Supplementary material for active nonlinear acoustic sensing of an object with sum or difference frequency fields

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1 First order time harmonic scattering from an object with possible radiation induced centroidal motion

For a time harmonic problem at angular frequency ω, first order quantities can be written as

$$q_1(\mathbf{r},t) = \Re\{Q_1(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1a}$$
$$p_1(\mathbf{r},t) = \Re\{P_1(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1b}$$
$$p_{\mathrm{I1}}(\mathbf{r},t) = \Re\{P_{\mathrm{I1}}(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1c}$$
$$p_{\mathrm{S1}}(\mathbf{r},t) = \Re\{P_{\mathrm{S1}}(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1d}$$
$$p_{\mathrm{D1}}(\mathbf{r},t) = \Re\{P_{\mathrm{D1}}(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1e}$$
$$p_{\mathrm{R1}}(\mathbf{r},t) = \Re\{P_{\mathrm{R1}}(\mathbf{r})e^{-\mathrm{i}\omega t}\}, \tag{S1f}$$
$$\mathbf{u}_{\mathrm{c1}}(t) = \Re\{\mathbf{U}_{\mathrm{c1}}e^{-\mathrm{i}\omega t}\}, \tag{S1g}$$
$$\mathbf{f}_1^{\mathrm{excit}}(t) = \Re\{\mathbf{F}_1^{\mathrm{excit}}e^{-\mathrm{i}\omega t}\}, \tag{S1h}$$

where capital symbols are the complex amplitudes of the corresponding physical quantities denoted by lower case symbols. Substituting p_1 and q_1 into equation (2.2) yields the Helmholtz equation

$$(\nabla^2 + k^2)P_1 = -Q_1, \tag{S2}$$

where $k = \omega/c_0$. Substituting p_{S1} into equation (2.8) and integrating over t_0 gives

$$P_{\mathrm{S1}}(\mathbf{r},t) = \iint dS_0 \mathbf{n}(\mathbf{r}_0) \cdot \left[P_{\mathrm{S1}}(\mathbf{r}_0)\nabla_0 G(\mathbf{r}|\mathbf{r}_0) - G(\mathbf{r}|\mathbf{r}_0)\nabla_0 P_{\mathrm{S1}}(\mathbf{r}_0)\right], \tag{S3}$$

where the frequency domain Green function is $G(\mathbf{r}|\mathbf{r}_0) = e^{\mathrm{i}kR}/(4\pi R)$. Solution for P_{S1} for pressure release or rigid immovable objects can be obtained from equation (S3) with boundary conditions

$$\text{pressure release:} \qquad P_{\mathrm{S1}} = -P_{\mathrm{I1}} \quad \text{on } \bar{S} \tag{S4}$$

$$\text{rigid immovable:} \qquad \mathbf{V}_{\mathrm{S1}} \cdot \mathbf{n} = -\mathbf{V}_{\mathrm{I1}} \cdot \mathbf{n} \quad \text{on } \bar{S} \tag{S5}$$

where $\mathbf{V}_{\mathrm{S1}} = (\mathrm{i}\omega\rho_0)^{-1}\nabla P_{\mathrm{S1}}$ and $\mathbf{V}_{\mathrm{I1}} = (\mathrm{i}\omega\rho_0)^{-1}\nabla P_{\mathrm{I1}}$.

For rigid movable objects, P_{D1} is the same as P_{S1} for the rigid immovable object and P_{R1} can be obtained from equation (S3) with $P_{\mathrm{S1}} = P_{\mathrm{R1}}$ and boundary condition

$$\mathbf{V}_{\mathrm{R1}} \cdot \mathbf{n} = \mathbf{U}_{\mathrm{c1}} \cdot \mathbf{n}, \tag{S6}$$

where $\mathbf{V}_{\mathrm{R1}} = (\mathrm{i}\omega\rho_0)^{-1}\nabla P_{\mathrm{R1}}$,

$$\mathbf{U}_{\mathrm{c1}} = [\boldsymbol{Z}_m(\omega) + \boldsymbol{Z}_r(\omega)]^{-1}\mathbf{F}_1^{\mathrm{excit}}, \tag{S7}$$

and

$$\mathbf{F}_1^{\mathrm{excit}} = \iint_{\bar{S}} (P_{\mathrm{I1}} + P_{\mathrm{D1}})\mathbf{n} dS. \tag{S8}$$

Here $\boldsymbol{Z}_m(\omega) = -\mathrm{i}\omega M$ is the Fourier transform of $\boldsymbol{z}_m$ and $\boldsymbol{Z}_r(\omega)$ is the Fourier transform of $\boldsymbol{z}_r$, which is sometimes referred to as the radiation impedance [1, 2, 3].

If the rigid movable object is attached to a spring and damper in the direction of motion, equation (S7) becomes

$$\mathbf{U}_{\mathrm{c1}} = \left[\boldsymbol{Z}_m(\omega) + \boldsymbol{Z}_r(\omega) + \frac{M\omega_n^2}{-\mathrm{i}\omega} + 2\zeta M\omega_n\right]^{-1} \mathbf{F}_1^{\mathrm{excit}}, \tag{S9}$$

where ω_n is the (undamped) natural frequency, and ζ is the dimensionless damping ratio.

To find $\boldsymbol{Z}_r(\omega)$, three radiated fields $\hat{P}_{\mathrm{R},j}(\omega)$, $j = x, y, z$ at frequency ω need to be determined from the boundary conditions

$$\hat{\mathbf{V}}_{\mathrm{R},j} \cdot \mathbf{n} = \hat{\mathbf{U}}_{\mathrm{c},j} \cdot \mathbf{n}, \quad \text{for} \quad j = x, y, z \tag{S10}$$

where $\hat{\mathbf{V}}_{\mathrm{R},j} = (\mathrm{i}\omega\rho_0)^{-1}\nabla\hat{P}_{\mathrm{R},j}$, $j = x, y, z$, given unit velocities $\hat{\mathbf{U}}_{\mathrm{c},x} = (1, 0, 0)$, $\hat{\mathbf{U}}_{\mathrm{c},y} = (0, 1, 0)$ and $\hat{\mathbf{U}}_{\mathrm{c},z} = (0, 0, 1)$. These radiated fields can be determined from the integral equation (S3), in a similar manner as that for P_{S1}. In terms of $\hat{\mathbf{P}}_{\mathrm{R}} = (\hat{P}_{\mathrm{R},x}, \hat{P}_{\mathrm{R},y}, \hat{P}_{\mathrm{R},z})$, $\boldsymbol{Z}_r(\omega)$ is determined from its elements by

$$\{Z_r(\omega)\}_{ij} = -\iint_{\bar{S}} n_i \hat{P}_{\mathrm{R},j}(\omega) dS, \quad \text{for} \quad i,j = x,y,z. \tag{S11}$$

As a special case, for a rigid sphere with radius a in an ideal fluid with density ρ_0 and sound speed c_0 , the radiation impedance is $\{Z_r(\omega)\}_{ij} = 4\pi \mathrm{i}\rho_0 c_0 a^2 h_1(ka)/h_1'(ka)/3\delta_{ij}$, where h_1 is the spherical Hankel function of the first kind, h_1' is the derivative of h_1 with respect to its argument and δ_{ij} is the Kronecker delta.

For an object oscillating with centroidal velocity $\mathbf{U}_{\mathrm{c}} = U_{\mathrm{c},x}\hat{\mathbf{U}}_{\mathrm{c},x} + U_{\mathrm{c},y}\hat{\mathbf{U}}_{\mathrm{c},y} + U_{\mathrm{c},z}\hat{\mathbf{U}}_{\mathrm{c},z}$, the radiated field is

$$P_{\mathrm{R}}(\omega) = \hat{\mathbf{P}}_{\mathrm{R}}(\omega) \cdot \mathbf{U}_{\mathrm{c}}(\omega). \tag{S12}$$

This relation holds in both first and second order.

2 Jones and Beyer's experiment

Jones and Beyer [4, 5] measured the sum frequency second order pressure due to an object in water insonified by two perpendicular incident beams at 7 MHz and 5 MHz respectively. The total second order pressure is determined via equations (50), (59), (77)-(80) as a function of angle at a fixed range about the forward direction of the 7 MHz beam (figure S1) and found to be consistent with these measurements as shown in figure 2 of the main text.

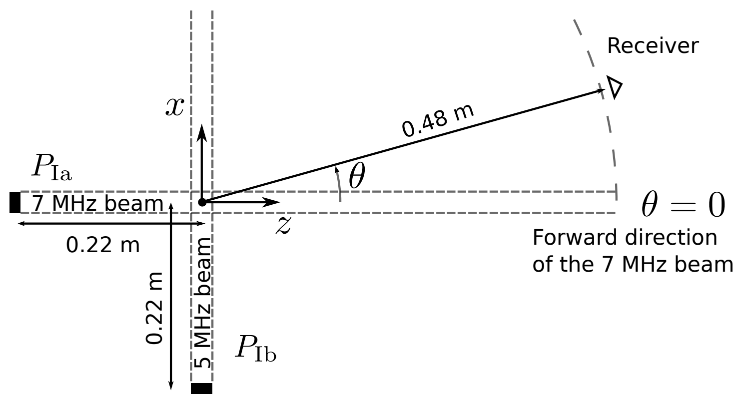


Figure S1: Geometry of the Jones and Beyer measurement, where a rigid sphere is located at the center of the overlap region of the two incident beams. Not to scale.

For quantitative comparisons between theory X_i and measured data Y_i for second order pressure amplitude at the sum frequency, the correlation coefficients and mean square error are determined, where $i = 1, 2, \cdots, N$ for N

angular positions. The correlation coefficient is defined as

$$r_{X,Y} = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{N}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}} \tag{S13}$$

where $\bar{X}$ and $\bar{Y}$ are the average values for X_i and Y_i, respectively. The correlation coefficient $r_{X,Y}$ is found to be 0.98 for the $N = 21$ points measured by Jones and Beyer between ± 0.2 rad. The mean square error (MSE) is defined in decibels as

$$\mathrm{MSE} = 10\log_{10}\left[1 + \frac{\sum_{i=1}^{N}(X_i - Y_i)^2}{\sum_{i=1}^{N} Y_i^2}\right] \text{ dB}, \tag{S14}$$

which is found to be 0.23 dB.

Since the incident waves used by Jones and Beyer were narrow-band pulses of 15 μs, we use the harmonic wave approximation to estimate the second order pressure amplitude at the sum frequency. The $P_{\mathrm{SS}+}$ component is found to be dominant, with the $P_{\mathrm{IS}+} + P_{\mathrm{SI}+}$ amplitude no more than roughly 3% that of $P_{\mathrm{SS}+}$ according to our calculations. This is due to the narrowness of the incident beams generated by Jones and Beyer's circular transducers (radius 0.9525 cm) [4], which makes a plane wave a poor approximation to the spatially varying incident field necessary to calculate $P_{\mathrm{IS}+}$ and $P_{\mathrm{SI}+}$ by the extended IS interaction dictated by their experimental geometry. Jones and Beyer, however, showed that the measured primary scattered fields are an excellent match to those due to incident plane waves [4, 5], which is consistent with fact that the object was well within the beams. Given this and the fact that they did not specify the transducer tapers which determine details of the beams, we use their suggested equivalent plane wave fields at the object to calculate the primary scattered fields required for determination of $P_{\mathrm{SS}+}$ in figure 2. When evaluating the volume integrals for $P'_{\mathrm{SS}+}$, $P'_{\mathrm{IS}+}$ and $P'_{\mathrm{SI}+}$, of equations (78)-(80), a spatial step size of 1/20 of the sum frequency wavelength is used to ensure convergence and accuracy. The amplitudes of the plane waves for SS computations are made to be consistent with the primary scattered fields reported by Jones and Beyer, while the amplitudes of the beams for IS and SI calculations are made to be consistent with measured values (2.45×10^5 Pa and 3.36×10^5 Pa) that were specified at 0.2189 m from the sources for the 7 MHz and 5 MHz beams respectively [5], averaged over a 0.2 cm [4] radius to account for the finite receiver size [6]. Calculations using equation (50) show the second order scattered field $P_{\mathrm{S2}+}$ be on the order of 1 Pa at the receiver, which is negligibly small compared to $P_{\mathrm{SS}+}$ (figure 2).

3 Derivation of the second-order field arising from the nonlinear interaction of plane waves of arbitrary time dependence

Let the first order incident field be the sum of two incident waves of arbitrary time dependence, $p_1(\mathbf{r},t) = \Re\{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t) + \tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)\}$, then

$$p_1^2 = \frac{1}{2}\Re\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ia}} + \tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ia}}^* + \tilde{p}_{\mathrm{Ib}}\tilde{p}_{\mathrm{Ib}} + \tilde{p}_{\mathrm{Ib}}\tilde{p}_{\mathrm{Ib}}^*\} + \Re\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}} + \tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}}^*\}, \tag{S15}$$

and the solution to equation (63) can then be decomposed as

$$p'_{\mathrm{II}} = p'_{\mathrm{II},aa} + p'_{\mathrm{II},aa^*} + p'_{\mathrm{II},bb} + p'_{\mathrm{II},bb^*} + p'_{\mathrm{II},ab} + p'_{\mathrm{II},ab^*} \tag{S16}$$

where each component satisfies

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},aa} = -\frac{\beta}{2Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ia}}^2\} \tag{S17a}$$

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},aa^*} = -\frac{\beta}{2Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ia}}^*\} \tag{S17b}$$

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},bb} = -\frac{\beta}{2Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ib}}^2\} \tag{S17c}$$

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},bb^*} = -\frac{\beta}{2Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ib}}\tilde{p}_{\mathrm{Ib}}^*\} \tag{S17d}$$

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},ab} = -\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}}\} \tag{S17e}$$

$$\left(\nabla - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p'_{\mathrm{II},ab^*} = -\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\Re\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}}^*\} \tag{S17f}$$

where $p'_{\mathrm{II},aa}$, p'_{II,aa^*}, $p'_{\mathrm{II},bb}$ and p'_{II,bb^*} correspond to self-intersections of p_{Ia} or p_{Ib}, and $p'_{\mathrm{II},ab}$ and p'_{II,ab^*} correspond to cross-interactions between p_{Ia} and p_{Ib}.

Self-interaction component $p'_{\mathrm{II},\alpha\alpha}$ for $\alpha = a, b$ contains sum frequencies of all frequency components in wave $p_{\mathrm{I}\alpha}$. If $p_{\mathrm{I}\alpha}$ is harmonic at frequency ω_α, then $p'_{\mathrm{II},\alpha\alpha}$ contains the double frequency $2\omega_\alpha$. Self-interaction component $p'_{\mathrm{II},\alpha\alpha^*}$ for $\alpha = a, b$ contains difference frequencies of all frequency components in wave $p_{\mathrm{I}\alpha}$ which will include a zero-frequency component from the difference of each frequency with itself. If $p_{\mathrm{I}\alpha}$ is harmonic at frequency ω_α, then $p'_{\mathrm{II},\alpha\alpha^*}$ is at zero frequency and it is zero.

The cross-interaction component $p'_{\mathrm{II},ab}$ due to $\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}}$ contains sum frequencies of all frequency components between p_{Ia} and p_{Ib}. If p_{Ia} and p_{Ib} are harmonic at frequencies ω_a and ω_b respectively, then $p'_{\mathrm{II},ab}$ is at the sum frequency $\omega_a + \omega_b$. The cross-interaction component p'_{II,ab^*} due to $\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}}^*$ contains difference frequencies of all frequency components between p_{Ia} and p_{Ib}. If p_{Ia} and p_{Ib} are harmonic at frequencies ω_a and ω_b, then p'_{II,ab^*} is at the difference frequency $\omega_a - \omega_b$.

For the collinear case, we solve for the cross-interactions $p'_{\mathrm{II},ab}$ and p'_{II,ab^*} in section 3.1. Self-interaction components $p'_{\mathrm{II},aa}$ and $p'_{\mathrm{II},bb}$ can be obtained from $p_{\mathrm{II},ab}$ by letting $\tilde{p}_{\mathrm{Ib}} = \tilde{p}_{\mathrm{Ia}}$, and letting $\tilde{p}_{\mathrm{Ia}} = \tilde{p}_{\mathrm{Ib}}$, respectively in equation (S33a). Self-interaction components p'_{II,aa^*} and p'_{II,bb^*} can be obtained from p'_{II,ab^*} by letting $\tilde{p}^*_{\mathrm{Ib}} = \tilde{p}^*_{\mathrm{Ia}}$, and letting $\tilde{p}^*_{\mathrm{Ia}} = \tilde{p}^*_{\mathrm{Ib}}$, respectively in equation (S33b).

For the non-collinear case, we solve for the cross-interactions $p'_{\mathrm{II},ab}$ and p'_{II,ab^*} in section 3.2. Self-interaction components in this case are the same as those in the collinear case given in section 3.1. Specifically, $p'_{\mathrm{II},aa}$, $p'_{\mathrm{II},bb}$ can be obtained from equation (S33a) and p'_{II,aa^*} and p'_{II,bb^*} can be obtained from equation (S33b).

