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Review

Defects in Static Elasticity of Quasicrystals

Qin Xu [1], Jing Lu [1,2,*] and Wu Li [1,*]

[1] Institute of Science, Taiyuan University of Technology, Taiyuan 030024, China; xuqin0123456@126.com
[2] Shanxi meteorological Administration department, Chinese Meteorological Administration, Taiyuan 030024, China
* Correspondence: lujing19810629@hotmail.com (J.L.); liwu@tyut.edu.cn (W.L.)

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Abstract: A review on mathematical elasticity of quasicrystals is given. In this review, the focus is on various defects of quasicrystals. Dislocation and crack are two classes of typical topological defects, while their existence has great influence on the mechanical behavior of quasicrystals. The analytic and numerical solutions of dislocations and crack in quasicrystals are the core of the static and dynamic elasticity theory, and this paper gives a comprehensive review on the solutions for dislocations and crack with different configurations in different various important quasicrystalline systems. We review some results in linear elasticity of quasicrystals, referring to different boundary value problems. We also add some new achievements.

Keywords: quasicrystals; elasticity; crack; stress intensity factor; Dugdale model

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1. Introduction

Quasicrystals are inter-metallic solids characterized by quasi-periodic long-range translational symmetry and absence of the standard periodicity of crystals. Their existence was first recognized in 1984 by Shechtman et al. by evaluating diffraction patterns obtained on specimens of rapidly solidified alloys. This discovery was awarded the Nobel Prize of Chemistry 2011 [1]. Soon after the discovery of the novel matter, elasticity, dynamics of elasticity, defects and other subjects concerning mechanical behavior of quasicrystalline materials become an intensive subject of researches in theory and experiment and attracted considerable attention of researchers [2–7]. Quasicrystal is deformable under external loads, and certain thermal loads. About the elasticity of quasicrystals, the theoretical physicists have provided various descriptions for quasicrystals. The phenomenological theory of Landau and Lifshitz on the elementary excitation of condensed matters was essentially regarded as the physical basis of elasticity of quasicrystals, and two types of excitations, phonons and phasons, were considered for quasi-periodicity of materials [8–15]. In essence, the quasicrystal mechanical behavior falls within the general framework for building models in the mechanics of complex materials: inner degrees of freedom are attributed to every material element besides the degrees of freedom in two- or three-dimensional ambient space. By exploiting a general model-building framework for the mechanics of complex continua [16–18], they are deemed that there are two displacement fields $\mathbf{u}$ and $\mathbf{w}$, in which the former $\mathbf{u}$ named as the phonon field is similar to that in crystals; the latter $\mathbf{w}$ named as the phase field is defined over the body. Thus, the total displacement field in a quasicrystal can be expressed by

$$\overline{\mathbf{u}} = \mathbf{u}^{\parallel} \oplus \mathbf{u}^{\perp} = \mathbf{u} \oplus \mathbf{w} \tag{1}$$

where $\mathbf{u}$ is in the parallel space or the physical space; $\mathbf{w}$ is in the complement space or the perpendicular space, which is an internal space; and $\oplus$ denotes the direct sum. Based on the above physical framework and the extended methodology in mathematical physics from classical elasticity, the independent elastic constants for different symmetries of quasicrystals can be determined [19–24]. Then, the mathematical elasticity theory of quasicrystals has been developed rapidly. In 2004, Fan and Mai give a review based on the static elasticity theory of various quasicrystals [25]. Fan and his workers devoted to development of a mathematical elasticity theory of quasicrystals and its applications [26]. First Levine and Lubensky et al. obtained many solutions for the elasticity and dislocations in pentagonal and icosahedral quasicrystals [27]. Then, Li provided two analytical solutions for a decagonal quasicrystal with a Griffith crack and a straight dislocation [28,29]. Based on one-dimensional hexagonal quasicrystals, Chen studied a three-dimensional elastic problem and gave a general solution for this problem [30]. Liu et al. obtained the governing equations for plane elasticity of one-dimensional quasicrystals and the general solutions [31]. Based on the stress potential function, Li accomplished notch problem of two-dimensional quasicrystals [32]. Wang and Gao et al. obtained some solutions for some elastic problems of some one-, two-dimensional quasicrystal [33–35]. Coddens deduced the elasticity and dynamics of the phason and phonon in some quasicrystals [36]. Wang et al. and Guo et al. discussed the phonon- and phason-type inclusions in icosahedral quasicrystals and an elliptical inclusion in hexagonal quasicrystals [37,38]. More recently, the phonon–phason elasticity of quasicrystals attracted a lot of attention. For example, Radi and Mariano focused their attention on the straight cracks and dislocations in two-dimensional quasicrystals, and described linear elasticity of quasicrystals and obtained some profound results for [39–41]. Li and his workers induced fundamental solutions for thermo-elasticity of one-dimensional hexagonal quasicrystals with half infinite plane cracks and obtained some solutions [42–44]. Of course, elastic theory of quasicrystals has been developed by some researchers (for example, Li and Chai [45] and Sladek et al. [46]). A mass of experimental observations have been provided for plastic deformation of quasicrystals based on dislocation mechanism by Wollgarten et al., Feuerbacher et al., and Messerschmidt et al. [47–49]. Through a series of experimental observations, many studies have also been executed based on the deformed Al-Pd-Mn single quasicrystals [50–53].

