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The Path of Terror Attacks

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Abstract: This paper derives a dynamic path of ongoing terror attacks as a function of terrorists' capacity and a target government's counterterror capacity. The analysis provides several novel insights and characterizations. First, the effect of counterterror policy is limited. Second, proactive counterterror policy affects the depreciation (fatigue) of terrorists' capacity, and defensive counterterror policy limits the worst-case scenario. Third, fluctuations in the time path of attacks are a function of terrorists' time preferences and adjustment costs of changing tactics, which are policy invariant. Indeed, in our model, the oscillations of terror attacks occur irrespective of the government's counterterror stance. Fourth, collective action inefficiencies associated with the underprovision of proactive counterterror policies and overprovision of defensive ones are further exacerbated by our finding that proactive counterterror policy is the more effective of the two. Hence, the more effective policy is underprovided.

Keywords: terror cycles; terror paths; counterterror policy; conflict dynamics; asymmetric conflict

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1. Introduction

Terrorist attacks follow cyclical paths, in terms of both tactics used and measurable consequences such as casualties (e.g., Enders et al. [1] (1992); Enders and Sandler [2] (2000); Feichtinger et al. [3] (2001); Faria [4] (2003); Das [5] (2008); Feichtinger and Novak [6] (2008)). Knowledge of the path's determinants is essential for designing efficient counterterror policies, since it identifies both the main variables and parameters associated with government counterterror tactics, and the terrorists' rational use of resources, and gives a time horizon for terror campaigns and the duration of terror organizations. We introduce and analyze a game-theoretic dynamic model generating an explicit cyclical path of terror attacks and the time adjustment for changing tactics.

The stocks of terror and counterterror capacities co-determine the outcome of the necessarily asymmetric conflict between terrorists and target government (The concept of an organization's stock of terror capacity abstracts from issues regarding recruitment and training of militants explicitly examined in Faria and Arce [7,8] (2005, 2012a), Calkins et al. [9] (2008), Udwadia, Leitman, and Lambertini [10] (2006), and Faria [11] (2014). Following Kaplan et al. [12] (2005), the stock of terror capacity is a broad notion constituting the human, physical, and monetary resources used to launch terror attacks. "A terror organization's stock of terror capacity can usefully be viewed as an accumulation of the potential to plan and carry out terror attacks" (Keohane and Zeckhauser [13] (2003, p. 204)). Terrorists' threat capacity includes anything of value to the terrorist, including, but not limited to, its organization, its possessions, a physical or nonphysical commodity, and an information set (Hausken [14] (2008)).

The targeted government's interests lie not so much in terrorists' capacity as in eliminating its effects. In Keohane and Zeckhauser [13] (2003), the analysis of terror capacity is multi-period; however, only the government acts, as the stock of terror capacity is assumed to follow Brownian motion with positive drift. In Hausken's [14] (2008) analysis of terrorists' resource capacity, both terrorists and the government interact strategically, albeit for two periods. Here, the interaction between terrorists and government is both strategic and ongoing.

Terrorism is a form of asymmetric conflict, with asymmetry appearing in our model in two ways. First, we adopt a Stackelberg or leader–follower framework where the government leads and terrorists follow. Consequently, the target government maximizes its payoff, understanding terrorists' strategy will be a best reply to its counterterror policy. Second, the costs of terror and counterterror actions respect Richardson–Lanchester line-of-fire (in)efficiencies associated with asymmetric dynamic conflict (e.g., Avenhaus and Fichtner [15] (1984); Strickland [16] (2011); MacKay [17] (2015); Kress [18] (2020)). Specifically, terrorists' costs of investing in capacity are a function of their investment only, because terrorists have no problems in identifying or locating the target government. That is to say, the interdependence between terrorists and government does not arise on the cost side of terrorists' payoffs. Instead, government counterterror policy affects the benefit side of terrorists' payoffs by partially determining the probability and consequences of a successful attack. By contrast, the government's costs are proportional to the product of its level of investment and terrorists' capacity because the clandestine nature of terror operations creates an asymmetry in the terrorists' favor. In particular, terrorists must be found before their capacity for terror can be targeted.

2. The Model

The model is essentially a stock of (counter)terror capacity competition between the terrorists and targeted government; therefore, the natural framework is dynamic game theory. Define k as the stock of terror capacity and K the stock of counterterror capacity. The stock of terror capacity includes resources terrorists accumulate to support their cause: "a network of supporters; financial capacity; weapons, explosives, and materiel; destructive know-how; a communications network; the tacit approval or even active encouragement of a state or states; trained personnel; and a sufficient number of recruits willing to risk prison or death. The mix of resources may vary greatly from organization to organization, but some accumulated capacity is essential for terror activity" (Keohane and Zeckhauser [13] (2003, pp. 203–4).

