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Article

The Existence of Perfect Equilibrium in Discontinuous Games

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Abstract: We prove the existence of a trembling-hand perfect equilibrium within a class of compact, metric, and possibly discontinuous games. Our conditions for existence are easily verified in a variety of economic games.

Keywords: trembling-hand perfect equilibrium; discontinuous game; infinite normal-form game; payoff security

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1. Introduction

A Nash equilibrium is trembling-hand perfect if it is robust to the players' choice of unintended strategies through slight trembles. That is, in a world where agents make slight mistakes, trembling-hand perfection requires that there exist at least one perturbed model of low-probability errors with an equilibrium that is close to the original equilibrium, which is then thought of as an approximate description of "slightly constrained" rational behavior, or what could be observed if the players were to interact within the perturbed game. In this regard, a Nash equilibrium that is not trembling-hand perfect cannot be a good prediction of equilibrium behavior under *any* "conceivable" theory of (improbable, but not impossible) imperfect choice.

Ever since it was coined by Selten [1], trembling-hand perfection has been a popular solution concept. However, the fact that Selten's treatment is valid only for finite games poses a problem, since many strategic settings are most naturally modeled as games with a continuum of actions (e.g., models of price and spatial competition (Bertrand [2], Hotelling [3]), auctions (Milgrom and Weber [4]), and patent races (Fudenberg *et al*. [5])).

There have been attempts to use the notion of trembling-hand perfection in infinite economic games to rule out undesirable equilibria (examples include provision of public goods (Bagnoli and Lipman [6]), credit markets with adverse selection (Broecker [7]), budget-constrained sequential auctions (Pitchik and Schotter [8]), and principal-agent problems (Allen [9])).[1] However, absent a theory of trembling-hand perfection for infinite games (and given that a well-accepted formulation for finite games has long been available), there has been a general tendency to study limits of sequences of trembling-hand perfect equilibria in discretized, successively larger versions of the original (infinite) game at hand (e.g., Bagnoli and Lipman [6], Broecker [7]).[2] While this is a legitimate approach to trembling-hand perfection in infinite games, Simon and Stinchcombe [10] have shown that similar *limit-of-finite* approaches have limitations as general solution concepts even in continuous games. Moreover, since there are alternative formulations of trembling-hand perfection for infinite games, confining attention to a *limit-of-finite* approach, without any comparison with other concepts, seems unsatisfactory.

For continuous games, Simon and Sinchcombe [10] offer several notions of trembling-hand perfection and compare their properties. However, infinite economic games often exhibit discontinuities in their payoffs, and a treatment for this kind of games is not available. For instance, most of the above references feature discontinuous games. In the presence of discontinuities, existence of trembling-hand perfect equilibria is not guaranteed by standard arguments. By adapting arguments from Carbonell-Nicolau [11], this paper addresses the issue of existence for an infinite-game extension of Selten's [1] original notion of trembling-hand perfection. This extension corresponds to Simon and Stinchcombe's [10] strong approach when the universe of games is restricted by continuity of the players' payoffs. Building on the existence results obtained here, a companion paper, Carbonell-Nicolau [12], compares the properties of various notions of trembling-hand perfection within families of discontinuous games, and states the analogue of the standard characterization of trembling-hand perfection for finite games (e.g., van Damme [13], p. 28), in terms of the strong approach and other formulations. This characterization is restated in Section 2.

We first illustrate that the existence of trembling-hand perfect equilibria depends crucially on the existence of Nash equilibria in Selten perturbations. Selten perturbations are perturbed games in which the players choose any strategy in their action space with positive probability. The strategy spaces in Selten perturbations of infinite, discontinuous games exhibit peculiarities that prevent a straightforward application of the results available in the literature on the existence of Nash equilibria. In fact, in Section 2 we show that Selten perturbations need not inherit Reny's [14] *better-reply security* from the original game. Even the available strengthenings of better-reply security—*payoff security* or *uniform payoff security*, along with upper semicontinuity of the sum of payoffs—do not generally give better-reply security (or some of its generalizations) in Selten perturbations. Thus, one must either rely on an *appropriate* generalization of the main existence theorem of Reny [14] or impose a suitable strengthening of better-reply security. We seek conditions on the payoffs of a game that prove useful in applications and imply better-reply security—and hence the existence of Nash equilibria—in Selten perturbations. Ideally, to avoid dealing with expected payoffs (defined on mixed strategies) and the weak

[1] For instance, sometimes the Nash equilibrium concept is too weak to sustain a given result, and the notion of trembling-hand perfection constitutes a natural refinement of the set of Nash equilibria. Beyond its intuitive appeal, trembling-hand perfection is weaker than other refinements, and therefore permits more general theories.

[2] Allen [9] and Pitchik and Schotter [8] finitize their respective games at the outset, rather than approaching an infinite game by a series of successively larger finite games. However, their models are most conveniently analyzed in terms of continua of actions.

convergence of measures, one would like to have conditions that can be verified using the payoffs of the *original* game, rather than its mixed extension.

Carbonell-Nicolau [11] introduces a condition—termed Condition (A)—that is used to prove the existence of a pure-strategy trembling-hand perfect equilibrium. This condition is used here to establish the existence of a mixed-strategy trembling-hand perfect equilibrium. While the current paper adapts arguments from [11], the results obtained here are not implied by those of [11]. We shall provide a detailed comparison with the results in [11] in Section 2.

Roughly speaking, Condition (A) is satisfied when there exists, for each player i, a measurable map $f : X_i \to X_i$, where X_i represents player i's action space, with the following two properties: (1) for each pure strategy x_i of player i, there is an alternative pure strategy $f(x_i)$ such that given any pure action profile y_{-i} of the other players, the action $f(x_i)$ almost guarantees the payoff player i receives at (x_i, y_{-i}), even if the other players slightly deviate from y_{-i}; and (2) given any pure action profile y_{-i} of the other players, there is a subset of generic elements of X_i (which may depend upon y_{-i}) such that given any generic pure strategy x_i of player i, the action profile (x_i, z_{-i}), where z_{-i} is a slight deviation from y_{-i}, almost guarantees the payoff player i receives at $(f(x_i), z_{-i})$.

We show that this condition gives payoff security of certain Selten perturbations (Lemma 2). We then combine this finding with known results to establish the existence of a trembling-hand perfect equilibrium in discontinuous games (Theorem 2). In addition, we derive (as in Carbonell-Nicolau [11]) corollaries of these results in terms of two independent conditions—*generic entire payoff security* and *generic local equi-upper semicontinuity*—that imply the existence of a map f with the above properties. In applications, verifying the two independent conditions can prove easier than checking Condition (A), for Condition (A) typically requires constructing a measurable map and verifying two conditions that depend on one another (via the said measurable map).[3] The alternative hypothesis does not explicitly require the measurability of the map f, and proves easy to verify in applications.

The hypotheses of the main existence theorems are satisfied in many economic games and are often rather simple to verify. This is exemplified in Section 3.

2. Perturbed Games and Perfect Equilibria

A ***metric game*** is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, each X_i is a nonempty metric space, and each $u_i : X \to \mathbb{R}$ is bounded and Borel measurable with domain $X := \times_{i=1}^N X_i$. If in addition each X_i is compact, G is called a ***compact metric game***.

In the sequel, by X_{-i} we mean the set $\times_{j\neq i} X_j$, and, given i, $x_i \in X_i$, and

$$x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in X_{-i}$$

we slightly abuse notation and represent the point $(x_1, ..., x_N)$ as (x_i, x_{-i}).

The ***mixed extension*** of G is the game

$$\overline{G} = (M_i, U_i)_{i=1}^N$$

[3] Constructing a measurable map can sometimes be cumbersome, especially if pure strategies are, say, maps between metric spaces rather than points in Euclidean space.

where each M_i represents the set of Borel probability measures on X_i, endowed with the weak* topology, and $U_i : M \to \mathbb{R}$ is defined by

$$U_i(\mu) := \int_X u_i \mathrm{d}\mu$$

where $M := \times_{i=1}^N M_i$.

Henceforth, the set $\times_{j \neq i} M_j$ is denoted as M_{-i}, and given i, $\mu_i \in M_i$, and

$$\mu_{-i} = (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_N) \in M_{-i}$$

we sometimes represent the point $(\mu_1, ..., \mu_N)$ as (μ_i, μ_{-i}).

Given $x_i \in X_i$, let δ_{x_i} be the Dirac measure on X_i with support $\{x_i\}$. We sometimes write, by a slight abuse of notation, x_i in place of δ_{x_i}. For $\delta \in [0, 1]$ and $(\mu_i, \nu_i) \in M_i^2$

$$(1 - \delta)\nu_i + \delta\mu_i$$

denotes the member σ_i of M_i for which $\sigma_i(B) = (1 - \delta)\nu_i(B) + \delta\mu_i(B)$ for every Borel set B. When $\nu_i = \delta_{x_i}$ for some $x_i \in X_i$, we sometimes write $(1 - \delta)x_i + \delta\mu_i$ for $(1 - \delta)\nu_i + \delta\mu_i$. Similarly, given $(\nu, \mu) \in M^2$

$$(1 - \delta)\nu + \delta\mu$$

denotes the point

$$((1 - \delta)\nu_1 + \delta\mu_1, ..., (1 - \delta)\nu_N + \delta\mu_N)$$

where $\nu = (\nu_1, ..., \nu_N)$ and $\mu = (\mu_1, ..., \mu_N)$.