We have cited 53 references in this review article. We also obtain some new achievements. For example, the crack opening displacement of an extended Dugdale model for anti-plane quasicrystals and the stress intensity factor of an elastic problem of plane problem of two-dimensional quasicrystals with elliptical hole with double cracks are deduced by using potential theory, which are expressed in terms of closed form.

2. Fundamental Equations of Quasicrystals

Referring to a quasicrystal, in which phonon and phason fields exist simultaneously. Based on the cut-and-projection theory, quasicrystals can be understood as a three-dimensional projection of a higher-dimensional periodic space, with a huge number of field variables and field equations involving its elasticity. The displacement vectors are labeled as $\mathbf{u} = (u_1, u_2, u_3)$ for phonon field and $\mathbf{w} = (w_1, w_2, w_3)$ for phason field, respectively, which are both dependent on the spatial point $\mathbf{x} = (x_1, x_2, x_3)$ in the real space. Similar to classical elasticity, there are giving rise to two displacement fields, so they are named the elastic strain tensors ε_{ij} and w_{ij} can be expressed by equation 2 respectively.

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), w_{ij} = \frac{\partial w_i}{\partial x_j} \tag{2}$$

We reduce our analyses in small strain setting and linear elastic behavior. For the sake of simplicity, we do not consider inertial effects, body forces, and phason self-actions so that, with these restrictions, the balance equations with absent of the body force (for a complete derivation of them in large strain regime, see [41]) read

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \frac{\partial H_{ij}}{\partial x_j} = 0 \tag{3}$$

When quasicrystals are in the case of small deformation, the phonon stresses σ_{ij} and phason stresses H_{ij} obey generalized Hooke's law. Namely, they are linearly dependent upon both the phonon strains and phason strains, that is it can be expressed by the following generalized Hooke's law [25,26]:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl}, H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl} \tag{4}$$

where σ_{ij} and ε_{ij} phonon stresses and strains; u_i and w_i are phonon and phason displacements; H_{ij} and w_{ij} phason stresses and strains; and C_{ijkl}, K_{ijkl} and R_{ijkl} the phonon, phason, phonon–phason coupling elastic constants, respectively. This leads to a mathematical complexity in solving associated boundary value problems. The elasticity equations for a variety of one-, two-, and three-dimensional quasicrystals are concluded in [26]. To avoid duplication, here we no longer display these equations.

In light of the huge number of the field variables and their equations, the solution for three elasticity problems of quasicrystals is not easily obtained. One commonly used method to solve the elasticity problem is to eliminate the number of the field variables and equations above mentioned. Similar to classical mathematical physics, we can utilize the eliminating element method in the elasticity problem of quasicrystals. We can find the equations can be simplified for some meaningful cases physically. The elasticity problem of quasicrystals includes in-plane mode case and anti-plane mode case. For example, some field variables disappear while the materials are in plane mode case. Meanwhile, these displacements and relevant strains and stresses are independent of the periodic direction. For example, supposing there is a straight dislocation line along the periodic direction, a crack penetrates periodic direction, etc., then, this fact implies $\partial/\partial x_3 (= \partial/\partial z) = 0$. Then,

$$\frac{\partial \mu_i}{\partial z} = 0, \frac{\partial w_z}{\partial_z} = 0, \ (i = 1, 2, 3) \tag{5}$$