For failed states (e.g., Barros et al. [19] (2008) and George [20] (2016)), such as Somalia, $K < k$ holds, while for poor and disorganized regions (e.g., Faria [21] (2008)), such as Russia's interaction with Chechenia, $K \approx k$; for Europe and North America, $K > k$. We focus on the asymmetric case where $K > k$.

2.1. *Deriving the Path of Terrorists' Capacity*

Our solution concept for each period of our infinite horizon game is Stackelberg equilibrium. Dockner et al. [22] (2000, pp. 135–141) provide an overview of this solution. As the follower, in each state, t, terrorists observe the government's counterterror strategy. As such, the leader (government) anticipates that the follower selects its best reply to the leader's strategy and the leader maximizes its payoff accordingly. Therefore, we start by analyzing the terrorists' problem first in order to derive the terrorist's best reply to the government's strategy at time t.

The terrorists rationally employ their resources to efficiently attack the target government. Function $A(\cdot)$ measures the consequence of terror attacks, including the logistical likelihood of success. Arce [23] (2019) provides measurements of $A(\cdot)$ in terms of disability-adjusted lives lost to terror tactics ranging from suicide bombings to combined firearms/explosives attacks to vehicular assaults. Moreover, the target government's investment in counterterror capacity in the previous period, ΔK_{t-1}, reduces the number or lethality of terror attacks in the current period; i.e., $A(\cdot)$ decreases in ΔK_{t-1}. The benefit portion of the terrorists' payoff therefore takes the form $A(\Delta K_{t-1})k_t$, where $A'(\Delta K_{t-1}) < 0$, and k_t is the attack resources stockpiled by terrorists (e.g., Hausken and Zhuang [24,25] 2011a, b). Coefficient $A(\Delta K_{t-1})$ on k_t indicates the efficacy of the terrorists' stock is a negative function of the government's counterterror policy (Berman and Gavious [26] (2007) analyze a model in which the government chooses cities in which to maximize security, through the location choice of facilities that provide support in case of a terrorist attack.). As such, strategic interdependence arises on the benefit side of terrorists' payoffs.

Turning to the terrorists' costs, they face increasing costs of adjusting their stock of terror capacity, an assumption present in dynamic models of capital investment dating back to at least Gould [27] (1968). Hence, $c(i_t)$, $c'(i_t) > 0$, and $c''(i_t) > 0$, where i_t is the gross investment in k. Adjustment costs correspond to the implicit opportunity costs of foregone terrorism owing to the use of an organization's resources to alter its terror capacity. Such a cost structure implies investment in terror capacity can result in a spectacular attack, but the size or number of terrorist attacks is not unlimited. From the perspective of Richardson's [28] (1939) dynamics of conflict, terrorists cannot engage in an arms race with governments.

The rate of change of the stock of terrorists' capacity is given by

$$\Delta k_t \equiv k_{t+1} - k_t = i_t - \delta k_t \tag{1}$$

where $\delta < 1$ is the terrorists' rate of capacity depreciation ("fatigue" in the parlance of Richardson's (1939) conflict dynamics).

We assume the purchase price of a unit of stock of terrorism capacity is constant and equal to 1. The terrorists' payoff at a point in time is $A(\Delta K_{t-1})k_t - i_t - c(i_t)$. As foreshadowed in the introduction, asymmetric conflict is captured by terrorists' costs being a function of the terrorists' investment, i_t, and not the counterterror investment of the government, I_t, or the government's stock of counterterror capacity, K_t. Moreover, in Richardson–Lanchester approaches to asymmetric dynamic conflict, government targets are "in the open" for terrorists (e.g., MacKay [17] (2015)). By contrast, the government's cost structure, given below, reflects the fact that terrorists are rarely in the open.