The p''_{II} component is given in equation (61). When p_{Ia} and p_{Ib} are plane waves propagating in directions $\hat{\mathbf{i}}_a$ and $\hat{\mathbf{i}}_b$ respectively, the first order velocity is

$$\mathbf{v}_1(\mathbf{r},t) = \frac{1}{\rho_0 c_0}\Re\left\{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)\hat{\mathbf{i}}_a + \tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)\hat{\mathbf{i}}_b\right\}, \tag{S18}$$

then

$$\begin{aligned} v_1^2 = \mathbf{v}_1\cdot\mathbf{v}_1 = &\frac{1}{2\rho_0^2 c_0^2}\Re\left\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ia}} + \tilde{p}_{\mathrm{Ia}}\tilde{p}^*_{\mathrm{Ia}} + \tilde{p}_{\mathrm{Ib}}\tilde{p}_{\mathrm{Ib}} + \tilde{p}_{\mathrm{Ib}}\tilde{p}^*_{\mathrm{Ib}}\right\} \\ &+ \frac{1}{\rho_0^2 c_0^2}\Re\left\{\tilde{p}_{\mathrm{Ia}}\tilde{p}_{\mathrm{Ib}} + \tilde{p}_{\mathrm{Ia}}\tilde{p}^*_{\mathrm{Ib}}\right\}\cos\theta, \end{aligned} \tag{S19}$$

where $\cos\theta = \hat{\mathbf{i}}_a\cdot\hat{\mathbf{i}}_b$, and

$$\begin{aligned} \frac{\partial p_1}{\partial t}\int_{-\infty}^{t} p_1 d\tau = &\frac{1}{2}\Re\left\{\frac{\partial\tilde{p}_{\mathrm{Ia}}}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}d\tau + \frac{\partial\tilde{p}_{\mathrm{Ia}}}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ia}}d\tau\right\} \\ &+ \frac{1}{2}\Re\left\{\frac{\partial\tilde{p}_{\mathrm{Ib}}}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ib}}d\tau + \frac{\partial\tilde{p}_{\mathrm{Ib}}}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ib}}d\tau\right\} \\ &+ \frac{1}{2}\Re\left\{\frac{\partial\tilde{p}_{\mathrm{Ia}}}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ib}}d\tau + \frac{\partial\tilde{p}_{\mathrm{Ib}}}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}d\tau\right\} \\ &+ \frac{1}{2}\Re\left\{\frac{\partial\tilde{p}_{\mathrm{Ia}}}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ib}}d\tau + \frac{\partial\tilde{p}^*_{\mathrm{Ib}}}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}d\tau\right\}, \end{aligned} \tag{S20}$$

Then components of p''_{II} due to self-interactions and cross-interactions are

$$p''_{\mathrm{II}} = p''_{\mathrm{II},aa} + p''_{\mathrm{II},aa^*} + p''_{\mathrm{II},bb} + p''_{\mathrm{II},bb^*} + p''_{\mathrm{II},ab} + p''_{\mathrm{II},ab^*}, \tag{S21}$$

where

$$p''_{\mathrm{II},aa}(\mathbf{r},t) = -\Re\left\{\frac{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)}{2A}\right\} - \Re\left\{\frac{1}{2A}\frac{\partial\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}(\mathbf{r},\tau)d\tau\right\}, \tag{S22}$$

$$p''_{\mathrm{II},aa^*}(\mathbf{r},t) = -\Re\left\{\frac{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)\tilde{p}^*_{\mathrm{Ia}}(\mathbf{r},t)}{2A}\right\} - \Re\left\{\frac{1}{2A}\frac{\partial\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ia}}(\mathbf{r},\tau)d\tau\right\}, \tag{S23}$$

$$p''_{\mathrm{II},bb}(\mathbf{r},t) = -\Re\left\{\frac{\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)}{2A}\right\} - \Re\left\{\frac{1}{2A}\frac{\partial\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ib}}(\mathbf{r},\tau)d\tau\right\}, \tag{S24}$$

$$p''_{\mathrm{II},bb^*}(\mathbf{r},t) = -\Re\left\{\frac{\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)\tilde{p}^*_{\mathrm{Ib}}(\mathbf{r},t)}{2A}\right\} - \Re\left\{\frac{1}{2A}\frac{\partial\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ib}}(\mathbf{r},\tau)d\tau\right\}, \tag{S25}$$

$$\begin{aligned} p''_{\mathrm{II},ab}(\mathbf{r},t) = &-\Re\left\{\frac{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)(1+\cos\theta)}{2A}\right\} \\ &-\Re\left\{\frac{1}{2A}\left(\frac{\partial\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ib}}(\mathbf{r},\tau)d\tau + \frac{\partial\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}(\mathbf{r},\tau)d\tau\right)\right\}, \end{aligned} \tag{S26}$$

$$\begin{aligned} p''_{\mathrm{II},ab^*}(\mathbf{r},t) = &-\Re\left\{\frac{\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)\tilde{p}^*_{\mathrm{Ib}}(\mathbf{r},t)(1+\cos\theta)}{2A}\right\} \\ &-\Re\left\{\frac{1}{2A}\left(\frac{\partial\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}^*_{\mathrm{Ib}}(\mathbf{r},\tau)d\tau + \frac{\partial\tilde{p}^*_{\mathrm{Ib}}(\mathbf{r},t)}{\partial t}\int_{-\infty}^{t}\tilde{p}_{\mathrm{Ia}}(\mathbf{r},\tau)d\tau\right)\right\}. \end{aligned} \tag{S27}$$

For the collinear case, $p''_{\mathrm{II},aa}$, p''_{II,aa^*}, $p''_{\mathrm{II},bb}$, p''_{II,bb^*}, $p''_{\mathrm{II},ab}$ and p''_{II,ab^*} for both self-interactions and cross-interactions appear as plane waves propagating with the primary plane waves. For the non-collinear case, self-interaction components $p''_{\mathrm{II},\alpha\alpha}$ and $p''_{\mathrm{II},\alpha\alpha^*}$ for $\alpha = a, b$ are the same as those in the collinear case, while cross-interaction components $p''_{\mathrm{II},ab}$ and p''_{II,ab^*}, are nonzero only in the union of two primary waves through which intersection occurred.

3.1 Second order field for collinear plane waves of arbitrary time dependence, equation (84)

Let $\tilde{p}_{\mathrm{Ia}}$ and $\tilde{p}_{\mathrm{Ib}}$ be two collinear plane waves of arbitrary time dependence propagating in the positive $\hat{\mathbf{i}}_z$ direction

$$\tilde{p}_{\mathrm{Ia}}(\mathbf{r},t) = \tilde{p}_{\mathrm{Ia}}(t - z/c_0), \tag{S28}$$

$$\tilde{p}_{\mathrm{Ib}}(\mathbf{r},t) = \tilde{p}_{\mathrm{Ib}}(t - z/c_0). \tag{S29}$$

The second order fields $p'_{\mathrm{II},ab}(x,z',t) = \Re\left\{\tilde{p}'_{\mathrm{II},ab}(x,z',t)\right\}$ and $p'_{\mathrm{II},ab^*}(x,z',t) =$

$\Re\left\{\tilde{p}'_{\mathrm{II},ab^*}(x,z',t)\right\}$ due to cross-interaction must satisfy

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\tilde{p}'_{\mathrm{II},ab}(z,t)=-\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\right], \tag{S30a}$$

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\tilde{p}'_{\mathrm{II},ab^*}(z,t)=-\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\right]. \tag{S30b}$$

Let $\tilde{p}_{\mathrm{Ia}}(t)\Leftrightarrow\Psi_a(\omega)$ and $\tilde{p}_{\mathrm{Ib}}(t)\Leftrightarrow\Psi_b(\omega)$, so that $\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\Leftrightarrow e^{\mathrm{i}\omega z/c_0}\Psi_a(\omega)$, $\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\Leftrightarrow e^{\mathrm{i}\omega z/c_0}\Psi_b(\omega)$ and $\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\Leftrightarrow e^{\mathrm{i}\omega z/c_0}\Psi^*_b(-\omega)$. Fourier transformation of equations (S30a) and (S30b) leads to the Helmholtz equations

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c_0^2}\right)\tilde{P}'_{\mathrm{II},ab}(z,\omega)=\frac{\beta\omega^2}{Ac_0^2}e^{\mathrm{i}\omega z/c_0}\Psi_a(\omega)\star\Psi_b(\omega), \tag{S31a}$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c_0^2}\right)\tilde{P}'_{\mathrm{II},ab^*}(z,\omega)=\frac{\beta\omega^2}{Ac_0^2}e^{\mathrm{i}\omega z/c_0}\Psi_a(\omega)\star\Psi^*_b(-\omega), \tag{S31b}$$

where $\tilde{P}'_{\mathrm{II},ab}(z,\omega)\Leftrightarrow\tilde{p}'_{\mathrm{II},ab}(z,t)$, $\tilde{P}'_{\mathrm{II},ab^*}(z,\omega)\Leftrightarrow\tilde{p}'_{\mathrm{II},ab^*}(z,t)$, and $\star$ is the convolution over frequency ω. The solutions to equations (S31a) and (S31b) can be found as

$$\tilde{P}'_{\mathrm{II},ab}(z,\omega)=-\frac{\mathrm{i}\omega\beta}{2Ac_0}\Psi_a(\omega)\star\Psi_b(\omega)ze^{\mathrm{i}\omega z/c_0}, \tag{S32a}$$

$$\tilde{P}'_{\mathrm{II},ab^*}(z,\omega)=-\frac{\mathrm{i}\omega\beta}{2Ac_0}\Psi_a(\omega)\star\Psi^*_b(-\omega)ze^{\mathrm{i}\omega z/c_0}. \tag{S32b}$$

Taking the inverse Fourier transform of equations (S32a) and (S32b) and keeping the real part yields

$$p'_{\mathrm{II},ab}(z,t)=\Re\left\{\frac{\beta z}{2Ac_0}\frac{\partial}{\partial t}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\right]\right\}, \tag{S33a}$$

$$p'_{\mathrm{II},ab^*}(z,t)=\Re\left\{\frac{\beta z}{2Ac_0}\frac{\partial}{\partial t}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\right]\right\}. \tag{S33b}$$

By adding equations (S26) and (S27) into equations (S33a) and (S33b) respectively, we obtain the complete solution for the cross interaction of collinear plane waves of arbitrary time dependence as

$$
\begin{aligned}
p_{\mathrm{II},ab}(z,t) = & \Re\left\{\frac{\beta z}{2Ac_0}\frac{\partial}{\partial t}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\right]\right\} \\
& -\Re\left\{\frac{1}{2A}\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\right\} \\
& -\Re\left\{\frac{1}{2A}\left(\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\right)\int_{-\infty}^{t-z/c_0}\tilde{p}_{\mathrm{Ib}}(\tau)d\tau\right\} \\
& -\Re\left\{\frac{1}{2A}\left(\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z/c_0)\right)\int_{-\infty}^{t-z/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right\}, \qquad \text{(S34a)}
\end{aligned}
$$

$$
\begin{aligned}
p_{\mathrm{II},ab^*}(z,t) = & \Re\left\{\frac{\beta z}{2Ac_0}\frac{\partial}{\partial t}\left[\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\right]\right\} \\
& -\Re\left\{\frac{1}{2A}\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\right\} \\
& -\Re\left\{\frac{1}{2A}\left(\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-z/c_0)\right)\int_{-\infty}^{t-z/c_0}\tilde{p}^*_{\mathrm{Ib}}(\tau)d\tau\right\} \\
& -\Re\left\{\frac{1}{2A}\left(\frac{\partial}{\partial t}\tilde{p}^*_{\mathrm{Ib}}(t-z/c_0)\right)\int_{-\infty}^{t-z/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right\}, \qquad \text{(S34b)}
\end{aligned}
$$

When boundary conditions are given, additional homogeneous solutions (plane waves) need to be added to the above solutions, but these homogeneous solutions are not due to the nonlinear interactions.

When $\tilde{p}_{\mathrm{Ia}}(t)$ and $\tilde{p}_{\mathrm{Ib}}(t)$ are narrow-band plane waves with compact support between $t=0$ and $t=T$, they can be written as the product of harmonic plane waves with a real moving window $w_1(t-z/c_0)$ as

$$\tilde{p}_{\mathrm{Ia}}(t-z/c_0) = P_{a0}e^{-\mathrm{i}\omega_a(t-z/c_0)}w_1(t-z/c_0), \qquad \text{(S35)}$$

$$\tilde{p}_{\mathrm{Ib}}(t-z/c_0) = P_{b0}e^{-\mathrm{i}\omega_b(t-z/c_0)}w_1(t-z/c_0). \qquad \text{(S36)}$$

For a sufficiently long and smooth window w_1, its time derivatives are negligible, such that

$$\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-z/c_0) \approx -\mathrm{i}\omega_a P_{a0}e^{\mathrm{i}(k_a z-\omega_a t)}w_1(t-z/c_0), \qquad \text{(S37)}$$

$$\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z/c_0) \approx -\mathrm{i}\omega_b P_{b0}e^{\mathrm{i}(k_b z-\omega_b t)}w_1(t-z/c_0), \qquad \text{(S38)}$$

and the time integrals of $\tilde{p}_{\mathrm{Ia}}$ and $\tilde{p}_{\mathrm{Ib}}$ can be approximated by the contributions from the end point $t-z/c_0$ [7], such that

$$\int_{-\infty}^{t-z/c_0} e^{-\mathrm{i}\omega_a\tau}w_1(\tau)d\tau \approx \frac{1}{-\mathrm{i}\omega_a}w_1(t-z/c_0)e^{-\mathrm{i}\omega_a(t-z/c_0)}, \qquad \text{(S39)}$$

$$\int_{-\infty}^{t-z/c_0} e^{-\mathrm{i}\omega_b\tau}w_1(\tau)d\tau \approx \frac{1}{-\mathrm{i}\omega_b}w_1(t-z/c_0)e^{-\mathrm{i}\omega_b(t-z/c_0)}. \qquad \text{(S40)}$$

Substituting equations (S37) - (S40) into equations (S33a) and (S33b) yields

$$p_{\mathrm{II},ab}(z,t) \approx \Re\left\{-\frac{P_{a0}P_{b0}}{2A}\left[\mathrm{i}\beta k_+ z + \frac{\omega_+^2}{\omega_a\omega_b}\right] e^{\mathrm{i}(k_+ z-\omega_+ t)} w_1^2(t-z/c_0)\right\}, \quad \text{(S41a)}$$

$$p_{\mathrm{II},ab^*}(z,t) \approx \Re\left\{-\frac{P_{a0}P_{b0}^*}{2A}\left[\mathrm{i}\beta k_+ z - \frac{\omega_-^2}{\omega_a\omega_b}\right] e^{\mathrm{i}(k_- z-\omega_- t)} w_1^2(t-z/c_0)\right\}. \quad \text{(S41b)}$$

where the terms linear in z in equations (S41a) and (S41b) correspond to Lamb's respective sum and difference frequency second order fields caused by the interaction of two collinear harmonic plane waves [8].

By setting $\tilde{p}_{\mathrm{Ib}} = \tilde{p}_{\mathrm{Ia}}$ in equations (S33a) and (S33b) then dividing the results by two, we can obtain the self-interaction components $p_{\mathrm{II},aa}$ and p_{II,aa^*} respectively. Similarly by setting $\tilde{p}_{\mathrm{Ia}} = \tilde{p}_{\mathrm{Ib}}$ in equations (S33a) and (S33b) then dividing by 2, we can obtain the self-interaction components $p_{\mathrm{II},bb}$ and p_{II,bb^*}, respectively. It can be seen that all self-interactions components are non-zero only inside the compact support of the corresponding primary waves.

3.2 Second order field for non-collinear plane waves of arbitrary time dependence, equation (86)

Let $\tilde{p}_{\mathrm{Ia}}$ and $\tilde{p}_{\mathrm{Ib}}$ be two non-collinear plane waves of arbitrary time dependence propagating in $\hat{\mathbf{i}}_x$ and $\hat{\mathbf{i}}_{z'}$ directions respectively,

$$\tilde{p}_{\mathrm{Ia}}(x,z',t) = \tilde{p}_{\mathrm{Ia}}(t-x/c_0) \quad \text{(S42)}$$

$$\tilde{p}_{\mathrm{Ib}}(x,z',t) = \tilde{p}_{\mathrm{Ib}}(t-z'/c_0), \quad \text{(S43)}$$

where $z' = x\cos\theta + z\sin\theta$ and $\theta \neq 0$ is the angle between $\hat{\mathbf{i}}_x$ and $\hat{\mathbf{i}}_{z'}$.

The second order fields $p'_{\mathrm{II},ab}(x,z',t) = \Re\left\{\tilde{p}'_{\mathrm{II},ab}(x,z',t)\right\}$ and $p'_{\mathrm{II},ab^*}(x,z',t) = \Re\left\{\tilde{p}'_{\mathrm{II},ab^*}(x,z',t)\right\}$ due to cross-interactions must satisfy

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\tilde{p}'_{\mathrm{II},ab}(x,z',t) = -\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\left[\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right], \quad \text{(S44a)}$$

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\tilde{p}'_{\mathrm{II},ab^*}(x,z',t) = -\frac{\beta}{Ac_0^2}\frac{\partial^2}{\partial t^2}\left[\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right]. \quad \text{(S44b)}$$

Let $\tilde{p}_{\mathrm{Ia}}(t) \Leftrightarrow \Psi_a(\omega)$ and $\tilde{p}_{\mathrm{Ib}}(t) \Leftrightarrow \Psi_b(\omega)$, it follows that $\tilde{p}^*_{\mathrm{Ib}}(t) \Leftrightarrow \Psi^*(-\omega)$. Fourier transformation of equations (S44a) and (S44b) leads to the Helmholtz equations

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right)\tilde{P}'_{\mathrm{II},ab}(x,z',\omega) = \frac{\beta\omega^2}{Ac_0^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b[(\omega-\Omega)]d\Omega, \tag{S45a}$$

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right)\tilde{P}'_{\mathrm{II},ab^*}(x,z',\omega) = \frac{\beta\omega^2}{Ac_0^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b^*[-(\omega-\Omega)]d\Omega, \tag{S45b}$$

where $\tilde{P}'_{\mathrm{II},ab}(x,z',\omega) \Leftrightarrow \tilde{p}'_{\mathrm{II},ab}(x,z',t)$, $\tilde{P}'_{\mathrm{II},ab^*}(x,z',\omega) \Leftrightarrow \tilde{p}'_{\mathrm{II},ab^*}(x,z',t)$, and the convolution theorem is used. Equations (S45a) and (S45b) can be solved analytically. Spatial Fourier transformation of (S45a) and (S45b) leads to

$$\left(-k_x^2 - k_z^2 + \frac{\omega^2}{c_0^2}\right)\iint_{-\infty}^{\infty}\tilde{P}'_{\mathrm{II},ab}(x,z',\omega)e^{\mathrm{i}k_x x+\mathrm{i}k_z z}dxdz = \frac{\beta\omega^2}{Ac_0^2}\frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b[(\omega-\Omega)]d\Omega e^{\mathrm{i}k_x x+\mathrm{i}k_z z}dxdz, \tag{S46a}$$

$$\left(-k_x^2 - k_z^2 + \frac{\omega^2}{c_0^2}\right)\iint_{-\infty}^{\infty}\tilde{P}'_{\mathrm{II},ab^*}(x,z',\omega)e^{\mathrm{i}k_x x+\mathrm{i}k_z z}dxdz = \frac{\beta\omega^2}{Ac_0^2}\frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b^*[-(\omega-\Omega)]d\Omega e^{\mathrm{i}k_x x+\mathrm{i}k_z z}dxdz. \tag{S46b}$$

Dividing by $(-k_x^2 - k_z^2 + \omega^2/c_0^2)$ and taking the inverse spatial Fourier transform of equations (S46a) and (S46b) leads to

$$\int_{-\infty}^{\infty}\tilde{p}'_{\mathrm{II},ab}(x,z',t)e^{\mathrm{i}\omega t}dt = \frac{\beta\omega^2}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b[(\omega-\Omega)]d\Omega}{\omega^2-[\Omega+(\omega-\Omega)\cos\theta]^2-(\omega-\Omega)^2\sin^2\theta}, \tag{S47a}$$

$$\int_{-\infty}^{\infty}\tilde{p}'_{\mathrm{II},ab^*}(x,z',t)e^{\mathrm{i}\omega t}dt = \frac{\beta\omega^2}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}\Omega x/c_0}\Psi_a(\Omega)e^{\mathrm{i}(\omega-\Omega)z'/c_0}\Psi_b^*[-(\omega-\Omega)]d\Omega}{\omega^2-[\Omega+(\omega-\Omega)\cos\theta]^2-(\omega-\Omega)^2\sin^2\theta}. \tag{S47b}$$

With the aid of partial fractions, the above expressions become

$$\int_{-\infty}^{\infty} \tilde{p}'_{\mathrm{II},ab}(x,z',t)e^{\mathrm{i}\omega t}dt = \frac{\beta\omega e^{\mathrm{i}\omega z'/c_0}}{2A(1-\cos\theta)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\Psi_a(\Omega)\Psi_b(\omega-\Omega)\left(\frac{1}{\Omega}+\frac{1}{\omega-\Omega}\right)d\Omega, \tag{S48a}$$

$$\int_{-\infty}^{\infty} \tilde{p}'_{\mathrm{II}\pm}(x,z',t)e^{\mathrm{i}\omega t}dt = \frac{\beta\omega e^{\mathrm{i}\omega z'/c_0}}{2A(1-\cos\theta)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\Psi_a(\Omega)\Psi_b^*[-(\omega-\Omega)]\left(\frac{1}{\Omega}+\frac{1}{\omega-\Omega}\right)d\Omega. \tag{S48b}$$

Taking the inverse Fourier transform and keeping the real part yields

$$p'_{\mathrm{II},ab}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A(1-\cos\theta)}\frac{\partial}{\partial t}[I^{(1)}_{ab}(x,z',t)+I^{(2)}_{ab}(x,z',t)]\right\}, \tag{S49a}$$

$$p'_{\mathrm{II},ab^*}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A(1-\cos\theta)}\frac{\partial}{\partial t}[I^{(1)}_{ab^*}(x,z',t)+I^{(2)}_{ab^*}(x,z',t)]\right\}, \tag{S49b}$$

where

$$I^{(1)}_{ab}(x,z',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\frac{\Psi_a(\Omega)}{\Omega}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_b(\omega-\Omega)e^{-\mathrm{i}\omega(t-z'/c_0)}d\omega\right]d\Omega, \tag{S50a}$$

$$I^{(2)}_{ab}(x,z',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\Psi_a(\Omega)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_b(\omega-\Omega)}{\omega-\Omega}e^{-\mathrm{i}\omega(t-z'/c_0)}d\omega\right]d\Omega, \tag{S50b}$$

$$I^{(1)}_{ab^*}(x,z',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\frac{\Psi_a(\Omega)}{\Omega}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_b^*[-(\omega-\Omega)]e^{-\mathrm{i}\omega(t-z'/c_0)}d\omega\right]d\Omega, \tag{S50c}$$

$$I^{(2)}_{ab^*}(x,z',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\Omega(x-z')/c_0}\Psi_a(\Omega)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_b^*[-(\omega-\Omega)]}{\omega-\Omega}e^{-\mathrm{i}\omega(t-z'/c_0)}d\omega\right]d\Omega, \tag{S50d}$$

after switching the order of the integrations.

