $t = 0$, implying that $\tilde{I}_G(t) = \tilde{I}_G$ for all $t \geq 0$. Public consumption is assumed not to change: $\tilde{C}_G(t) = 0$. The time path of public capital can be written as:

$$\tilde{K}_G(t) \equiv \begin{cases} \left[1 - e^{-\sigma_G t}\right] \tilde{I}_G & 0 < \delta_G \ll \infty \\ \tilde{I}_G & \delta_G \to \infty \end{cases}, \tag{48}$$

where $\sigma_G \equiv \dfrac{I_G \Phi'_G(\cdot)}{K_G} > 0$ is the elasticity of the public capital installation function. The latter can be derived using the Laplace transform method applied to the log-linearized version of (43):

$$s\mathcal{L}\{\tilde{K}_G, s\} = \sigma_G \left[\mathcal{L}\{\tilde{I}_G, s\} - \mathcal{L}\{\tilde{K}_G, s\}\right], \tag{49}$$

where we have used:

$$\mathcal{L}\{\dot{\tilde{K}}_G, s\} = s\mathcal{L}\{\tilde{K}_G(t), s\} - \tilde{K}_G(0), \tag{50}$$

and the following definition:

$$\begin{aligned} \mathbf{A}\left(\sigma_G, t\right) &\equiv 1 - e^{-\sigma_G t}, \\ \mathcal{L}\left\{\mathbf{A}(\sigma_G, t)\right\} &= \frac{\sigma_G}{s(s+\sigma_G)}. \end{aligned} \tag{51}$$

Equation (48) has two interpretations: (i) the *stock* interpretation (i.e., $0 < \delta_G \ll \infty$); and (ii) the *flow* interpretation (i.e., $\delta_G \to \infty$). The paper focuses on the stock interpretation. See Heijdra and Meijdam (2002, p. 716) for an application of the flow interpretation.

Lump-sum taxes must adjust in response to the public investment shock, but can do so with a delay of k periods:

$$\tilde{T}(t) = \tilde{T}u(t-k) \tag{52}$$

where $u(t-k)$ is an indicator variable assuming the value 0 if $t < k$, and 1 if $t \geq k$. The tax change, $\tilde{T}(t)$, is determined so as to satisfy (42), given the initial conditions and the intended paths of investment spending. Noting that $\int_0^\infty u(\tau - k)e^{-r\tau}d\tau = e^{-kr}/r$, the time-$k$ tax adjustment is thus given by

$$\tilde{T} = e^{kr}\omega_G^I \tilde{I}_G. \tag{53}$$

3.3 The Static System

Equations (TA2.8)-(TA2.10) can be written in matrix notation as:

$$\begin{bmatrix} \frac{1}{\sigma} & -\frac{1}{\sigma} & -1 \\ 1 & -\theta_L & 0 \\ 0 & 1 & -\omega_{LL} \end{bmatrix} \begin{bmatrix} \tilde{Y}(t) \\ \tilde{L}(t) \\ \tilde{w}(t) \end{bmatrix} = \begin{bmatrix} \frac{1-\sigma}{\sigma}\eta_L \tilde{K}_G(t) \\ \tilde{K}^*(t) \\ -\omega_{LL}\tilde{X}(t) \end{bmatrix}, \tag{54}$$

where $K^*(t) \equiv K(t)^{\theta_K} K_G(t)^{\theta_G}$ denotes'broad capital' as, so that $\tilde{K}^*(t)$ is given by:

$$\tilde{K}^*(t) = \theta_K \tilde{K}(t) + \theta_G \tilde{K}_G(t). \tag{55}$$

Using (48), one finds

$$\tilde{K}^*(t) = \theta_K \tilde{K}(t) + \theta_G \left[1 - e^{-\sigma_G t}\right] \tilde{I}_G. \tag{56}$$

The solution of the system is:

$$\begin{bmatrix} \tilde{Y}(t) \\ \tilde{L}(t) \\ \tilde{w}(t) \end{bmatrix} = \Omega \begin{bmatrix} \frac{1-\sigma}{\sigma}\eta_L \tilde{K}_G(t) \\ \tilde{K}^*(t) \\ -\omega_{LL}\tilde{X}(t) \end{bmatrix}, \tag{57}$$

where:

$$\Omega \equiv \begin{bmatrix} \frac{1}{\sigma} & -\frac{1}{\sigma} & -1 \\ 1 & -\theta_L & 0 \\ 0 & 1 & -\omega_{LL} \end{bmatrix}^{-1} = \frac{\sigma}{\sigma + \omega_{LL}\theta_K} \begin{bmatrix} -\theta_L\omega_{LL} & 1 + \frac{\omega_{LL}}{\sigma} & \theta_L \\ -\omega_{LL} & \frac{\omega_{LL}}{\sigma} & 1 \\ -1 & \frac{1}{\sigma} & -\frac{\theta_K}{\sigma} \end{bmatrix}.$$

Therefore, the solution can also be also written as

$$\begin{bmatrix} \tilde{Y}(t) \\ \tilde{L}(t) \\ \tilde{w}(t) \end{bmatrix} = \begin{bmatrix} \theta_L \omega_{LL} \\ \omega_{LL} \\ 1 \end{bmatrix} \frac{(\sigma-1)\eta_L(1-e^{-\sigma_G t})\tilde{I}_G}{\sigma+\omega_{LL}\theta_K}$$
$$+ \begin{bmatrix} \sigma+\omega_{LL} \\ \omega_{LL} \\ 1 \end{bmatrix} \frac{\theta_K \tilde{K}(t) + \theta_G \left(1-e^{-\sigma_G t}\right) \tilde{I}_G}{\sigma+\omega_{LL}\theta_K} - \begin{bmatrix} \sigma\theta_L \\ \sigma \\ -\theta_K \end{bmatrix} \frac{\omega_{LL}\tilde{X}(t)}{\sigma+\omega_{LL}\theta_K}. \tag{58}$$

For future reference, write the quasi-reduced form expression as

$$\begin{bmatrix} \tilde{Y}(t) \\ \tilde{L}(t) \\ \tilde{w}(t) \end{bmatrix} = \begin{bmatrix} \xi_{yk} & \xi_{yx} & \xi_{yg} \\ \xi_{lk} & \xi_{lx} & \xi_{lg} \\ \xi_{wk} & \xi_{wx} & \xi_{wg} \end{bmatrix} \begin{bmatrix} \tilde{K}(t) \\ \tilde{X}(t) \\ \left(1-e^{-\sigma_G t}\right)\tilde{I}_G \end{bmatrix}, \tag{59}$$

where the ξ_{ij} coefficients can be recovered from (58). The most interesting coefficients are those for output:

$$\xi_{yk} \equiv \frac{\theta_K(\sigma+\omega_{LL})}{\sigma+\omega_{LL}\theta_K}, \quad \xi_{yx} \equiv -\frac{\theta_L\omega_{LL}\sigma}{\sigma+\omega_{LL}\theta_K}, \quad \xi_{yg} \equiv \frac{\theta_G(\sigma+\omega_{LL})-(1-\sigma)\omega_{LL}\theta_L\eta_L}{\sigma+\omega_{LL}\theta_K}.$$

For employment, the coefficients are given by:

$$\xi_{lk} \equiv \frac{\theta_K\omega_{LL}}{\sigma+\omega_{LL}\theta_K}, \qquad \xi_{lx} \equiv -\frac{\sigma\omega_{LL}}{\sigma+\omega_{LL}\theta_K}, \qquad \xi_{lg} \equiv \frac{\omega_{LL}[\theta_G+\eta_L(\sigma-1)]}{\sigma+\omega_{LL}\theta_K}.$$

Finally, for the wage rate the coefficients are:

$$\xi_{wk} \equiv \frac{\theta_K}{\sigma+\omega_{LL}\theta_K}, \qquad \xi_{wx} \equiv \frac{\omega_{LL}\theta_K}{\sigma+\omega_{LL}\theta_K}, \qquad \xi_{wg} \equiv \frac{\theta_G+\eta_L(\sigma-1)}{\sigma+\omega_{LL}\theta_K}.$$

3.4 The Dynamic System

The dynamic system can now be written in terms of one matrix equation of the form:

$$\begin{bmatrix} \dot{\tilde{K}}(t) \\ \dot{\tilde{q}}(t) \\ \dot{\tilde{X}}(t) \\ \dot{\tilde{A}}(t) \end{bmatrix} = \Delta \begin{bmatrix} \tilde{K}(t) \\ \tilde{q}(t) \\ \tilde{X}(t) \\ \tilde{A}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma_q(t) \\ 0 \\ \gamma_A(t) \end{bmatrix}, \tag{60}$$

where Δ, with typical element $\bar{\delta}_{ij}$, is given by:

$$\Delta \equiv \begin{bmatrix} 0 & \frac{r\omega_I}{\sigma_A\omega_A} & 0 & 0 \\ \frac{r\theta_K}{\sigma\omega_A}(1-\xi_{yk}) & r & -\frac{r\theta_K}{\sigma\omega_A}\xi_{yx} & 0 \\ 0 & 0 & r-\alpha & -\frac{r-\alpha}{\omega_A} \\ r\omega_w\xi_{wk} & 0 & r(\omega_w\xi_{wx}-\omega_X) & r \end{bmatrix}. \tag{61}$$

The shock terms are defined as $\gamma_q(t)$ and $\gamma_A(t)$:

$$\gamma_q(t) \equiv \frac{r\theta_K}{\sigma\omega_A}\big[\xi_{yg}+(\sigma-1)\eta_K\big](1-e^{-\sigma_G t})\tilde{I}_G, \tag{62}$$

$$\gamma_A(t) \equiv -r\Big[\omega_w\xi_{wg}(1-e^{-\sigma_G t})-e^{kr}\omega_G^I\Big]\tilde{I}_G. \tag{63}$$