The present value of the terrorists' payoffs is

$$\sum_{t=0}^{\infty} \frac{1}{(1+\tau)^t} [A(\Delta K_{t-1})k_t - i_t - c(i_t)] \tag{2}$$

where τ is the terrorists' rate of time preference or its impatience. Terrorists choose the level of investment over time, i_t, to maximize (2) subject to (1), taking the path of the government's counterterror capacity, K_t, as given. The Lagrangian, $\mathcal{L}$, for the terrorists' maximization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+\tau)^t} \{A(\Delta K_{t-1})k_t - i_t - c(i_t) + q_t[i_t + (1-\delta)k_t - k_{t+1}]\} \tag{3}$$

where the ${q_t}'s$ are the Lagrange multipliers corresponding to the identity in Equation (1) for the evolution of the stock of terror capacity, Δk_t, given i_t for $t = 0, 1, 2, \ldots, \infty$. We denote the Lagrange multipliers by the lower-case q, rather than the more common λ or μ, owing to their relationship to Tobin's (marginal) q. Tobin's q measures the internal value capacity generates for an organization relative to its replacement cost. When $q > 1$, the returns on investment in capacity exceed its costs. Here, q_t is the value to the terrorists of an additional unit of capacity at time t; q_t is the shadow price of Δk_t at the end of period t. Note that the constraint is contained within the braces of $\mathcal{L}$, implying shadow prices, q_t, $t = 0, 1, 2, \ldots, \infty$, are measured in t-period values rather than in present values. Each q_t is discounted by $\frac{1}{(1+\tau)^t}$ in $\mathcal{L}$.

The first-order conditions for the terrorists with respect to i_t are

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Longrightarrow q_t = 1 + c'(i_t) \tag{4}$$

For period $t+1$ this is

$$q_{t+1} = 1 + c'(i_{t+1}). \tag{5}$$

Equations (4) and (5) are Tobin's q for terrorists' investment in terror capacity. In the steady state, the value of q corresponds to the cost of acquiring a unit of capacity (fixed at 1) plus marginal adjustment costs. Given the prices of a unit of capacity, P_{k_t} and $P_{k_{t+1}}$, the investment rule associated with Tobin's q is as follows: investment takes place, $i_t > 0$, if $q_t / P_{k_t} > 1$ (similarly, $i_{t+1} > 0$ if $q_{t+1}/P_{k_{t+1}} > 1$). With prices P_{k_t} and $P_{k_{t+1}}$ normalized to 1, Equations (4) and (5) are consistent with the terrorists' positive level of investment in capacity.

The first-order condition for the path of terror capacity is

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\frac{1}{(1+\tau)^t} q_t + \frac{1}{(1+\tau)^{t+1}} [A(\Delta K_t) + (1-\delta) q_{t+1}] = 0 \tag{6}$$

Inserting the values of q_t and q_{t+1} from (4) and (5) into this relationship yields

$$1 + c'(i_t) = \frac{A(\Delta K_t) + (1-\delta)[1 + c'\ (i_{t+1})]}{(1+\tau)} \tag{7}$$

Equation (7) is the Euler equation for terrorists' capacity investment, capturing the optimal choice between investment today and investment tomorrow when both investments are interior. The terrorists equate the cost of an additional unit of terror capacity in the current period, which is fixed at 1, plus the adjustment costs, to the discounted value of the sum of (i) the return on increased capacity in the next period, $A(\Delta K_t)$, and (ii) the fatigued (depreciated) unit of additional capacity in the next period, along with the associated savings in adjustment costs. The term $A(\Delta K_t)$ reflects the strategic interdependence of the value of an attack, as it is a function of the government's counterterror policy, ΔK_t.

Assuming a simple convex adjustment cost function, $c(i_t) = 0.5 c i_t^2$, from Equations (4), (5), and (7) we have

$$1 + c i_{t+1} = (1-\delta)^{-1} [(1+\tau)(1 + c i_t) - A(\Delta K_t)] \tag{8}$$

Inserting Equation (1) for $i_t = k_{t+1} - k_t(1-\delta)$ into (8) yields

$$k_{t+2} - k_{t+1} = [\tau + \delta - \delta c (1-\delta) k_{t+1} + (1+\tau) c (k_{t+1} - (1-\delta) k_t) - A(\Delta K_t)] \tag{9}$$

Equation (9) characterizes the optimal path of terrorists' capacity as a function of the government's counterterror policy, ΔK_t.

2.2. Examining the Government's Counterterror Policy

We now address the government's problem. Safety from terrorism is a public good, and a government's constituents often hold it accountable for the provision of this public good or lack thereof (Müller [29] (2011); Arce [30] (2020)). Consequently, the government maximizes society's net safety, given by the difference between total safety, $S(K_t)$, and cost term $k_t I_t$. The product $k_t I_t$ captures governments' difficulties in targeting terrorists. As terrorists are clandestine by definition, the cost of targeting terrorists is proportional to the stock of terror capacity. Terrorists must be found prior to being targeted, as is the case in models of dynamic conflict where governments "fire blindly" into an "area" defined by k_t. Kress [18] (2020) calls it, "firing into the brown." By contrast, government targets have to be in the open for terrorism to influence an audience beyond the immediate victims, in agreement with the objective of terrorism in standard definitions of the phenomenon.