The bracketed integrals in equations (S50a) and (S50c) are inverse Fourier transforms that can be written as $e^{-\mathrm{i}\Omega(t-z'/c_0)}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)$ and $e^{-\mathrm{i}\Omega(t-z'/c_0)}\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)$, respectively. Then $I^{(1)}_{ab}$ and $I^{(1)}_{ab^*}$ become

$$I^{(1)}_{ab}(x,z',t) = \tilde{p}_{\mathrm{Ib}}(t-z'/c_0)J_a(t-x/c_0), \tag{S51a}$$

$$I^{(1)}_{ab^*}(x,z',t) = \tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)J_a(t-x/c_0), \tag{S51b}$$

where

$$J_a(t-x/c_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_a(\Omega)}{\Omega}e^{-\mathrm{i}\Omega(t-x/c_0)}d\Omega \tag{S52}$$

is the inverse Fourier transform of $\Psi_a(\Omega)/\Omega$, evaluated at $t - x/c_0$. It follows from the integration property of Fourier transforms [9] that

$$J_a(t - x/c_0) = -\mathrm{i}\int_{-\infty}^{t-x/c_0} \tilde{p}_{\mathrm{Ia}}(\tau)d\tau + \frac{\mathrm{i}\Psi_a(0)}{2}, \tag{S53}$$

and the time derivative of J_a is

$$\frac{\partial}{\partial t}J_a(t - x/c_0) = -\mathrm{i}\tilde{p}_{\mathrm{Ia}}(t - x/c_0). \tag{S54}$$

The procedure to simplify $I^{(2)}_{ab}(t)$ and $I^{(2)}_{ab^*}(t)$ is similar. Substituting $\eta = \omega - \Omega$ in the bracketed integrals of equations (S50b) and (S50d) leads to

$$I^{(2)}_{ab}(x, z', t) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_a(\Omega)e^{-\mathrm{i}\Omega(t-x/c_0)}d\Omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi_b(\eta)}{\eta}e^{-\mathrm{i}\eta(t-z'/c_0)}d\eta\right], \tag{S55a}$$

$$I^{(2)}_{ab^*}(x, z', t) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_a(\Omega)e^{-\mathrm{i}\Omega(t-x/c_0)}d\Omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi^*_b(-\eta)}{\eta}e^{-\mathrm{i}\eta(t-z'/c_0)}d\eta\right], \tag{S55b}$$

which simplify to

$$I^{(2)}_{ab}(x, z', t) = \tilde{p}_{\mathrm{Ia}}(t - x/c_0)J_b(t - z'/c_0), \tag{S56a}$$

$$I^{(2)}_{ab^*}(x, z', t) = \tilde{p}_{\mathrm{Ia}}(t - x/c_0)J_{b^*}(t - z'/c_0), \tag{S56b}$$

where

$$J_b(t - z'/c_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi_b(\eta)}{\eta}e^{-\mathrm{i}\eta(t-z'/c_0)}d\eta, \tag{S57a}$$

$$J_{b^*}(t - z'/c_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Psi^*_b(-\eta)}{\eta}e^{-\mathrm{i}\eta(t-z'/c_0)}d\eta. \tag{S57b}$$

It can be seen that J_b has the same form as J_a in equation (S52) and $J^*_b = -J_{b^*}$. With the integration property of Fourier transforms, equations (S57a) and (S57b) become

$$J_b(t - z'/c_0) = -\mathrm{i}\int_{-\infty}^{t-z'/c_0} \tilde{p}_{\mathrm{Ib}}(\tau)d\tau + \frac{\mathrm{i}\Psi_b(0)}{2}, \tag{S58a}$$

$$J_{b^*}(t - z'/c_0) = -\mathrm{i}\int_{-\infty}^{t-z'/c_0} \tilde{p}^*_{\mathrm{Ib}}(\tau)d\tau + \frac{\mathrm{i}\Psi^*_b(0)}{2}. \tag{S58b}$$

The time derivatives of J_b and J_{b^*} are

$$\frac{\partial}{\partial t}J_b(t - z'/c_0) = -\mathrm{i}\tilde{p}_{\mathrm{Ib}}(t - z'/c_0), \tag{S59a}$$

$$\frac{\partial}{\partial t}J_{b^*}(t - z'/c_0) = -\mathrm{i}\tilde{p}^*_{\mathrm{Ib}}(t - z'/c_0), \tag{S59b}$$

Substituting equations (S51), (S53), (S56) and (S58) into equations (S49b) and (S49a) yields particular solutions

$$p'_{\mathrm{II},ab}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau + \frac{\mathrm{i}\Psi_a(0)}{2}\right]\right\}$$
$$+\Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-z'/c_0}\tilde{p}_{\mathrm{Ib}}(\tau)d\tau + \frac{\mathrm{i}\Psi_b(0)}{2}\right]\right\}$$
$$+\Re\left\{\frac{\beta}{A}\frac{1}{1-\cos\theta}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right\}, \tag{S60a}$$

$$p'_{\mathrm{II},ab^*}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau + \frac{\mathrm{i}\Psi_a(0)}{2}\right]\right\}$$
$$+\Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-z'/c_0}\tilde{p}^*_{\mathrm{Ib}}(\tau)d\tau + \frac{\mathrm{i}\Psi^*_b(0)}{2}\right]\right\}$$
$$+\Re\left\{\frac{\beta}{A}\frac{1}{1-\cos\theta}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right\}. \tag{S60b}$$

The physical solutions for $p'_{\mathrm{II},ab}(x,z',t)$ and $p'_{\mathrm{II},ab^*}(x,z',t)$ are obtained by adding homogeneous solutions $\Re\left\{\psi^{(1)}_{ab}(t-x/c_0)\right\}$ and $\Re\left\{\psi^{(2)}_{ab}(t-z'/c_0)\right\}$ to equation (S60a), and $\Re\left\{\psi^{(1)}_{ab^*}(t-x/c_0)\right\}$ and $\Re\left\{\psi^{(2)}_{ab^*}(t-z'/c_0)\right\}$ to equation and (S60b) respectively, and imposing the causality condition that the second order field from intersecting plane waves must be zero in space before any wave arrives there. The homogeneous solutions are determined as

$$\psi^{(1)}_{ab}(t-x/c_0) = -\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\frac{\mathrm{i}\Psi_b(0)}{2}\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0), \tag{S61}$$
$$\psi^{(2)}_{ab}(t-z'/c_0) = -\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\frac{\mathrm{i}\Psi_a(0)}{2}\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0), \tag{S62}$$
$$\psi^{(1)}_{ab^*}(t-x/c_0) = -\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\frac{\mathrm{i}\Psi^*_b(0)}{2}\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0), \tag{S63}$$
$$\psi^{(2)}_{ab^*}(t-z'/c_0) = -\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\frac{\mathrm{i}\Psi_a(0)}{2}\frac{\partial}{\partial t}\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0), \tag{S64}$$

such that the physical solutions for $p'_{\mathrm{II},ab}(x,z',t)$ and $p'_{\mathrm{II},ab^*}(x,z',t)$ are

$$
\begin{aligned}
p'_{\mathrm{II},ab}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-z'/c_0}\tilde{p}_{\mathrm{Ib}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{\beta}{A}\frac{1}{1-\cos\theta}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right\},
\end{aligned}
\tag{S65a}
$$

$$
\begin{aligned}
p'_{\mathrm{II},ab^*}(x,z',t) = \Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{\mathrm{i}\beta}{2A}\frac{1}{1-\cos\theta}\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[-\mathrm{i}\int_{-\infty}^{t-z'/c_0}\tilde{p}^*_{\mathrm{Ib}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{\beta}{A}\frac{1}{1-\cos\theta}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right\}.
\end{aligned}
\tag{S65b}
$$

By adding equations (S26) and (S27) into equations (S65a) and (S65b) respectively, we obtain the complete solution for the cross interaction of non-collinear plane waves of arbitrary time dependence

$$
\begin{aligned}
p_{\mathrm{II},ab}(x,z',t) = \Re\left\{\frac{1}{2A}\left[\frac{\beta}{1-\cos\theta}-1\right]\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right]\left[\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{1}{2A}\left[\frac{\beta}{1-\cos\theta}-1\right]\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[\int_{-\infty}^{t-z'/c_0}\tilde{p}_{\mathrm{Ib}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{1}{2A}\left[\frac{2\beta}{1-\cos\theta}-(1+\cos\theta)\right]\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}_{\mathrm{Ib}}(t-z'/c_0)\right\},
\end{aligned}
\tag{S66a}
$$

$$
\begin{aligned}
p_{\mathrm{II},ab^*}(x,z',t) = \Re\left\{\frac{1}{2A}\left[\frac{\beta}{1-\cos\theta}-1\right]\left[\frac{\partial}{\partial t}\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right]\left[\int_{-\infty}^{t-x/c_0}\tilde{p}_{\mathrm{Ia}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{1}{2A}\left[\frac{\beta}{1-\cos\theta}-1\right]\left[\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\right]\left[\int_{-\infty}^{t-z'/c_0}\tilde{p}^*_{\mathrm{Ib}}(\tau)d\tau\right]\right\} \\
+\Re\left\{\frac{1}{2A}\left[\frac{2\beta}{1-\cos\theta}-(1+\cos\theta)\right]\tilde{p}_{\mathrm{Ia}}(t-x/c_0)\tilde{p}^*_{\mathrm{Ib}}(t-z'/c_0)\right\}.
\end{aligned}
\tag{S66b}
$$

When boundary conditions are given, additional homogeneous solutions need to be added to the above solutions, but these homogeneous solutions are not due to the nonlinear interactions.

It can be seen from equations (S66a) and (S66b) that sum and difference frequency components due to cross-interaction between two intersecting plane waves $p_{\mathrm{Ia}}(t-x/c_0)$ and $p_{\mathrm{Ib}}(t-z'/c_0)$ with compact support between $t=0$ and $t=T$ only exist in the region of compact support intersection (purple region in figure S2), so that no scattering of sound by sound at sum or difference frequencies is found outside the region of compact support intersection to second order. There will be a component at the primary frequency of a given plane wave (p_{Ia} or p_{Ib}) due to cross-interaction only where the given primary field exists and the intersecting plane wave (p_{Ib} or p_{Ia}) has passed through (hatched region in figure S2), if the intersecting plane wave has a non-zero zero-frequency component.

There will be non-zero $p_{\mathrm{II},aa}$ and p_{II,aa^*} inside the compact support of p_{Ia} due to self-interaction, which gives rise to sum and difference frequency components from p_{Ia}. Similarly, there will be non-zero $p_{\mathrm{II},bb}$ and p_{II,bb^*} inside the compact support of p_{Ib} due to self-interaction, which gives rise to sum and difference frequency components from p_{Ib}. There is no scattering of sound by sound due to cross-interaction or self-interaction of two finite-duration plane waves at any frequency outside the region of compact support union through which compact support intersection occurred.

When $p_{\mathrm{Ia}}(t-x/c_0)$ and $p_{\mathrm{Ib}}(t-z'/c_0)$ are narrow-band plane waves with compact support, they can be written as the product of harmonic plane waves and real moving windows $w_1(t-x/c_0)$ or $w_1(t-z'/c_0)$ as

$$\tilde{p}_{\mathrm{Ia}}(t-x/c_0) = P_{a0}e^{-\mathrm{i}\omega_a(t-x/c_0)}w_1(t-x/c_0), \tag{S67}$$

$$\tilde{p}_{\mathrm{Ib}}(t-z'/c_0) = P_{b0}e^{-\mathrm{i}\omega_b(t-z'/c_0)}w_1(t-z'/c_0). \tag{S68}$$

For a sufficiently long and smooth window w_1, its time derivatives are negligible, such that

$$\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ia}}(t-x/c_0) \approx -\mathrm{i}\omega_a P_{a0}e^{\mathrm{i}(k_a x-\omega_a t)}w_1(t-x/c_0), \tag{S69}$$

$$\frac{\partial}{\partial t}\tilde{p}_{\mathrm{Ib}}(t-z'/c_0) \approx -\mathrm{i}\omega_b P_{b0}e^{\mathrm{i}(k_b z'-\omega_b t)}w_1(t-z'/c_0), \tag{S70}$$

and the time integrals of $\tilde{p}_{\mathrm{Ia}}$ and $\tilde{p}_{\mathrm{Ib}}$ can be approximated by the contributions from the respective end points $t-x/c_0$ and $t-z'/c_0$ [7], such that

$$\int_{-\infty}^{t-x/c_0} e^{-\mathrm{i}\omega_a\tau}w_1(\tau)d\tau \approx \frac{1}{-\mathrm{i}\omega_a}w_1(t-x/c_0)e^{-\mathrm{i}\omega_a(t-x/c_0)}, \tag{S71}$$

$$\int_{-\infty}^{t-z'/c_0} e^{-\mathrm{i}\omega_b\tau}w_1(\tau)d\tau \approx \frac{1}{-\mathrm{i}\omega_b}w_1(t-z'/c_0)e^{-\mathrm{i}\omega_b(t-z'/c_0)}. \tag{S72}$$

Substituting equations (S69) - (S72) into equations (S66a) and (S66b) yields

$$p_{\mathrm{II},ab}(x,z',t) \approx \Re\left\{ \frac{P_{a0}P_{b0}}{2A}\left[\left(\frac{\beta}{1-\cos\theta}-1\right)\frac{\omega_+^2}{\omega_a\omega_b}+1-\cos\theta\right] \right.$$
$$\left. \times\, e^{\mathrm{i}(k_a x + k_b z' - \omega_+ t)} w_1(t-x/c_0) w_1(t-z'/c_0) \right\}. \qquad \text{(S73a)}$$
$$p_{\mathrm{II},ab^*}(x,z',t) \approx \Re\left\{ \frac{P_{a0}P_{b0}^*}{2A}\left[-\left(\frac{\beta}{1-\cos\theta}-1\right)\frac{\omega_-^2}{\omega_a\omega_b}+1-\cos\theta\right] \right.$$
$$\left. \times\, e^{\mathrm{i}(k_a x - k_b z' - \omega_- t)} w_1(t-x/c_0) w_1(t-z'/c_0) \right\}. \qquad \text{(S73b)}$$

The sum and difference frequency components due to cross-interaction of two intersecting narrow-band plane wave pulses are again seen to only exist in the region of compact support intersection of the two plane waves (purple region in figure S2), where they agree with Westervelt's respective sum and difference frequency second order fields resulting from the interaction of two non-collinear time-harmonic plane waves [10]. This shows that a time-harmonic approximation can be made in the region of compact support intersection for narrow-band finite-duration plane waves with sufficiently long and smooth windows.

3.3 Collinearity and dispersion relation

For two plane waves with vector wave number $\mathbf{k}_a$ and $\mathbf{k}_b$ at angular frequencies ω_a and ω_b propagating in a medium with sound speed c_0, they satisfy the dispersion relations $\omega_a = |\mathbf{k}_a|c_0$ and $\omega_b = |\mathbf{k}_b|c_0$ respectively. The sum ($\omega_+ = \omega_a + \omega_b$) and difference ($\omega_- = |\omega_a - \omega_b|$) frequency second order fields due to their cross-interaction have vector wavenumbers $\mathbf{k}_\pm = \mathbf{k}_a \pm \mathbf{k}_b$. These second order fields propagate when the dispersion relations are satisfied, i.e. $\omega_\pm = |\mathbf{k}_\pm|c_0$, which only occurs when the two plane waves are collinear, as illustrated in figure S3.

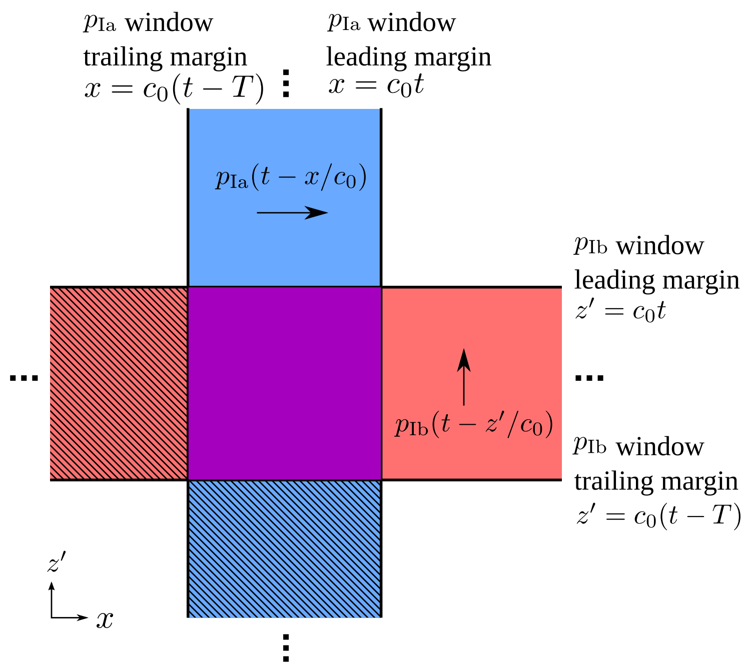


Figure S2: Intersecting non-collinear plane waves of compact support duration T. The $p_{\mathrm{Ia}}(t-x/c_0)$ wave (blue) is propagating to the right and the $p_{\mathrm{Ib}}(t-z'/c_0)$ wave (red) is propagating upward. The sum and difference frequency components of $p_{\mathrm{II},ab}$ and p_{II,ab^*} are only nonzero in the region of compact support intersection (purple). There are no conditions under which there is second order sound outside the union of the primary windows. The second order field $p_{\mathrm{II},ab}$ or p_{II,ab^*} will have primary frequency components where the primary field exists and the intersecting plane wave has passed through (hatched) if the intersecting field has a non-zero zero-frequency spectral component. For narrow-band plane waves with sufficiently long and smooth windows, a harmonic approximation can be made for the sum or difference frequency field in the region of compact support intersection.