For future reference, the shock terms are written in the following compact form:

$$\gamma_i(t) = \pi_{ip}+\pi_{it}e^{-\sigma_G t}, \qquad \text{for } i=q,A, \tag{64}$$

with:

$$\begin{aligned} \pi_{qp} &\equiv \frac{r\theta_K}{\sigma\omega_A}\left[\xi_{yg}+(\sigma-1)\eta_K\right]\tilde{I}_G, \\ \pi_{qt} &\equiv -\frac{r\theta_K}{\sigma\omega_A}[\xi_{yg}+(\sigma-1)\eta_K]\tilde{I}_G, \\ \pi_{Ap} &\equiv -r\left(\omega_w\xi_{wg}-e^{kr}\omega_G^I\right)\tilde{I}_G, \\ \pi_{At} &\equiv r\omega_w\xi_{wg}\tilde{I}_G. \end{aligned}$$

Note that equation (61) embeds two important special cases. First, exogenous labor supply (i.e., $\omega_{LL}=0$), yields $\bar{\delta}_{23}=0$, implying that the $[\tilde{q}(t),\tilde{K}(t)]$ system can be solved independent of the $[\tilde{X}(t),\tilde{A}(t)]$ system. Second, infinitely lived households (i.e., $r=\alpha$) imply that the third row of (61) consists of zeros only. The knife-edge condition $r = \alpha$ yields a hysteretic steady state. The four characteristic roots in this case are: $h_1^* = 0$, $-h_2^* = (r - \sqrt{r^2 + 4\delta_{12}\delta_{21}})/2$, $r_1^* = r$, and $r_2^* = (r + \sqrt{r^2 + 4\delta_{12}\delta_{21}})/2$.

3.5 Stability Issues

Solving the system (60) gives rise to a characteristic polynomial of the fourth order:

$$P(s) \equiv |s\mathrm{I} - \Delta| = \phi(s)\,\psi(s) - \bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}\bar{\delta}_{41} = 0, \tag{65}$$

where I is the identity matrix and $\phi(s)$ and $\psi(s)$ are:

$$\phi(s) \equiv \left(s - \bar{\delta}_{33}\right)\left(s - \bar{\delta}_{22}\right) - \bar{\delta}_{34}\bar{\delta}_{43}, \tag{66}$$

$$\psi(s) \equiv s\left(s - \bar{\delta}_{22}\right) - \bar{\delta}_{12}\bar{\delta}_{21}. \tag{67}$$

$P(s)$ can be written as

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0, \tag{68}$$

where the $a_i's$ are defined as:

$$a_3 \equiv -\mathrm{tr}(\Delta) = -(2\bar{\delta}_{22} + \bar{\delta}_{33}) < 0 \tag{69}$$

$$a_2 \equiv \bar{\delta}_{22}^2 - \bar{\delta}_{12}\bar{\delta}_{21} + 2\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43} \tag{70}$$

$$a_1 \equiv \bar{\delta}_{12}\bar{\delta}_{21}(\bar{\delta}_{22} + \bar{\delta}_{33}) + \bar{\delta}_{22}\left[\bar{\delta}_{34}\bar{\delta}_{43} - \bar{\delta}_{22}\bar{\delta}_{33}\right] \tag{71}$$

$$a_0 \equiv |\Delta| \tag{72}$$

where the determinant of Δ is equal to:

$$\begin{aligned}
|\Delta| &\equiv -(r-\alpha)\frac{r^3\omega_I\theta_K}{\sigma\sigma_A\omega_A^2}\begin{vmatrix} 1-\xi_{yk} & -\xi_{yx} & 0 \\ 0 & 1 & -\frac{1}{\omega_A} \\ \omega_w\xi_{wk} & \omega_w\xi_{wx}-\omega_X & 1 \end{vmatrix} \\
&= -(r-\alpha)\frac{r^3\omega_I\theta_K}{\sigma\sigma_A\omega_A^2}\left\{1-\xi_{yk}+\frac{1}{\omega_A}\left[(1-\xi_{yk})(\omega_w\xi_{wx}-\omega_X)+\omega_w\xi_{yx}\xi_{wk}\right]\right\} \\
&= -(r-\alpha)\frac{r^3\omega_I\theta_K(1-\xi_{yk})}{\sigma\sigma_A\omega_A^3}\left[\omega_A+\omega_w\xi_{wx}-\omega_X+\omega_w\left(\frac{\xi_{yx}\xi_{wk}}{1-\xi_{yk}}\right)\right] \\
&= (r-\alpha)\frac{r^3\omega_I\theta_K\theta_L}{\sigma_A\omega_A^3(\sigma+\omega_{LL}\theta_K)}(\omega_X-\omega_A)>0, \qquad (73)
\end{aligned}$$

where use was made of $1-\xi_{yk}=\frac{\theta_L\sigma}{\sigma+\omega_{LL}\theta_K}$ and $\frac{\xi_{yx}\xi_{wk}}{1-\xi_{yk}}=-\xi_{wx}=-\frac{\theta_K\omega_{LL}}{\sigma+\omega_{LL}\theta_K}$, in going from the third to the last line.

The positive determinant may either indicate two positive roots and two negative roots or four positive roots (in which case the system is unstable). The case of four negative roots—giving rise to an indeterminate steady state—is excluded because of the positive trace of Δ (i.e., $\text{tr}(\Delta)>0$). The model has a unique and locally saddle-path stable steady state, featuring four characteristic roots. All roots are real. The two stable (negative) roots are denoted by $-h_1^*<0$ and $-h_2^*<0$; the two unstable (positive) roots are denoted by $r_1^*>0$ and $r_2^*>0$.

4 Solving for the Comparative Dynamics

4.1 The Reduced-Form Model

By taking the Laplace transform of (60) and noting that $\tilde{K}(0)=0$ and recognizing that $\tilde{A}(0)\neq 0$ due to unanticipated capital gains/losses (i.e. $\tilde{A}(0)=\omega_A\tilde{q}(0)$) we obtain:

$$\Lambda(s)\begin{bmatrix}\mathcal{L}\{\tilde{K},s\} \\ \mathcal{L}\{\tilde{q},s\} \\ \mathcal{L}\{\tilde{X},s\} \\ \mathcal{L}\{\tilde{A},s\}\end{bmatrix}=\begin{bmatrix}0 \\ \tilde{q}(0)-\mathcal{L}\{\gamma_q(t),s\} \\ \tilde{X}(0) \\ \omega_A\tilde{q}(0)-\mathcal{L}\{\gamma_A(t),s\}\end{bmatrix}, \qquad (74)$$

where $\Lambda(s)\equiv s\mathrm{I}-\Delta$. We know that:

$$\Lambda(s)^{-1}\equiv\frac{1}{(s+h_1^*)(s+h_2^*)(s-r_1^*)(s-r_2^*)}\operatorname{adj}\Lambda(s), \qquad (75)$$

where $\operatorname{adj}\Lambda(s)$ is the adjoint matrix of $\Lambda(s)$. By pre-multiplying both sides of (74) by $\Lambda(s)^{-1}$ and rearranging we obtain the following expression in Laplace transforms:

$$(s+h_1^*)(s+h_2^*)\begin{bmatrix}\mathcal{L}\{\tilde{K},s\}\\ \mathcal{L}\{\tilde{q},s\}\\ \mathcal{L}\{\tilde{X},s\}\\ \mathcal{L}\{\tilde{A},s\}\end{bmatrix}=\frac{\operatorname{adj}\Lambda(s)\begin{bmatrix}0\\ \tilde{q}(0)-\mathcal{L}\{\gamma_q(t),s\}\\ \tilde{X}(0)\\ \omega_A\tilde{q}(0)-\mathcal{L}\{\gamma_A(t),s\}\end{bmatrix}}{(s-r_1^*)(s-r_2^*)}. \tag{76}$$

The $\operatorname{adj}\Lambda(s)$ matrix is equal to:

$$\operatorname{adj}\Lambda(s)\equiv\begin{bmatrix}\left(s-\bar{\delta}_{22}\right)\phi(s) & \bar{\delta}_{12}\phi(s) & \bar{\delta}_{12}\bar{\delta}_{23}\left(s-\bar{\delta}_{22}\right) & \bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}\\ \bar{\delta}_{21}\phi(s)+\bar{\delta}_{23}\bar{\delta}_{34}\bar{\delta}_{41} & s\phi(s) & \bar{\delta}_{23}s\left(s-\bar{\delta}_{22}\right) & \bar{\delta}_{23}\bar{\delta}_{34}s\\ \bar{\delta}_{34}\bar{\delta}_{41}\left(s-\bar{\delta}_{22}\right) & \bar{\delta}_{12}\bar{\delta}_{34}\bar{\delta}_{41} & \left(s-\bar{\delta}_{22}\right)\psi(s) & \bar{\delta}_{34}\psi(s)\\ \bar{\delta}_{41}\left(s-\bar{\delta}_{22}\right)\left(s-\bar{\delta}_{33}\right) & \bar{\delta}_{12}\bar{\delta}_{41}\left(s-\bar{\delta}_{33}\right) & \bar{\delta}_{43}\psi(s)+\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{41} & \left(s-\bar{\delta}_{33}\right)\psi(s)\end{bmatrix}. \tag{77}$$

The following useful results can be established.