A number of definitions of trembling-hand perfection for infinite normal-form games have been proposed (*cf.* Simon and Stinchcombe [10], Al-Najjar [15]). For continuous games, the refinement specification considered here is equivalent to the strong approach in [10] and to the formulation in [15]. In this paper, we focus on the issue of existence. In passing, we also illustrate certain limitations of what appears to be a natural approach to the question of existence of trembling-hand perfect equilibria in discontinuous games. This is done more transparently if we frame our discussion in terms of just one notion of trembling-hand perfection. A companion paper, Carbonell-Nicolau [12], compares the various notions of trembling-hand perfection and studies their properties, and contains the analogue of the standard three-way characterization of trembling-hand perfection for finite games (e.g., van Damme [13], p. 28), which will be stated here after several definitions.

Before presenting the formal definition of trembling-hand perfection, we need some terminology.

A Borel probability measure μ_i on X_i is said to be ***strictly positive*** if $\mu_i(O) > 0$ for every nonempty open set O in X_i.

For each i, let $\widehat{M_i}$ stand for the set of all strictly positive members of M_i. Set $\widehat{M} := \times_{i=1}^N \widehat{M_i}$. For $\nu_i \in \widehat{M_i}$ and $\delta = (\delta_1, ..., \delta_N) \in [0, 1)^N$, define

$$M_i(\delta_i\nu_i) := \{\mu_i \in M_i : \mu_i \geq \delta_i\nu_i\}$$

and $M(\delta\nu) := \times_{i=1}^N M_i(\delta_i\nu_i)$. Given $\delta = (\delta_1, ..., \delta_N) \in [0, 1)^N$ and $\nu = (\nu_1, ..., \nu_N) \in \widehat{M}$, the game

$$\overline{G}_{\delta\nu} = \left(M_i(\delta_i\nu_i), U_i|_{M(\delta\nu)}\right)_{i=1}^N$$

is called a ***Selten perturbation*** of G. We often work with perturbations $\overline{G}_{\delta\nu}$ satisfying $\delta_1 = \cdots = \delta_N$. When referring to these objects, we simply write $\overline{G}_{\delta\nu}$ with $\delta = \delta_1 = \cdots = \delta_N$.

Definition 1. A strategy profile $x = (x_1, ..., x_N) \in X$ is a ***Nash equilibrium*** of G if for each i, $u_i(x) \geq u_i(y_i, x_{-i})$ for every $y_i \in X_i$.

Given a game $G = (X_i, u_i)_{i=1}^N$, a Nash equilibrium of the mixed extension $\overline{G}$ is called a ***mixed-strategy Nash equilibrium*** of G. By a slight abuse of terminology, we sometimes refer to a mixed-strategy Nash equilibrium of G simply as a Nash equilibrium of G.

Definition 2. A strategy profile $\mu \in M$ is a ***trembling-hand perfect (thp) equilibrium*** of G if there are sequences (δ^n), (ν^n), and (μ^n) with $(0,1)^N \ni \delta^n \to 0$, $\nu^n \in \widehat{M}$, and $\mu^n \to \mu$, where each μ^n is a Nash equilibrium of the perturbed game $\overline{G}_{\delta^n \nu^n}$.

In words, μ is a *thp* equilibrium of G if it is the limit of some sequence of exact equilibria of neighboring Selten perturbations of G. Intuitively, Selten perturbations of G may be interpreted as "models of mistakes", *i.e.*, formal descriptions of strategic interactions where any player may "tremble" and play any one of her actions. The requirement that μ be the limit of some sequence of equilibria of perturbations of G says that there exists at least one model of (low-probability) mistakes that has at least one equilibrium close to μ, so that μ is an approximate description of what the players would do (at the said equilibrium) were they to interact in the perturbed game.

Remark 1. Note that, in Definition 2, we do not require that μ be a Nash equilibrium of G. It is well-known that, for continuous games, the fact that a strategy profile μ is the limit of some sequence of equilibria of Selten perturbations of G guarantees that μ is a Nash equilibrium of G. While we do not impose continuity of payoff functions, we shall show that our conditions also ensure that the limit point is an equilibrium.[4]

For $\mu \in M$, let $Br_i(\mu)$ denote player i's set of best responses in M_i to the vector of strategies μ:

$$Br_i(\mu) := \left\{ \sigma_i \in M_i : U_i(\sigma_i, \mu_{-i}) \geq \sup_{\varrho_i \in M_i} U_i(\varrho_i, \mu_{-i}) \right\}$$

Consider the following distance function between members of M_i:

$$\rho_i^s(\mu, \nu) := \sup_B |\mu(B) - \nu(B)|$$

Definition 3 (Simon and Stinchcombe [10])**.** Given $\epsilon > 0$, a ***strong*** ϵ***-perfect equilibrium*** of G is a vector $\mu^\epsilon \in \widehat{M}$ such that for each i

$$\rho_i^s\left(\mu_i^\epsilon, Br_i(\mu^\epsilon)\right) < \epsilon$$

A strategy profile in G is a ***strong perfect equilibrium*** of G if it is the weak* limit as $\epsilon^n \to 0$ of strong ϵ^n-perfect equilibria.

[4] In Definition 2, each μ^n is an exact equilibrium of the perturbed game $\overline{G}_{\delta^n \nu^n}$. Should one insist upon requiring that these equilibria be exact? While letting each μ^n be an ϵ^n-equilibrium (with $(\epsilon^n, \delta^n) \to 0$) would still give a (weak) refinement of Nash equilibrium, *any* Nash equilibrium would survive this weakening of Definition 2. In fact, given a Nash equilibrium μ of G, take $\nu \in \widehat{M}$ and a sequence $(0,1) \ni \delta^n \to 0$, and observe that each

$$(1-\delta^n)\mu + \delta^n \nu = ((1-\delta^n)\mu_1 + \delta^n \nu_1, ..., (1-\delta^n)\mu_N + \delta^n \nu_N)$$

is an ϵ^n-equilibrium of $\overline{G}_{\delta^n \nu}$ for some $\epsilon^n \to 0$, and we have $(1-\delta^n)\mu + \delta^n \nu \to \mu$.

The following result is taken from Carbonell-Nicolau [12] and establishes the relationship between trembling-hand perfection and strong perfection in the presence of payoff discontinuities. The equivalence of (1)-(3) is analogous to the standard characterization of trembling-hand perfect equilibria for finite games (e.g., van Damme [13], p. 28).

Theorem 1. *For a metric game, the following three conditions are equivalent.*

(1) μ *is a trembling-hand perfect equilibrium of* G*.*

(2) μ *is a strong perfect equilibrium of* G*.*

(3) μ *is the limit of a sequence* (μ^n) *in* $\widehat{M}$ *with the property that for each* i *and every* $\epsilon > 0$

$$\mu_i^n\left(\left\{x_i \in X_i : U_i(x_i, \mu_{-i}^n) \geq \sup_{y_i \in X_i} U_i(y_i, \mu_{-i}^n)\right\}\right) \geq 1 - \epsilon$$

for any sufficiently large n*.*

The following example illustrates that the set of *thp* equilibria of an infinite game may well be a strict refinement of the set of Nash equilibria.

Example 1. Consider the two-player game $G = ([0,1],[0,1],u_1,u_2)$, where u_1 and u_2 are defined by $u_1(x_1,x_2) := x_1(1-2x_2)$ and $u_2(x_1,x_2) := -x_1x_2^2$.

It is easily seen that the strategy profile $(0,1)$ is a Nash equilibrium of G. Note however that

$$u_2(x_1,0) \geq u_2(x_1,x_2), \quad \text{for all } (x_1,x_2) \in [0,1]^2$$

and that the inequality is strict if $x_1 > 0$. Therefore, player 2's best response to any tremble of player 1 in any Selten perturbation of G cannot be the action 1. Thus, the equilibrium $(0,1)$ is not *thp*.

The *graph* of G is the set

$$\Gamma_G := \left\{(x,\alpha) \in X \times \mathbb{R}^N : u_i(x) = \alpha_i, \text{ for each } i\right\}$$

The graph of the mixed extension $\overline{G}$, $\Gamma_{\overline{G}}$, is defined analogously. The closures of Γ_G and $\Gamma_{\overline{G}}$ are denoted by $\overline{\Gamma}_G$ and $\overline{\Gamma}_{\overline{G}}$ respectively.

Given $\{A,B\} \subseteq \mathbb{R} \ni \varepsilon$, we write

$$A > \varepsilon \quad \text{and} \quad A > B - \varepsilon$$

if $a > \varepsilon$, for all $a \in A$, and $a > b - \varepsilon$, for all $(a,b) \in A \times B$, respectively. The definitions of $A \geq \varepsilon$ and $A \geq B - \varepsilon$ are analogous.

The following definition is taken from Reny [14].

Definition 4. The game G is ***better-reply secure*** if for every $(x,\alpha) \in \overline{\Gamma}_G$ such that x is not a Nash equilibrium of G, there exist i, $y_i \in X_i$, a neighborhood $O_{x_{-i}}$ of x_{-i}, and $\beta \in \mathbb{R}$ such that $u_i(y_i, O_{x_{-i}}) \geq \beta > \alpha_i$.

The following proposition is analogous to Proposition 1 in Carbonell-Nicolau [11].[5] It suggests that the existence of Nash equilibria surviving trembling-hand perfection depends crucially on the existence of Nash equilibria in Selten perturbations of G.