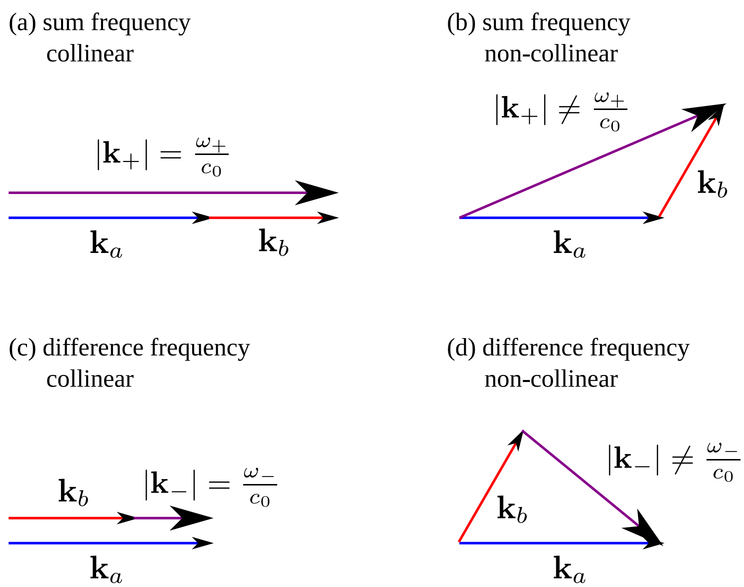


Figure S3: Sum and difference frequency second order fields due to the cross-interaction of two plane waves propagate only when the two plane waves are collinear, such that the dispersion relation $\omega_\pm = |\mathbf{k}_\pm|c_0$ can be satisfied. Here $\mathbf{k}_\pm = \mathbf{k}_a \pm \mathbf{k}_b$, $\omega_\pm = \omega_a \pm \omega_b = (|\mathbf{k}_a| \pm |\mathbf{k}_b|)c_0$.

4 Fourier transform pairs

The Fourier transform pairs $F(\omega) \Leftrightarrow f(t)$ are defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}\omega t}dt, \tag{S74}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\mathrm{i}\omega t}d\omega. \tag{S75}$$

For this normalization, the convolution theorem is

$$f(t)h(t) \Leftrightarrow F(\omega) \star H(\omega), \tag{S76}$$

$$F(\omega)H(\omega) \Leftrightarrow f(t) \star h(t), \tag{S77}$$

where $f(t) \Leftrightarrow F(\omega)$, $h(t) \Leftrightarrow H(\omega)$, and $\star$ denotes convolution operation which is different in frequency domain and time domain

$$F(\omega) \star H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)H(\omega - \Omega)d\Omega, \tag{S78}$$

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau. \tag{S79}$$

5 Asymptotic solution for SS interaction, equation (90)

We begin with a spherical object assumption to illustrate how general asymptotic expressions can be obtained for the SS interaction in terms of the far field scatter functions of any object. For a spherical object, the two scattered fields can be expressed as

$$P_{\mathrm{Sa}}(\mathbf{r}) = P_{a0}\sum_{l=0}^{l_{\max}}\sum_{m=-l}^{l} a_{lm}h_l(k_a r)Y_l^m(\theta,\phi), \tag{S80}$$

$$P_{\mathrm{Sb}}(\mathbf{r}) = P_{b0}\sum_{k=0}^{k_{\max}}\sum_{n=-k}^{k} b_{kn}h_k(k_b r)Y_k^n(\theta,\phi), \tag{S81}$$

where a_{lm} and b_{kn} are constant coefficients determined by the boundary condition. The product $P_{\mathrm{Sa}}P_{\mathrm{Sb}}^{(*)}$ can be expanded in spherical harmonics:

$$P_{\mathrm{Sa}}(\mathbf{r})P_{\mathrm{Sb}}^{(*)}(\mathbf{r}) = P_{a0}P_{b0}^{(*)}\sum_{L=0}^{L_{\max}}\sum_{M=-L}^{L} Q_L^{M(*)}(k_a r, k_b r)Y_L^M(\theta,\phi), \tag{S82}$$

where Q_L^{M*} is a quadruple summation involving Clebsch-Gordan coefficients for every degree L and order M [11]. When the expansions for P_{Sa} and P_{Sb} are truncated at $l_{\max}$ and $k_{\max}$ respectively, the expansion for $P_{\mathrm{Sa}}P_{\mathrm{Sb}}^{(*)}$ is limited to $L_{\max} = l_{\max} + k_{\max}$. The free space Green function can also be expanded as [2]

$$G_\pm(\mathbf{r}|\mathbf{r}_0) = \mathrm{i}k_\pm \sum_{K=0}^{\infty} \sum_{N=-K}^{K} Y_K^{N*}(\theta_0, \phi_0) Y_K^N(\theta, \phi) j_K(k_\pm r_<) h_K(k_\pm r_>), \tag{S83}$$

where j_K is the spherical Bessel function of order K, $r_< = \min(r, r_0)$ and $r_> = \max(r, r_0)$. With the orthogonal property of the spherical harmonics, $P'_{\mathrm{SS}\pm}$ in equation (78) can be written as

$$\begin{aligned} P'_{\mathrm{SS}\pm}(\mathbf{r}) = &-\frac{\mathrm{i}\beta k_\pm^3 P_{a0} P_{b0}^{(*)}}{A} \sum_{L=0}^{L_{\max}} \sum_{M=-L}^{L} Y_L^M(\theta, \phi) \\ &\times \left[h_L(k_\pm r) \int_a^r Q_L^{M(*)} j_L(k_\pm r_0) r_0^2 dr_0 + j_L(k_\pm r) \int_r^\infty Q_L^{M(*)} h_L(k_\pm r_0) r_0^2 dr_0 \right], \end{aligned} \tag{S84}$$

where the first integral represents the interaction within range r and the second integral represents the interaction beyond range r. Equation (S84) is exact and can be directly evaluated analytically or numerically, yet the number of terms in $Q_L^{M(*)}$ increases dramatically as $L_{\max}$ increases [11]. To proceed, we define a range $r_{\mathrm{ref}\pm} \le r$ and decompose the first integral in equation (S84) into two integrals: one from a to $r_{\mathrm{ref}\pm}$ and one from $r_{\mathrm{ref}\pm}$ to r, such that

$$P'_{\mathrm{SS}\pm} = P'^{(1)}_{\mathrm{SS}\pm} + P'^{(2)}_{\mathrm{SS}\pm} \tag{S85}$$

where

$$P'^{(1)}_{\mathrm{SS}\pm}(\mathbf{r}) = -\frac{\mathrm{i}\beta k_\pm^3 P_{a0} P_{b0}^{(*)}}{A} \sum_{L,M} Y_L^M(\theta, \phi) h_L(k_\pm r) \int_a^{r_{\mathrm{reff}\pm}} Q_L^{M(*)} j_L(k_\pm r_0) r_0^2 dr_0, \tag{S86}$$

and

$$\begin{aligned} P'^{(2)}_{\mathrm{SS}\pm}(\mathbf{r}) = &-\frac{\mathrm{i}\beta k_\pm^3 P_{a0} P_{b0}^{(*)}}{A} \sum_{L,M} Y_L^M(\theta, \phi) \\ &\times \left[h_L(k_\pm r) \int_{r_{\mathrm{ref}\pm}}^r Q_L^{M(*)} j_L(k_\pm r_0) r_0^2 dr_0 + j_L(k_\pm r) \int_r^\infty Q_L^{M(*)} h_L(k_\pm r_0) r_0^2 dr_0 \right]. \end{aligned} \tag{S87}$$

The $P'^{(1)}_{\mathrm{SS}\pm}$ term of equation (S86) can be numerically evaluated because the integration domain is finite (from a to $r_{\mathrm{ref}\pm}$). For any finite $r_{\mathrm{ref}\pm}$, $P'^{(1)}_{\mathrm{SS}\pm}$ falls off by r^{-1} as $r \to \infty$ so becoming small compared to $P'^{(2)}_{\mathrm{SS}\pm}$.

5.1 Spherical wave expansion and approximation to $P'^{(2)}_{\mathbf{SS}\pm}$

Let r_a and r_b be the far field ranges for the primary scattered fields P_{Sa} and P_{Sb} respectively, such that beyond r_a and r_b, far field approximations [12] apply

$$P_{\mathrm{Sa}}(\mathbf{r}) = P_{a0}\frac{S_a(\hat{\mathbf{i}}_r)}{k_a}\frac{e^{ik_a r}}{r}, \tag{S88}$$

$$P_{\mathrm{Sb}}(\mathbf{r}) = P_{b0}\frac{S_b(\hat{\mathbf{i}}_r)}{k_b}\frac{e^{ik_b r}}{r}, \tag{S89}$$

where $r_{a,b} = l^2/\lambda_{a,b}$, $k_a r_a, k_b r_b \gg 1$, l is the length scale of the object, $\lambda_{a,b}$ are the wavelengths of the incident fields, P_{a0} and P_{b0} are the amplitudes of the incident fields, S_a and S_b are the far field scatter functions, and $\hat{\mathbf{i}}_r = \mathbf{r}/r$.

The product $S_a S_b^{(*)}$ can be expanded in spherical harmonics with coefficients $q_L^{M(*)}$ as

$$S_a(\theta,\phi)S_b^{(*)}(\theta,\phi) = \sum_{L=0}^{\infty}\sum_{M=-L}^{L} q_L^{M(*)} Y_L^M(\theta,\phi). \tag{S90}$$

Comparing equation (S82) with equation (S90) for $r \geq r_{\mathrm{ref}\pm}$, we have

$$Q_L^{M(*)} = \frac{e^{\mathrm{i}k_\pm r}}{k_a k_b r^2} q_L^{M(*)}, \qquad \text{for } L = 0, 1, \cdots, L_{\max}. \tag{S91}$$

When $k_\pm r_{\mathrm{ref}\pm} \gg 1$ the spherical Bessel and Hankel functions follow their asymptotic behaviors [13] for $r > r_{\mathrm{ref}\pm}$

$$j_L(k_\pm r) \approx \frac{e^{\mathrm{i}(k_\pm r - L\pi/2)} - e^{-\mathrm{i}(k_\pm r - L\pi/2)}}{2\mathrm{i}k_\pm r}, \tag{S92}$$

$$h_L(k_\pm r) \approx \frac{e^{\mathrm{i}(k_\pm r - L\pi/2)}}{\mathrm{i}k_\pm r} \tag{S93}$$

which are exact for $L = 0$. For sufficiently small objects such that the L=0 case completely describes the radial dependence, $r_{\mathrm{ref}\pm}$ should be taken as the object radius a.

Substituting equations (S91)-(S93) into equation (S87) yields

$$\begin{aligned} P'^{(2)}_{\mathrm{SS}\pm}(\mathbf{r}) = {} & \frac{\mathrm{i}\beta k_\pm^2 P_{a0} P_{b0}^{(*)}}{2A k_a k_b k_\pm r} \\ & \times \left[e^{\mathrm{i}k_\pm r} \int_{r_{\mathrm{ref}\pm}}^{\infty} \frac{e^{\mathrm{i}2k_\pm r_0}}{r_0} dr_0 \left(\sum_{L,M} q_L^{M(*)} Y_L^M(\theta,\phi)(-1)^L \right) \right. \\ & \quad - e^{\mathrm{i}k_\pm r} \int_{r_{\mathrm{ref}\pm}}^{r} \frac{dr_0}{r_0} \left(\sum_{L,M} q_L^{M(*)} Y_L^M(\theta,\phi) \right) \\ & \quad \left. - e^{-\mathrm{i}k_\pm r} \int_{r}^{\infty} \frac{e^{\mathrm{i}2k_\pm r_0}}{r_0} dr_0 \left(\sum_{L,M} q_L^{M(*)} Y_L^M(\theta,\phi) \right) \right]. \end{aligned} \tag{S94}$$

Since $\sum_{L,M} q_L^{M(*)} Y_L^M(\theta,\phi) = S_a(\theta,\phi) S_b^{(*)}(\theta,\phi) = S_a(\hat{\mathbf{i}}_r) S_b^{(*)}(\hat{\mathbf{i}}_r)$ as given by equation (S90) where $\hat{\mathbf{i}}_r = \mathbf{r}/r$, it can be shown, by using the relation $P_L^M(-x) = (-1)^{L+M} P_L^M(x)$ for associated Legendre function P_L^M, that $\sum\limits_{L,M} (-1)^L q_L^{M(*)} Y_L^M(\theta,\phi) = S_a(\pi-\theta,\pi+\phi) S_b^{(*)}(\pi-\theta,\pi+\phi) = S_a(-\hat{\mathbf{i}}_r) S_b^{(*)}(-\hat{\mathbf{i}}_r)$. Equation (S94) then becomes identical to equation (90).

5.2 Stationary phase approximation to $P'^{(2)}_{\mathbf{SS}\pm}$

Without loss of generality, we choose a spherical coordinate system (r_0, α_0, β_0) such that the zenith direction coincides with $\mathbf{r}$. In equation (S87), $P'^{(2)}_{\mathrm{SS}\pm}$ can be rewritten in this coordinate system as

$$P'^{(2)}_{\mathrm{SS}\pm}(\mathbf{r}) = -\frac{\beta k_\pm^2}{A} \iiint_{|\mathbf{r}_0| \geq r_{\mathrm{ref}\pm}} P_{\mathrm{Sa}}(r_0,\alpha_0,\beta_0) P_{\mathrm{Sb}}^{(*)}(r_0,\alpha_0,\beta_0) \times \frac{e^{\mathrm{i}k_\pm R}}{4\pi R} r_0^2 dr_0 \sin\alpha_0 d\alpha_0 d\beta_0. \tag{S95}$$

where $\exp(\mathrm{i}k_\pm R)/(4\pi R)$ is the free space Green function and $R = \sqrt{r^2 + r_0^2 - 2 r r_0 \cos\alpha_0}$.

To remove the singularity in the Green function when $R = 0$, we use R instead of α_0 as dummy variable, then

$$P'^{(2)}_{\mathrm{SS}\pm}(\mathbf{r}) = -\frac{\beta k_\pm^2}{4\pi A r} \int_{r_{\mathrm{ref}\pm}}^{\infty} r_0 dr_0 \int_0^{2\pi} d\eta_0 \times \int_{|r-r_0|}^{|r+r_0|} \hat{P}_{\mathrm{Sa}}(r_0,R,\beta_0) \, \hat{P}_{\mathrm{Sb}}^{(*)}(r_0,R,\beta_0) \, e^{\mathrm{i}k_\pm R} dR. \tag{S96}$$

Assuming $k_\pm r \geq k_\pm r_{\mathrm{ref}\pm} \gg 1$, and $\hat{P}_{\mathrm{Sa}}$ and $\hat{P}_{\mathrm{Sb}}$ are slow varying functions in R, we find that the integrand for the ξ_0 integral in equation (S96) is rapidly oscillating with a linear function of R in its phase, suggesting that the leading order contribution comes from the two end points [7] at $R = |r - r_0|$ and $R = |r + r_0|$. Integrating over R then β_0 yields

$$\begin{aligned} P'^{(2)}_{\mathrm{SS}\pm}(\mathbf{r}) \approx -\frac{\beta k_\pm}{2\mathrm{i}Ar} \Bigg[& e^{\mathrm{i}k_\pm r} \int_{r_{\mathrm{ref}\pm}}^{\infty} P_{\mathrm{Sa}}(r_0, -\hat{\mathbf{i}}_r) P_{\mathrm{Sb}}^{(*)}(r_0, -\hat{\mathbf{i}}_r) e^{\mathrm{i}k_\pm r_0} r_0 dr_0 \\ & - e^{\mathrm{i}k_\pm r} \int_{r_{\mathrm{ref}\pm}}^{r} P_{\mathrm{Sa}}(r_0, \hat{\mathbf{i}}_r) P_{\mathrm{Sb}}^{(*)}(r_0, \hat{\mathbf{i}}_r) e^{-\mathrm{i}k_\pm r_0} r_0 dr_0 \\ & - e^{-\mathrm{i}k_\pm r} \int_{r}^{\infty} P_{\mathrm{Sa}}(r_0, \hat{\mathbf{i}}_r) P_{\mathrm{Sb}}^{(*)}(r_0, \hat{\mathbf{i}}_r) e^{\mathrm{i}k_\pm r_0} r_0 dr_0 \Bigg]. \end{aligned} \tag{S97}$$

If $r_{\mathrm{ref}\pm} \geq r_a, r_b$ so that the far field approximations for P_{Sa} and P_{Sb} apply, substituting equations (S88) and (S89) into equation (S97) yields equation (90).

6 Dean's solution

For two spherical waves P_{Sa} and P_{Sb} given by

$$P_{\mathrm{Sa}}(r) = P_{a0}\frac{e^{\mathrm{i}k_a r}}{k_a r} \quad \text{and} \quad P_{\mathrm{Sb}}(r) = P_{b0}\frac{e^{\mathrm{i}k_b r}}{k_b r}, \tag{S98}$$

Dean [10] provided a solution of the inhomogeneous Helmholtz equation corresponding to equation (3.4) for their cross-interaction outside a sphere of radius a

$$P_{\pm}^{\mathrm{Dean}} = -\frac{\mathrm{i}\beta k_{\pm}}{2Ak_a k_b} P_{a0} P_{b0}^{(*)} \frac{e^{\mathrm{i}k_{\pm} r}}{r}\left[\log\left(\frac{r}{a}\right) - e^{-2\mathrm{i}k_{\pm} r}\int_a^r \frac{e^{2\mathrm{i}k_{\pm} r_0}}{r_0} dr_0\right]. \tag{S99}$$

This solution does not satisfy the Sommerfeld radiation condition [2] in the far field since

$$br\left(\frac{\partial}{\partial r} - \mathrm{i}k_{\pm}\right) P_{\pm}^{\mathrm{Dean}} \propto e^{-\mathrm{i}k_{\pm} r}\int_a^r \frac{e^{2\mathrm{i}k_{\pm} r_0}}{r_0} dr_0 + O(r^{-1}) \tag{S100}$$

does not vanish but approaches a finite value as $r \to \infty$.

The difference between Dean's and Baxter's [14] solutions is a term proportional to the spherical Bessel function $j_0(k_{\pm} r)$. By adding $\alpha j_0(k_{\pm} r)$ to Dean's solution, the resulting solution $P_{\pm}^{\mathrm{Dean}} + \alpha j_0(k_{\pm} r)$ satisfies the Sommerfeld radiation condition with constant α given by

$$\alpha = -\frac{\beta k_{\pm}^2}{Ak_a k_b} P_{a0} P_{b0}^{(*)} \int_a^{\infty} \frac{e^{2\mathrm{i}k_{\pm} r_0}}{r_0} dr_0. \tag{S101}$$

The solution $P_{\pm}^{\mathrm{Dean}} + \alpha j_0(k_{\pm} r)$ is then identical to Baxter's solution.