Lemma 1. *Define $\phi(s)$ and $\psi(s)$ as in (66) and (67), respectively.*
Define $\zeta(s)\equiv\left(s-\bar{\delta}_{22}\right)\psi(s)$ and $\theta(s)\equiv\left(s-\bar{\delta}_{33}\right)\psi(s)$. Then it follows that:

$$\frac{\phi(s)-\phi(x)}{s-x} = s+x-\bar{\delta}_{22}-\bar{\delta}_{33}, \tag{78}$$
$$\frac{s\phi(s)-x\phi(x)}{s-x} = s^2+sx+x^2-\left(\bar{\delta}_{22}+\bar{\delta}_{33}\right)(s+x)+\bar{\delta}_{22}\bar{\delta}_{33}-\bar{\delta}_{34}\bar{\delta}_{43}, \tag{79}$$
$$\frac{\psi(s)-\psi(x)}{s-x} = s+x-\bar{\delta}_{22}, \tag{80}$$
$$\frac{\zeta(s)-\zeta(x)}{s-x} = s^2+sx+x^2-2\bar{\delta}_{22}(s+x)+\bar{\delta}_{22}^2-\bar{\delta}_{12}\bar{\delta}_{21}, \tag{81}$$
$$\frac{\theta(s)-\theta(x)}{s-x} = s^2+sx+x^2-\left(\bar{\delta}_{22}+\bar{\delta}_{33}\right)(s+x)+\bar{\delta}_{22}\bar{\delta}_{33}-\bar{\delta}_{12}\bar{\delta}_{21}, \tag{82}$$

which we label parts (i)–(v).

Proof: Part (i) can be written as:

$$\begin{aligned}\phi(s)-\phi(x) &= \left(s-\bar{\delta}_{33}\right)\left(s-\bar{\delta}_{22}\right)-\left(x-\bar{\delta}_{33}\right)\left(x-\bar{\delta}_{22}\right)\\ &= \left(s^2-x^2\right)-\left(\bar{\delta}_{22}+\bar{\delta}_{33}\right)(s-x)\\ &= (s-x)\left[s+x-\left(\bar{\delta}_{22}+\bar{\delta}_{33}\right)\right],\end{aligned}$$

where we have used $s^2 - x^2 = (s-x)(s+x)$. The proof of part (iii) is similar. For part (ii), we write:

$$\begin{aligned}
s\phi(s) - x\phi(x) &= \left(s^3 - x^3\right) - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)\left(s^2 - x^2\right) + \left[\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right](s-x) \\
&= (s-x)\left[s^2 + sx + x^2\right] - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)(s-x)(s+x) + \left[\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right](s-x) \\
&= (s-x)\left[s^2 + sx + x^2 - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)(s+x) + \bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right],
\end{aligned}$$

where we have used $s^3 - x^3 = (s-x)\left[s^2 + sx + x^2\right]$. Part (v) can be written as:

$$\begin{aligned}
\theta(s) - \theta(x) &= \left(s^3 - x^3\right) - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)\left(s^2 - x^2\right) + \left[\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{12}\bar{\delta}_{21}\right](s-x) \\
&= (s-x)\left[s^2 + sx + x^2 - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)(s+x) + \left[\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{12}\bar{\delta}_{21}\right]\right].
\end{aligned}$$

The proof of part (iv) is similar ($\bar{\delta}_{33}$ is replaced by $\bar{\delta}_{22}$). □

We know that:

$$\begin{aligned}
&\operatorname{adj}\Delta = -\operatorname{adj}\Lambda(0) \\
&= \begin{bmatrix}
\bar{\delta}_{22}\phi(0) & -\bar{\delta}_{12}\phi(0) & \bar{\delta}_{12}\bar{\delta}_{22}\bar{\delta}_{23} & -\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34} \\
-\bar{\delta}_{21}\phi(0) - \bar{\delta}_{23}\bar{\delta}_{34}\bar{\delta}_{41} & 0 & 0 & 0 \\
\bar{\delta}_{22}\bar{\delta}_{34}\bar{\delta}_{41} & -\bar{\delta}_{12}\bar{\delta}_{34}\bar{\delta}_{41} & \bar{\delta}_{22}\psi(0) & -\bar{\delta}_{34}\psi(0) \\
-\bar{\delta}_{22}\bar{\delta}_{33}\bar{\delta}_{41} & \bar{\delta}_{12}\bar{\delta}_{33}\bar{\delta}_{41} & -\bar{\delta}_{43}\psi(0) - \bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{41} & \bar{\delta}_{33}\psi(0)
\end{bmatrix},
\end{aligned} \tag{83}$$

which follows from using $s = 0$ in $\Lambda(s) = s\mathrm{I} - \Delta$.

4.2 Jumps

There are two predetermined variables [$\tilde{K}(t)$ and $\tilde{A}(t)$] and two jumping variables [$\tilde{q}(t)$ and $\tilde{X}(t)$], so that only two initial conditions need to be imposed. The system is conditionally stable (i.e., it is a saddle point). Instability originates from the unstable roots.[4] The jumps in $\tilde{X}(0)$ and $\tilde{q}(0)$ are such that the right-hand side of (76) is of the $0 \div 0$ type for both unstable roots, r_1^* and r_2^*.[5] Using the first row of $\operatorname{adj}\Lambda(s)$, for example, we get for $s = r_1^*$ and $s = r_2^*$:

$$\left[\phi(s) + \bar{\delta}_{23}\bar{\delta}_{34}\omega_A\right]\tilde{q}(0) + \bar{\delta}_{23}\left(s - \bar{\delta}_{22}\right)\tilde{X}(0) = \phi(s)\mathcal{L}\{\gamma_q, s\} + \bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A, s\}, \tag{84}$$

[4] Note that the two stable roots determine the speed of transition.

[5] The denominator on the right hand side of (76) is zero. The only way to obtain bounded solutions for the four key variables is that the numerator on the right hand side is also zero.

where we have divided by δ_{12} on both sides of the equation. All rows of $\operatorname{adj}\Lambda(s)$ give the same information (because the rank of $\operatorname{adj}\Lambda(s)$ equals 1, for $s = r_1^*, r_2^*$). In summary, for $s = r_1^*$ and $s = r_2^*$ we have:

$$\bar{\delta}_{12}\left[\phi\left(s\right)+\omega_A\bar{\delta}_{23}\bar{\delta}_{34}\right]\tilde{q}\left(0\right)+\bar{\delta}_{12}\bar{\delta}_{23}\left(s-\bar{\delta}_{22}\right)\tilde{X}\left(0\right)=\bar{\delta}_{12}\phi\left(s\right)\mathcal{L}\{\gamma_q,s\}$$
$$+\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,s\}. \tag{85}$$

Taking the second row of $\operatorname{adj}\Lambda(s)$:

$$\left[\phi\left(s\right)+\omega_A\bar{\delta}_{23}\bar{\delta}_{34}\right]\tilde{q}\left(0\right)+\bar{\delta}_{23}\left(s-\bar{\delta}_{22}\right)\tilde{X}\left(0\right)=\phi\left(s\right)\mathcal{L}\{\gamma_q,s\}$$
$$+\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,s\}. \tag{86}$$

The third row of $\operatorname{adj}\Lambda(s)$ yields:

$$\bar{\delta}_{34}\left[\bar{\delta}_{12}\bar{\delta}_{41}+\omega_A\psi\left(s\right)\right]\tilde{q}\left(0\right)+\left(s-\bar{\delta}_{22}\right)\psi\left(s\right)\tilde{X}\left(0\right)=\bar{\delta}_{12}\bar{\delta}_{34}\bar{\delta}_{41}\mathcal{L}\{\gamma_q,s\}$$
$$+\bar{\delta}_{34}\psi\left(s\right)\mathcal{L}\{\gamma_A,s\}, \tag{87}$$

and the fourth row:

$$\left[\bar{\delta}_{12}\bar{\delta}_{41}+\omega_A\psi\left(s\right)\right]\left(s-\bar{\delta}_{33}\right)\tilde{q}\left(0\right)+\left[\bar{\delta}_{43}\psi\left(s\right)+\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{41}\right]\tilde{X}\left(0\right)=\bar{\delta}_{12}\bar{\delta}_{41}\left(s-\bar{\delta}_{33}\right)\mathcal{L}\{\gamma_q,s\}$$
$$+\left(s-\bar{\delta}_{33}\right)\psi\left(s\right)\mathcal{L}\{\gamma_A,s\}. \tag{88}$$

Taking the first row, we thus have two independent equations in two unknowns, that is, the jumping variables $\tilde{X}(0)$ and $\tilde{q}\,(0)$. Hence, we get:

$$\begin{bmatrix} \phi\left(r_1^*\right)+\bar{\delta}_{23}\bar{\delta}_{34}\omega_A & \bar{\delta}_{23}\left(r_1^*-\bar{\delta}_{22}\right) \\ \phi\left(r_2^*\right)+\bar{\delta}_{23}\bar{\delta}_{34}\omega_A & \bar{\delta}_{23}\left(r_2^*-\bar{\delta}_{22}\right) \end{bmatrix}\begin{bmatrix} \tilde{q}\left(0\right) \\ \tilde{X}\left(0\right) \end{bmatrix}=\begin{bmatrix} \phi\left(r_1^*\right)\mathcal{L}\{\gamma_q,r_1^*\}+\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,r_1^*\} \\ \phi\left(r_2^*\right)\mathcal{L}\{\gamma_q,r_2^*\}+\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,r_2^*\} \end{bmatrix}, \tag{89}$$

or:

$$\begin{bmatrix} \tilde{q}(0) \\ \tilde{X}(0) \end{bmatrix}=\begin{bmatrix} \phi\left(r_1^*\right)+\bar{\delta}_{23}\bar{\delta}_{34}\omega_A & \bar{\delta}_{23}\left(r_1^*-\bar{\delta}_{22}\right) \\ \phi\left(r_2^*\right)+\bar{\delta}_{23}\bar{\delta}_{34}\omega_A & \bar{\delta}_{23}\left(r_2^*-\bar{\delta}_{22}\right) \end{bmatrix}^{-1}\begin{bmatrix} \phi\left(r_1^*\right)\mathcal{L}\{\gamma_q,r_1^*\}+\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,r_1^*\} \\ \phi\left(r_2^*\right)\mathcal{L}\{\gamma_q,r_2^*\}+\bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A,r_2^*\} \end{bmatrix}. \tag{90}$$

4.3 Transitional Dynamics

4.3.1 Relevant Lemmas

Several results can be derived from the generic shock expression (64).