The present value of the government's objective function is

$$\textstyle\sum_{t=0}^{\infty} \frac{1}{(1+\gamma)^t} [S(K_t) - k_t I_t] \tag{10}$$

where γ is the government's rate of time preference (impatience).

In a Stackelberg solution, in each state, t, the government takes the terrorists' best reply function, given by Equation (9), and the rate of change of its stock of counterterror capacity:

$$K_t = I_t - \overline{\delta}K_t \quad (11)$$

as dynamic constraints. Where, $\overline{\delta}$ is the depreciation rate (fatigue) of the government's stock of counterterror capacity.

The Lagrangian, Γ, for the government's maximization problem is

$$\Gamma = \sum_{t=0}^{\infty} \frac{1}{(1+\gamma)^t}\{S(K_t) - k_t I_t + Q_t[I_t + (1-\overline{\delta})K_t - K_{t+1}] + \mu_t[(1+\tau)(1+ci_t) - A(\Delta K_t) - (1+\delta)(1+ci_{t+1})]\} \quad (12)$$

where the $Q_t{}'s$ are the Lagrange multipliers corresponding to the value to the government of an additional unit of counterterror capacity formation at time t, as given in Equation (11). The use of the letter Q for the Lagrange multipliers for the capacity formation constraints in Equation (12) is again as an indicator of the relationship between these Lagrange multipliers and Tobin's marginal q for government investment in counterterror capacity. The $\mu_t{}'s$ are the Lagrange multipliers measuring the effect on the government of an additional unit of terrorists' capacity formation by terrorists at time t, as given in in Equation (8). As such, the $\mu_t{}'s$ are expected to take negative values because additional terror capacity is detrimental to safety. Once again, the Lagrange multipliers, $Q_0, Q_1, Q_2, \ldots; \mu_0, \mu_1, \mu_2, \ldots$, and associated constraints are measured as t-period values (i.e., contained within the braces of Γ). They are discounted each period by $\frac{1}{(1+\gamma)^t}$.

The first-order condition for the government's maximization problem with respect to I_t is

$$\frac{\partial \Gamma}{\partial I_t} = 0 \Rightarrow Q_t = k_t \ (\Rightarrow \text{ for period } t+1 : Q_{t+1} = k_{t+1}) \quad (13)$$

Unlike Tobin's q for terrorists, Tobin's q for the government, Q_t, exhibits strategic interdependence because it is a function of terrorists' current capacity, k_t. Intuitively, from Equation (13), if $k_t = 0$ then $Q_t = 0$ and, by Tobin's q, the government does not invest in counterterror capacity: $I_t = 0$. This is similarly the case for k_{t+1}, Q_{t+1}, and I_{t+1}. Moreover, by the investment rule for Tobin's q, investment takes place only if $Q_t > 1$. Consequently, if k_t is nominal, i.e., $k_t < 1$, then the government does not invest in counterterror capacity either $(k_t < 1 \Rightarrow Q_t < 1 \Rightarrow I_t = 0)$, because the government's (shadow) cost of reducing terror capacity exceeds terrorists' current capacity. As a result, small terror groups/capacities fly below the government's radar as they do not trigger a government response. Continuing:

$$\frac{\partial \Gamma}{\partial i_t} = 0 \Rightarrow \mu_{t+1} = \frac{(1-\delta)(1+\gamma)}{(1+\tau)}\mu_t; \text{ and} \quad (14)$$

$$\frac{\partial \Gamma}{\partial K_{t+1}} = 0 \Rightarrow S'(K_{t+1}) + Q_{t+1}\left(1-\overline{\delta}\right) + \mu_{t+1}\left(A'(\Delta K_{t+1})\right) = (1+\gamma)[Q_t + \mu_t A'(\Delta K_t)] \quad (15)$$

To simplify, we assume safety function $S(K_t) = \overline{S}K_t{}^{\sigma}$. The properties of adjustment costs, $c(i_t)$, allow for spectacular terror attacks, but asymmetry precludes terrorists from engaging in an arms race with the government. As such, let $\overline{A}$ be the maximum potential effect of a terrorist attack, and the parameter g the marginal efficiency of the growth of government's counterterror capacity in curbing terror attacks. It follows that $A(\Delta K_t) = \overline{A} - g\Delta K_t$ and $A'(\Delta K_t) = -g$. Equation (15) becomes

$$\sigma\overline{S}K_{t+1}^{\sigma-1} + Q_{t+1}\left(1-\overline{\delta}\right) - \mu_{t+1}g = (1+\gamma)[Q_t - \mu_t g] \quad (16)$$

2.3. Elimination of Terrorists' Threat

Equation (16) can be further reduced by applying $Q_t = k_t$ and $Q_{t+1} = k_{t+1}$ from Equation (13) and steady-state conditions $K_{t+1} = K_t = K^*$; $\mu_{t+1} = \mu_t = \mu^*$. As discussed above, μ^* must be negative, since an increase in the terror capacity must decrease the optimal value of the government's objective function. Without loss of generality, let $\mu^* = -1$.