Dean did not provide any information on how to derive his solution. Here we present a derivation based on the variation of parameters method [15]. We consider the inhomogeneous Helmholtz equation for the interaction of two spherical waves P_{Sa} and P_{Sb} given by equations (S98),

$$\left(\nabla^2 + k_{\pm}\right) P'_{\pm} = -Q'_{\pm}(r), \tag{S102}$$

where

$$Q'_{\pm}(r) = -\frac{\beta k_{\pm} P_{a0} P_{b0}^{(*)}}{A} \frac{e^{\mathrm{i}k_{\pm} r}}{k_a k_b r^2}. \tag{S103}$$

This problem is spherically symmetric. The Helmholtz equation is in fact a second order inhomogeneous ordinary differential equation. According to the method of variation of parameter, a particular solution $\hat{P}'_{\pm}(r)$ can be constructed from two known linearly independent homogeneous solutions $P_{\pm}^{(1)}(r)$ and $P_{\pm}^{(2)}(r)$, as [15]

$$\hat{P}'_{\pm}(r) = P_{\pm}^{(1)}(r)\int \frac{P_{\pm}^{(2)}(r) Q'_{\pm}(r)}{W_{\pm}(r)} dr - P_{\pm}^{(2)}(r)\int \frac{P_{\pm}^{(1)}(r) Q'_{\pm}(r)}{W_{\pm}(r)} dr \tag{S104}$$

where $W_\pm(r)$ is the Wronskian of the two homogeneous solutions,

$$W_\pm(r) = P_\pm^{(1)}(r)\frac{dP_\pm^{(2)}(r)}{dr} - P_\pm^{(2)}(r)\frac{dP_\pm^{(1)}(r)}{dr}. \tag{S105}$$

We choose the spherical Hankel functions of the first and second kind as the homogeneous solutions,

$$P_\pm^{(1)}(r) = h_0^{(1)}(k_\pm r), \tag{S106}$$

$$P_\pm^{(2)}(r) = h_0^{(2)}(k_\pm r), \tag{S107}$$

and their Wronskian is $W_\pm(r) = -2\mathrm{i}(k_\pm r)^{-2}$. Substituting equations (S103), (S105), (S106) and (S107) into equation (S104), and changing the indefinite integral to a definite integral from a to r, we obtain

$$\hat{P}'_\pm(r) = -\mathrm{i}\frac{\beta k_\pm P_{a0}P_{b0}^{(*)}}{2Ak_a k_b}\frac{e^{\mathrm{i}k_\pm r}}{r}\int_a^r \frac{1}{r_0}dr_0 + \mathrm{i}\frac{\beta k_\pm P_{a0}P_{b0}^{(*)}}{2Ak_a k_b}\frac{e^{-\mathrm{i}k_\pm r}}{r}\int_a^r \frac{e^{2\mathrm{i}k_\pm r_0}}{r_0}dr_0, \tag{S108}$$

which is identical to Dean's solution in equation (S99). We note that the particular solution $\hat{P}'_\pm$ obtained by this method is not unique and it depends on the choice of the homogeneous solutions $P_\pm^{(1)}$ and $P_\pm^{(2)}$. As we discussed above, the choice of $P_\pm^{(1)} = h_0^{(1)}$ and $P_\pm^{(2)} = h_0^{(2)}$ leads to Dean's solution that violates the Sommerfeld radiation condition.

7 Asymptotic solution for IS and SI interactions, equations (92)-(94)

Consider equations (79) and (80) with incident waves given by equation (66) and scattered waves given by equation (88) in the far field.

7.1 Forward direction

We first consider the case when the receiver at range r is in the forward direction of the incident wave and focus on the spherically symmetric scattered waves ($S_a = S_b = 1$). Generalization to arbitrary scattered waves valid in the far field are provided later in terms of the their far field scatter functions. We define a spherical coordinate system (r_0, θ_0, ϕ_0) so that the zenith direction coincides with $\hat{\mathbf{i}}_a$ or $\hat{\mathbf{i}}_b$. Equations (79) and (80) become

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \iiint e^{\mathrm{i}k_a r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_b r_0}}{k_b r_0} \frac{e^{\mathrm{i}k_+ R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S109}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \iiint e^{\mathrm{i}k_b r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_a r_0}}{k_a r_0} \frac{e^{\mathrm{i}k_+ R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S110}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \iiint e^{\mathrm{i}k_a r_0 \cos\theta_0} \frac{e^{-\mathrm{i}k_b r_0}}{k_b r_0} \frac{e^{\mathrm{i}k_- R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S111}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \iiint e^{-\mathrm{i}k_b r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_a r_0}}{k_a r_0} \frac{e^{\mathrm{i}k_- R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0. \tag{S112}$$

In the (r_0, θ_0, ϕ_0) coordinate system, $R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta_0}$, so $r_0\cos\theta_0 = (r^2 + r_0^2 - R^2)/(2r)$, $r_0 \sin\theta_0 d\theta_0 = RdR/r$, and the integrals of (S109) - (S112) become

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_a r/2}}{2k_b r} \iint_D e^{\mathrm{i}k_a \frac{r_0^2 - R^2}{2r} + \mathrm{i}k_b r_0 + \mathrm{i}k_+ R} dR dr_0, \tag{S113}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_b r/2}}{2k_a r} \iint_D e^{\mathrm{i}k_a r_0 + \mathrm{i}k_b \frac{r_0^2 - R^2}{2r} + \mathrm{i}k_+ R} dR dr_0, \tag{S114}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_a r/2}}{2k_b r} \iint_D e^{\mathrm{i}k_a \frac{r_0^2 - R^2}{2r} - \mathrm{i}k_b r_0 + \mathrm{i}k_- R} dR dr_0, \tag{S115}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{-\mathrm{i}k_b r/2}}{2k_a r} \iint_D e^{\mathrm{i}k_a r_0 - \mathrm{i}k_b \frac{r_0^2 - R^2}{2r} + \mathrm{i}k_- R} dR dr_0, \tag{S116}$$

where the integration domain is $D = \{r_0, R : |r - r_0| \le R \le |r + r_0|; r_0 \ge a\}$ (figure S5).

We define a new coordinate system (ξ_1, ξ_2) as a rotation of (r_0, R), such that

$$r_0 = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \tag{S117}$$

$$R = \frac{1}{\sqrt{2}}(-\xi_1 + \xi_2), \tag{S118}$$

$$r_0^2 - R^2 = 2\xi_1\xi_2, \tag{S119}$$

$$dr_0 dR = d\xi_1 d\xi_2. \tag{S120}$$

Equations (S113) - (S116) can be written as

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2\beta P_{a0}P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_a r/2}}{2k_b r}\iint_D e^{\mathrm{i}\left(k_a\frac{\xi_1\xi_2}{r}+k_b\frac{\xi_1+\xi_2}{\sqrt{2}}+k_+\frac{-\xi_1+\xi_2}{\sqrt{2}}\right)}d\xi_1 d\xi_2 \tag{S121}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2\beta P_{a0}P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_b r/2}}{2k_a r}\iint_D e^{\mathrm{i}\left(k_a\frac{\xi_1+\xi_2}{\sqrt{2}}+k_b\frac{\xi_1\xi_2}{r}+k_+\frac{-\xi_1+\xi_2}{\sqrt{2}}\right)}d\xi_1 d\xi_2, \tag{S122}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2\beta P_{a0}P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_a r/2}}{2k_b r}\iint_D e^{\mathrm{i}\left(k_a\frac{\xi_1\xi_2}{r}-k_b\frac{\xi_1+\xi_2}{\sqrt{2}}+k_-\frac{-\xi_1+\xi_2}{\sqrt{2}}\right)}d\xi_1 d\xi_2, \tag{S123}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2\beta P_{a0}P_{b0}^*}{Ac_0^2}\frac{e^{-\mathrm{i}k_b r/2}}{2k_a r}\iint_D e^{\mathrm{i}\left(k_a\frac{\xi_1+\xi_2}{\sqrt{2}}-k_b\frac{\xi_1\xi_2}{r}+k_-\frac{-\xi_1+\xi_2}{\sqrt{2}}\right)}d\xi_1 d\xi_2. \tag{S124}$$

Since the leading order contribution comes from the region near the line segment $r_0 + R = r$ between $r_0 = a$ and $r_0 = r$, we approximate domain D in the (ξ_1, ξ_2) coordinate system as $D' = \{\xi_1, \xi_2 : |\xi_1| \leq r/\sqrt{2}; \xi_2 \geq r/\sqrt{2}\}$, which introduces an error proportional to $1/r$ which rapidly becomes negligible as r increases. Integrals (S121) - (S124) can then be evaluated by integrating over ξ_1 first then ξ_2. Integrating over ξ_1 leads to

$$\begin{aligned}P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) &= -\frac{\omega_+^2\beta P_{a0}P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_a r/2}}{2k_b r}\\ &\times\lim_{L\to\infty}\int_{r/\sqrt{2}}^{L}\frac{e^{\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}-e^{-\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}}{\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})}e^{\mathrm{i}(\frac{k_a}{\sqrt{2}}+k_b\sqrt{2})\xi_2}d\xi_2,\end{aligned} \tag{S125}$$

$$\begin{aligned}P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) &= -\frac{\omega_+^2\beta P_{a0}P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_b r/2}}{2k_a r}\\ &\times\lim_{L\to\infty}\int_{r/\sqrt{2}}^{L}\frac{e^{\mathrm{i}k_b(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}-e^{-\mathrm{i}k_b(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}}{\mathrm{i}k_b(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})}e^{\mathrm{i}(k_a\sqrt{2}+\frac{k_b}{\sqrt{2}})\xi_2}d\xi_2,\end{aligned} \tag{S126}$$

$$\begin{aligned}P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) &= -\frac{\omega_-^2\beta P_{a0}P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_a r/2}}{2k_b r}\\ &\times\lim_{L\to\infty}\int_{r/\sqrt{2}}^{L}\frac{e^{\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}-e^{-\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})\frac{r}{\sqrt{2}}}}{\mathrm{i}k_a(\frac{\xi_2}{r}-\frac{1}{\sqrt{2}})}e^{\mathrm{i}(\frac{k_a}{\sqrt{2}}-k_b\sqrt{2})\xi_2}d\xi_2,\end{aligned} \tag{S127}$$

$$\begin{aligned}P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) &= -\frac{\omega_-^2\beta P_{a0}P_{b0}^*}{Ac_0^2}\frac{e^{-\mathrm{i}k_b r/2}}{2k_a r}\\ &\times\lim_{L\to\infty}\int_{r/\sqrt{2}}^{L}\frac{e^{\mathrm{i}k_b(\frac{1}{\sqrt{2}}-\frac{\xi_2}{r})\frac{r}{\sqrt{2}}}-e^{-\mathrm{i}k_b(\frac{1}{\sqrt{2}}-\frac{\xi_2}{r})\frac{r}{\sqrt{2}}}}{\mathrm{i}k_b(\frac{1}{\sqrt{2}}-\frac{\xi_2}{r})}e^{\mathrm{i}(k_a\sqrt{2}-\frac{k_b}{\sqrt{2}})\xi_2}d\xi_2.\end{aligned} \tag{S128}$$

Let $\xi_2/r - 1/\sqrt{2} = \eta$, then $d\xi_2 = rd\eta$ and equations (S125) - (S128) become

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_z) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_a r/2}}{2k_b r} \times \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}k_a \eta \frac{r}{\sqrt{2}}} - e^{-\mathrm{i}k_a \eta \frac{r}{\sqrt{2}}}}{\mathrm{i}k_a \eta} e^{\mathrm{i}(\frac{k_a}{\sqrt{2}} + k_b\sqrt{2})(r\eta + \frac{r}{\sqrt{2}})} r d\eta, \tag{S129}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_z) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_b r/2}}{2k_a r} \times \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}k_b \eta \frac{r}{\sqrt{2}}} - e^{-\mathrm{i}k_b \eta \frac{r}{\sqrt{2}}}}{\mathrm{i}k_b \eta} e^{\mathrm{i}(k_a\sqrt{2} + \frac{k_b}{\sqrt{2}})(r\eta + \frac{r}{\sqrt{2}})} r d\eta, \tag{S130}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_z) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_a r/2}}{2k_b r} \times \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}k_a \eta \frac{r}{\sqrt{2}}} - e^{-\mathrm{i}k_a \eta \frac{r}{\sqrt{2}}}}{\mathrm{i}k_a \eta} e^{\mathrm{i}(\frac{k_a}{\sqrt{2}} - k_b\sqrt{2})(r\eta + \frac{r}{\sqrt{2}})} r d\eta, \tag{S131}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_z) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{-\mathrm{i}k_b r/2}}{2k_a r} \times \lim_{L\to\infty} \int_0^{L/r} \frac{e^{-\mathrm{i}k_b \eta \frac{r}{\sqrt{2}}} - e^{\mathrm{i}k_b \eta \frac{r}{\sqrt{2}}}}{-\mathrm{i}k_b \eta} e^{\mathrm{i}(k_a\sqrt{2} - \frac{k_b}{\sqrt{2}})(r\eta + \frac{r}{\sqrt{2}})} r d\eta, \tag{S132}$$

which can be written as

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}\sqrt{2}k_+ r\eta} - e^{\mathrm{i}\sqrt{2}k_b r\eta}}{\mathrm{i}\eta} d\eta, \tag{S133}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}\sqrt{2}k_+ r\eta} - e^{\mathrm{i}\sqrt{2}k_a r\eta}}{\mathrm{i}\eta} d\eta, \tag{S134}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}\sqrt{2}k_- r\eta} - e^{-\mathrm{i}\sqrt{2}k_b r\eta}}{\mathrm{i}\eta} d\eta, \tag{S135}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L/r} \frac{e^{\mathrm{i}\sqrt{2}k_- r\eta} - e^{\mathrm{i}\sqrt{2}k_a r\eta}}{-\mathrm{i}\eta} d\eta. \tag{S136}$$

Let $\sqrt{2}r\eta = \zeta$, then equations (S133) - (S136) become

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L} \frac{e^{\mathrm{i}k_+ \zeta} - e^{\mathrm{i}k_b \zeta}}{\mathrm{i}\zeta} d\zeta, \tag{S137}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L} \frac{e^{\mathrm{i}k_+ \zeta} - e^{\mathrm{i}k_a \zeta}}{\mathrm{i}\zeta} d\zeta, \tag{S138}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L} \frac{e^{\mathrm{i}k_- \zeta} - e^{-\mathrm{i}k_b \zeta}}{\mathrm{i}\zeta} d\zeta, \tag{S139}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \lim_{L\to\infty} \int_0^{L} \frac{e^{\mathrm{i}k_a \zeta} - e^{\mathrm{i}k_- \zeta}}{\mathrm{i}\zeta} d\zeta. \tag{S140}$$

The limits in equations (S137) - (S140) can be determined analytically with the cosine integral $\mathrm{Ci}(x)$ [13], as

$$
\begin{aligned}
&\lim_{L\to\infty}\int_0^L \frac{e^{\mathrm{i}k_+\zeta}-e^{\mathrm{i}k_b\zeta}}{\mathrm{i}\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_+\zeta}{\zeta}d\zeta - \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_b\zeta}{\zeta}d\zeta + \int_0^\infty \frac{\sin k_-\zeta}{\zeta}d\zeta - \int_0^\infty \frac{\sin k_b\zeta}{\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_+L)-\mathrm{Ci}(k_+L)] - \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_bL)-\mathrm{Ci}(k_bL)] + \pi/2-\pi/2 \\
&\quad= \mathrm{i}\log(k_+/k_b), \qquad \text{(S141)}\\
&\lim_{L\to\infty}\int_0^L \frac{e^{\mathrm{i}k_+\zeta}-e^{\mathrm{i}k_a\zeta}}{\mathrm{i}\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_+\zeta}{\zeta}d\zeta - \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_a\zeta}{\zeta}d\zeta + \int_0^\infty \frac{\sin k_-\zeta}{\zeta}d\zeta - \int_0^\infty \frac{\sin k_a\zeta}{\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_+L)-\mathrm{Ci}(k_+L)] - \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_aL)-\mathrm{Ci}(k_aL)] + \pi/2-\pi/2 \\
&\quad= \mathrm{i}\log(k_+/k_a), \qquad \text{(S142)}\\
&\lim_{L\to\infty}\int_0^L \frac{e^{\mathrm{i}k_-\zeta}-e^{-\mathrm{i}k_b\zeta}}{\mathrm{i}\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_-\zeta}{\zeta}d\zeta - \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_b\zeta}{\zeta}d\zeta + \int_0^\infty \frac{\sin k_-\zeta}{\zeta}d\zeta + \int_0^\infty \frac{\sin k_b\zeta}{\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_-L)-\mathrm{Ci}(k_-L)] - \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_bL)-\mathrm{Ci}(k_bL)] + \pi/2+\pi/2 \\
&\quad= \mathrm{i}\log(k_-/k_b)+\pi, \qquad \text{(S143)}\\
&\lim_{L\to\infty}\int_0^L \frac{e^{\mathrm{i}k_a\zeta}-e^{\mathrm{i}k_-\zeta}}{\mathrm{i}\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_a\zeta}{\zeta}d\zeta - \mathrm{i}\lim_{L\to\infty}\int_0^L \frac{1-\cos k_-\zeta}{\zeta}d\zeta + \int_0^\infty \frac{\sin k_a\zeta}{\zeta}d\zeta - \int_0^\infty \frac{\sin k_-\zeta}{\zeta}d\zeta \\
&\quad= \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_aL)-\mathrm{Ci}(k_aL)] - \mathrm{i}\lim_{L\to\infty}[\gamma+\log(k_-L)-\mathrm{Ci}(k_-L)] + \pi/2-\pi/2 \\
&\quad= -\mathrm{i}\log(k_-/k_a), \qquad \text{(S144)}
\end{aligned}
$$

where γ is the Euler-Mascheroni constant [13].