Lemma 2. *Consider $\phi(s)$, $\psi(s)$, and $\gamma_i(t)$ as defined in (66), (67), and (64), respectively. Write $\theta(s) \equiv \left(s-\bar{\delta}_{33}\right)\psi(s)$. Then the following results can be established:*

$$
\begin{aligned}
\mathcal{L}\{\gamma_i, s\} &= \frac{\pi_{ip}}{s}+\frac{\pi_{it}}{s+\sigma_G}, && (91)\\
\frac{\mathcal{L}\{\gamma_i, x\}-\mathcal{L}\{\gamma_i, s\}}{s-x} &= \frac{\pi_{ip}}{sx}+\frac{\pi_{it}}{(x+\sigma_G)(s+\sigma_G)}, && (92)\\
\frac{\left(s-\bar{\delta}_{33}\right)\mathcal{L}\{\gamma_i, s\}-\left(x-\bar{\delta}_{33}\right)\mathcal{L}\{\gamma_i, x\}}{s-x} &= \frac{\pi_{ip}\bar{\delta}_{33}}{sx}+\frac{\pi_{it}\left(\bar{\delta}_{33}+\sigma_G\right)}{(x+\sigma_G)(s+\sigma_G)}, && (93)\\
\frac{\phi(x)\mathcal{L}\{\gamma_i, x\}-\phi(s)\mathcal{L}\{\gamma_i, s\}}{s-x} &= -\pi_{ip}\left[1+\frac{\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}}{xs}\right] && (94)\\
&\quad +\pi_{it}\left[\frac{\phi(-\sigma_G)}{(x+\sigma_G)(s+\sigma_G)}-1\right], \\
\frac{\psi(x)\mathcal{L}\{\gamma_i, x\}-\psi(s)\mathcal{L}\{\gamma_i, s\}}{s-x} &= -\pi_{ip}\left[1+\frac{\bar{\delta}_{12}\bar{\delta}_{21}}{xs}\right] \\
&\quad +\pi_{it}\left[\frac{\psi(-\sigma_G)}{(x+\sigma_G)(s+\sigma_G)}-1\right], && (95)\\
\frac{\theta(x)\mathcal{L}\{\gamma_i, x\}-\theta(s)\mathcal{L}\{\gamma_i, s\}}{s-x} &= \left[-s-x+\bar{\delta}_{22}+\bar{\delta}_{33}+\frac{\bar{\delta}_{12}\bar{\delta}_{21}\bar{\delta}_{33}}{xs}\right]\pi_{ip} \\
&\quad +\Bigg[-s-x+\bar{\delta}_{22}+\bar{\delta}_{33}+\sigma_G \\
&\quad -\frac{\left(\sigma_G+\bar{\delta}_{33}\right)\psi(-\sigma_G)}{(x+\sigma_G)(s+\sigma_G)}\Bigg]\pi_{it}. && (96)
\end{aligned}
$$

Proof: Parts (i)–(ii) are obvious. For part (iii) we write:

$$
\begin{aligned}
\left(x-\bar{\delta}_{33}\right)\mathcal{L}\{\gamma_i, x\} &- \left(s-\bar{\delta}_{33}\right)\mathcal{L}\{\gamma_i, s\} = \frac{\pi_{it}x}{x+\sigma_G}-\frac{\pi_{it}s}{s+\sigma_G}-\bar{\delta}_{33}\left[\mathcal{L}\{\gamma_i, x\}-\mathcal{L}\{\gamma_i, s\}\right]\\
&= \pi_{it}\left(\frac{x}{x+\sigma_G}-\frac{s}{s+\sigma_G}\right)-\bar{\delta}_{33}(s-x)\left[\frac{\pi_{ip}}{sx}+\frac{\pi_{it}}{(x+\sigma_G)(s+\sigma_G)}\right]\\
&= \pi_{it}\frac{x(s+\sigma_G)-s(x+\sigma_G)}{(x+\sigma_G)(s+\sigma_G)}-\bar{\delta}_{33}(s-x)\left[\frac{\pi_{ip}}{sx}+\frac{\pi_{it}}{(x+\sigma_G)(s+\sigma_G)}\right]\\
&= -(s-x)\left(\frac{\pi_{it}\sigma_G}{(x+\sigma_G)(s+\sigma_G)}+\bar{\delta}_{33}\left[\frac{\pi_{ip}}{sx}+\frac{\pi_{it}}{(x+\sigma_G)(s+\sigma_G)}\right]\right)\\
&= -(s-x)\left(\frac{\pi_{ip}\bar{\delta}_{33}}{sx}+\frac{\pi_{it}\left(\bar{\delta}_{33}+\sigma_G\right)}{(x+\sigma_G)(s+\sigma_G)}\right).
\end{aligned}
$$

For part (iv), we write $\phi(s)\,\mathcal{L}\{\gamma_i, s\}$ as:

$$\begin{aligned}
\phi(s)\,\mathcal{L}\{\gamma_i, s\} &= \frac{\pi_{ip}}{s}\left[s^2 - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)s + \left(\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right)\right] \\
&\quad + \frac{\pi_{it}}{s+\sigma_G}\Big[\,(s+\sigma_G)^2 - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)(s+\sigma_G) + \left(\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right) \\
&\quad + \sigma_G\left(\bar{\delta}_{22} + \bar{\delta}_{33} + \sigma_G - 2\left(s+\sigma_G\right)\right)\Big] \\
&= \pi_{ip}\left[s - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right) + \left(\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right)\frac{1}{s}\right] \\
&\quad + \pi_{it}\left[(s+\sigma_G) - \left(\bar{\delta}_{22} + \bar{\delta}_{33} + 2\sigma_G\right) + \frac{\phi(-\sigma_G)}{s+\sigma_G}\right],
\end{aligned}$$

where $\phi(-\sigma_G)$ is given by:

$$\phi(-\sigma_G) \equiv \sigma_G^2 + \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)\sigma_G + \bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}. \tag{97}$$

Hence, it follows that:

$$\begin{aligned}
\phi(x)\,\mathcal{L}\{\gamma_i, x\} - \phi(s)\,\mathcal{L}\{\gamma_i, s\} &= \pi_{ip}\left[x - s + \left(\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}\right)\left(\frac{1}{x} - \frac{1}{s}\right)\right] \\
&\quad + \pi_{it}\left[x - s + \phi(-\sigma_G)\left(\frac{1}{x+\sigma_G} - \frac{1}{s+\sigma_G}\right)\right] \\
&= (s-x)\,\pi_{ip}\left[-1 + \frac{\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{34}\bar{\delta}_{43}}{xs}\right] \\
&\quad + (s-x)\,\pi_{it}\left[\frac{\phi(-\sigma_G)}{(x+\sigma_G)(s+\sigma_G)} - 1\right].
\end{aligned}$$

The proof of part (v) is similar and $\psi(-\sigma_G)$ is given by:

$$\psi(-\sigma_G) \equiv \sigma_G^2 + \bar{\delta}_{22}\sigma_G - \bar{\delta}_{12}\bar{\delta}_{21}. \tag{98}$$

To prove part (vi) we write $\theta(s)\,\mathcal{L}\{\gamma_i, s\}$ as:

$$\begin{aligned}
\theta(s)\,\mathcal{L}\{\gamma_i, s\} &= \pi_{ip}\left[s^2 - \left(\bar{\delta}_{22} + \bar{\delta}_{33}\right)s + \left(\bar{\delta}_{22}\bar{\delta}_{33} - \bar{\delta}_{12}\bar{\delta}_{21}\right) + \frac{\bar{\delta}_{12}\bar{\delta}_{21}\bar{\delta}_{33}}{s}\right] \\
&\quad + \pi_{it}\left[s^2 - \bar{\delta}_{22}s - \bar{\delta}_{12}\bar{\delta}_{21} - \left(\bar{\delta}_{33} + \sigma_G\right)\frac{s^2 - \bar{\delta}_{22}s - \bar{\delta}_{12}\bar{\delta}_{21}}{s+\sigma_G}\right].
\end{aligned}$$

We know that:

$$
\begin{aligned}
s^2 - x^2 &= (s-x)(s+x), \\
\frac{1}{s+\sigma_G} - \frac{1}{x+\sigma_G} &= -(s-x)\frac{1}{(x+\sigma_G)(s+\sigma_G)} \\
\frac{s}{s+\sigma_G} - \frac{x}{x+\sigma_G} &= (s-x)\frac{\sigma_G}{(x+\sigma_G)(s+\sigma_G)}, \\
\frac{s^2}{s+\sigma_G} - \frac{x^2}{x+\sigma_G} &= \frac{x^2+(s-x)(s+x)}{s+\sigma_G} - \frac{x^2}{x+\sigma_G} \\
&= (s-x)\left[1+\frac{x-\sigma_G}{s+\sigma_G} - \frac{x^2}{(x+\sigma_G)(s+\sigma_G)}\right] \\
&= (s-x)\left[1-\frac{\sigma_G^2}{(x+\sigma_G)(s+\sigma_G)}\right].
\end{aligned}
$$