Our first major result is highlighted by the following proposition:

Proposition 1. *Only governments who are more impatient than terrorists (i.e., $\gamma > \tau$) find it in their interests to attempt to fully eliminate terrorists' capacity.*

Proof of Proposition 1. The long-run equilibrium level of K^* necessary to fully eliminate terrorists' capacity, i.e., the value of K^* yielding $k = 0$, is obtained from Equation (16) in the steady state and is given by:

$$K^*_{k=0} = \left(\frac{\sigma \overline{S}}{g\gamma}\right)^{\frac{1}{1-\sigma}} \tag{17}$$

For an interior solution, $K^*_{k=0} > 0$, by Equation (14), the steady state where $\mu_{t+1} = \mu_t = \mu^*$ yields $\frac{1+\tau}{1+\gamma} = 1 - \delta < 1$. Consequently, only governments who are more impatient than terrorists (i.e., $\gamma > \tau$) find it in their interests to attempt to fully eliminate terrorists' capacity. □

Yet, it unlikely that targeted governments are less impatient than terrorists because the lifespan of terrorist organizations is relatively short (Vittori [31] (2009; Faria and Arce [32] (2012b); Gaibulloev and Sandler [33] (2013)). Consequently, when governments are the more patient of the two, they do not find it optimal to set K^* such that $k = 0$. Instead, patient governments treat terrorism as an ongoing phenomenon. The ongoing interaction between governments and terrorists is the subject of the following section.

3. Ongoing Terrorism and the Dynamic Path of Terror Capacity

An ongoing terrorist threat occurs when the government is more patient than terrorists. In order to characterize the dynamic path of terror capacity, we analyze the terrorists' (follower's) response by substituting Equation (17) into Equation (9). From Equation (11), when $I_t \neq K^*_{k=0}$, terrorists' capacity follows the following dynamic path:

$$k_{t+2} - k_{t+1} = [c(1-\delta)]^{-1}\left[\tau + \delta - \delta c(1-\delta)k_{t+1} + (1+\tau)c(k_{t+1} - (1-\delta)k_t) - \overline{A} + g(I_t - \overline{\delta}K^*)\right] \tag{18}$$

The presence of terms k_{t+2}, k_{t+1}, and k_t in Equation (18) implies the path of terrorists' capacity is a second-order linear difference equation. Solving the equation involves dividing it into two parts: a particular solution and a homogenous solution. We begin by deriving a solution particular to the steady state: $k_{t+2} = k_{t+1} = k_t \neq 0$. The *particular solution*, k_p, is

$$k_p = \left(\frac{\overline{A} - g(I_t - K^*) - \tau - \delta}{c\delta(\tau + \delta)}\right) \tag{19}$$

The second part of the solution to Equation (18) is the *homogenous solution*, so named because it corresponds to the case where the constant term in Equation (18), $\overline{A} - g(I_t - K^*) - \tau - \delta$, is zero. The trivial solution takes the form $k_{t+2} = k_{t+1} = k_t = 0$. The nontrivial solution is typically derived by assuming the homogenous solution takes the same form as the solution to a first-order difference equation: $k_t = \lambda^t$, where $\lambda \neq 0$ is an unknown constant interpreted as an eigenvalue (characteristic root). Setting $\overline{A} - g(I_t - K^*) - \tau - \delta = 0$ and substituting $k_t = \lambda^t$ into Equation (18) gives the following:

$$\lambda^t\left[\lambda^2 - \left(1 - \delta + \frac{1+\tau}{1-\delta}\right)\lambda + (1+\tau)c\right] = 0 \tag{20}$$

where the term in brackets is the characteristic equation for Equation (18). The homogeneous solution has two roots:

$$\lambda_1 = \frac{\left(1-\delta+\frac{1+\tau}{1-\delta}\right)+\sqrt{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2-4(1+\tau)c}}{2} \tag{21}$$

$$\lambda_2 = \frac{\left(1-\delta+\frac{1+\tau}{1-\delta}\right)-\sqrt{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2-4(1+\tau)c}}{2} \tag{22}$$