Proposition 1. *Suppose that G is a compact, metric game. If $\overline{G}$ is better-reply secure and there exists $(\alpha, \mu) \in (0, 1) \times \widehat{M}$ such that $\overline{G}_{\delta\mu}$ has a Nash equilibrium for every $\delta \in (0, \alpha]$, then G possesses a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

Proof. Let (α, μ) be as in the statement of the proposition. Then, for large n, each $\overline{G}_{n^{-1}\mu}$ possesses a Nash equilibrium ϱ^n. Because $\varrho^n \in M$ and M is sequentially compact, we may write (passing to a subsequence if necessary) $\varrho^n \to \varrho$ for some $\varrho \in M$. Therefore, ϱ is a *thp* equilibrium of G.

To see that all *thp* equilibria of G are Nash, suppose that ϱ is a *thp* equilibrium of G, and let ϱ^n be the corresponding sequence of equilibria in Selten perturbations, *i.e.*, each ϱ^n is a Nash equilibrium of $\overline{G}_{\delta^n\mu^n}$, where $\delta^n \to 0$, $\mu^n \in \widehat{M}$, and $\varrho^n \to \varrho$. We wish to show that ϱ is a (mixed-strategy) Nash equilibrium of G. To this end, we assume that ϱ is not an equilibrium and derive a contradiction.

Because $\varrho^n \to \varrho$ and each u_i is bounded, we may write (passing to a subsequence if necessary)

$$(\varrho^n, (U_1(\varrho^n), ..., U_N(\varrho^n))) \to (\varrho, (\alpha_1, ..., \alpha_N)) \tag{1}$$

for some $\alpha := (\alpha_1, ..., \alpha_N) \in \mathbb{R}^N$. Consequently, $(\varrho, \alpha) \in \overline{\Gamma}_{\overline{G}}$, so if ϱ is not a Nash equilibrium of G, then, since $\overline{G}$ is better-reply secure, some player i can secure a payoff strictly above α_i at ϱ. That is, for some $\sigma_i \in M_i$, some neighborhood $O_{\varrho_{-i}}$ of ϱ_{-i}, and some $\gamma > 0$

$$U_i(\sigma_i, \sigma_{-i}) \geq \alpha_i + \gamma, \quad \text{for all } \sigma_{-i} \in O_{\varrho_{-i}}$$

We therefore have, in view of (1)

$$U_i(\sigma_i, \varrho^n_{-i}) > U_i(\varrho^n) + \beta$$

for any sufficiently large n and some $\beta > 0$. Consequently, because $\delta^n \to 0$, for large enough n we have

$$U_i\left(\left(1 - \delta^n_i\right)\sigma_i + \delta^n_i \mu^n_i, \varrho^n_{-i}\right) > U_i\left(\varrho^n\right)$$

thereby contradicting that ϱ^n is a Nash equilibrium in $\overline{G}_{\delta^n\mu^n}$. ∎

In light of Proposition 1, it is only natural to ask whether the machinery developed within the literature on the existence of Nash equilibria in discontinuous games can be employed to show that Selten perturbations of G possess Nash equilibria. Reny ([14], Theorem 3.1) proves that a compact, metric, quasiconcave, and better-reply secure game possesses a Nash equilibrium.[6] If G is a compact, metric game, then, for $(\delta, \mu) \in [0, 1) \times \widehat{M}$, $\overline{G}_{\delta\mu}$ is a compact, metric game.[7] In addition, $\overline{G}_{\delta\mu}$ is easily

[5] The reader is referred to the discussion following the statement of Theorem 2 for a comparison between Proposition 1 and Proposition 1 in [11].

[6] A game $G = (X_i, u_i)_{i=1}^N$ is *quasiconcave* if each X_i is a convex subset of a vector space and for each i and every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasiconcave on X_i.

[7] If X_i is compact and metric, the weak* topology on M_i coincides with the topology induced by the Prokhorov metric on M_i. Hence, if X_i is nonempty, compact, and metric, then $M_i(\delta_i\mu_i)$ is nonempty and metric. In addition, if X_i is nonempty, compact, and metric, $M_i(\delta\mu_i)$ is a nonempty convex subset of the weakly* compact set M_i. It is easy to check that $M_i(\delta\mu_i)$ is strongly closed, and therefore (Dunford and Schwartz ([16], Theorem V.3.13, p. 422)) weakly* closed, so $M_i(\delta\mu_i)$ is weakly* compact.

seen to be quasiconcave. Consequently, a Selten perturbation $\overline{G}_{\delta\mu}$ possesses a Nash equilibrium if it is better-reply secure. This observation, together with Proposition 1, gives the following lemma.

Lemma 1. *If G is a compact, metric game and there exists $(\alpha, \mu) \in (0,1) \times \widehat{M}$ such that $\overline{G}_{\delta\mu}$ is better-reply secure for every $\delta \in [0, \alpha]$, then G possesses a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

In general, verifying the existence of $(\alpha, \mu) \in (0,1) \times \widehat{M}$ such that $\overline{G}_{\delta\mu}$ is better-reply secure for every $\delta \in [0, \alpha]$ is cumbersome, for it entails dealing with expected payoffs, defined on mixed strategies, and the weak* convergence of measures. Consequently, rather than imposing better-reply security directly on $\overline{G}_{\delta\mu}$, one would like to have conditions on the payoffs of the *original* game G that (1) prove useful in applications and (2) imply better-reply security in perturbations of G.

Unfortunately, $\overline{G}_{\delta\mu}$ need not inherit better-reply security from G, and even standard strengthenings of better-reply security—*payoff security* or *uniform payoff security* (to be defined below), along with upper semicontinuity of $\sum_{i=1}^{N} u_i$—do not generally give the desired property in $\overline{G}_{\delta\mu}$.

The following definition is taken from Reny [14].

Definition 5. The game G is ***payoff secure*** if for each $\varepsilon > 0$, $x \in X$, and i, there exists $y_i \in X_i$ such that $u_i(y_i, O_{x_{-i}}) > u_i(x) - \varepsilon$ for some neighborhood $O_{x_{-i}}$ of x_{-i}.

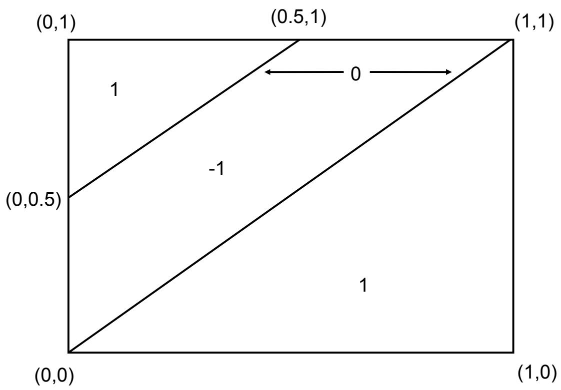
It is well-known (Reny [14], Proposition 3.2) that payoff security of G and upper semicontinuity of $\sum_{i=1}^{N} u_i$ ensure better-reply security of G. However, payoff security of G and upper semicontinuity of $\sum_{i=1}^{N} u_i$ need not give better-reply security of the mixed extension $\overline{G}$. The following example illustrates this point.

Example 2 (Sion and Wolfe [17])**.** Consider the game $G = ([0,1], [0,1], u_1, u_2)$, where

$$u_1(x_1, x_2) := \begin{cases} -1 & \text{if } x_1 < x_2 < x_1 + \frac{1}{2}, \\ 0 & \text{if } x_1 = x_2 \text{ or } x_2 = x_1 + \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

and $u_2 := -u_1$ (Figure 1).

Figure 1. Example 2: The payoff functions of *G*.



This game is zero-sum (and so $\sum_{i=1}^{N} u_i$ is constant) and payoff secure (Carmona [18], Proposition 4). Moreover, as shown by Sion and Wolfe [17], G has no mixed-strategy Nash equilibria. Hence, by Corollary 5.2 of Reny [14], $\overline{G}$ fails better-reply security.

Now consider the following strengthening of payoff security (*cf.* Monteiro and Page [19]).

Definition 6. Given $Y_i \subseteq X_i$ for each i, the game G is ***uniformly payoff secure over*** $\times_{i=1}^{N} Y_i$ if for each i, $\varepsilon > 0$, and $x_i \in Y_i$, there exists $y_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(y_i, O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.

The game G is ***uniformly payoff secure*** if it is uniformly payoff secure over X.

Uniform payoff security of G yields payoff security of the mixed extension $\overline{G}$ ([19], Theorem 1). By standard arguments, this means that uniform payoff security of G, together with upper semicontinuity of $\sum_{i=1}^{N} u_i$, implies better-reply security of $\overline{G}$. Nonetheless, these two conditions need not lead to better-reply security of $\overline{G}_{\delta\mu}$, as illustrated by the following example.[8]

Example 3. Let (α^n) be a sequence in $(\frac{1}{2}, 1)$ with $\alpha^n \nearrow 1$. Let (f^n) be a sequence of functions $f^n : [0,1] \to \mathbb{R}$ with the following properties:

$$f^n(x) = \begin{cases} 1 & \text{if } x \in [1-\alpha^n, \alpha^n] \cup \{0,1\}, \\ 0 & \text{elsewhere.} \end{cases}$$

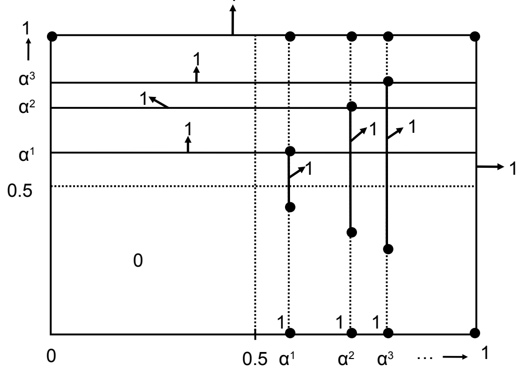
for all n.