Hence,

$$\varepsilon_{zz} = w_{zz} = 0, \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\frac{\partial \mu_z}{\partial y}, \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\frac{\partial \mu_z}{\partial x} \tag{6}$$

$$\frac{\partial \sigma_{ij}}{\partial z} = 0, \frac{\partial H_{ij}}{\partial z} = 0 \tag{7}$$

2.1. The Elasticity of One-Dimensional Hexagonal Quasicrystals

There are many kinds of one-dimensional quasicrystals; we only discuss the simplest system of one-dimensional hexagonal quasicrystals. For this type of quasicrystal, phonon displacements are u_x, u_x, u_x, phason displacement is w_z (because $w_x = w_y = 0$), the corresponding strains are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{8a}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}), \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}), \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \tag{8b}$$

$$w_{zx} = \frac{\partial w_z}{\partial x}, w_{zy} = \frac{\partial w_z}{\partial y}, w_{zz} = \frac{\partial w_z}{\partial z} \tag{8c}$$

and other $w_{ij} = 0$. The strain components are nine in the lump. Formulas (8a–c) are the same with all one-dimensional quasicrystals. The nine strains can be represented by vectors with nine components, i.e.,

$$[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}, w_{33}, w_{31}, w_{32}] \tag{9}$$

or

$$[\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{zx}, 2\varepsilon_{xy}, w_{zz}, w_{zx}, w_{zy}]. \tag{10}$$

The corresponding stresses are

$$[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}, \sigma_{zz}, \sigma_{zx}, \sigma_{zy}]. \tag{11}$$

The elastic constant matrix is

$$[CKR] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & R_2 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & 0 & R_3 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ R_1 & R_1 & R_2 & 0 & 0 & 0 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_3 & 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & R_3 & 0 & 0 & 0 & 0 & K_2 \end{bmatrix} \tag{12}$$

where the four indexes of phonon elastic constants are reduced to two quotas, i.e., quotas $11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 31 \to 5, 12 \to 6,$ thus C_{ijkl} can be denoted as C_{pq}:

$$C_{11} = C_{1111} = C_{2222}, C_{12} = C_{1122}, C_{33} = C_{3333}, C_{44} = C_{2323} = C_{3131}$$

$$C_{13} = C_{1133} = C_{2323}, C_{66} = \frac{C_{11} - C_{12}}{2} = \frac{C_{1111} - C_{1122}}{2}$$

That fact means that the number of independent phonon elastic constants is five. Furthermore, the phason elastic constants $K_1 = K_{3333}, K_2 = K_{3131} = K_{3232}$, i.e., there are only two independent phason elastic constants. Finally, we also have $R_1 = R_{1133} = R_{2233}, R_2 = R_{3333}, R_3 = R_{2332} = R_{3131}$, i.e., obviously, there are three phonon–phason coupling elastic constants.

We may learn from the elastic constants matrix that the corresponding stress–strain relations are:

$$\begin{cases} \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + R_1 w_{zz}, \\ \sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + R_1 w_{zz}, \\ \sigma_{zz} = C_{13}\varepsilon_{xx} + C_{13}\varepsilon_{yy} + C_{33}\varepsilon_{zz} + R_2 w_{zz}, \\ \sigma_{yz} = \sigma_{zy} = 2C_{44}\varepsilon_{yz} + R_3 w_{zy}, \\ \sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx} + R_3 w_{zx}, \\ \sigma_{xy} = \sigma_{yx} = 2C_{66}\varepsilon_{xy}, \\ H_{zz} = R_1(\varepsilon_{xx} + \varepsilon_{yy}) + R_2\varepsilon_{zz} + K_1 w_{zz}, \\ H_{zx} = 2R_3\varepsilon_{zx} + K_2 w_{zx}, \\ H_{zy} = 2R_3\varepsilon_{yz} + K_2 w_{zy} \end{cases} \tag{13}$$

and other $H_{ij} = 0$.