By using these results we obtain:

$$
\begin{aligned}
\theta(x)\mathcal{L}\{\gamma_i, x\} - \theta(s)\mathcal{L}\{\gamma_i, s\} &= \pi_{ip}(s-x)\left[-(s+x)+\bar{\delta}_{22}+\bar{\delta}_{33}+\frac{\bar{\delta}_{12}\bar{\delta}_{21}\bar{\delta}_{33}}{xs}\right] \\
&\quad +\pi_{it}(s-x)\Bigg[-(s+x)+\bar{\delta}_{22}+\bar{\delta}_{33}+\sigma_G \\
&\quad -\left(\bar{\delta}_{33}+\sigma_G\right)\frac{\psi(-\sigma_G)}{(x+\sigma_G)(s+\sigma_G)}\Bigg].
\end{aligned}
$$

This establishes the result. □

Below, we will need some inverse Laplace transforms. The first transform we need the inverse of is:

$$
\frac{1}{(s+h_1^*)(s+h_2^*)} = \frac{1}{h_2^*-h_1^*}\left(\frac{1}{s+h_1^*} - \frac{1}{s+h_2^*}\right). \tag{99}
$$

The inverse of the Laplace transform is then a stable transition term of the form:

$$
\mathcal{L}^{-1}\left\{\frac{1}{(s+h_1^*)(s+h_2^*)}\right\} = \frac{1}{h_2^*-h_1^*}\left(e^{-h_1^* t} - e^{-h_2^* t}\right),
$$

as summarized for convenience in the following definition:

Lemma 3. *The temporary transition term* $\mathbf{T_1}(h_1^*, h_2^*, t)$ *is given by:*

$$
\begin{aligned}
\mathbf{T_1}(h_1^*, h_2^*, t) &\equiv \frac{e^{-h_1^* t} - e^{-h_2^* t}}{h_2^*-h_1^*}, \\
\mathcal{L}\{\mathbf{T_1}(h_1^*, h_2^*, t)\} &= \frac{1}{(s+h_1^*)(s+h_2^*)}.
\end{aligned}
$$

Properties: (i) $\mathbf{T_1}(h_1^*, h_2^*, 0) = 0$*; and (ii)* $\lim_{t\to\infty}\mathbf{T_1}(h_1^*, h_2^*, t) = 0$.

The second transform we need the inverse of is:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s+h_1^*)(s+h_2^*)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{h_2^*-h_1^*}\left(\frac{s}{s+h_1^*}-\frac{s}{s+h_2^*}\right)\right\} \\
&= \frac{1}{h_2^*-h_1^*}\left(-h_1^* e^{-h_1^* t}+h_2^* e^{-h_2^* t}\right),
\end{aligned}$$

which we define in the following lemma:

Lemma 4. *The temporary transition term* $\mathbf{T_2}\left(h_1^*, h_2^*, t\right)$ *is:*

$$\begin{aligned}
\mathbf{T_2}\left(h_1^*, h_2^*, t\right) &\equiv \frac{h_2^* e^{-h_2^* t}-h_1^* e^{-h_1^* t}}{h_2^*-h_1^*}=\frac{d\mathbf{T_1}\left(h_1^*, h_2^*, t\right)}{dt}, \\
\mathcal{L}\left\{\mathbf{T_2}(h_1^*, h_2^*, t)\right\} &= \frac{s}{(s+h_1^*)(s+h_2^*)}.
\end{aligned}$$

Properties: (i) $\mathbf{T_2}\left(h_1^*, h_2^*, 0\right)=1$*; and (ii)* $\lim_{t\to\infty}\mathbf{T_2}\left(h_1^*, h_2^*, t\right)=0$.

The third inverse we need is:

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{(s+h_1^*)(s+h_2^*)(s+\sigma_G)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{h_2^*-h_1^*}\left[\frac{1}{(s+h_1^*)\left(s+\sigma_G\right)}-\frac{1}{(s+h_2^*)\left(s+\sigma_G\right)}\right]\right\} \\
&=\mathcal{L}^{-1}\left\{\frac{1}{h_2^*-h_1^*}\left[\frac{1}{\sigma_G-h_1^*}\left(\frac{1}{s+h_1^*}-\frac{1}{s+\sigma_G}\right)-\frac{1}{\sigma_G-h_2^*}\left(\frac{1}{s+h_2^*}-\frac{1}{s+\sigma_G}\right)\right]\right\} \\
&=\frac{1}{h_2^*-h_1^*}\left[\frac{1}{\sigma_G-h_1^*}\left(e^{-h_1^* t}-e^{-\sigma_G t}\right)-\frac{1}{\sigma_G-h_2^*}\left(e^{-h_2^* t}-e^{-\sigma_G t}\right)\right],
\end{aligned} \tag{100}$$

which we summarize in the following definition:

Lemma 5. *The temporary transition term* $\mathbf{T_3}\left(h_1^*, h_2^*, \sigma_G, t\right)$ *is defined as:*

$$\begin{aligned}
\mathbf{T_3}\left(h_1^*, h_2^*, \sigma_G, t\right) &\equiv \frac{1}{h_2^*-h_1^*}\left[\frac{e^{-h_1^* t}-e^{-\sigma_G t}}{\sigma_G-h_1^*}-\frac{e^{-h_2^* t}-e^{-\sigma_G t}}{\sigma_G-h_2^*}\right] \\
&= \frac{1}{h_2^*-h_1^*}\left[\mathbf{T_1}\left(h_1^*, \sigma_G, t\right)-\mathbf{T_1}\left(h_2^*, \sigma_G, t\right)\right], \\
\mathcal{L}\left\{\mathbf{T_3}(h_1^*, h_2^*, \sigma_G, t)\right\} &= \frac{1}{(s+h_1^*)(s+h_2^*)(s+\sigma_G)}.
\end{aligned}$$

Properties: (i) $\mathbf{T_3}\left(h_1^*, h_2^*, \sigma_G, 0\right)=0$*; and (ii)* $\lim_{t\to\infty}\mathbf{T_3}\left(h_1^*, h_2^*, \sigma_G, t\right)=0$.

Finally, we need the following inverse:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+h_1^*)(s+h_2^*)s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{h_2^*-h_1^*}\left[\frac{1}{(s+h_1^*)\,s} - \frac{1}{(s+h_2^*)\,s}\right]\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{h_2^*-h_1^*}\left[-\frac{1}{h_1^*}\left(\frac{1}{s+h_1^*} - \frac{1}{s}\right) + \frac{1}{h_2^*}\left(\frac{1}{s+h_2^*} - \frac{1}{s}\right)\right]\right\}$$
$$= \frac{1}{h_2^*-h_1^*}\left[\frac{1}{h_1^*}\left(1-e^{-h_1^* t}\right) - \frac{1}{h_2^*}\left(1-e^{-h_2^* t}\right)\right], \tag{101}$$

where we note that (101) is just a special case of (100) with $\sigma_G = 0$. This gives rise to the *adjustment* term described in the following lemma:

Lemma 6. *The adjustment term* $\mathbf{A}\left(h_1^*, h_2^*, t\right)$ *is:*

$$\begin{aligned}
\mathbf{A}\left(h_1^*, h_2^*, t\right) &\equiv \frac{1}{h_2^*-h_1^*}\left[\frac{1-e^{-h_1^* t}}{h_1^*} - \frac{1-e^{-h_2^* t}}{h_2^*}\right] \\
&= \frac{1}{h_2^*-h_1^*}\left[\mathbf{T_1}\left(h_1^*, 0, t\right) - \mathbf{T_1}\left(h_2^*, 0, t\right)\right] \\
&= \mathbf{T_3}\left(h_1^*, h_2^*, 0, t\right), \\
\mathcal{L}\left\{\mathbf{A}(h_1^*, h_2^*, t)\right\} &= \frac{1}{(s+h_1^*)(s+h_2^*)s}.
\end{aligned}$$

Properties: (i) $\mathbf{A}\left(h_1^*, h_2^*, 0\right) = 0$*; and (ii)* $\lim_{t\to\infty} \mathbf{A}\left(h_1^*, h_2^*, t\right) = \frac{1}{h_1^* h_2^*}$.

We are now fully equipped to obtain the transition paths for all the variables in the dynamic system.