As δ is the rate of terror capacity depreciation (fatigue), it follows that $\delta < 1$ and both roots are complex. The complex roots can be written as conjugate pairs $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where, from Equation (20), $\alpha = \frac{\left(\delta-1-\frac{1+\tau}{1-\delta}\right)}{2}$ and $\beta = \frac{\sqrt{4(1+\tau)c-\left(\delta-1-\frac{1+\tau}{1-\delta}\right)^2}}{2}$. The complex conjugate pairs correspond to the solutions $k_t = \lambda_1^t = (\alpha+i\beta)^t$ and $k_t = \lambda_2^t = (\alpha-i\beta)^t$. Adopting polar representations, $\lambda_1 = \alpha+i\beta = \sqrt{\alpha^2+\beta^2}\cdot(\cos\theta + i\sin\theta)$ and $\lambda_2 = \alpha - i\beta = \sqrt{\alpha^2+\beta^2}\cdot(\cos\theta - i\sin\theta)$, where $\sin\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ and $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$.

For these complex roots, the two corresponding homogenous solutions, $k_h^{(1)}$ and $k_h^{(2)}$, are

$$k_h^{(1)} = \left(\sqrt{(1+\tau)c}\right)^t(\cos\theta t + i\sin\theta t) \text{ and } k_h^{(2)} = \left(\sqrt{(1+\tau)c}\right)^t(\cos\theta t - i\sin\theta t) \tag{23}$$

By the superposition principle, if $k_h^{(1)}$ and $k_h^{(2)}$ are solutions to a homogenous difference equation, then so is

$$k_h = C_1 k_h^{(1)} + C_2 k_h^{(2)} \tag{24}$$

where C_1 and C_2 are arbitrary constants.

Since k_t must be a real number, the homogenous solution must be a real number. As $k_h^{(1)}$ and $k_h^{(2)}$ are imaginary, then it must be the case that C_1 and C_2 are imaginary as well. Hence, C_1 and C_2 can also be expressed as complex conjugates:

$$C_1 = \hat{\alpha}+i\hat{\beta} = \sqrt{\hat{\alpha}^2+\hat{\beta}^2}\cdot(\cos\theta+i\sin\theta);\ C_2 = \hat{\alpha}-i\hat{\beta} = \sqrt{\hat{\alpha}^2+\hat{\beta}^2}\cdot(\cos\theta - i\sin\theta) \tag{25}$$

By substituting these values for C_1 and C_2 into Equation (24), Goldberg [34] (1986, p. 140) provides the steps for reducing Equation (24) to

$$k_h = 2\hat{C}_1\left(\sqrt{(1+\tau)c}\right)^t\cos(\theta t + \hat{C}_2), \tag{26}$$

where $\hat{C}_1$ and $\hat{C}_2$ are arbitrary real constants and θ is the same as before. Interestingly, the homogenous solution depends only on the terrorists' impatience, τ, and adjustment costs, c.

We now present the main result characterizing the dynamic path of terrorists' capacity accumulation. The complete solution for the path of terrorists' capacity requires combining the homogenous solution, k_h, with the particular solution, k_p (Intuitively, if k_t and k_p are solutions to a difference equation with nonzero constant term, then $k_t - k_p$ is a solution to the homogenous version of the difference equation). That is, $k_t = k_h + k_p$, yielding

$$k_t = 2\hat{C}_1\left(\sqrt{(1+\tau)c}\right)^t\cos(\theta t+\hat{C}_2) + \underbrace{\left(\frac{\overline{A}-g(I_t-K^*)-\tau-\delta}{c\delta(\tau+\delta)}\right)}_{k_p} \tag{27}$$

As the cosine function oscillates, the path of terror capacity exhibits a fluctuating pattern periodic in nature. The path is a stepped fluctuation of discrete points (it is only smooth for continuous time), oscillating between values above and below $k_p = \left(\frac{\overline{A}-g(I_t-K^*)-\tau-\delta}{c\delta(\tau+\delta)}\right)$. What matters in our context is convergence, as determined by the term $\left(\sqrt{(1+\tau)c}\right)^t$. Three possibilities emerge.

Case 1: $\sqrt{(1+\tau)c} > 1$. Here, k_t oscillates with ever-increasing amplitude, implying a divergent and explosive path of terrorists' capacity accumulation. Such an outcome is not possible because terrorists' limited resources are the defining feature of terrorism as asymmetric conflict; i.e., $K_t \gg k_t$. Alternatively, for the situation of failed states, $K_t < k_t$, this case identifies when terrorists win.