The equilibrium equations are:

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \\ \frac{\partial H_{zx}}{\partial x} + \frac{\partial H_{zy}}{\partial y} + \frac{\partial H_{zz}}{\partial z} = 0. \end{cases} \tag{14}$$

Here are four displacements, nine strains and nine stresses, which added up to 22 field variables. The corresponding field equations are 22, consisting of four for equilibrium equations, nine equations of deformation geometry, and nine stress–strain relation. It is evident that the elastic equilibrium problem of one-dimensional hexagonal quasicrystals is a little complicated. In the following we give a simplified treatment. If there is a straight dislocation or a Griffith crack along the direction of the atom quasi-periodic arrangement, the deformation is independent of the z-axis, therefore

$$\frac{\partial}{\partial z} = 0 \tag{15}$$

This fact leads to

$$\frac{\partial u_i}{\partial z} = 0, \frac{\partial w_z}{\partial z} = 0, \quad (i = 1, 2, 3) \tag{16a}$$

$$\varepsilon_{zz} = w_{zz} = 0, \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\frac{\partial u_z}{\partial y}, \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\frac{\partial u_z}{\partial x} \tag{16b}$$

$$\frac{\partial \sigma_{ij}}{\partial z} = 0, \frac{\partial H_{ij}}{\partial z} = 0 \tag{16c}$$

The generalized Hooke's law is simplified as

$$\begin{cases} \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy}, \\ \sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy}, \\ \sigma_{xy} = \sigma_{yx} = 2C_{66}\varepsilon_{xy}, \\ \sigma_{zz} = C_{13}(\varepsilon_{xx} + \varepsilon_{yy}), \\ \sigma_{yz} = \sigma_{zy} = 2(C_{44}\varepsilon_{yz} + R_3 w_{zy}), \\ \sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx} + R_3 w_{zx}, \\ H_{zz} = R_1(\varepsilon_{xx} + \varepsilon_{yy}), \\ H_{zx} = 2R_3\varepsilon_{zx} + K_2 w_{zx}, \\ H_{zy} = 2R_3\varepsilon_{yz} + K_2 w_{zy}. \end{cases} \tag{17}$$

In the absence of the body force and generalized body force, the equilibrium equations are

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \\ \frac{\partial H_{zx}}{\partial x} + \frac{\partial H_{zy}}{\partial y} = 0 \end{cases} \tag{18}$$

Equations (15)–(18) define two uncoupled problems. The first of them is

$$\begin{cases} \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy}, \\ \sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy}, \\ \sigma_{xy} = (C_{11} - C_{12})\varepsilon_{xy}, \\ \sigma_{zz} = C_{13}(\varepsilon_{xx} + \varepsilon_{yy}), \\ H_{zz} = R_1(\varepsilon_{xx} + \varepsilon_{yy}), \\ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \\ \varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{xy} = \frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) \end{cases} \tag{19}$$

This is the classical plane elasticity of conventional hexagonal crystals. The problem has been studied, extensively using the stress function approach, e.g. it introduces

$$\sigma_{xx} = \frac{\partial^2 u}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 u}{\partial x^2}, \sigma_{xy} = -\frac{\partial^2 u}{\partial x \partial y} \tag{20}$$

Then, equations are reduced to solve

$$\nabla^2\nabla^2 u = 0 \tag{21}$$

The problem is considerably discussed in the elasticity of crystals, so we do not consider it here. We are concerned with the problem of phonon–phason coupling anti-plane elasticity. We illustrate the basis equations of anti-plane elasticity for one-dimensional hexagonal quasicrystals. The stress tensors are related to the strain tensors repressed by

$$\begin{cases} \sigma_{yz} = \sigma_{zy} = 2C_{44}\varepsilon_{yz} + R_3 w_{zy} \\ \sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx} + R_3 w_{zx} \\ H_{zx} = 2R_3\varepsilon_{zx} + K_2 w_{zx} \\ H_{zy} = 2R_3\varepsilon_{yz} + K_2 w_{zy} \end{cases} \tag{22}$$

The deformation geometry relations are

$$\begin{cases} \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\frac{\partial u_z}{\partial x}, \varepsilon_{zy} = \varepsilon_{yz} = \frac{1}{2}\frac{\partial u_z}{\partial y} \\ w_{zx} = \frac{\partial w_z}{\partial x}, w_{zy} = \frac{\partial w_z}{\partial y} \end{cases} \tag{23}$$

The equilibrium equations are

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0, \frac{\partial H_{zx}}{\partial x} + \frac{\partial H_{zy}}{\partial y} = 0 \tag{24}$$

Substituting the deformation geometry relations into the stress–strain relations, and then into the equilibrium equations, yields the final governing equations such as

$$\begin{cases} C_{44}\nabla^2 u_z + R_3\nabla^2 w_z = 0 \\ R_3\nabla^2 u_z + K_2\nabla^2 w_z = 0 \end{cases} \tag{25}$$

Considering $C_{44}K_2 - R_3^2 \neq 0$, we have

$$\nabla^2 u_z = 0, \nabla^2 w_z = 0 \tag{26}$$

where $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$, so u_z and w_z are two harmonic functions.