4.3.2 Private Capital Stock

The first row of (76) can be written as $(s+h_1^*)(s+h_2^*)\mathcal{L}\{\tilde{K}, s\} = \Gamma_k$, where Γ_k is equal to:

$$\Gamma_k = \bar{\delta}_{12}\frac{\left[\phi\left(s\right) + \bar{\delta}_{23}\bar{\delta}_{34}\omega_A\right]\tilde{q}\left(0\right) + \bar{\delta}_{23}\left(s-\bar{\delta}_{22}\right)\tilde{X}\left(0\right) - \phi\left(s\right)\mathcal{L}\{\gamma_q, s\} - \bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A, s\}}{(s-r_1^*)(s-r_2^*)}. \tag{102}$$

By writing

$$\frac{1}{(s-r_1^*)(s-r_2^*)} = \frac{1}{r_1^*-r_2^*}\left[\frac{1}{s-r_1^*} - \frac{1}{s-r_2^*}\right], \tag{103}$$

we can rewrite the above expression to yield:

$$\Gamma_k = \frac{\bar{\delta}_{12}}{r_1^* - r_2^*} \left[\frac{\left[\phi(s) + \bar{\delta}_{23}\bar{\delta}_{34}\omega_A\right] \tilde{q}(0) + \bar{\delta}_{23}\left(s - \bar{\delta}_{22}\right) \tilde{X}(0) - \phi(s) \mathcal{L}\{\gamma_q, s\} - \bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A, s\}}{s - r_1^*} \right.$$
$$\left. - \frac{\left[\phi(s) + \bar{\delta}_{23}\bar{\delta}_{34}\omega_A\right] \tilde{q}(0) + \bar{\delta}_{23}\left(s - \bar{\delta}_{22}\right) \tilde{X}(0) - \phi(s) \mathcal{L}\{\gamma_q, s\} - \bar{\delta}_{23}\bar{\delta}_{34}\mathcal{L}\{\gamma_A, s\}}{s - r_2^*} \right]. \quad (104)$$

Using (85) for the two roots yields:

$$[\phi(s) - \phi(r_1^*)]\, \tilde{q}(0) + \bar{\delta}_{23}(s - r_1^*)\tilde{X}(0) + \phi(r_1^*)\mathcal{L}\{\gamma_q, r_1^*\} - \phi(s)\mathcal{L}\{\gamma_q, s\} + \bar{\delta}_{23}\bar{\delta}_{34}\left[\mathcal{L}\{\gamma_A, r_1^*\} - \mathcal{L}\{\gamma_A, s\}\right],$$

$$[\phi(s) - \phi(r_2^*)]\, \tilde{q}(0) + \bar{\delta}_{23}(s - r_2^*)\tilde{X}(0) + \phi(r_2^*)\mathcal{L}\{\gamma_q, r_2^*\} - \phi(s)\mathcal{L}\{\gamma_q, s\} + \bar{\delta}_{23}\bar{\delta}_{34}\left[\mathcal{L}\{\gamma_A, r_2^*\} - \mathcal{L}\{\gamma_A, s\}\right],$$

where the general form of (85) (which is not equal to zero) is subtracted from (85) with the respective root plugged in (yielding an expression equal to zero). We thus have subtracted zero from both expressions.

Plugging these expressions into (104)

$$\Gamma_k = \frac{\bar{\delta}_{12}}{r_1^* - r_2^*} \left[\frac{[\phi(s) - \phi(r_1^*)]\, \tilde{q}(0) + \bar{\delta}_{23}(s - r_1^*)\, \tilde{X}(0) + \bar{\delta}_{23}\bar{\delta}_{34}\left[\mathcal{L}\{\gamma_A, r_1^*\} - \mathcal{L}\{\gamma_A, s\}\right]}{s - r_1^*} \right.$$
$$+ \frac{\phi(r_1^*)\, \mathcal{L}\{\gamma_q, r_1^*\} - \phi(s)\, \mathcal{L}\{\gamma_q, s\}}{s - r_1^*} - \frac{\phi(r_2^*)\, \mathcal{L}\{\gamma_q, r_2^*\} - \phi(s)\, \mathcal{L}\{\gamma_q, s\}}{s - r_2^*}$$
$$\left. - \frac{[\phi(s) - \phi(r_2^*)]\, \tilde{q}(0) + \bar{\delta}_{23}(s - r_2^*)\, \tilde{X}(0) + \bar{\delta}_{23}\bar{\delta}_{34}\left[\mathcal{L}\{\gamma_A, r_2^*\} - \mathcal{L}\{\gamma_A, s\}\right]}{s - r_2^*} \right]. \quad (105)$$

By using Lemma 1(i) in (105) we obtain:

$$\frac{\phi(s) - \phi(r_1^*)}{s - r_1^*} = s + r_1^* - \bar{\delta}_{22} - \bar{\delta}_{33},$$
$$\frac{\phi(s) - \phi(r_2^*)}{s - r_2^*} = s + r_2^* - \bar{\delta}_{22} - \bar{\delta}_{33}, \quad (106)$$

which yields:

$$\frac{\phi(s) - \phi(r_1^*)}{s - r_1^*} - \frac{\phi(s) - \phi(r_2^*)}{s - r_2^*} = r_1^* - r_2^*, \quad (107)$$

so that:

$$\Gamma_k = \bar{\delta}_{12}\tilde{q}(0) + \frac{\bar{\delta}_{12}}{r_1^* - r_2^*} \left(\frac{\phi(r_1^*)\, \mathcal{L}\{\gamma_q, r_1^*\} - \phi(s)\, \mathcal{L}\{\gamma_q, s\}}{s - r_1^*} - \frac{\phi(r_2^*)\, \mathcal{L}\{\gamma_q, r_2^*\} - \phi(s)\, \mathcal{L}\{\gamma_q, s\}}{s - r_2^*} \right)$$
$$+ \frac{\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}}{r_1^* - r_2^*} \left(\frac{\mathcal{L}\{\gamma_A, r_1^*\} - \mathcal{L}\{\gamma_A, s\}}{s - r_1^*} - \frac{\mathcal{L}\{\gamma_A, r_2^*\} - \mathcal{L}\{\gamma_A, s\}}{s - r_2^*} \right). \quad (108)$$

By using Lemma 2(ii) and (iv) in (108) we can rewrite Γ_k as follows:

$$\begin{aligned}
\Gamma_k &= \bar{\delta}_{12}\tilde{q}(0) \\
&+\frac{\bar{\delta}_{12}}{r_1^*-r_2^*}\Bigg[-\pi_{qp}\left(1+\frac{\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}}{r_1^* s}\right)+\pi_{qt}\left(\frac{\phi(-\sigma_G)}{(r_1^*+\sigma_G)(s+\sigma_G)}-1\right) \\
&+\pi_{qp}\left(1+\frac{\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}}{r_2^* s}\right)-\pi_{qt}\left(\frac{\phi(-\sigma_G)}{(r_2^*+\sigma_G)(s+\sigma_G)}-1\right)\Bigg] \\
&+\frac{\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}}{r_1^*-r_2^*}\left[\frac{\pi_{Ap}}{sr_1^*}+\frac{\pi_{At}}{(r_1^*+\sigma_G)(s+\sigma_G)}-\frac{\pi_{Ap}}{sr_2^*}-\frac{\pi_{At}}{(r_2^*+\sigma_G)(s+\sigma_G)}\right].
\end{aligned}$$

After simplifying we obtain:

$$\begin{aligned}
\Gamma_k &= \bar{\delta}_{12}\tilde{q}(0)+\bar{\delta}_{12}\left[\pi_{qp}\frac{\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}}{r_1^* r_2^* s}-\pi_{qt}\frac{\phi(-\sigma_G)}{(r_1^*+\sigma_G)(r_2^*+\sigma_G)(s+\sigma_G)}\right] \\
&-\bar{\delta}_{12}\bar{\delta}_{23}\bar{\delta}_{34}\left[\frac{\pi_{Ap}}{r_1^* r_2^* s}+\frac{\pi_{At}}{(r_1^*+\sigma_G)(r_2^*+\sigma_G)(s+\sigma_G)}\right].
\end{aligned}$$

It follows that:

$$\begin{aligned}
(s+h_1^*)(s+h_2^*)\mathcal{L}\{\tilde{K},s\} &= \bar{\delta}_{12}\tilde{q}(0)+\bar{\delta}_{12}\frac{\pi_{qp}\left(\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}\right)-\pi_{Ap}\bar{\delta}_{23}\bar{\delta}_{34}}{r_1^* r_2^* s} \\
&-\bar{\delta}_{12}\frac{\pi_{qt}\phi(-\sigma_G)+\pi_{At}\bar{\delta}_{23}\bar{\delta}_{34}}{(r_1^*+\sigma_G)(r_2^*+\sigma_G)(s+\sigma_G)}. \qquad (109)
\end{aligned}$$

Hence, it follows that the path for the private capital stock is given by:

$$\begin{aligned}
\tilde{K}(t) &= \bar{\delta}_{12}\tilde{q}(0)\,\mathbf{T_1}(h_1^*,h_2^*,t)+\bar{\delta}_{12}\frac{\pi_{qp}\left(\bar{\delta}_{34}\bar{\delta}_{43}-\bar{\delta}_{22}\bar{\delta}_{33}\right)-\pi_{Ap}\bar{\delta}_{23}\bar{\delta}_{34}}{r_1^* r_2^*}\mathbf{A}(h_1^*,h_2^*,t) \\
&-\bar{\delta}_{12}\frac{\pi_{qt}\phi(-\sigma_G)+\pi_{At}\bar{\delta}_{23}\bar{\delta}_{34}}{(r_1^*+\sigma_G)(r_2^*+\sigma_G)}\mathbf{T_3}(h_1^*,h_2^*,t). \qquad (110)
\end{aligned}$$

4.3.3 Tobin's *q*

By using (86) we can write the second row of (76) as $(s+h_1^*)(s+h_2^*)\mathcal{L}\{\tilde{q},s\}=\Gamma_q$, where Γ_q is equal to:

$$\begin{aligned}
\Gamma_q &= \frac{1}{r_1^*-r_2^*}\Bigg[\left(\frac{s\phi(s)-r_1^*\phi(r_1^*)}{s-r_1^*}-\frac{s\phi(s)-r_2^*\phi(r_2^*)}{s-r_2^*}\right)\tilde{q}(0)+\delta_{23}(r_1^*-r_2^*)\tilde{X}(0) \\
&+\left(\frac{r_1^*\phi(r_1^*)\mathcal{L}\{\gamma_q,r_1^*\}-s\phi(s)\mathcal{L}\{\gamma_q,s\}}{s-r_1^*}-\frac{r_2^*\phi(r_2^*)\mathcal{L}\{\gamma_q,r_2^*\}-s\phi(s)\mathcal{L}\{\gamma_q,s\}}{s-r_2^*}\right) \\
&+\delta_{23}\delta_{34}\left(\frac{r_1^*\mathcal{L}\{\gamma_A,r_1^*\}-s\mathcal{L}\{\gamma_A,s\}}{s-r_1^*}-\frac{r_2^*\mathcal{L}\{\gamma_A,r_2^*\}-s\mathcal{L}\{\gamma_A,s\}}{s-r_2^*}\right)\Bigg]. \qquad (111)
\end{aligned}$$