Case 2: $\sqrt{(1+\tau)c} = 1$. The solution is an equilibrium solution. Here, k_t oscillates (about k_p) with constant amplitude.

Case 3: $\sqrt{(1+\tau)c} < 1$. Here, k_t oscillates with monotonic-decreasing amplitude and converges to k_p as $t \to \infty$. This holds iff $c < \frac{1}{1+\tau}$.

Cases 2 and 3 are relevant for the present study because terrorists generally do not have the resources to engage in an arm's race with targeted governments. Several novel observations arise from the characterization of the dynamics of the capacity accumulation path given in Equation (27). First, $c \leq \frac{1}{1+\tau}$ relates terrorists' adjustment (opportunity) cost of investing in new terror capacity (foregone terrorism) to terrorists' discount factor. In particular, patient terrorists can exhibit a variety of tactics over their lifespan because their patience (low τ) allows for the associated higher adjustment costs. By contrast, impatient terrorists will not forestall attacks in order to accommodate the adjustment costs associated with a portfolio of tactics. No direct counterterror policy prescription follows from the $c \leq \frac{1}{1+\tau}$ characterization, as neither τ nor c are policy variables. They are, instead, the terrorists' primitives. Impatience term τ stems from the terrorists' time preferences, and counterterror policy has no effect on adjustment costs, c, which are measured in terms of the attacks terrorists are willing to forgo to adjust their stock of terror capacity.

Second, a lull in terror activity need not be indicative of successful counterterror policy. Instead, it can be due to patient terrorists undergoing the adjustment costs associated with a forthcoming wave of new tactics. For example, Enders and Sandler [2] (2000, p. 323) employ time series analysis to show that "authorities should focus on anticipating upturns in incidents involving casualties following fairly length lulls of greater than two years". Moreover, the authors identify the period immediately prior to the yet-to-occur events of September 11, 2001 as being the longest lull on record. The tactical innovation of simultaneous coordinated skyjackings to employ airliners as weapons during 9/11 reveals Al Qaeda's willingness to undertake the adjustment costs stemming from its meticulous preparations prior to the attacks. By contrast, the spate of vehicular assaults incited by ISIS during the late 2010′s required little in the way of adjustment costs; as instructions were distributed online, many of the vehicles were stolen or rented, and the operatives were at arms-length (Siqueira and Arce [35] 2020).

Proposition 2. *Fluctuations in the time path of attacks are a function of terrorists' time preferences and adjustment costs of changing tactics, which are policy-invariant.*

Proof of Proposition 2. The government has no control over the oscillatory component of the time path of terrorism, as the terms surrounding the cosine function in Equation (27), c and τ, are the terrorists' primitives.

Fourth, the time path characterized in Equation (27) is akin to a time-variant system with k_p as the input and k_t the output. This begs the question as to the government's degree of control (over k_p). In the steady state (i.e., $\Delta K_t = 0 \Rightarrow I_t = K^*$), k_p reduces to

$$k_p^* = \frac{\overline{A}-\tau-\delta}{c\delta(\tau+\delta)} \tag{28}$$

Counterterror policies traditionally fall into two broad categories: proactive and defensive (e.g., Frey [36] (2004); Arce and Sandler [37] (2005); Sandler and Siqueira [38] (2006); Bandyopadhyay and Sandler [39] (2011); Bier and Hausken [40] (2011)). Proactive policies include attacking terrorists' training grounds and freezing the assets of supporting organizations. Proactive policies directly target the stock of terror assets, i.e., they increase fatigue term δ. By contrast, defensive policies, such as hardening targets and controlling the inflow of potential terrorists' immigrants or refugees, limit the upper bound on terrorists' capacity, $\overline{A}$ (For alternatives to defensive strategies see (Frey and Luechinger [41] 2003)). Moreover, the literature on the collective action problems associated with proactive and defensive policies most often treats terrorists as a passive third party. Equation (28) provides the direct link between proactive and defensive counterterror policies and the actions of terrorists themselves. In particular, $k_p = 0$ when the mix of defensive and proactive counterterror policies satisfies $\overline{A} - \delta = \tau$. *Counterterror policy is formulated with reference to terrorists' impatience*. Under these circumstances, terrorists' capacity is not zero but instead oscillates about $k_p = 0$ if the government has the requisite winningness, resources, intelligence, etc., to set $\overline{A} - \delta = \tau$ by decreasing $\overline{A}$ via defensive policy and increasing δ via proactive policy. □