It is well known that the two-dimensional harmonic functions u_z and w_z can be a real part or an imaginary part of any analytic functions $\phi(\mathrm{t})$ and $\psi(t)$ of complex variable $z = x + iy, i = \sqrt{-1}$, respectively, i.e.,

$$\begin{cases} u_z(x,y) = \mathrm{Re}\phi_1(z) \\ w_z(x,y) = \mathrm{Re}\psi_1(z) \end{cases} \tag{27}$$

In this version, Equation (26) should be identically satisfied.

Substituting Equation (27) into Equation (23), we have

$$\begin{cases} \sigma_{zx} - i\sigma_{zy} = C_{44}\phi_1' + R_3\psi_1' \\ H_{zx} - iH_{zy} = K_2\psi_1' + R_3\phi' \end{cases} \tag{28}$$

in which $\phi_1' = \frac{d\phi_1}{dz}, \psi_1' = \frac{d\psi_1}{dz}$.

Based on Equation (28), we get

$$\begin{cases} \sigma_{zy} = \sigma_{yz} = -\frac{1}{2i}[C_{44}(\phi_1' - \overline{\phi_1'}) + R_3(\psi_1' - \overline{\psi_1'})] \\ H_{zy} = -\frac{1}{2i}[K_2(\psi_1' - \overline{\psi'}) + R_3(\phi_1' - \overline{\phi_1'})] \end{cases} \tag{29}$$

2.2. The Elasticity of Two-Dimensional Quasicrystals

As everyone knows, the five-fold and ten-fold symmetries quasicrystals of point groups $5, \overline{5}$ and $10, \overline{10}$ are different in plane elasticity from that of point groups 5 m and 10 mm [26]. The difference lies in only the phonon–phason coupling elastic constants, in which the former has two coupling elastic constants R_1 and R_2 rather than one constant R. In this section, we mainly discuss linear theory for decagonal quasicrystals of point group $\overline{10}$.

We assume that the atom arrangement is periodic along the z-direction and quasiperiodic along the plane, and denote $x = x_1, y = x_2$, and $z = x_3$ for a two-dimensional decagonal quasicrystal. According to quasicrystal elasticity theory, the equations of deformation geometry are

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),\ w_{ij} = \frac{\partial w_i}{\partial x_j} \tag{30a}$$

When the body force is neglected, the equilibrium equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0,\ \frac{\partial H_{ij}}{\partial x_j} = 0 \tag{30b}$$

and the concrete generalized Hooke's law of decagonal quasicrystals of point group $\overline{10}$ is

$$\begin{cases} \sigma_{xx} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{xx} + R_1(w_{xx} + w_{yy}) + R_2(w_{xy} - w_{yx}) \\ \sigma_{yy} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{yy} - R_1(w_{xx} + w_{yy}) - R_2(w_{xy} - w_{yx}) \\ \sigma_{xy} = \sigma_{yx} = 2M\varepsilon_{xy} + R_1(w_{yx} - w_{xy}) + R_2(w_{xx} + w_{yy}) \\ H_{xx} = K_1 w_{xx} + K_2 w_{yy} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy} \\ H_{yy} = K_1 w_{yy} + K_2 w_{xx} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy} \\ H_{xy} = K_1 w_{xy} - K_2 w_{yx} - 2R_1\varepsilon_{xy} + R_2(\varepsilon_{xx} - \varepsilon_{yy}) \\ H_{yx} = K_1 w_{yx} - K_2 w_{xy} + 2R_1\varepsilon_{xy} - R_2(\varepsilon_{xx} - \varepsilon_{yy}) \end{cases} \tag{31}$$

where the elastic constants $L = C_{12}$, $M = C_{66} = (C_{11} - C_{12})/2$; u_i and w_i denote the phonon displacement and the phason displacement; σ_{ij} and ε_{ij} the phonon stress and the phonon strain; H_{ij} and w_{ij} phason stress and strain; and C_{ijkl}, K_{ijkl} and R_{ijkl} the phonon, phason and phonon–phason coupling elastic constants, respectively. We assume that a plane notch penetrates through the solid along the period direction. In this case, it is evident that all the field variables are independent of z.