Consider the two-player game $G = ([0,1], [0,1], u_1, u_2)$, where

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } x_2 = \alpha^n, n = 1, 2, ..., \\ f^n(x_2) & \text{if } x_1 = \alpha^n, n = 1, 2, ..., \\ 1 & \text{if } x_1 = 1 \text{ or } x_2 = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

and $u_2(x_1, x_2) := u_1(x_2, x_1)$ (Figure 2).

Figure 2. Example 3: The payoff function for player 1.



[8]Even the generalized notion of better-reply security of Barelli and Soza [20] or the conditions for existence of Baye *et al.* [21] need not hold for the perturbation $\overline{G}_{\delta\mu}$ when G is uniformly payoff secure and $\sum_{i=1}^{N} u_i$ is upper semicontinuous.

It is easy to verify that $\sum_{i=1}^{N} u_i$ is upper semicontinuous and G is uniformly payoff secure. However, $\overline{G}_{\delta\mu}$ fails payoff security whenever $\mu \in \widehat{M}$ and $\delta \in (0,1)$. To see this, fix $\mu = (\mu_1, \mu_2) \in \widehat{M}$ and $\delta \in (0,1)$. We need to show that there exist $\varepsilon > 0$, i, and $\nu \in M(\delta\mu)$ such that for all $\sigma_i \in M_i(\delta\mu_i)$ there is a point $\sigma_{-i} \in \times_{j\neq i} M_j(\delta\mu_j)$ arbitrarily close to ν_{-i} for which $U_i(\sigma_i, \sigma_{-i}) \leq U_i(\nu) - \varepsilon$. Thus, it suffices to establish the following for $\varepsilon > 0$ sufficiently small: there is an n such that for each neighborhood $O_{(1-\delta)\alpha^n+\delta\mu_2}$ of $(1-\delta)\alpha^n + \delta\mu_2$ and every $y_1 \in [0,1]$

$$U_1((1-\delta)y_1 + \delta\mu_1, \nu_2) \leq U_1((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2) - \varepsilon \tag{2}$$

for some $\nu_2 \in O_{(1-\delta)\alpha^n+\delta\mu_2} \cap M_2(\delta\mu_2)$.

Choose $\varepsilon > 0$ with the property that for any large enough n

$$\delta(1-\delta)\left(\mu_1(\{1\}) + \sum_{m=n+1}^{\infty} \mu_1(\{\alpha^m\})\right) \leq \delta(1-\delta)\left(\mu_2(\{0,1\}) + \mu_2([1-\alpha^n, \alpha^n])\right) - \varepsilon \tag{3}$$

Take any neighborhood $O_{(1-\delta)\alpha^n+\delta\mu_2}$ of $(1-\delta)\alpha^n + \delta\mu_2$ and any $y_1 \in [0,1]$. Clearly, we may pick some $y_2 \in (\alpha^n, \alpha^{n+1})$ sufficiently close to α^n to ensure that $(1-\delta)y_2 + \delta\mu_2 \in O_{(1-\delta)\alpha^n+\delta\mu_2}$. By linearity of U_1 we have

$$\begin{aligned} &U_1((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2) \\ &\qquad = (1-\delta)^2 U_1(y_1, y_2) + (1-\delta)\delta U_1(y_1, \mu_2) + \delta(1-\delta)U_1(\mu_1, y_2) + \delta^2 U_1(\mu) \end{aligned}$$

Therefore, because $U_1(y_1, y_2) \leq 1 \geq U_1(y_1, \mu_2)$ and $U_1(\mu_1, y_2) \leq \mu_1(\{1\}) + \sum_{m=n+1}^{\infty} \mu_1(\{\alpha^m\})$

$$\begin{aligned} U_1((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2) &\leq (1-\delta)^2 \\ &\quad + (1-\delta)\delta + \delta(1-\delta)\Big(\mu_1(\{1\}) + \sum_{m=n+1}^{\infty} \mu_1(\{\alpha^m\})\Big) + \delta^2 U_1(\mu) \end{aligned} \tag{4}$$

On the other hand, we have

$$\begin{aligned} &U_1((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2) - \varepsilon \\ &\qquad = (1-\delta)^2 U_1(\alpha^n, \alpha^n) + (1-\delta)\delta U_1(\alpha^n, \mu_2) + \delta(1-\delta)U_1(\mu_1, \alpha^n) + \delta^2 U_1(\mu) - \varepsilon \\ &\qquad = (1-\delta)^2 + (1-\delta)\delta\left(\mu_2(\{0,1\}) + \mu_2([1-\alpha^n, \alpha^n])\right) + \delta(1-\delta) + \delta^2 U_1(\mu) - \varepsilon \end{aligned}$$

and since the right-hand side of this equation is, in light of (3), greater than or equal to the right-hand side of (4), the desired inequality (2), follows. We conclude that $\overline{G}_{\delta\mu}$ is not payoff secure.

The perturbation $\overline{G}_{\delta\mu}$ also fails better-reply security. To see this, choose $\mu = (\mu_1, \mu_2) \in \widehat{M}$ and $\delta \in (0,1)$, and observe that

$$(((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2), (\gamma_1^n, \gamma_2^n))$$

where

$$\gamma_1^n = U_1\left((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\right)$$

and

$$\gamma_2^n = U_2\left((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\right)$$

belongs to $\overline{\Gamma}_{\overline{G}_{\delta\mu}}$. Moreover, the strategy profile

$$((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2)$$

is not a Nash equilibrium in $\overline{G}_{\delta\mu}$, for

$$U_1\left((1-\delta)1 + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\right) > U_1\left((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\right)$$

Reasoning as in the previous paragraph one can show that for large enough n there is no $\nu_i \in M_i(\delta\mu_i)$ for which $U_i(\nu_i, O_{(1-\delta)\alpha^n+\delta\mu_{-i}}) > \gamma_i^n$ for some neighborhood $O_{(1-\delta)\alpha^n+\delta\mu_{-i}}$ of $(1-\delta)\alpha^n + \delta\mu_{-i}$. It follows that $\overline{G}_{\delta\mu}$ is not better-reply secure.[9]

In light of Example 3, any condition on the payoff functions of G leading to the hypothesis of Lemma 1 (when combined with upper semicontinuity of $\sum_{i=1}^N u_i$) must be stronger than uniform payoff security.[10]

The following condition appears in Carbonell-Nicolau [11].

Condition (A). There exists $(\mu_1, ..., \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that

$$u_i\left(f(x_i), O_{y_{-i}}\right) > u_i(x_i, y_{-i}) - \varepsilon$$

(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.[11]

It is clear that (A) strengthens the concept of uniform payoff security.

Remark 2. The following implications are immediate:

$$\begin{aligned} \text{continuity} &\Rightarrow \text{(A)} \\ &\Rightarrow \text{uniform payoff security} \\ &\Rightarrow \text{payoff security.} \end{aligned}$$

[9] While G is quasi-symmetric in the sense of Reny [14], and so an appropriate choice of μ renders $\overline{G}_{\delta\mu}$ quasi-symmetric, $\overline{G}_{\delta\mu}$ also violates diagonal better-reply security (as defined in [14]).

[10] This means that the machinery developed in the literature on the existence of Nash equilibria cannot be employed to establish the existence of a Nash equilibrium in $\overline{G}_{\delta\mu}$ under the assumption that G is uniformly payoff secure and $\sum_{i=1}^N u_i$ is upper semicontinuous. We ignore if uniform payoff security of G and upper semicontinuity of $\sum_{i=1}^N u_i$ implies the existence of a *thp* equilibrium in G. If this were true, its proof would require an *appropriate* generalization of the main theorem of Reny [14].

[11] The following generalization of Condition (A) leaves all of our results intact.

Condition (A'). There exists $(\mu_1, ..., \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:

(a) For each k, $x_i \in X_i$, and $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f_k(x_i), O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.

(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for each $x_i \in Y_i$ and every sufficiently large k, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_k(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

We can establish payoff security of a Selten perturbation of G from Condition (A).

Lemma 2. *Suppose that a compact, metric game G satisfies Condition* (A). *Then there exists $\mu \in \widehat{M}$ such that $\overline{G}_{\delta\mu}$ is payoff secure for every $\delta \in [0,1)$.*

This result plays a central role in the proof of the main results of this paper.[12] The proof of Lemma 2 can be found in Section 4.

Lemma 2 can be combined with known results to prove an existence theorem. In fact, under the hypothesis of Lemma 2, we obtain payoff security of $\overline{G}_{\delta\mu}$ for any $\delta \in [0,1)$. If in addition $\sum_{i=1}^{N} u_i$ is upper semicontinuous, since upper semicontinuity of $\sum_{i=1}^{N} u_i$ gives upper semicontinuity of $\sum_i U_i$ (Reny [14], Proposition 5.1), it follows that $\overline{G}_{\delta\mu}$ is better-reply secure for any $\delta \in [0,1)$ (Reny [14], Proposition 3.2). Applying Lemma 1 gives a *thp* equilibrium in G.