We then obtain the second order nonlinear fields in the forward directions due to interaction between a plane wave and a spherical wave, as

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_+ r}}{2k_a k_b}\left[\mathrm{i}\log\left(\frac{k_+}{k_b}\right)\right], \tag{S145}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) = -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{Ac_0^2}\frac{e^{\mathrm{i}k_+ r}}{2k_a k_b}\left[\mathrm{i}\log\left(\frac{k_+}{k_a}\right)\right], \tag{S146}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_- r}}{2k_a k_b}\left[\mathrm{i}\log\left(\frac{k_-}{k_b}\right) + \pi\right], \tag{S147}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_- r}}{2k_a k_b}\left[-\mathrm{i}\log\left(\frac{k_-}{k_a}\right)\right]. \tag{S148}$$

As seen in equations (S145) - (S148), the IS and SI field magnitudes are constant with range. Unlike the interaction of collinear plane waves where growth is found along the propagation path, collinearity between planar and spherical wavefronts within an equivalent Fresnel area about the forward direction, together with spreading of the spherical wave, balances out second-order wave growth. For $P'_{\mathrm{IS}-}$, there is an additional contribution from a stationary phase point at range rk_b/k_a in the forward direction. To see this, we rewrite equation (S115) for $P'_{\mathrm{IS}-}$ as

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_z) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_a r/2}}{2k_b r}\iint_D e^{\mathrm{i}k_- r\varphi_{\mathrm{IS}-}^{\mathrm{forward}}(r_0,R)}dRdr_0, \tag{S149}$$

where

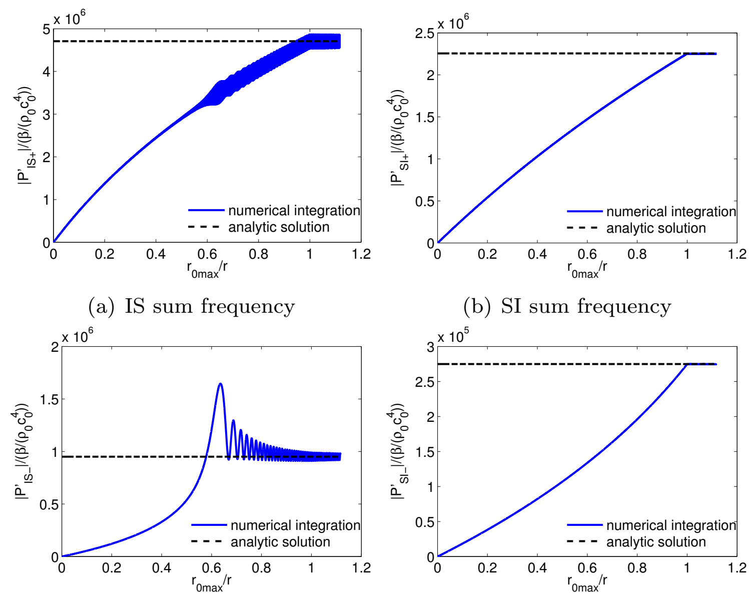
$$\varphi_{\mathrm{IS}-}^{\mathrm{forward}}(r_0,R) = \frac{k_a}{k_-}\frac{r_0^2 - R^2}{2r^2} - \frac{k_b}{k_-}\frac{r_0}{r} + \frac{R}{r}. \tag{S150}$$

The stationary phase point can be found by letting $(\partial/\partial r_0)\varphi_{\mathrm{IS}-}^{\mathrm{forward}} = 0$ and $(\partial/\partial R)\varphi_{\mathrm{IS}-}^{\mathrm{forward}} = 0$, which gives $(r_0, R) = (rk_b/k_a, rk_-/k_a)$. Applying a two-dimensional stationary phase approximation [16] to equation (S149) for $P'_{\mathrm{IS}-}$ leads to

$$P'^{\,\mathrm{stationary\ phase}}_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{Ac_0^2}\frac{e^{\mathrm{i}k_- r}}{2k_a k_b}\pi, \tag{S151}$$

which corresponds to the π term contribution of the full solution for $P'_{\mathrm{IS}-}$ in equation (S147). The stationary phase points for $P'_{\mathrm{IS}+}$, $P'_{\mathrm{SI}+}$ and $P'_{\mathrm{SI}-}$ can be found in the same way but they are outside of the domain D, i.e. they do not exist in physical domain. The leading order contributions to $P'_{\mathrm{IS}+}$, $P'_{\mathrm{SI}+}$ and $P'_{\mathrm{SI}-}$ come from an equivalent Fresnel width along the forward path between the object and the receiver, as found in equations (S145), (S146) and (S148).

Analytic solutions (S145) - (S148) are verified by direct numerical integration of equations (S109) - (S112), as shown in figure S4. We truncate the numerical integration at radius $r_{0\max}$. It can be seen that the numerical results agree very well with the analytic solutions as long as the integrations are truncated beyond the receiver radius r. It can also be seen the integral for $P'_{\mathrm{IS}-}$ has a significant contribution near $0.6r$, which corresponds to the stationary phase point at $r_0 = rk_b/k_a$. The physical parameters are $\omega_a/2\pi = 500$ kHz, $\omega_b/2\pi = 300$ kHz, $a = 1$ mm and $r = 1$ m.



(a) IS sum frequency (b) SI sum frequency

(c) IS difference frequency (d) SI difference frequency

Figure S4: Verification of analytic solutions (S145) - (S148) with direct numerical integration of equations (S109) - (S112). Excellent agreements are found as long as the integrations are truncated at a range $r_{0\max}$ that is larger than receiver range r. The physical parameters are $\omega_a/2\pi = 500$ kHz, $\omega_b/2\pi = 300$ kHz, $a = 1$ mm and $r = 1$ m.

If the scattered waves are not spherically symmetric the solutions in (S145) - (S148) for sufficiently large range become

$$P'_{\mathrm{IS}+}(r\hat{\mathbf{i}}_a) \approx -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \left[\mathrm{i} \log \left(\frac{k_+}{k_b} \right) \right] S_b(\hat{\mathbf{i}}_a), \tag{S152}$$

$$P'_{\mathrm{SI}+}(r\hat{\mathbf{i}}_b) \approx -\frac{\omega_+^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{\mathrm{i}k_+ r}}{2k_a k_b} \left[\mathrm{i} \log \left(\frac{k_+}{k_a} \right) \right] S_a(\hat{\mathbf{i}}_b), \tag{S153}$$

$$P'_{\mathrm{IS}-}(r\hat{\mathbf{i}}_a) \approx -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \left[\mathrm{i} \log \left(\frac{k_-}{k_b} \right) + \pi \right] S_b^*(\hat{\mathbf{i}}_a), \tag{S154}$$

$$P'_{\mathrm{SI}-}(r\hat{\mathbf{i}}_b) \approx -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_- r}}{2k_a k_b} \left[-\mathrm{i} \log \left(\frac{k_-}{k_a} \right) \right] S_a(\hat{\mathbf{i}}_b). \tag{S155}$$

which are scaled by the corresponding far field scatter functions S_a and S_b of the scattered waves in the respective forward directions $\hat{\mathbf{i}}_b$ and $\hat{\mathbf{i}}_a$.

The second order field components $P_{\mathrm{IS}\pm}$ and $P_{\mathrm{SI}\pm}$ due to IS and SI interactions in the forward directions can then be approximated by equations (S152) - (S155) for large range because $P''_{\mathrm{IS}\pm}$ and $P''_{\mathrm{SI}\pm}$ all fall off by r^{-1}.

7.2 Backscatter direction

Now consider the case when the receiver at range r is in the backscatter direction $-\hat{\mathbf{i}}_a$ or $-\hat{\mathbf{i}}_b$ of the incident waves for plane wave and spherical wave interaction. We define another spherical coordinate system (r_0, θ_0, ϕ_0) so that the zenith direction coincides with $-\hat{\mathbf{i}}_a$ or $-\hat{\mathbf{i}}_b$. Equations (79) and (80) become

$$P'_{\mathrm{IS}+}(-r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}}{A c_0^2} \iiint e^{-\mathrm{i}k_a r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_b r_0}}{k_b r_0} \frac{e^{\mathrm{i}k_+ R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S156}$$

$$P'_{\mathrm{SI}+}(-r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}}{A c_0^2} \iiint e^{-\mathrm{i}k_b r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_a r_0}}{k_a r_0} \frac{e^{\mathrm{i}k_+ R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S157}$$

$$P'_{\mathrm{IS}-}(-r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \iiint e^{-\mathrm{i}k_a r_0 \cos\theta_0} \frac{e^{-\mathrm{i}k_b r_0}}{k_b r_0} \frac{e^{\mathrm{i}k_- R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0, \tag{S158}$$

$$P'_{\mathrm{SI}-}(-r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \iiint e^{\mathrm{i}k_b r_0 \cos\theta_0} \frac{e^{\mathrm{i}k_a r_0}}{k_a r_0} \frac{e^{\mathrm{i}k_- R}}{4\pi R} r_0^2 \sin\theta_0 dr_0 d\theta_0 d\phi_0. \tag{S159}$$

In the (r_0, θ_0, ϕ_0) coordinate system, $R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta_0}$, so $r_0 \cos\theta_0 = (r^2 + r_0^2 - R^2)/(2r)$, $r_0 \sin\theta_0 d\theta_0 = RdR/r$, and the integrals of (S156) - (S159) become

$$P'_{\mathrm{IS}+}(-r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{-\mathrm{i}k_a r/2}}{2k_b r} \iint_D e^{\mathrm{i}k_+ r \varphi_{\mathrm{IS}+}^{\mathrm{back}}(r_0, R)} dR dr_0, \tag{S160}$$

$$P'_{\mathrm{SI}+}(-r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}}{A c_0^2} \frac{e^{-\mathrm{i}k_b r/2}}{2k_a r} \iint_D e^{\mathrm{i}k_+ r \varphi_{\mathrm{SI}+}^{\mathrm{back}}(r_0, R)} dR dr_0, \tag{S161}$$

$$P'_{\mathrm{IS}-}(-r\hat{\mathbf{i}}_a) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{-\mathrm{i}k_a r/2}}{2k_b r} \iint_D e^{\mathrm{i}k_- r \varphi_{\mathrm{IS}-}^{\mathrm{back}}(r_0, R)} dR dr_0, \tag{S162}$$

$$P'_{\mathrm{SI}-}(-r\hat{\mathbf{i}}_b) = -\frac{\omega_-^2 \beta P_{a0} P_{b0}^*}{A c_0^2} \frac{e^{\mathrm{i}k_b r/2}}{2k_a r} \iint_D e^{\mathrm{i}k_- r \varphi_{\mathrm{SI}-}^{\mathrm{back}}(r_0, R)} dR dr_0, \tag{S163}$$

where the integration domain is $D = \{r_0, R : |r - r_0| \leq R \leq |r + r_0|; r_0 \geq a\}$

(figure S5), and

$$\varphi_{\mathrm{IS+}}^{\mathrm{back}}(r_0,R) = -\frac{k_a}{k_+}\frac{r_0^2-R^2}{2r^2} + \frac{k_b}{k_+}\frac{r_0}{r} + \frac{R}{r}, \tag{S164}$$

$$\varphi_{\mathrm{SI+}}^{\mathrm{back}}(r_0,R) = \frac{k_a}{k_+}\frac{r_0}{r} - \frac{k_b}{k_+}\frac{r_0^2-R^2}{2r^2} + \frac{R}{r}, \tag{S165}$$

$$\varphi_{\mathrm{IS-}}^{\mathrm{back}}(r_0,R) = -\frac{k_a}{k_-}\frac{r_0^2-R^2}{2r^2} - \frac{k_b}{k_-}\frac{r_0}{r} + \frac{R}{r}, \tag{S166}$$

$$\varphi_{\mathrm{SI-}}^{\mathrm{back}}(r_0,R) = \frac{k_a}{k_-}\frac{r_0}{r} + \frac{k_b}{k_-}\frac{r_0^2-R^2}{2r^2} + \frac{R}{r}. \tag{S167}$$

The stationary points in $\varphi_{\mathrm{IS+}}^{\mathrm{back}}$, $\varphi_{\mathrm{SI+}}^{\mathrm{back}}$, $\varphi_{\mathrm{IS-}}^{\mathrm{back}}$, and $\varphi_{\mathrm{SI-}}^{\mathrm{back}}$ are

$$(r_0,R) = \begin{cases} (rk_b/k_a, -rk_+/k_a) & \text{for } \varphi_{\mathrm{IS+}}^{\mathrm{back}}, \\ (rk_a/k_b, -rk_+/k_b) & \text{for } \varphi_{\mathrm{SI+}}^{\mathrm{back}}, \\ (-rk_b/k_a, -rk_-/k_a) & \text{for } \varphi_{\mathrm{IS-}}^{\mathrm{back}}, \\ (-rk_a/k_b, rk_-/k_b) & \text{for } \varphi_{\mathrm{SI-}}^{\mathrm{back}}, \end{cases} \tag{S168}$$

none of which are inside domain D. Unlike the case in the forward direction, the phases $\varphi_{\mathrm{IS+}}^{\mathrm{back}}$, $\varphi_{\mathrm{SI+}}^{\mathrm{back}}$, $\varphi_{\mathrm{IS-}}^{\mathrm{back}}$, and $\varphi_{\mathrm{SI-}}^{\mathrm{back}}$ also vary along the path between the object and the receiver in the backscatter direction. The leading order contribution to the integrals in equations (S160) - (S163) then comes from the all finite corners in domain D for $k_- r \gg 1$ [17]. As shown in figure S5, there are three corners: (1) $(r_0,R) = (a, r+a)$, (2) $(r_0,R) = (a, r-a)$, and (3) $(r_0,R) = (r,0)$. As an example, when P_{Sb} is a spherical wave, corner contributions to the integral (S162) can be determined via

$$\iint_D e^{\mathrm{i}k_- r\varphi_{\mathrm{IS-}}^{\mathrm{back}}} dR dr_0 \approx \sum_{\text{corners } 1,2,3} \frac{2\cot\alpha e^{\mathrm{i}k_- r\varphi_{\mathrm{IS-}}^{\mathrm{back}}}}{k_-^2 r^2[(\frac{\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}}{\partial r_0}\cos\beta + \frac{\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}}{\partial R}\sin\beta)^2 - (\cot\alpha)^2(-\frac{\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}}{\partial r_0}\sin\beta + \frac{\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}}{\partial R}\cos\beta)^2]}, \tag{S169}$$

where the variables being summed are α, β, $\varphi_{\mathrm{IS-}}^{\mathrm{back}}$, $\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}/\partial r_0$ and $\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}/\partial R$, whose values are listed in Table S1 for each corner. As $r\to\infty$, $\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}/\partial r_0$ and $\partial\varphi_{\mathrm{IS-}}^{\mathrm{back}}/\partial R$ behave as r^{-1} as seen in Table S1, so the integral (S169) approaches constant magnitude for large r. The backscatter magnitude $P'_{\mathrm{IS-}}(-r\hat{\mathbf{i}}_a)$ in equation (S162) then falls off by r^{-1} as $r\to\infty$. Results for $P'_{\mathrm{IS+}}$, $P'_{\mathrm{SI+}}$ and $P'_{\mathrm{SI-}}$ can be obtained in a similar manner using the formulas in Ref. [17]. When including $P''_{\mathrm{IS\pm}}$ and $P_{\mathrm{SI\pm}}$ that also fall off by r^{-1}, the second order field components due to IS and SI interactions in the backscatter direction have an overall range dependence of r^{-1}, which become insignificant for large range because the dominant component $P_{\mathrm{SS\pm}}$ due to SS interaction has $\log(r)/r$ range dependence.

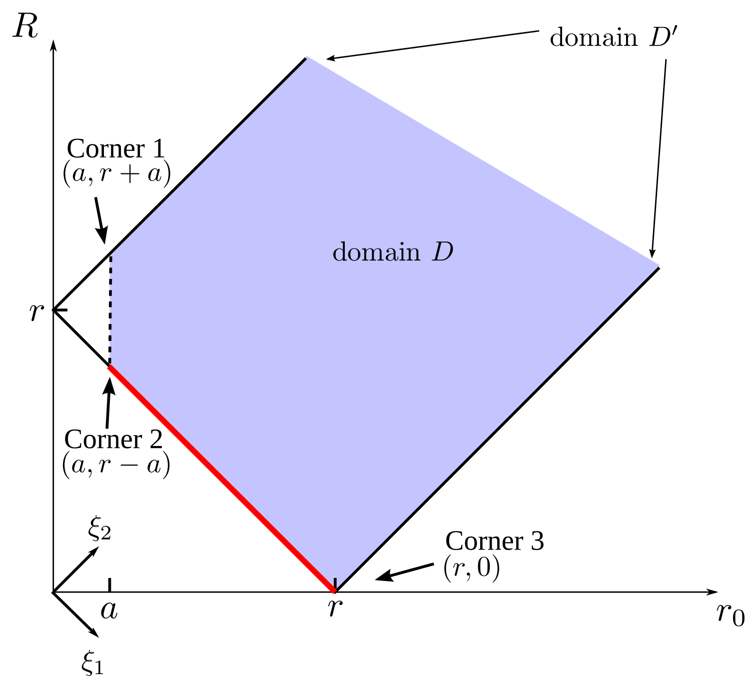


Figure S5: Integration domain D (blue shaded) in equations (S113) - (S116) and (S160) - (S163) for the IS and SI interactions. For a receiver in the forward direction, the dominant contribution comes from the region near the line segment between the object to the receiver (corresponding to the red segment). The difference between domain D and D' (enclosed by the solid boundary) become negligible as $r \to \infty$. For a receiver in the backscatter direction, the dominant contribution comes from the three corners, which can be determined via equation (S169) and Table S1 for $P'_{\mathrm{IS}-}(-r\hat{\mathbf{i}}_a)$.

	α	β	$\varphi_{\mathrm{IS}-}^{\mathrm{back}}$	$(\partial/\partial r_0)\varphi_{\mathrm{IS}-}^{\mathrm{back}}$	$(\partial/\partial R)\varphi_{\mathrm{IS}-}^{\mathrm{back}}$
corner 1	$3\pi/8$	$11\pi/8$	$\frac{k_a}{2k_-} + \frac{2a}{r} + 1$	$-\frac{k_a a}{k_- r^2} - \frac{k_b}{k_- r}$	$\frac{k_a}{k_-}\frac{r+a}{r^2} + \frac{1}{r}$
corner 2	$3\pi/8$	$13\pi/8$	$\frac{k_a}{2k_-} - \frac{2k_b a}{k_- r} + 1$	$-\frac{k_a a}{k_- r^2} - \frac{k_b}{k_- r}$	$\frac{k_a}{k_-}\frac{r-a}{r^2} + \frac{1}{r}$
corner 3	$\pi/4$	0	$-\frac{k_a+2k_b}{2k_-}$	$-\frac{k_a+k_b}{k_- r}$	$\frac{1}{r}$

Table S1: Quantities to determine corner contributions of $P'_{\mathrm{IS}-}(\mathbf{r}_B)$ in the backscatter direction.

8 More on space-time isolation of sum and difference frequency field components containing object information (Appendix E)

For the collinear case, the II overlap region is an infinite slab moving in the propagating direction of the primary incident waves in 3-D; for the perpendicular case, it is the intersection of two slabs and it moves diagonally. These II overlap regions (grey) are projected on a 2-D plane in figure A2 of the main text. The SS overlap region is a 3-D spherical shell that appears as a 2-D ring (blue) in the 2-D plane in figure 4 of the main text. IS or SI overlap region is a spherical cap moving in the direction of the incident wave in 3-D, which is shown as a circular segment in green in figure A2 of the main text.

The following window function $w_1(t)$ is used in the computation:

$$w_1(t) = \begin{cases} \frac{t}{t_1} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{t_1}\right), & 0 < t \le t_1 \\ 1, & t_1 < t \le T - t_1 \\ \frac{T-t}{t_1} - \frac{1}{2\pi}\sin\left[\frac{2\pi(T-t)}{t_1}\right], & T - t_1 < t \le T \\ 0, & \text{otherwise}, \end{cases} \tag{S170}$$

where $T = 20\pi/w_-$ is the duration of the window, which contains 10 difference frequency cycles, and $t_1 = 4\pi/w_-$ is the duration of the transition regions, which contain 2 difference frequency cycles.