Fifth, in the absence of the requirements sufficient to set $\overline{A} - \delta = \tau$, government control over the time path of terror capacity via counterterror policy is characterized by

$$\frac{\partial k_p^*}{\partial \overline{A}} = \frac{1}{c\delta(\tau+\delta)} \tag{29}$$

$$\frac{\partial k_p^*}{\partial \delta} = \frac{-c\delta(\tau+\delta) - \left(\overline{A} - \tau - \delta\right)[c(\tau+\delta) + c\delta\tau]}{[c\delta(\tau+\delta)]^2} \tag{30}$$

An increase in defensive counterterror policy decreases $\overline{A}$, leading to a decrease in the stock of terror capacity given by Equation (29). Similarly, an increase in proactive counterterror policy increases δ, leading to a decrease in the stock of terror capacity given by Equation (30).

Sixth, from the perspective of international collective action and the coordination of counterterror policy (e.g., Faria et al. [42] 2020), defensive counterterror policies are strategic complements and proactive ones are strategic substitutes (Sandler and Siqueira [38] (2006); Faria et al. [43] (2017).) Accordingly, governments overuse defensive policies and underprovide proactive ones. The characterizations given in Equations (29) and (30) shed further light on these inefficiencies, giving rise to the following proposition.

Proposition 3. *Proaction is both underprovided and more effective compared with defensive counterterror policy.*

Proof of Proposition 3. From Equations (29) and (30), $\left|\frac{\partial k_p^*}{\partial \delta}\right| > \left|\frac{\partial k_p^*}{\partial \overline{A}}\right|$. □

Corollary. *At the same time, neither proactive nor defensive counterterror policies affect the ebb and flow of terrorists' actions; the amplitude of terrorism is determined by terrorists' primitives c and τ. Hence, the fluctuations in our model are consistent with terrorists deliberately undertaking what appear to be uncertain (time-variant) actions on their part.*

4. Conclusions

This paper considers a dynamic game of terror and counterterror capacity accumulation between terrorists and a target government in order to provide a full characterization of their ongoing asymmetric conflict in terms of the time path of terrorists' capacity. The term "ongoing conflict" is used because the necessary condition for governments' willingness to attempt to fully eliminate terrorists' capacity is for the government to be more impatient than the terrorists. As it is well known that terrorist groups are short-lived relative to their target governments (excluding failed or organizationally disadvantage states), the necessary condition on relative time preferences is unlikely to be met. Consequently, targeted governments purposefully treat terrorism as an ongoing phenomenon.

Within this context, an advantage of dynamic models producing time paths of terror activity is they can be "tested, evaluated, and improved upon through the use of actual field data" (Strickland [16] (2011, p. 161)). While such an exercise is a future research direction, it is beyond the scope of the present analysis.

At the same time, the analysis provides several novel insights. For example, governments' resignation to terrorism's persistence is not akin to an "optimal negative externality" argument, such as occurs for pollution abatement. In the case of pollution abatement, the presence of diminishing marginal social benefits and increasing marginal social costs leads to a positive level of (optimal) pollution (Mishan [44] 1974). By contrast, only the base accumulation of terror capacity around which oscillations occur is determined by policy. We characterize how proactive counterterror capacity affects the depreciation (fatigue) of terrorists' capacity and how defensive counterterror policy limits the worst-case scenario. The effectiveness of such policies is a function of terrorists' primitives (time preferences and adjustment costs of changing tactics), which are policy-invariant.

Accordingly, the effect of counterterror policy is limited. Oscillations in the time path of terror capacity are a function of terrorists' primitives. Consequently, the amplitude of terrorism only converges to zero in the long run. Once again, such dampening is determined by terrorists' primitives, rather than counterterror policy. As such, terrorists' willingness to make the impatience–adjustment cost tradeoff is the root determinant of their longevity.

Counterterror policy is therefore plagued by inefficiencies and paradoxes. The ebb and flow of terror tactics results from terrorists trading impatience for improved tactics and their associated adjustment costs. Such lulls in activity and changes in tactics are observationally equivalent to terrorists substituting tactics in response to defensive counterterror policies (e.g., security screening in airports) and yet they may have absolutely nothing to do with counterterror tactics. Indeed, in our model, the oscillations occur *irrespective* of the government's counterterror stance. In addition, collective action inefficiencies associated with the underprovision of proactive counterterror policies and overprovision of defensive ones are further exacerbated by our finding that proactive counterterror policy is the more effective of the two. Hence, the more effective policy is underprovided. This is a novel characterization of counterterror policy relative to the extant literature.

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