The following statement summarizes this finding.

Theorem 2. *Suppose that a compact, metric game G satisfies Condition* (A). *Suppose further that $\sum_{i=1}^{N} u_i$ is upper semicontinuous. Then G has a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

Lemma 2 is similar to Lemma 1 in Carbonell-Nicolau [11]. Lemma 1 in [11] states that if a compact, metric game satisfies Condition (A), then there exists $\mu \in \widehat{M}$ such that the game $G_{(\delta,\mu)}$ is payoff secure for every $\delta \in [0,1)$, where $G_{(\delta,\mu)}$ is defined as

$$G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^{N}$$

and $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta)x_1 + \delta\mu_1, ..., (1-\delta)x_N + \delta\mu_N\right)$$

The statement of this lemma differs from that of Lemma 2 in that $G_{(\delta,\mu)}$ and $\overline{G}_{\delta\mu}$ are distinct objects. In fact, the latter can be shown to be homeomorphic to the mixed extension of the former. Consequently, since payoff security of a game does not generally imply payoff security of its mixed extension, Lemma 2 is not implied by Lemma 1 in [11]. On the other hand, it should be noted that Theorem 2 is not implied by Theorem 3 or Theorem 4 in [11]. In fact, both the hypothesis and the conclusion are weaker for Theorem 2.[13]

The remainder of this section derives a corollary of Theorem 2 in terms of two independent conditions introduced in [11]—*generic entire payoff security* and *generic local equi-upper semicontinuity*—that imply Condition (A). While stronger, these conditions prove useful in applications: they apply in a variety of economic games and do not explicitly require the measurability of the map f in Condition (A). Both Theorem 2 and its corollary (Corollary 1, in terms of generic entire payoff security and generic local equi-upper semicontinuity) are illustrated in Section 3.[14]

[12] Lemma 2 is similar to Lemma 1 in Carbonell-Nicolau [11]. We provide a comparison between these two results after the statement of Theorem 2.

[13] The hypothesis is weaker because it does not assume concavity or quasiconcavity-like conditions, while the conclusion is weaker because trembling-hand perfect equilibria may be in mixed strategies.

[14] The relationship between Corollary 1 and Corollaries 1 and 3 in [11] is similar to that between Theorem 2 and Theorems 3 and 4 in [11]. In particular, Corollary 1 is not implied by the results in [11].

Let A_i be the set of all accumulation points of X_i (*i.e.*, the set A_i of points $x_i \in X_i$ such that $(V \setminus \{x_i\}) \cap A_i \neq \emptyset$ for each neighborhood V of x_i). Since X_i is compact and metric, it can be written as a disjoint union $A_i \cup K_i$, where A_i is closed and dense in itself (*i.e.*, with no isolated points) and K_i is countable.

Let $\widetilde{M}_i$ be the set of measures μ_i in M_i such that $\mu_i(\{x_i\}) = 0$ and $\mu_i(N_\epsilon(x_i)) > 0$ for each $x_i \in A_i$ and every $\epsilon > 0$, and $\mu_i(\{x_i\}) > 0$ for every $x_i \in K_i$. Define $\widetilde{M} := \times_{i=1}^N \widetilde{M}_i$.

Clearly, $\widetilde{M}_i$ is a subset of $\widehat{M}_i$. Moreover, $\widetilde{M}_i$ is nonempty. In fact, it is not difficult to show that $\widetilde{M}_i$ is dense in M_i for each i.

Definition 7. Given $Y_i \subseteq X_i$ for each i, we say that G is ***entirely payoff secure over*** $\times_{i=1}^N Y_i$ if for each i, $\varepsilon > 0$, and $x_i \in Y_i$, and for every neighborhood O of x_i, there exist $y_i \in O$ and a neighborhood O_{x_i} of x_i such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i(y_i, O_{y_{-i}}) > u_i(O_{x_i}, y_{-i}) - \varepsilon$.

We say that G is ***entirely payoff secure*** if it is entirely payoff secure over X.

Definition 8. Given $Y_i \subseteq X_i$ for each i, we say that the game G is ***generically entirely payoff secure over*** $\times_{i=1}^N Y_i$ if there is, for each i, a set $Z_i \subseteq Y_i$ with $Y_i \setminus Z_i$ countable such that G is uniformly payoff secure over $\times_{i=1}^N (Y_i \setminus Z_i)$ and entirely payoff secure over $\times_{i=1}^N Z_i$.

A game G is ***generically entirely payoff secure*** if it is entirely payoff secure over $\times_{i=1}^N K_i$ and generically entirely payoff secure over $\times_{i=1}^N A_i$ (recall that $X_i = A_i \cup K_i$, where A_i is closed and dense in itself and K_i is countable).

Remark 3. The following implications are immediate:

$$\begin{aligned} \text{continuity} &\Rightarrow \text{entire payoff security} \\ &\Rightarrow \text{generic entire payoff security} \\ &\Rightarrow \text{uniform payoff security} \\ &\Rightarrow \text{payoff security.} \end{aligned}$$

Definition 9. The game G is ***locally equi-upper semicontinuous*** if for each i, $x_{-i} \in X_{-i}$, and $x_i \in X_i$, and for each $\varepsilon > 0$, there exists a neighborhood O_{x_i} of x_i such that for every $y_i \in O_{x_i}$ there exists a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

Definition 10. The game G is ***generically locally equi-upper semicontinuous*** if there exists $(\mu_1, ..., \mu_N) \in \widetilde{M}$ such that for each i and $x_{-i} \in X_{-i}$, there exists $Y_i \subseteq X_i$ with $\mu_i(Y_i) = 1$ such that for each $x_i \in Y_i$ and $\varepsilon > 0$, there exists a neighborhood O_{x_i} of x_i such that for every $y_i \in O_{x_i}$ there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

It turns out that generic entire payoff security and generic local equi-upper semicontinuity imply Condition (A).

Lemma 3 (Carbonell-Nicolau [11], Lemma 4)**.** *Suppose that G is generically entirely payoff secure and generically locally equi-upper semicontinuous. Then G satisfies Condition* (A).

Lemma 3, combined with Theorem 2, gives the following result.

Corollary 1 (to Theorem 2)**.** *Suppose that G is compact, metric, generically entirely payoff secure, and generically locally equi-upper semicontinuous. Suppose further that $\sum_{i=1}^{N} u_i$ is upper semicontinuous. Then G has a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

3. Applications

The hypotheses of our main results are often satisfied in applications. This is illustrated by the following economic games.

Example 4 (*Bertrand competition with discontinuous demand*)**.** Consider a two-player Bertrand game $G = ([0,4],[0,4],u_1,u_2)$, where

$$u_i(p_i, p_{-i}) := \begin{cases} \pi(p_i) & \text{if } p_i < p_{-i}, \\ \frac{1}{2}\pi(p_i) & \text{if } p_i = p_{-i}, \\ 0 & \text{if } p_i > p_{-i}. \end{cases}$$

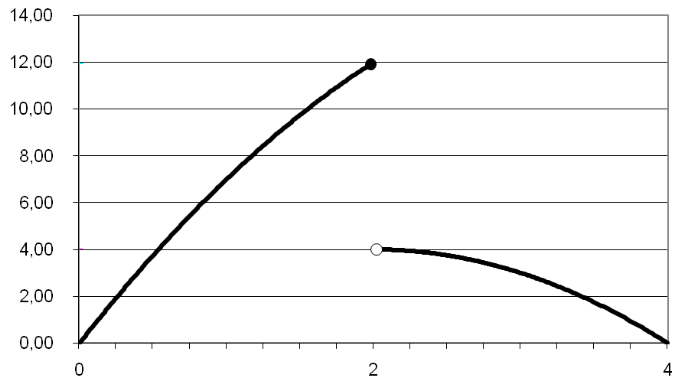
and

$$\pi(p) := \begin{cases} p(8-p) & \text{if } 0 \le p \le 2, \\ p(4-p) & \text{if } 2 < p \le 4. \end{cases}$$

The map $\pi(p)$ represents the operating profits that a monopolist charging a price p would earn (Figure 3). The two (identical) firms produce at zero costs, and the associated demand function is

$$D(p) = \begin{cases} 8-p & \text{if } 0 \le p \le 2, \\ 4-p & \text{if } 2 < p \le 4. \end{cases}$$

Figure 3. Example 4: Operating profit as a function of price.



Similar duopoly games can be found in Baye and Morgan [22]. See also [22] for a discussion on economic phenomena that explain demand discontinuities.