For the interaction of waves of compact support, time domain Green theorem solutions (68)-(70) are numerically evaluated for p'_{II,ab^*}, p'_{SS,ab^*} and $p'_{\mathrm{IS}ab^*}$. The integrations are over finite volumes in space defined by the compact support of w_{II,ab^*}, w_{SS,ab^*} and w_{IS,ab^*} of equations (72)-(74), respectively. These volumes are functions of time. For example, the volume corresponding to the SS interaction is defined by $w_1(t - R/c_0 - r_0/c_0)$. It is empty for $t < r_R/c_0$, a line segment connecting the origin and the receiver for $t = r_R/c_0$, a prolate spheroid for $r_R/c_0 < t \le r_R/c_0 + T$ and a prolate spheroidal shell for $t > r_R/c_0$. The total fields p_{II,ab^*}, p_{SS,ab^*} and $p_{\mathrm{IS},ab^*}$are obtained by including p''_{II,ab^*}, p''_{SS,ab^*} and $p''_{\mathrm{IS}ab^*}$ from equation (61) with appropriate products of the primary fields.

Frequency domain Green theorem solutions (77)-(79) are used in the harmonic wave approximations. For the II, SS and forward direction IS interactions, we can either integrate the infinite space where the primary fields exist, or integrate over a finite space defined by the compact support of w_{II,ab^*}, w_{SS,ab^*} and w_{IS,ab^*} at a time instance $\bar{t}$ between $t = r_R/c_0$ and $t = r_R/c_0 + T$. The time $\bar{t} = r_R/c_0 + T/2$ was used in our computation, but it is found that the results are insensitive to the choice of $\bar{t}$ as long as the window duration T is sufficiently long and $\bar{t}$ is at the center of the constant region within the window $w_1(t - r_R/c_0)$. For the backscatter direction IS interaction if the incident and scattered waves never overlap at the receiver, p_{IS,ab^*} is generated by the IS interaction that took place earlier between $t = 0$ and $t = T$ when the incident and scattered waves overlapped near the object. We have to integrate over the finite space defined by the compact support of w_{IS,ab^*} at time instance $\bar{t}$ between $t = r_R/c_0$ and $t = r_R/c_0 + T$ in the harmonic wave approximation. When using equation (S169) for this case, we only sum over the contributions from corners 1 and 2. The total fields $P_{\mathrm{II}-}$, $P_{\mathrm{SS}-}$ and $P_{\mathrm{IS}-}$ are obtained by including $P''_{\mathrm{II}-}$, $P''_{\mathrm{SS}-}$ and $P''_{\mathrm{IS}-}$ from equation (61) with appropriate products of the primary fields.

From the sensing perspective, the p_{II} is undesirable because it contains no information about the object and it can mask out other field components. If the receiver is placed near the backscatter directions, as shown in figure A1 of the main text, it is possible to separate the II component. There is, however, one more constraint that requires attention. Ideally, the time interval between $t = r_R/c_0$ and $t = r_R/c_0 + T$ is arrival time for p_{SS}, p_{IS} and p_{S2}, which is also the available window for these field components. For a very long window or at very close range, during the early part of this measuring window, the receiver may still be inside the II overlap region. If this is the case, the available measuring window is reduced to between $t = -r_R/c_0 + T$ and $t = r_R/c_0 + T$, with a duration $\Delta T = 2r_R/c_0$. To measure steady state response at the difference or sum frequency, ΔT should contain at least a few difference or sum frequency cycles, which imposes a lower limit on the normalized receiver range $k_\pm r_R$. For example if $N = 10$ difference or sum frequency cycles are desired, then $\Delta T > 2\pi N/\omega_\pm$ or $k_\pm r_R > N\pi$.

9 Varying Beta in Figures 3-8

Results in figures 3-8 are calculated using $\beta = 3.6$, which is a typical value for water. Special attention is needed to properly scale the results to different values of β. As seen in the decomposition in equation (60), p'_2 is linearly proportional to β (i.e. depends on the nonlinear property of the medium), while p''_2 is independent of β. The portion of the second order incident field p'_2 and its associated scattered field can be scaled by β. The portion of the second order incident field p''_2 and its associated scattered field, as well as the effects from the body-wave interactions (i.e. quadratic terms in the second order boundary conditions (33) and (36) and the quadratic term in the second order wave-exciting force (39) should not be scaled. Following this scaling principle, results for air with $\beta = 1.2$ and for solid earth with $\beta = 1000$ can be obtained. Results in air corresponding to figure 3b are shown in figure S6 and results in solid earth corresponding to figure 6c are shown in S7.

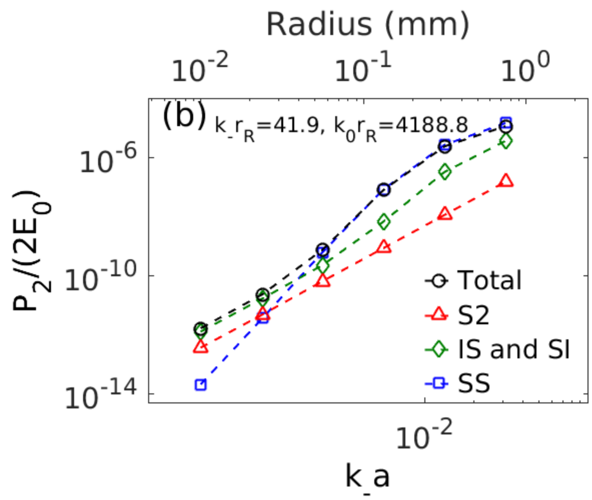


Figure S6: Same as figure 3b in the main text except that the nonlinear parameter is $\beta = 1.2$.

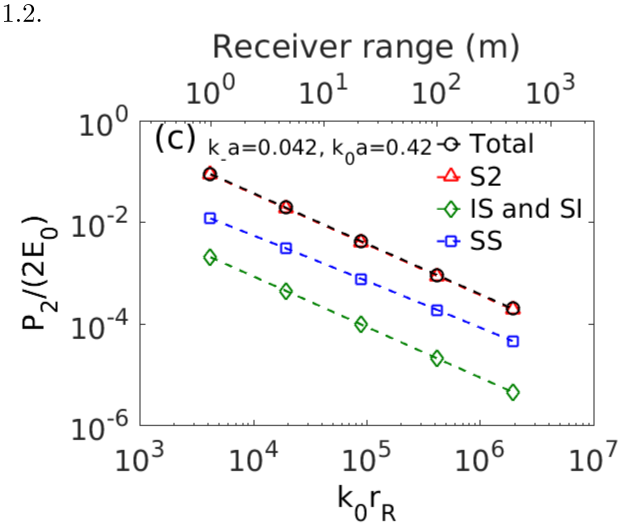


Figure S7: Same as figure 6c in the main text except that the nonlinear parameter is $\beta = 1000$.

10 Noise

10.1 In air

In the air, the 1/3 octave band noise spectrum given in Ref.[18] is shown in figure S8. At 400 Hz, the spectral level is about 8 dB re 20 μPa in Hermit Basin, Grand Canyon National Park and about 40 dB re 20 μPa in quiet residential environments. This is a 93 Hz frequency band between 356 Hz and 449 Hz. If we assume the spectral density is flat within this band, we can estimate the SPL for a 40 Hz frequency band centered at 400 Hz to be 4 dB re 20 μPa in Hermit Basin and 36 dB re 20 μPa in quiet residential areas.

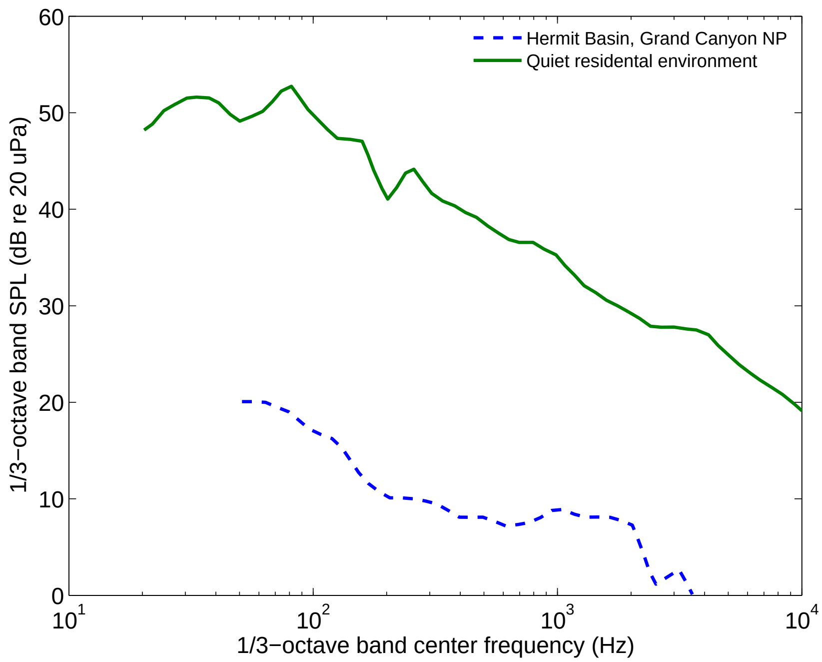


Figure S8: Ambient noise spectrum in air, extracted from figure 3 in Ref.[18].

10.2 In water

Figure S9 shows the nominal high and low ambient noise in the ocean. The values are extracted from figure 2 in Ref.[18].

At 10 kHz, the spectral density level is about 27 dB re 1 $\mu\text{Pa}^2/\text{Hz}$ for the nominal low noise condition according to figure S9. The noise level for a 1 kHz band centered at 10 kHz is estimated to be 57 dB re 1 μPa with the following formula

$$10\log_{10}\left(10^{27/10} \times 10^8 \times \int_{10512}^{9512} \frac{1}{f^2} df\right) \approx 57 \text{ dB re } 1\ \mu\text{Pa}. \tag{S171}$$

Medical ultrasound imaging experiments are usually conducted in water tanks. But there are very few noise spectral measurements available for water tanks near 10 kHz. One example is shown in Ref.[19], where the noise floor for the frequency band between 1 kHz and 50 kHz was measured to be 86 dB re 1 μPa. According to figure S9, ocean noise in the same frequency band is

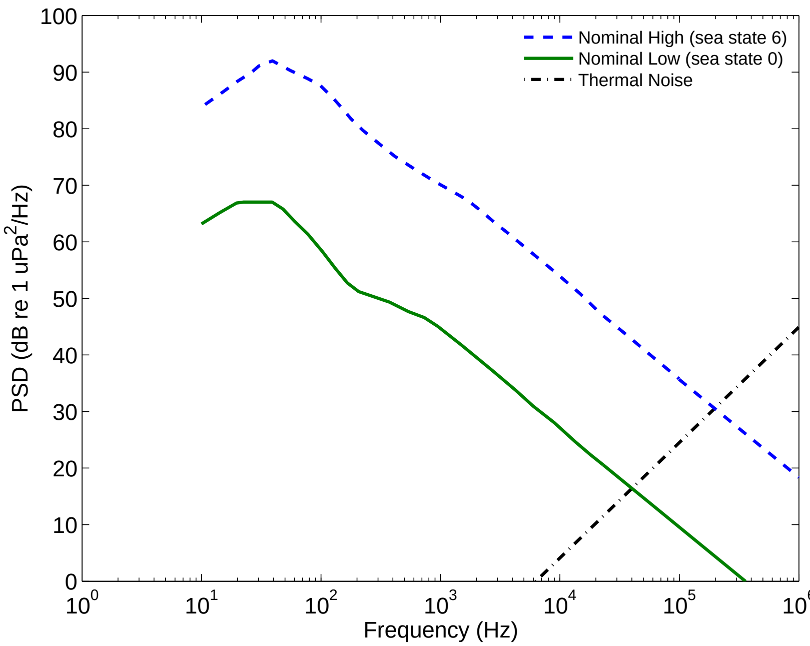


Figure S9: Ambient noise spectral density in the ocean [18].

between 77 dB (nominal low) and 104 dB re 1μPa (nominal high). The noise level in Ref.[19] is higher than this low level ocean noise, which may be due to contamination by other mechanical or urban sources in tank or low sensitivity hydrophones. We use 57 dB re 1 μPa as an estimation of the noise level in the ocean as well as in the water tank.

At 100 kHz, thermal noise due to random motion of water molecules becomes dominant when the environmental noise is low, as shown in the lower right corner of figure S9. The power spectral density for the thermal noise increases as 6 dB/octave [20]. At 100 kHz, the spectral density level for thermal noise is about 25 dB re 1 μPa^2/Hz. For a 10 kHz band centered at 100 kHz, the thermal noise SPL is estimated to be 65 dB re 1 μPa with the following formula

$$10\log_{10}\left(\frac{10^{25/10}}{10^{10}} \times \int_{95125}^{105125} f^2 df\right) \approx 65 \text{ dB re 1 } \mu\text{Pa}. \tag{S172}$$

10.3 In solid earth

Figure S10 shows the ambient seismic noise spectral densities from Ref.[21], which are referred to as the New High Noise Model (NHNM) and New Low Noise Model (NLNM). The noise in terms of surface velocity for a 1 Hz frequency band between 9 Hz and 10 Hz is found to be between 7×10^{-11} m/s and 4×10^{-7} m/s. With the plane wave impedance relation and assuming $\rho = 3000$ kg/m^3

and $c_0 = 3000$ m/s, we estimate the seismic ambient noise in pressure (normal stress) to be between 0.0006 Pa and 4 Pa, or equivalently between 56 dB and 132 dB re 1 μ Pa.

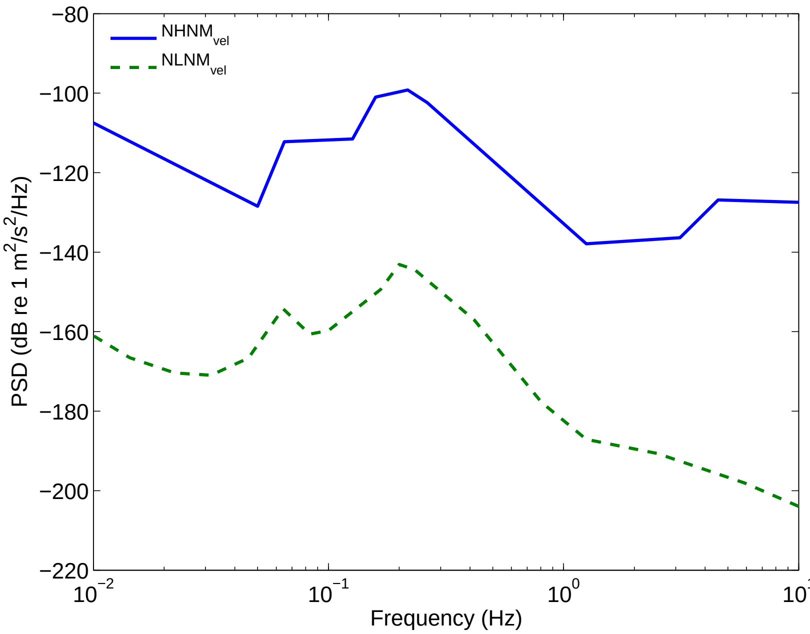


Figure S10: Ambient seismic noise spectral density. The New High Noise Model (NHNM) and New Low Noise Model (NLNM) are given in terms of surface velocity square per Hz.

11 Supporting Examples for §5

A number of examples describing applications discussed in §5 of the main text are provided.

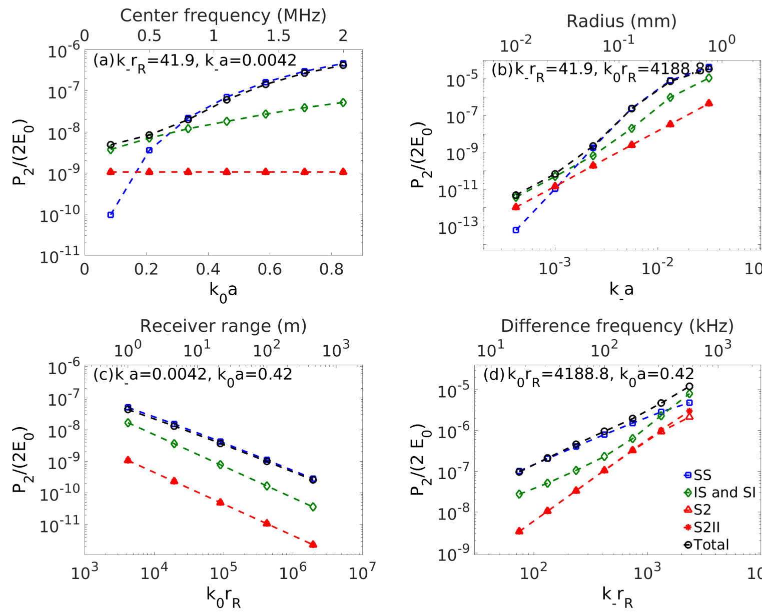


Figure S11: Same as figure 3 except p_{S2II} is also shown.

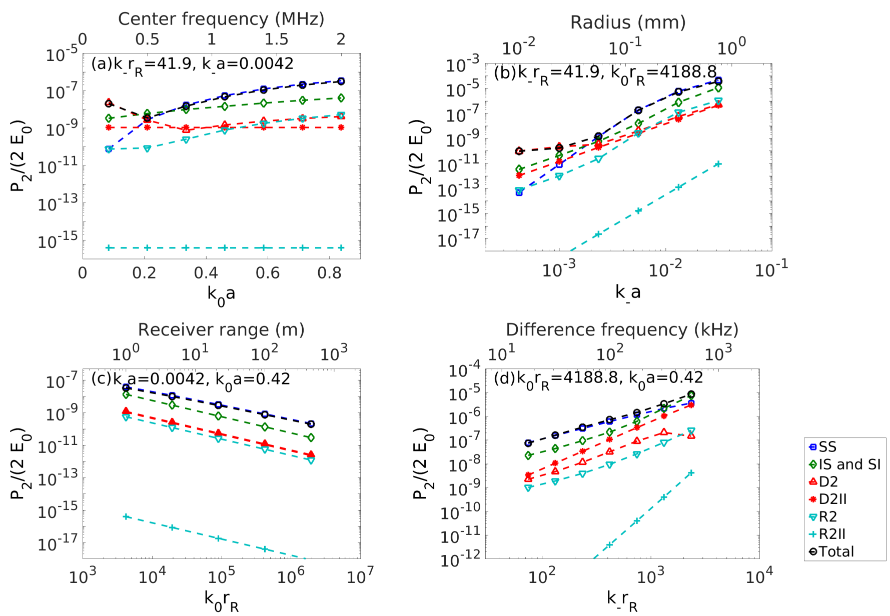


Figure S12: Same as figure 4 except p_{S2II} is also shown.