By using Lemma 1(ii), Lemma 2(iii), and 2(vi) (evaluated for $\bar{\delta}_{33}=0$) we can simplify this expression to:

$$\begin{aligned}\Gamma_q &= \left(s+r_1^*+r_2^*-\bar{\delta}_{22}-\bar{\delta}_{33}\right)\tilde{q}\,(0)+\bar{\delta}_{23}\tilde{X}\,(0)-(\pi_{qp}+\pi_{qt}) \\ &\quad +\frac{\pi_{qt}\sigma_G\psi\,(-\sigma_G)}{(r_1^*+\sigma_G)\,(r_2^*+\sigma_G)\,(s+\sigma_G)}+\bar{\delta}_{23}\bar{\delta}_{34}\frac{\sigma_G\pi_{At}}{(r_1^*+\sigma_G)\,(r_2^*+\sigma_G)\,(s+\sigma_G)}. \end{aligned} \tag{112}$$

It follows that $\mathcal{L}\{\tilde{q},s\}$ is equal to:

$$\begin{aligned}\mathcal{L}\{\tilde{q},s\} &= \frac{s+r_1^*+r_2^*-\bar{\delta}_{22}-\bar{\delta}_{33}}{(s+h_1^*)(s+h_2^*)}\tilde{q}\,(0)+\frac{\bar{\delta}_{23}\tilde{X}\,(0)-(\pi_{qp}+\pi_{qt})}{(s+h_1^*)(s+h_2^*)} \\ &\quad +\frac{\bar{\delta}_{23}\bar{\delta}_{34}\sigma_G\pi_{At}+\pi_{qt}\sigma_G\psi\,(-\sigma_G)}{(r_1^*+\sigma_G)\,(r_2^*+\sigma_G)}\frac{1}{(s+\sigma_G)\,(s+h_1^*)(s+h_2^*)}. \end{aligned} \tag{113}$$

Hence, the path for $\tilde{q}\,(t)$ is given by:

$$\begin{aligned}\tilde{q}\,(t) &= \left[\left(r_1^*+r_2^*-\bar{\delta}_{22}-\bar{\delta}_{33}\right)\tilde{q}\,(0)+\bar{\delta}_{23}\tilde{X}(0)-(\pi_{qp}+\pi_{qt})\right]\mathbf{T}_1\,(h_1^*,h_2^*,t) \\ &\quad \tilde{q}\,(0)\,\mathbf{T}_2\,(h_1^*,h_2^*,t)+\sigma_G\frac{\bar{\delta}_{23}\bar{\delta}_{34}\pi_{At}+\pi_{qt}\psi\,(-\sigma_G)}{(r_1^*+\sigma_G)\,(r_2^*+\sigma_G)}\mathbf{T_3}\,(h_1^*,h_2^*,\sigma_G,t)\,, \end{aligned} \tag{114}$$

where we have used Lemma 3, 4, and 5 to invert the Laplace transform. Note that there is never a permanent effect on q.

4.3.4 Private Consumption

Using (87), we can write the third row of (76) as $(s+h_1^*)(s+h_2^*)\mathcal{L}\{\tilde{X},s\}=\Gamma_x$, where Γ_x is equal to:

$$\begin{aligned}\Gamma_x &= \frac{1}{r_1^*-r_2^*}\left[\bar{\delta}_{34}\omega_A\left(\frac{\psi\,(s)-\psi\,(r_1^*)}{s-r_1^*}-\frac{\psi\,(s)-\psi\,(r_2^*)}{s-r_2^*}\right)\tilde{q}\,(0)\right. \\ &+ \left(\frac{\zeta\,(s)-\zeta\,(r_1^*)}{s-r_1^*}-\frac{\zeta\,(s)-\zeta\,(r_2^*)}{s-r_2^*}\right)\tilde{X}\,(0) \\ &+ \bar{\delta}_{12}\bar{\delta}_{34}\bar{\delta}_{41}\left(\frac{\mathcal{L}\{\gamma_q,r_1^*\}-\mathcal{L}\{\gamma_q,s\}}{s-r_1^*}-\frac{\mathcal{L}\{\gamma_q,r_2^*\}-\mathcal{L}\{\gamma_q,s\}}{s-r_2^*}\right) \\ &+ \left.\bar{\delta}_{34}\left(\frac{\psi\,(r_1^*)\,\mathcal{L}\{\gamma_A,r_1^*\}-\psi\,(s)\,\mathcal{L}\{\gamma_A,s\}}{s-r_1^*}-\frac{\psi\,(r_2^*)\,\mathcal{L}\{\gamma_A,r_2^*\}-\psi\,(s)\,\mathcal{L}\{\gamma_A,s\}}{s-r_1^*}\right)\right], \end{aligned} \tag{115}$$

where $\zeta(s) \equiv \left(s - \bar{\delta}_{22}\right)\psi(s)$. By using Lemma 1(iii), 1(iv), 2(ii), and 2(v) we can simplify (115) to:

$$\begin{aligned}
\Gamma_x &= \bar{\delta}_{34}\omega_A \tilde{q}(0) + \left(s + r_1^* + r_2^* - 2\bar{\delta}_{22}\right)\tilde{X}(0) \\
&\quad -\bar{\delta}_{12}\bar{\delta}_{34}\bar{\delta}_{41}\left[\frac{\pi_{qp}}{r_1^* r_2^* s} + \frac{\pi_{qt}}{(r_1^* + \sigma_G)(r_2^* + \sigma_G)(s + \sigma_G)}\right] \\
&\quad +\bar{\delta}_{34}\left[\frac{\pi_{Ap}\bar{\delta}_{12}\bar{\delta}_{21}}{r_1^* r_2^* s} - \frac{\pi_{At}\psi(-\sigma_G)}{(r_1^* + \sigma_G)(r_2^* + \sigma_G)(s + \sigma_G)}\right]. \qquad (116)
\end{aligned}$$

Hence, the Laplace transform for private consumption is:

$$\begin{aligned}
(s + h_1^*)(s + h_2^*)\mathcal{L}\{\tilde{X}, s\} &= \bar{\delta}_{34}\omega_A \tilde{q}(0) + \left[s + r_1^* + r_2^* - 2\bar{\delta}_{22}\right]\tilde{X}(0) \\
&\quad +\bar{\delta}_{12}\bar{\delta}_{34}\frac{\bar{\delta}_{21}\pi_{Ap} - \bar{\delta}_{41}\pi_{qp}}{r_1^* r_2^*}\frac{1}{s} \\
&\quad -\bar{\delta}_{34}\frac{\bar{\delta}_{12}\bar{\delta}_{41}\pi_{qt} + \pi_{At}\psi(-\sigma_G)}{(r_1^* + \sigma_G)(r_2^* + \sigma_G)}\frac{1}{s + \sigma_G}. \qquad (117)
\end{aligned}$$

Gathering the results together we find the path for consumption:

$$\begin{aligned}
\tilde{X}(t) &= \left[\bar{\delta}_{34}\omega_A \tilde{q}(0) + \left(r_1^* + r_2^* - 2\bar{\delta}_{22}\right)\tilde{X}(0)\right]\mathbf{T}_1(h_1^*, h_2^*, t) + \tilde{X}(0)\mathbf{T}_2(h_1^*, h_2^*, t) \\
&\quad -\bar{\delta}_{34}\frac{\bar{\delta}_{12}\bar{\delta}_{41}\pi_{qt} + \pi_{At}\psi(-\sigma_G)}{(r_1^* + \sigma_G)(r_2^* + \sigma_G)}\mathbf{T_3}(h_1^*, h_2^*, \sigma_G, t) \\
&\quad +\bar{\delta}_{12}\bar{\delta}_{34}\frac{\bar{\delta}_{21}\pi_{Ap} - \bar{\delta}_{41}\pi_{qp}}{r_1^* r_2^*}\mathbf{A}(h_1^*, h_2^*, t). \qquad (118)
\end{aligned}$$