It is readily seen that $\sum_{i=1}^{N} u_i$ is upper semicontinuous. Moreover, G is entirely payoff secure. To see this, fix i, $\varepsilon > 0$, $p_i \in [0,4]$, and a neighborhood O of p_i. We wish to show that there exist $a_i \in O$ and a neighborhood O_{p_i} around p_i such that for every $p_{-i} \in [0,4]$, there is a neighborhood $O_{p_{-i}}$ of p_{-i} for which

$$u_i(a_i, O_{p_{-i}}) > u_i(O_{p_i}, p_{-i}) - \varepsilon \tag{5}$$

This is clearly true if $p_i = 0$, for $u_i \geq 0$. Assume $p_i > 0$, and choose $a_i \in O$ with $0 < a_i < p_i$ sufficiently close to p_i to ensure that $\pi(a_i) > \pi(O_{p_i}) - \varepsilon$ for some neighborhood O_{p_i} of p_i satisfying $\{a_i\} \cap O_{p_i} = \emptyset$. Now fix $p_{-i} \in [0, 4]$, and pick a neighborhood $O_{p_{-i}}$ of p_{-i} satisfying the following:

- If $p_{-i} > a_i$, then $O_{p_{-i}} \cap \{a_i\} = \emptyset$.
- If $p_{-i} \leq a_i$, then $O_{p_{-i}} \cap O_{p_i} = \emptyset$.

It is straightforward to verify that (5) holds.

Finally, G is generically locally equi-upper semicontinuous. In fact, take i, $x_{-i} \in [0, 1]$, $x_i \in [0, 1] \setminus \{2, x_{-i}\}$, and $\varepsilon > 0$. We only consider the case when $x_i < x_{-i}$ and $x_i < 2$, for the other cases can be dealt with similarly. If $x_i < x_{-i}$ and $x_i < 2$, we have $u_i(y_i, y_{-i}) = y_i(8 - y_i)$ for all $(y_i, y_{-i}) \in V_{x_i} \times V_{x_{-i}}$ and for some neighborhoods V_{x_i} and $V_{x_{-i}}$ of x_i and x_{-i} respectively, so it is clear that there exists a neighborhood O_{x_i} of x_i such that for every $y_i \in O_{x_i}$ there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

Because G is generically locally equi-upper semicontinuous and entirely payoff secure, Corollary 1 can be invoked to establish the existence of a *thp* equilibrium.

Example 5 (*all-pay auction*)**.** There are N bidders competing for an object with a known value equal to 1. The highest bidder wins and every bidder pays his bid. Ties are broken via an equal probability rule. Given a profile of bids $(b_1, ..., b_N) \in [0, 1]^N$, the winning bid is $\max_{i \in \{1,...,N\}} b_i$.

This situation can be modeled as an N-person normal-form game $G = (X_i, u_i)_{i=1}^N$, where $X_i = [0, 1]$ and

$$u_i(b_1, ..., b_N) := \begin{cases} \frac{1}{\#W(b_1,...,b_N)} - b_i & \text{if } b_i = \max_{j \in \{1,...,N\}} b_j, \\ -b_i & \text{if } b_i < \max_{j \in \{1,...,N\}} b_j. \end{cases}$$

where $W(b_1, ..., b_N) := \left\{i : b_i = \max_{j \in \{1,...,N\}} b_j\right\}$.

This game is generically locally equi-upper semicontinuous. To see this, fix i and $b_{-i} \in X_{-i}$, and choose any $b_i \in [0, 1] \setminus \left\{\overline{b}_{-i}\right\}$ and any $\varepsilon > 0$, where $\overline{b}_{-i} := \max_{j \in \{1,...,N\} \setminus \{i\}} b_j$. We only consider the case when $b_i < \overline{b}_{-i}$, for the case when $b_i > \overline{b}_{-i}$ can be handled analogously. Take a neighborhood $(b_i - \delta, b_i + \delta)$ of b_i such that $(b_i - \delta, b_i + \delta) \cap \left\{\overline{b}_{-i}\right\} = \emptyset$ and $\delta < \varepsilon$. For each $a_i \in (b_i - \delta, b_i + \delta) \cap [0, 1]$ and for every $a_{-i} \in X_{-i}$ in a neighborhood $O_{b_{-i}}$ of b_{-i} such that

$$c_i < \max_{j \in \{1,...,N\} \setminus \{i\}} c_j, \quad \text{for all } (c_i, c_{-i}) \in (b_i - \delta, b_i + \delta) \times O_{b_{-i}}$$

we have

$$\begin{aligned} u_i(a_i, a_{-i}) &= -a_i \\ &< -b_i + \delta \\ &< -b_i + \varepsilon \\ &= u_i(b_i, a_{-i}) + \varepsilon \end{aligned}$$

We now show that G is generically entirely payoff secure.[15] Fix a player i, and choose $\varepsilon > 0$, $b_i \in (0, 1)$, and a neighborhood O of b_i (we omit the case when $b_i \in \{0, 1\}$, which is easy to handle).

[15] This game fails entire payoff security.

We wish to show that there exist $a_i \in O$ and a neighborhood O_{b_i} such that for all $b_{-i} \in [0,1]^{N-1}$, there is a neighborhood $O_{b_{-i}}$ of b_{-i} for which

$$u_i(a_i, O_{b_{-i}}) > u_i(O_{b_i}, b_{-i}) - \varepsilon \tag{6}$$

Choose $a_i \in O \cap (b_i, b_i + \varepsilon)$, and fix a neighborhood O_{b_i} of b_i such that $O_{b_i} \subseteq [0,1]$, $a_i \in [0,1] \cap (b_i, b_i + \varepsilon)$, and $\{a_i\} \cap O_{b_i} = \emptyset$. Pick any $b_{-i} \in [0,1]^{N-1}$, and let $O_{b_{-i}}$ be a neighborhood of b_{-i} with the following property: if $\max_{j \in \{1,\ldots,N\}} b_j \leq b_i$, then $O_{b_{-i}} \cap \{a_i\}^{N-1} = \emptyset$. It is easy to verify that the choices of a_i, O_{b_i}, and $O_{b_{-i}}$ yield Equation (6).

Finally, it is routine to verify that the sum of the bidders' payoffs is continuous. Hence, Corollary 1 gives a *thp* equilibrium.

Example 6 (*catalog games*)**.** Page and Monteiro [23] consider a common agency contracting game in which firms compete for the business of an agent of unknown type $t \in T$, where T is a Borel subset of a separable, complete, and metric space. The distribution of types is represented by a Borel probability measure μ defined on T. There are two firms competing simultaneously in prices and products. The set of products each firm can offer is represented by a compact metric space X, and it is assumed that X contains an element 0, which denotes "no contracting". The universe of prices that a firm can charge is denoted by $D := \left[0, \overline{d}\right]$, with $\overline{d} > 0$. The agent can only contract with one firm and can choose to abstain from contracting altogether. Given $i \in \{1,2\}$ and a closed subset X_i of X, let $K_i := X_i \times D$ be the feasible set of products and prices that a firm i can offer. Assume the existence of a fictitious firm $i = 0$ with feasible set $K_0 := \{(0,0)\}$. The agent chooses to abstain from contracting by choosing to contract with firm $i = 0$.

Each firm i competes by offering the agent a nonempty, closed subset $C_i \subseteq K_i$, a *catalog*, of products and prices. Thus, each firm i's action space is $\mathcal{P}(K_i)$, the compact, metric space of catalogs, equipped with the Hausdorff distance. The utility of a type t agent who chooses $(i,x,p) \in \{0,1,2\} \times C_i$ is denoted as $v_t(i,x,p)$; we have $v_t(i,x,p) := 0$ if $i = 0$ and $v_t(i,x,p) := u_t(i,x) - p$ if $i \in \{1,2\}$. It is assumed that utility is measurable in type t and continuous in contract choice (i,x,p). The agent's choice set given catalog profile (C_1, C_2) is given by

$$\Gamma(C_1, C_2) := \{(i,x,p) : i \in \{0,1,2\}, (x,p) \in C_i\}$$

A type t agent chooses $(i,x,p) \in \Gamma(C_1,C_2)$ to maximize her utility:

$$\max_{(i,x,p) \in \Gamma(C_1,C_2)} v_t(i,x,p)$$

Define

$$v^*(t, C_1, C_2) := \max_{(i,x,p) \in \Gamma(C_1,C_2)} v_t(i,x,p)$$

and

$$\Phi(t, C_1, C_2) := \arg\max_{(i,x,p) \in \Gamma(C_1,C_2)} v_t(i,x,p)^{16}$$

[16] It is shown in [23] that v^* is measurable in types and continuous in catalog profiles, while the correspondence Φ is jointly measurable in types and catalog profiles and upper hemicontinuous in catalog profiles.

The map $v^*(t, \cdot)$ represents a type t agent's indirect utility function over profiles of catalogs, while $\Phi(t, \cdot)$ gives the type t agent's best responses to each catalog profile. The j-th firm's profit function is given by

$$\pi_j(i, x, p) = \begin{cases} p - c_j(x) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

where the cost function $c_j(\cdot)$ is bounded and lower semicontinuous. Let

$$\pi_j^*(t, C_1, C_2) := \max_{(i,x,p) \in \Phi(t, C_1, C_2)} \pi_j(i, x, p)$$

Firm j's expected payoff under catalog profile (C_1, C_2) is

$$\Pi_j(C_1, C_2) := \int_T \pi_j^*(\cdot, C_1, C_2) \mathrm{d}\mu$$

The game $G = (\mathcal{P}(K_i), \Pi_i)$ is an upper semicontinuous, compact game. Moreover, an argument similar to that provided in the proof of Theorem 5 of [23] to establish uniform payoff security of G can be utilized to prove that G satisfies Condition (A). Consequently, by Theorem 2, the game possesses a *thp* equilibrium.