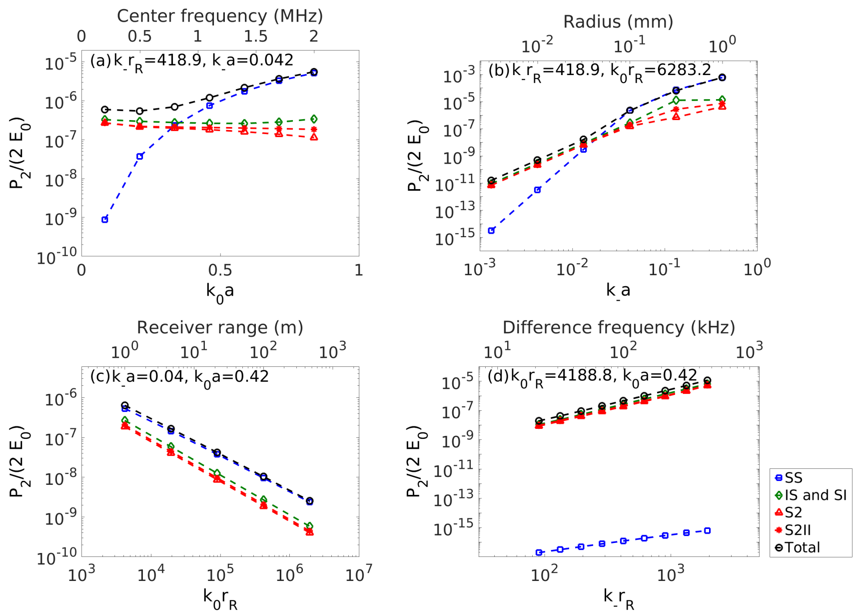


Figure S13: Same as figure 5 except p_{S2II} is also shown.

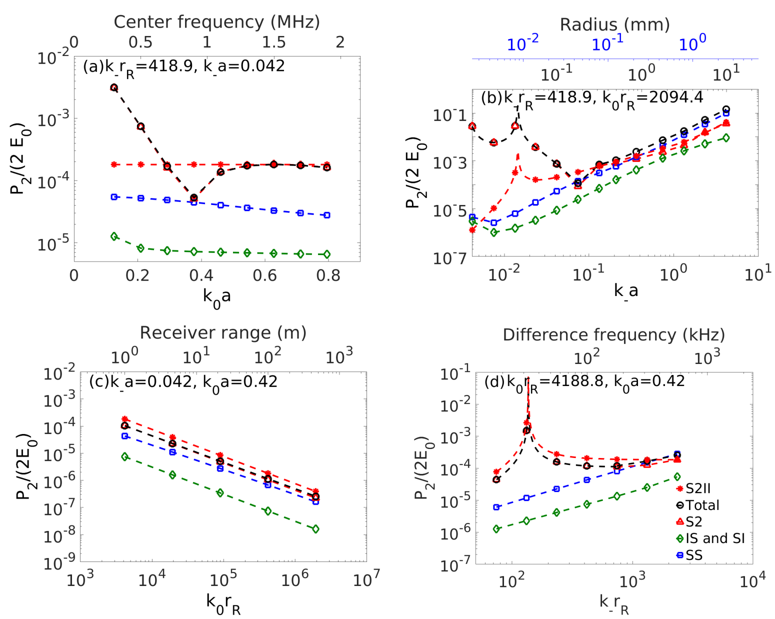


Figure S14: Same as figure 7 except p_{S2II} is also shown. p_{S2II} is the same as for the pressure release case but multiplied by $\alpha(\omega_-)$ of equation B21.

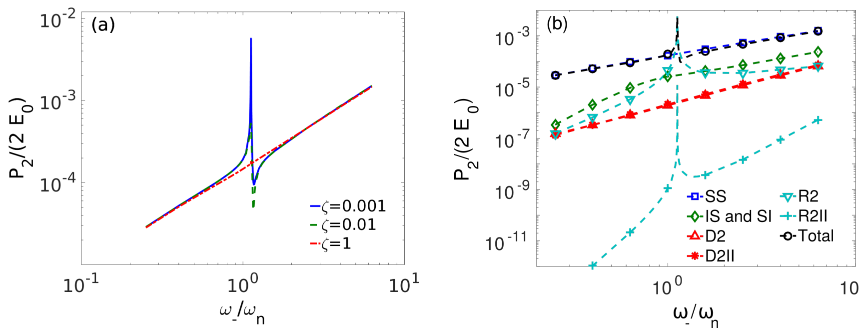


Figure S15: Same as figure 8 except p_{S2II} is also shown.

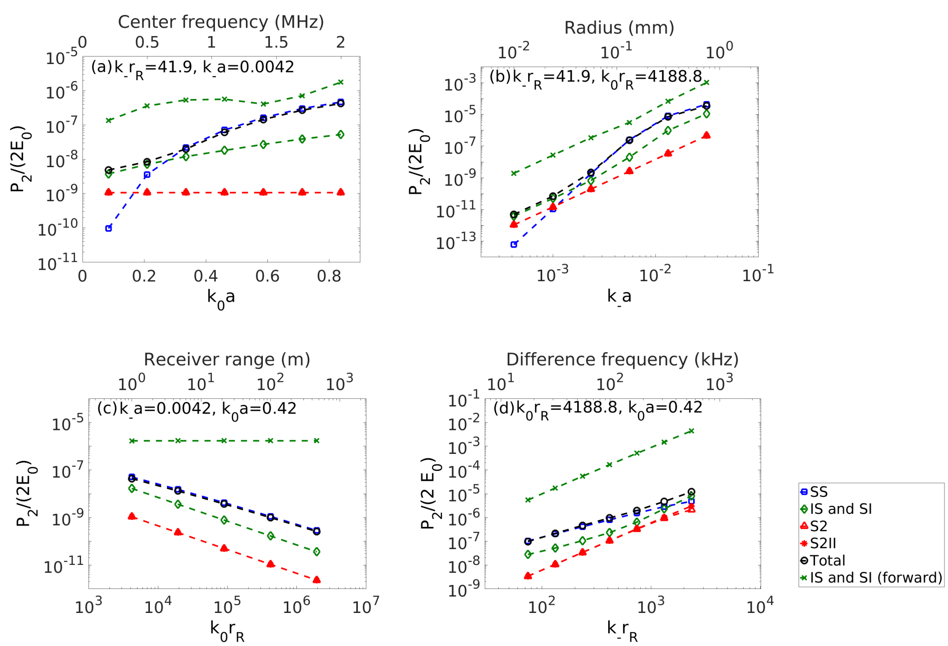


Figure S16: Same as figure 3 except p_{S2II} is also shown. Additionally, $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ is shown for a receiver in the forward direction at $\mathbf{r}_R = r_R\hat{\mathbf{i}}_z$.

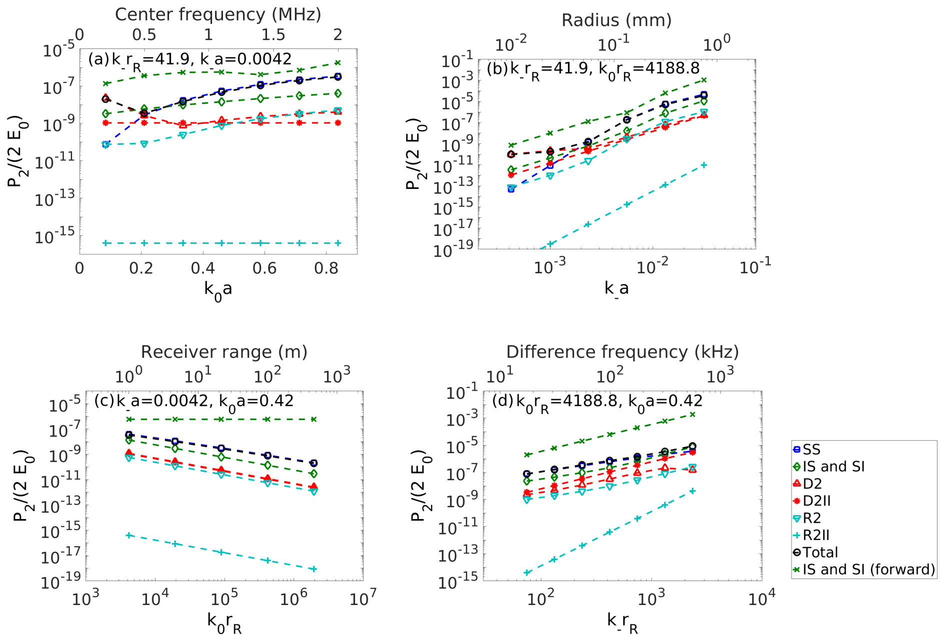


Figure S17: Same as figure 4 except p_{S2II} is also shown. Additionally, $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ is shown for a receiver in the forward direction at $\mathbf{r}_R = r_R \hat{\mathbf{i}}_z$.

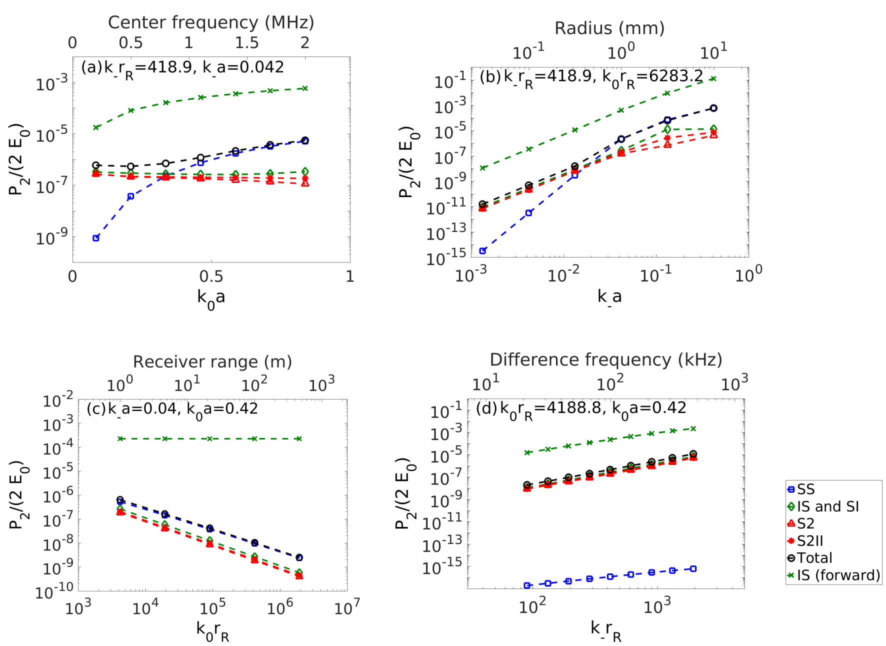


Figure S18: Same as figure 5 except p_{S2II} is also shown. Additionally, $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ is shown for a receiver in the forward direction at $\mathbf{r}_R = r_R \hat{\mathbf{i}}_z$.

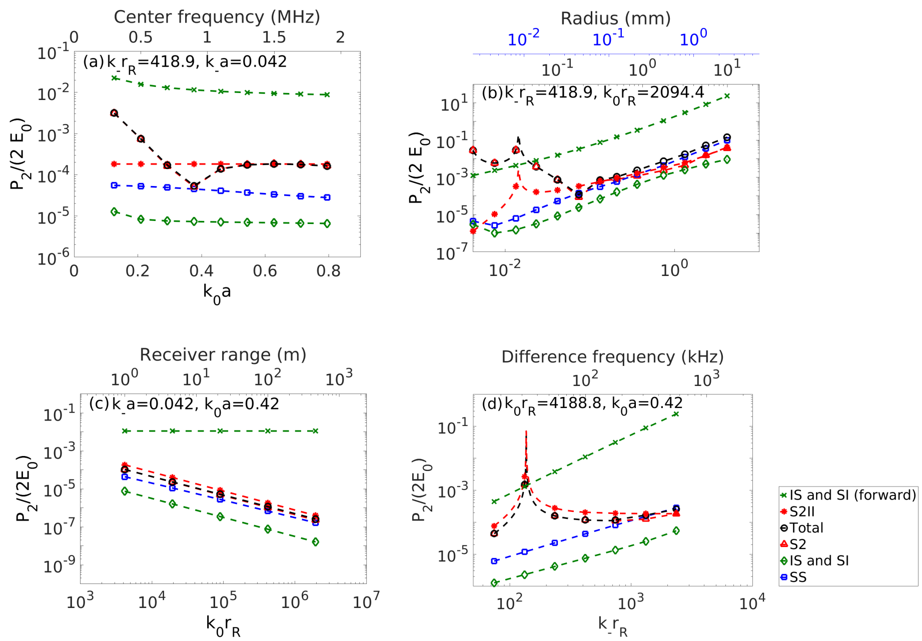


Figure S19: Same as figure 7 except p_{S2II} is also shown. Additionally, $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ is shown for a receiver in the forward direction at $\mathbf{r}_R = r_R \hat{\mathbf{i}}_z$.

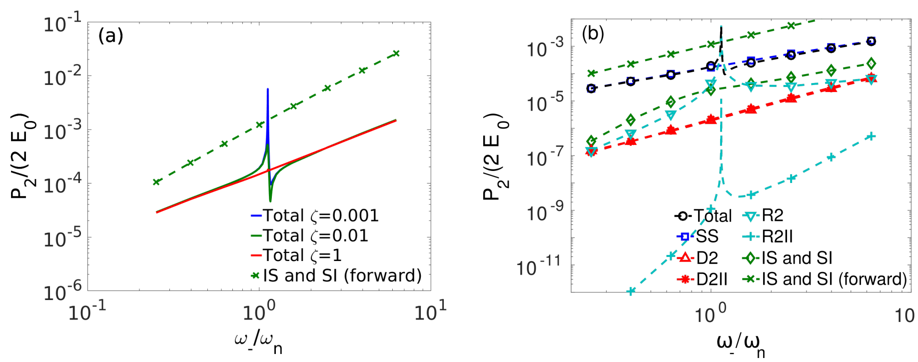


Figure S20: Same as figure 8 except p_{S2II} is also shown. Additionally, $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ is shown for a receiver in the forward direction at $\mathbf{r}_R = r_R \hat{\mathbf{i}}_z$.

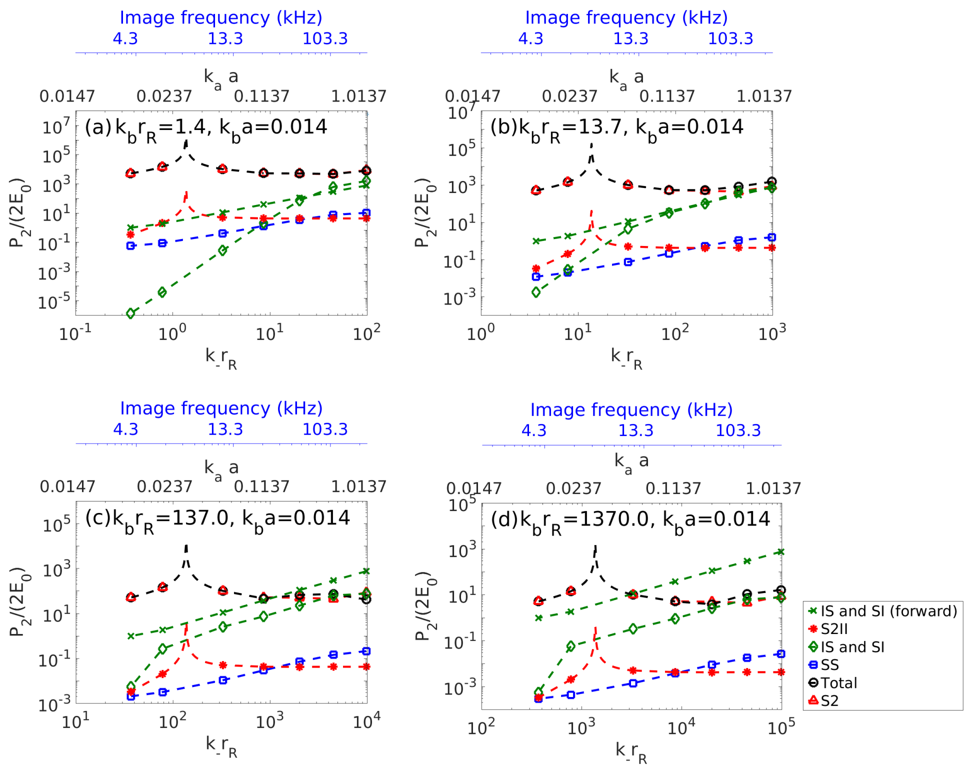


Figure S21: Same as figure 10 but with $p_{\mathrm{IS}} + p_{\mathrm{SI}}$ in the forward direction also included for $\mathbf{r}_R = r_R \hat{\mathbf{i}}_z$

12 Scattered field from a sphere

The spherical wave expansion is used to calculate the scattered field from a sphere of radius a. In a spherical coordinate system (r, θ, ϕ), the first or second order scattered field pressure and normal velocity can be written as

$$P_{\mathrm{S}}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} h_l(kr) Y_l^m(\theta, \phi), \tag{S173}$$

$$\mathbf{n} \cdot \mathbf{V}_{\mathrm{S}}(r, \theta, \phi) = \frac{1}{\mathrm{i}\rho_0 c_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} h_l'(kr) Y_l^m(\theta, \phi), \tag{S174}$$

where a_{lm} are the coefficients determined by the boundary condition, h_l is spherical Hankel function of the first kind of order l, h_l' is the derivative of h_l with respect to its argument, Y_l^m is the Laplace spherical harmonics of degree l and order m, $k = \omega/c_0$ is the wavenumber, ρ_0 and c_0 are the ambient density and sound speed.

When the boundary condition is given in terms of pressure as $P_{\mathrm{S}}(a, \theta, \phi) = \bar{P}_S(\theta, \phi)$, it is a Dirichlet boundary condition, which includes first and second order pressure release conditions (20) and (21). We expand the known $\bar{P}_S(\theta, \phi)$ in spherical harmonics as

$$\bar{P}_{\mathrm{S}}(\theta, \phi) = \sum_{l,m} c_{lm} Y_l^m(\theta, \phi), \tag{S175}$$

where

$$c_{lm} = \int_{2\pi} d\phi \int_{\pi} \bar{P}_{\mathrm{S}}(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin\theta d\theta. \tag{S176}$$

The coefficients a_{lm} for P_{S} can then be determined by equating (S175) with (S173) evaluated at $r = a$, as

$$a_{lm} = \frac{c_{lm}}{h_l(ka)}. \tag{S177}$$

When the boundary condition is given in terms of normal velocity as $\mathbf{n} \cdot \mathbf{V}_{\mathrm{S}}(a, \theta, \phi) = \bar{V}_{\mathrm{Sn}}(\theta, \phi)$, it is a Neumann boundary condition, which includes first and second order rigid boundary conditions (21), (22), (37) and (52). We expand the known function $\bar{V}_{\mathrm{Sn}}(\theta, \phi)$ in spherical harmonics as

$$\bar{V}_{\mathrm{Sn}}(\theta, \phi) = \sum_{l,m} d_{lm} Y_l^m(\theta, \phi) = \frac{1}{\mathrm{i}\rho_0 c_0} \sum_{l,m} a_{lm} h_l'(ka) Y_l^m(\theta, \phi), \tag{S178}$$

where

$$d_{lm} = \int_{2\pi} d\phi \int_{\pi} \bar{V}_{\mathrm{Sn}}(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin\theta d\theta. \tag{S179}$$

The coefficients a_{lm} for P_{S} can then be determined by equating equation (S178) with (S174) evaluated at $r = a$, as

$$a_{lm} = \frac{\mathrm{i}\rho_0 c_0 d_{lm}}{h_l'(ka)}. \tag{S180}$$

The expansion series of equation (S173) is truncated at $l_{\max}$ when evaluating the scattered fields numerically. An empirical formula $l_{\max} = ka + 4(ka)^{1/3} + 2$ [22] is found to give satisfactory convergence for wide range of ka from $ka \ll 1$ to $ka \gg 1$ in our applications.

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