4.3.5 Total Assets

Using (88), we can write the fourth row of (76) as $(s+h_1^*)(s+h_2^*)\mathcal{L}\{\tilde{A}, s\} = \Gamma_a$,where Γ_a is equal to:

$$\begin{aligned}
\Gamma_a &= \frac{1}{r_1^* - r_2^*}\Bigg[\omega_A\left(\frac{\theta(s) - \theta(r_1^*)}{s - r_1^*} - \frac{\theta(s) - \theta(r_2^*)}{s - r_2^*}\right)\tilde{q}(0) \\
&\quad +\bar{\delta}_{43}\left(\frac{\psi(s) - \psi(r_1^*)}{s - r_1^*} - \frac{\psi(s) - \psi(r_2^*)}{s - r_2^*}\right)\tilde{X}(0) \\
&\quad +\bar{\delta}_{12}\bar{\delta}_{41}\left(\frac{\left(r_1^* - \bar{\delta}_{33}\right)\mathcal{L}\{\gamma_q, r_1^*\} - \left(s - \bar{\delta}_{33}\right)\mathcal{L}\{\gamma_q, s\}}{s - r_1^*} - \frac{\left(r_2^* - \bar{\delta}_{33}\right)\mathcal{L}\{\gamma_q, r_2^*\} - \left(s - \bar{\delta}_{33}\right)\mathcal{L}\{\gamma_q, s\}}{s - r_2^*}\right) \\
&\quad +\left(\frac{\theta(r_1^*)\mathcal{L}\{\gamma_A, r_1^*\} - \theta(s)\mathcal{L}\{\gamma_A, s\}}{s - r_1^*} - \frac{\theta(r_2^*)\mathcal{L}\{\gamma_A, r_2^*\} - \theta(s)\mathcal{L}\{\gamma_A, s\}}{s - r_2^*}\right)\Bigg], \qquad (119)
\end{aligned}$$

where $\theta(s) \equiv \left(s - \bar{\delta}_{33}\right)\psi(s)$. By using Lemma 1(iii), 1(v), 2(iii), and 2(vi) we can simplify (119) to:

$$
\begin{aligned}
\Gamma_a &= \omega_A\left(s + r_1^* + r_2^* - \bar{\delta}_{22} - \bar{\delta}_{33}\right)\tilde{q}(0) + \bar{\delta}_{43}\tilde{X}(0) \\
&\quad + \bar{\delta}_{12}\bar{\delta}_{41}\left(\frac{\pi_{qp}\bar{\delta}_{33}}{r_1^* r_2^* s} + \frac{\pi_{qt}\left(\bar{\delta}_{33} + \sigma_G\right)}{\left(r_1^* + \sigma_G\right)\left(r_2^* + \sigma_G\right)\left(s + \sigma_G\right)}\right) \\
&\quad - \left(1 + \frac{\bar{\delta}_{33}\bar{\delta}_{12}\bar{\delta}_{21}}{r_1^* r_2^*}\frac{1}{s}\right)\pi_{Ap}. \\
&\quad - \left(1 - \frac{\left(\sigma_G + \bar{\delta}_{33}\right)\psi(-\sigma_G)}{\left(r_1^* + \sigma_G\right)\left(r_2^* + \sigma_G\right)}\frac{1}{s + \sigma_G}\right)\pi_{At}.
\end{aligned}
\tag{120}
$$

It follows from (120) that:

$$
\begin{aligned}
\mathcal{L}\{\tilde{A}, s\} &= \frac{\omega_A\left(r_1^* + r_2^* - \bar{\delta}_{22} - \bar{\delta}_{33}\right)\tilde{q}(0) + \bar{\delta}_{43}\tilde{X}(0) - \left(\pi_{Ap} + \pi_{At}\right)}{(s + h_1^*)(s + h_2^*)} \\
&\quad + \omega_A\tilde{q}(0)\frac{s}{(s + h_1^*)(s + h_2^*)} + \bar{\delta}_{12}\bar{\delta}_{33}\frac{\bar{\delta}_{41}\pi_{qp} - \bar{\delta}_{21}\pi_{Ap}}{r_1^* r_2^*}\frac{1}{s(s + h_1^*)(s + h_2^*)} \\
&\quad + \left(\sigma_G + \bar{\delta}_{33}\right)\frac{\bar{\delta}_{12}\bar{\delta}_{41}\pi_{qt} + \psi(-\sigma_G)\pi_{At}}{\left(r_1^* + \sigma_G\right)\left(r_2^* + \sigma_G\right)}\frac{1}{\left(s + \sigma_G\right)(s + h_1^*)(s + h_2^*)}.
\end{aligned}
\tag{121}
$$

By inverting the Laplace transform we obtain:

$$
\begin{aligned}
\tilde{A}(t) &= \left[\omega_A\left(r_1^* + r_2^* - \bar{\delta}_{22} - \bar{\delta}_{33}\right)\tilde{q}(0) + \bar{\delta}_{43}\tilde{X}(0) - \left(\pi_{Ap} + \pi_{At}\right)\right]\mathbf{T_1}\left(h_1^*, h_2^*, t\right) \\
&\quad + \omega_A\tilde{q}(0)\mathbf{T_2}\left(h_1^*, h_2^*, t\right) + \bar{\delta}_{12}\bar{\delta}_{33}\frac{\bar{\delta}_{41}\pi_{qp} - \bar{\delta}_{21}\pi_{Ap}}{r_1^* r_2^*}\mathbf{A}\left(h_1^*, h_2^*, t\right) \\
&\quad + \left(\sigma_G + \bar{\delta}_{33}\right)\frac{\bar{\delta}_{12}\bar{\delta}_{41}\pi_{qt} + \psi(-\sigma_G)\pi_{At}}{\left(r_1^* + \sigma_G\right)\left(r_2^* + \sigma_G\right)}\mathbf{T_3}\left(h_1^*, h_2^*, \sigma_G, t\right).
\end{aligned}
\tag{122}
$$

Table A1: Summary of Model

(a) Dynamic Equations:

$$\dot{K}(t) = \left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] K(t) \qquad \text{(TA1.1)}$$

$$\dot{q}(t) = q(t)\left[r + \delta - \Phi\left(\frac{I(t)}{K(t)}\right)\right] + \frac{I(t)}{K(t)} - (A_K(t))^{(\sigma-1)/\sigma}\left(\frac{Y(t)}{K(t)}\right)^{1/\sigma} \qquad \text{(TA1.2)}$$

$$\dot{X}(t) = (r-\alpha)X(t) - \beta(\alpha+\beta)A(t) \qquad \text{(TA1.3)}$$

$$\dot{A}(t) = rA(t) + w(t) - T(t) - X(t) \qquad \text{(TA1.4)}$$

$$\dot{B}(t) = rB(t) + I_G(t) + C_G(t) - T(t) \qquad \text{(TA1.5)}$$

$$\dot{K}_G(t) = \left[\Phi_G\left(\frac{I_G(t)}{K_G(t)}\right) - \delta_G\right] K_G(t) \qquad \text{(TA1.6)}$$

(b) Static Equations:

$$1 = q(t)\Phi'\left(\frac{I(t)}{K(t)}\right) \qquad \text{(TA1.7)}$$

$$w(t) = (A_L(t))^{(\sigma-1)/\sigma}\left(\frac{Y(t)}{L(t)}\right)^{1/\sigma} \qquad \text{(TA1.8)}$$

$$Y(t) = \left[[A_K(t)K(t)]^{\frac{\sigma-1}{\sigma}} + [A_L(t)L(t)]^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \qquad \text{(TA1.9)}$$

$$w(t)L(t) = w(t) - (1-\varepsilon)X(t) \qquad \text{(TA1.10)}$$

$$C(t) = \varepsilon X(t) \qquad \text{(TA1.11)}$$

$$F(t) = A(t) - q(t)K(t) - B(t) \qquad \text{(TA1.12)}$$

(c) Definitions:

$$P(t) \equiv \left(\frac{1}{\varepsilon}\right)^{\varepsilon}\left(\frac{w(t)}{1-\varepsilon}\right)^{1-\varepsilon} \qquad \text{(TA1.13)}$$

$$A_j(t) \equiv \rho_j K_G(t)^{\eta_j}, \qquad j = \{K, L\} \qquad \text{(TA1.14)}$$

Table A2: Summary of the Log-Linearized Model

(a) Dynamic Equations:

$$\dot{\tilde{K}}(t) = \frac{r\omega_I}{\omega_A}[\tilde{I}(t) - \tilde{K}(t)] \tag{TA2.1}$$

$$\dot{\tilde{q}}(t) = r\tilde{q}(t) - \frac{r\theta_K}{\sigma\omega_A}[\tilde{Y}(t) - \tilde{K}(t) + (\sigma - 1)\eta_K\tilde{K}_G(t)] \tag{TA2.2}$$

$$\dot{\tilde{X}}(t) = (r - \alpha)\left[\tilde{X}(t) - \frac{\tilde{A}(t)}{\omega_A}\right] \tag{TA2.3}$$

$$\dot{\tilde{A}}(t) = r\left[\tilde{A}(t) + \omega_w\tilde{w}(t) - \tilde{T}(t) - \omega_X\tilde{X}(t)\right] \tag{TA2.4}$$

$$\dot{\tilde{B}}(t) = r\left[\tilde{B}(t) + \omega_G^I\tilde{I}_G(t) + \omega_G^C\tilde{C}_G(t) - \tilde{T}(t)\right], \tag{TA2.5}$$

$$\dot{\tilde{K}}_G(t) = \sigma_G[\tilde{I}_G - \tilde{K}_G(t)] \tag{TA2.6}$$

(b) Static Equations:

$$\tilde{q}(t) = \sigma_A[\tilde{I}(t) - \tilde{K}(t)] \tag{TA2.7}$$

$$\tilde{w}(t) = \frac{1}{\sigma}\left[\tilde{Y}(t) - \tilde{L}(t) + (\sigma - 1)\eta_L\tilde{K}_G(t)\right] \tag{TA2.8}$$

$$\tilde{Y}(t) = \theta_K\tilde{K}(t) + \theta_L\tilde{L}(t) + \theta_G\tilde{K}_G(t), \tag{TA2.9}$$

$$\tilde{L}(t) = \omega_{LL}[\tilde{w}(t) - \tilde{X}(t)] \tag{TA2.10}$$

$$\tilde{C}(t) = \tilde{X}(t) \tag{TA2.11}$$

$$\tilde{F}(t) = \tilde{A}(t) - \omega_A[\tilde{q}(t) + \tilde{K}(t)] - \tilde{B}(t) \tag{TA2.12}$$

(c) Definitions:

$$\tilde{P}(t) = (1 - \varepsilon)\tilde{w}(t) \tag{TA2.13}$$

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