Example 7 (*provision of public goods*)**.** Bagnoli and Lipman [6] study the following contribution game. There are I finitely many agents. By a slight abuse of notation, the set of agents is denoted by I. Each agent $i \in I$ is endowed with an amount of wealth $w_i > 0$. A collective decision $d \in \{0, 1\}$ must be made (say, $d = 1$ designates the decision to provide streetlight, $d = 0$ represents the decision not to provide it). An outcome is a social decision together with an allocation of the private good (wealth) among the agents. The set of feasible outcomes is

$$\left\{(d, x) \in \{0, 1\} \times \mathbb{R}_+^I : \textstyle\sum_{i \in I} x_i \leq \sum_{i \in I} w_i - c(d)\right\}$$

The utility of agent i if outcome (d, x) is implemented is denoted by $v_i(d, x_i)$; here, each v_i is assumed strictly increasing in d and continuous and strictly increasing in x_i. The cost of adopting decision d is $c(d)$, where $c(0) = 0$ and $c(1) = c > 0$.

The agents simultaneously choose a contribution to the public project, each agent i's contribution being an element of $S_i := [0, w_i]$. Let w denote the vector of endowments. Given a profile $s = (s_i)_{i \in I}$ of contributions, the public project is undertaken if $\sum_{i \in I} s_i \geq c$, in which case the realized outcome is $(1, w - s)$; otherwise (*i.e.*, if $\sum_{i \in I} s_i < c$) the outcome $(0, w)$ obtains.

Let $S := \times_{i \in I} S_i$. The associated normal-form game is $G = (S_i, u_i)_{i \in I}$, where $u_i : S \to \mathbb{R}$ is defined by

$$u_i(s) := \begin{cases} v_i(1, w_i - s_i) & \text{if } \sum_i s_i \geq c, \\ v_i(0, w_i) & \text{if } \sum_i s_i < c. \end{cases}$$

Bagnoli and Lipman[6] uses an equilibrium concept, termed *undominated perfect equilibrium*, that eliminates the set of weakly dominated strategies in the original game and applies the notion of trembling-hand perfection to the resulting game. To avoid defining trembling-hand perfection in infinite games and dealing with the issue of existence, Bagnoli and Lipman work with approximating finite versions of G.

Specifically, assuming $v_i(0, w_i) = 0$ for each i (a normalization that does not affect generality) and $v_i(1, 0) < 0$ for each i (so that we do not need to consider cases when some agents would like to contribute more than their wealth), we can define a_i implicitly by $v_i(1, w_i - a_i) = 0$. Assume $\sum_{i \in I} w_i > c$. Clearly, the elimination of the interior of the set of weakly dominated strategies in G removes all $s_i \in S_i$ such that $s_i > a_i$. Consider the "subgame" g of G in which i's strategy space is restricted to $[0, a_i]$ and g is otherwise identical to G.

Bagnoli and Lipman replace each S_i by finite counterparts of varying grid sizes, and consider sequences of finite games in which the grid size converges to zero. They define an undominated perfect equilibrium in G as the limit of some sequence of undominated perfect equilibria of approximating finite versions of G.

The authors' main result is that the game form G fully implements the core of the associated economy in undominated perfect equilibrium (*i.e.*, any undominated perfect equilibrium of G induces a core allocation and vice versa). In view of our results, one may ask the following: Can one apply the characterization exercise conducted in [6] directly on the infinite game g? Can one obtain a similar theorem on the full implementation of the core in terms of trembling-hand perfection? While answering these questions requires a thorough analysis, Theorem 2 can be used to establish the existence of a *thp* equilibrium in g.

It is easily seen that the restriction of u_i to $[0, a_i]$ is upper semicontinuous, so the sum of payoffs for g is upper semicontinuous.

We now show that g is entirely payoff secure. Take i, $\varepsilon > 0$, $s_i \in [0, a_i]$, and a neighborhood O of s_i. We need to show that there exist $b_i \in O$ and a neighborhood O_{s_i} of s_i such that for every $s_{-i} \in \times_{j \neq i}[0, a_j]$, there is a neighborhood $O_{s_{-i}}$ for which

$$u_i(b_i, O_{s_{-i}}) > u_i(O_{s_i}, s_{-i}) - \varepsilon \tag{7}$$

The cases when $s_i \in \{0, a_i\}$ are easy to handle, so suppose that $s_i \in (0, a_i)$. Take $b_i \in O$ with $a_i > b_i > s_i$ close enough to s_i to ensure that

$$v_i(1, w_i - b_i) > v_i(1, w_i - O_{s_i}) - \varepsilon$$

for some sufficiently small neighborhood O_{s_i}. Given $s_{-i} \in \times_{j \neq i}[0, a_j]$, fix a neighborhood $O_{s_{-i}}$ with the following property: if $\sum_j s_j \geq c$, then $b_i + \sum_{j \neq i} \widetilde{s}_j \geq c$ for all $\widetilde{s}_{-i} \in O_{s_{-i}}$. Now, verifying that the choices of b_i, O_{s_i}, and $O_{s_{-i}}$ give (7) is straightforward.

Finally, we show that g is generically locally equi-upper semicontinuous. For each i, let μ_i be the normalized Lebesgue measure over $[0, a_i]$. Fix i and $s_{-i} \in \times_{j \neq i}[0, a_j]$. Consider the set of all $s_i \in [0, a_i]$ such that $\sum_j s_j \neq c$, a set that has full Lebesgue measure (*i.e.*, it has μ_i-measure 1), and take any s_i in this set, and $\varepsilon > 0$. We only consider the case when $\sum_j s_j > c$ (the case when $\sum_j s_j < c$ can be dealt with analogously). Clearly, we may choose a neighborhood O_{s_i} of s_i in $[0, a_i]$ such that $b_i + \sum_{j \neq i} s_j > c$ and $v_i(1, w_i - b_i) < v_i(1, w_i - s_i) + \varepsilon$ for all $b_i \in O_{s_i}$. Further, given $b_i \in O_{s_i}$, we may choose a neighborhood $O_{s_{-i}}$ of s_{-i} in $\times_{j \neq i}[0, a_j]$ such that

$$b_i + \sum_{j \neq i} b_j > c < s_i + \sum_{j \neq i} b_j, \quad \text{for all } b_{-i} \in O_{s_{-i}}$$

Consequently, for every $b_{-i} \in O_{s_{-i}}$, we have

$$u_i(b_i, b_{-i}) = v_i(1, w_i - b_i) < v_i(1, w_i - s_i) + \varepsilon = u_i(s_i, b_{-i}) + \varepsilon$$

In light of Theorem 2, therefore, we obtain the non-emptiness of the set of trembling-hand perfect equilibria in g.

4. Proof of Lemma 2

To begin, we state a number of intermediate results.

Given a metric space X and $Y \subseteq X$, $\mathbb{P}(Y)$ denotes the set of Borel probability measures on Y, and $\mathbb{P}_*(Y)$ is the subset of finitely supported measures in $\mathbb{P}(Y)$ that assign rational values to each Borel set.

Lemma 4 (Carbonell-Nicolau [11], Lemma 6)**.** *Let X be a compact metric space. Suppose that $f : X \to \mathbb{R}$ is bounded and Borel measurable. For each $\mu \in \mathbb{P}(X)$ and every $\varepsilon > 0$, there exists $\nu^* \in \mathbb{P}_*(X) \cap N_\varepsilon(\mu)$ such that $|\int_X f \mathrm{d}\mu - \int_X f \mathrm{d}\nu^*| < \varepsilon$.*

Lemma 5 (Carbonell-Nicolau [11], Lemma 7)**.** *Suppose that G is compact, metric, and satisfies Condition* (A)*. Then there exists $(\mu_1, ..., \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a map $f : X_i \to X_i$ such that the following is satisfied:*

(*i*) *For each $x_i \in X_i$ and every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that*

$$U_i(f(x_i), O_{\sigma_{-i}}) > U_i(x_i, \sigma_{-i}) - \varepsilon$$

(*ii*) *For every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) < U_i(\mu_i, p_{-i}) + \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$, where $\mu_i^f \in M_i$ is defined by $\mu_i^f(B) := \mu_i\left(f^{-1}(B \cap f(X_i))\right)$.*

Lemma 2. *Suppose that a compact, metric game G satisfies Condition* (A)*. Then there exists $\mu \in \widehat{M}$ such that $\overline{G}_{\delta\mu}$ is payoff secure for every $\delta \in [0, 1)$.*

Proof. Fix $\delta \in [0, 1)$, and let $\mu = (\mu_1, ..., \mu_N) \in \widehat{M}$ be the measure given by Condition (A). We fix $\varepsilon > 0$, $\sigma = (\sigma_1, ..., \sigma_N) \in M(\delta\mu)$, and i, and show that there exists $\nu_i \in M_i(\delta\mu_i)$ such that $U_i\left(\nu_i, O_{\sigma_{-i}}\right) > U_i(\sigma) - \varepsilon$ for some neighborhood $O_{\sigma_{-i}}$ of σ_{-i}.

Lemma 5 gives a Borel measurable map $f : X_i \to X_i$ satisfying the following:

(*i*) For every $y_i \in X_i$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that $U_i(f(y_i), O_{\sigma_{-i}}) > U_i(y_i, \sigma_{-i}) - \frac{\varepsilon}{4}$.

(*ii*) There is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) < U_i(\mu_i, p_{-i}) + \frac{\varepsilon}{2}$ for all $p_{-i} \in V_{\sigma_{-i}}$, where $\mu_i^f \in M_i$ is defined by

$$\mu_i^f(B) := \mu_i\left(f^{-1}(B \cap f(X_i))\right)$$

Claim 1. *There exists a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that*

$$\int_{X_i} U_i\left(f(\cdot), O_{\sigma_{-i}}\right) \mathrm{d}\sigma_i > \int_{X_i} U_i\left(\cdot, \sigma_{-i}\right) \mathrm{d}\sigma_i - \tfrac{\varepsilon}{2}$$

Proof. By (*i*), for every $y_i \in X_i$ there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i\left(f(y_i), O_{\sigma_{-i}}\right) > U_i(y_i, \sigma_{-i}) - \tfrac{\varepsilon}{4}$$

For each $n \in \mathbb{N}$, define

$$X_i^n := \bigcup_{\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i})} \left\{y_i \in X_i : U_i\left(f(y_i), \nu_{-i})\right) < U_i(y_i, \sigma_{-i}) - \tfrac{\varepsilon}{4}\right\}$$

Each X_i^n is Borel measurable. In fact, Lemma 4 gives

$$X_i^n = \bigcup_{\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i}) \cap \mathbb{P}_*(X_{-i})} X_i(\nu_{-i}) \tag{8}$$

where $X_i(\nu_{-i}) := \left\{y_i \in X_i : U_i\left(f(y_i), \nu_{-i})\right) < U_i(y_i, \sigma_{-i}) - \tfrac{\varepsilon}{4}\right\}$. Now, since u_i and f are Borel measurable, for each $\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i})$ the set $X_i(\nu_{-i})$ is Borel measurable. Therefore, each X_i^n is (by (8)) a countable union of Borel sets, and hence a Borel set itself.

Now observe that we have $\bigcap_n X_i^n = \emptyset$ and $X_i^1 \supseteq X_i^2 \supseteq \cdots$. Consequently, for any large enough n,

$$\sigma_i(X_i^n) \sup_{(\nu,\rho) \in M^2} [U_i(\nu) - U_i(\rho)] < \tfrac{\varepsilon}{4}$$

Hence, for any sufficiently large n,

$$\begin{aligned}
&\int_{X_i} U_i(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i}))\mathrm{d}\sigma_i \\
&\qquad = \int_{X_i \setminus X_i^n} U_i(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i}))\mathrm{d}\sigma_i + \int_{X_i^n} U_i(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i}))\mathrm{d}\sigma_i \\
&\qquad > \int_{X_i \setminus X_i^n} U_i(\cdot, \sigma_{-i})\mathrm{d}\sigma_i + \tfrac{\varepsilon}{4} + \int_{X_i^n} U_i(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i}))\mathrm{d}\sigma_i \\
&\qquad > U_i(\sigma_i, \sigma_{-i}) - \tfrac{\varepsilon}{2}
\end{aligned}$$

as desired. □

Because $\sigma \in M(\delta\mu)$, there exists, for each i, $\varrho_i \in M_i$ such that $\sigma_i = (1-\delta)\varrho_i + \delta\mu_i$. Define

$$p_i^f := (1-\delta)\varrho_i^f + \delta\mu_i \quad \text{and} \quad \upsilon_i^f := (1-\delta)\varrho_i^f + \delta\mu_i^f$$

where $\varrho_i^f \in M_i$ is defined by $\varrho_i^f(B) := \varrho_i\left(f^{-1}\left(B \cap f(X_i)\right)\right)$.

By (*ii*), there exists a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i, p_{-i}) > U_i(\mu_i^f, p_{-i}) - \tfrac{\varepsilon}{2}, \quad \text{for all } p_{-i} \in O_{\sigma_{-i}}$$

This, together with the definitions of p_i^f and υ_i^f, gives, for any p_{-i} in some neighborhood of σ_{-i}

$$\begin{aligned}
U_i(p_i^f, p_{-i}) &= (1-\delta)U_i(\varrho_i^f, p_{-i}) + \delta U_i(\mu_i, p_{-i}) \\
&> (1-\delta)U_i(\varrho_i^f, p_{-i}) + \delta U_i(\mu_i^f, p_{-i}) - \tfrac{\varepsilon}{2} \\
&= U_i(\upsilon_i^f, p_{-i}) - \tfrac{\varepsilon}{2}
\end{aligned} \tag{9}$$

In addition, the definition of υ_i^f and the equality $\sigma_i = (1-\delta)\varrho_i + \delta\mu_i$ entail

$$\begin{aligned} U_i(\upsilon_i^f, p_{-i}) &= \int_{X_i} U_i(\cdot, p_{-i}) \mathrm{d}\upsilon_i^f \\ &= (1-\delta)\int_{X_i} U_i(\cdot, p_{-i}) \mathrm{d}\varrho_i^f + \delta \int_{X_i} U_i(\cdot, p_{-i}) \mathrm{d}\mu_i^f \\ &= (1-\delta)\int_{X_i} U_i(f(\cdot), p_{-i}) \mathrm{d}\varrho_i + \delta \int_{X_i} U_i(f(\cdot), p_{-i}) \mathrm{d}\mu_i \\ &= \int_{X_i} U_i(f(\cdot), p_{-i}) \mathrm{d}\sigma_i \end{aligned} \tag{10}$$

Consequently, for every p_{-i} in some neighborhood of σ_{-i} we have

$$\begin{aligned} U_i(p_i^f, p_{-i}) &> U_i(\upsilon_i^f, p_{-i}) - \tfrac{\varepsilon}{2} \\ &= \int_{X_i} U_i(f(\cdot), p_{-i}) \mathrm{d}\sigma_i - \tfrac{\varepsilon}{2} \\ &> U_i(\sigma_i, \sigma_{-i}) - \varepsilon \end{aligned}$$

Here, the first inequality follows from (9), the second inequality is given by Claim 1, and the equality is a consequence of (10). Hence, because $p_i^f \in M_i(\delta\mu_i)$, $\overline{G}_{\delta\mu}$ is payoff secure. ■

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References

1. Selten, R. Reexamination of the perfectness concept for equilibrium points in extensive games. *Int. J. Game Theory* **1975**, *4*, 25-55.
2. Bertrand, J. Théorie mathématique de la richesse sociale. *J. Savants* **1883**, 499-508.
3. Hotelling, H. The stability of competition. *Econ. J.* **1929**, *39*, 41-57.
4. Milgrom, P.; Weber, R. A theory of auctions and competitive bidding. *Econometrica* **1982**, *50*, 1089-1122.
5. Fudenberg, D.; Gilbert, R.; Stiglitz, J.; Tirole, J. Preemption, leapfrogging, and competition in patent races. *Eur. Econ. Rev.* **1983**, *22*, 3-31.
6. Bagnoli, M.; Lipman, B.L. Provision of public goods: Fully implementing the core through private contributions. *Rev. Econ. Stud.* **1989**, *56*, 583-601.
7. Broecker, T. Credit-worthiness tests and interbank competition. *Econometrica* **1990**, *58*, 429-452.
8. Pitchik, C.; Schotter, A. Perfect equilibria in budget-constrained sequential auctions: An experimental study. *Rand J. Econ.* **1988**, *19*, 363-388.
9. Allen, B. Using trembling-hand perfection to alleviate the interlinked principal-agent problem. *Scand. J. Econ.* **1988**, *90*, 373-382.

10. Simon, L.K.; Stinchcombe, M.B. Equilibrium refinement for infinite normal-form games. *Econometrica* **1995**, *63*, 1421-1443.
11. Carbonell-Nicolau, O. On the existence of pure-strategy perfect equilibrium in discontinuous games. *Games Econ. Behav.* **2011**, *71*, 23-48.
12. Carbonell-Nicolau, O. Perfect and limit admissibe perfect equilibria in discontinuous games. *J. Math. Econ.* (forthcoming).
13. Van Damme, E.E.C. *Stability and Perfection of Nash Equilibria*; Springer-Berlag: Berlin, Germany, 2002.
14. Reny, P.J. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica* **1999**, *67*, 1029-1056.
15. Al-Najjar, N. Strategically stable equilibria in games with infinitely many pure strategies. *Math. Soc. Sci.* **1995**, *29*, 151-164.
16. Dunford, N.; Schwartz, J.T. *Linear Operators, Part I: General Theory*; John Wiley and Sons: New York, NY, USA, 1957.
17. Sion, M.; Wolfe, P. On a game without a value. In *Contributions to the Theory of Games, Volume III*; Dresher, M., Tucker, A.W., Wolfe, P., Eds.; Princeton University Press: Princeton, NJ, USA, 1957.
18. Carmona, G. On the existence of equilibria in discontinuous games: Three counterexamples. *Int. J. Game Theory* **2005**, *33*, 181-187.
19. Monteiro, P.K.; Page, F.H. Uniform payoff security and Nash equilibrium in compact games. *J. Econ. Theory* **2007**, *134*, 566-575.
20. Barelli, P.; Soza, I. On the existence of Nash equilibria in discontinuous and qualitative games. **2008**, (mimeo).
21. Baye, M.R.; Tian, G.; Zhou, J. Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. *Rev. Econ. Stud.* **1993**, *60*, 935-948.
22. Baye, M.R.; Morgan, J. Winner-take-all price competition. *Econ. Theory* **2002**, *19*, 271-282.
23. Page, F.H.; Monteiro, P.K. Three principles of competitive nonlinear pricing. *J. Math. Econ.* **2003**, *39*, 63-109.