According to the deformation geometry equation, deformation compatibility equations are as follows:

$$\begin{cases} \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\ \frac{\partial w_{xx}}{\partial y} = \frac{\partial w_{xy}}{\partial x} \\ \frac{\partial w_{yy}}{\partial x} = \frac{\partial w_{yx}}{\partial y} \end{cases} \tag{32}$$

The strain components ε_{ij} and w_{ij} can be expressed by the stress components σ_{ij} and H_{ij}, i.e.,

$$\begin{cases} \varepsilon_{xx} = \frac{1}{4(L+M)}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{4c}[(K_1 + K_2)(\sigma_{xx} - \sigma_{yy}) - 2R_1(H_{xx} + H_{yy}) - 2R_2(H_{xy} - H_{yx})] \\ \varepsilon_{yy} = \frac{1}{4(L+M)}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{4c}[(K_1 + K_2)(\sigma_{xx} - \sigma_{yy}) - 2R_1(H_{xx} + H_{yy}) - 2R_2(H_{xy} - H_{yx})] \\ \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2c}[(K_1 + K_2)\sigma_{xy} - R_2(H_{xx} + H_{yy}) + R_1(H_{xy} - H_{yx})] \\ w_{xx} = \frac{1}{2(K_1 - K_2)}(H_{xx} - H_{yy}) + \frac{1}{2c}[M(H_{xx} + H_{yy}) - R_1(\sigma_{xx} - \sigma_{yy}) - 2R_2\sigma_{xy}] \\ w_{yy} = -\frac{1}{2(K_1 - K_2)}(H_{xx} - H_{yy}) + \frac{1}{2c}[M(H_{xx} + H_{yy}) - R_1(\sigma_{xx} - \sigma_{yy}) - 2R_2\sigma_{xy}] \\ w_{xy} = \frac{1}{2c}[-R_2(\sigma_{xx} - \sigma_{yy}) + 2R_1\sigma_{xy}] + \frac{1}{2(K_1 - K_2)}(H_{xy} + H_{yx}) + \frac{M}{2c}(H_{xy} - H_{yx}) \\ w_{yx} = \frac{1}{2c}[R_2(\sigma_{xx} - \sigma_{yy}) - 2R_1\sigma_{xy}] + \frac{1}{2(K_1 - K_2)}(H_{xy} + H_{yx}) - \frac{M}{2c}(H_{xy} - H_{yx}) \end{cases} \tag{33}$$

with

$$c = M(K_1 + K_2) - 2(R_1^2 + R_2^2)$$

If the stress functions ϕ, ψ_1 and ψ_2 are introduced, such as

$$\begin{cases} \sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2}, \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \varphi}{\partial x \partial y} \\ H_{xx} = \frac{\partial \psi_1}{\partial y}, H_{xy} = -\frac{\partial \psi_1}{\partial x}, H_{yx} = -\frac{\partial \psi_2}{\partial y}, H_{yy} = \frac{\partial \psi_2}{\partial x} \end{cases} \tag{34}$$

then, the equilibrium Equation (30b) is automatically satisfied. Substituting above equations into Equation (33), the deformation compatibility equations expressed by stress components yields

$$\begin{cases} \frac{1}{2(L+M)}\nabla^2\nabla^2\varphi + \frac{K_1+K_2}{2c}\nabla^2\nabla^2\varphi + \frac{R_1}{c}(\frac{\partial}{\partial y}\Pi_1\psi_1 - \frac{\partial}{\partial x}\Pi_2\psi_2) + \\ \frac{R_2}{c}(\frac{\partial}{\partial x}\Pi_2\psi_1 + \frac{\partial}{\partial y}\Pi_1\psi_2) = 0 \\ (\frac{c}{K_1-K_2} + M)\nabla^2\psi_1 + R_1\frac{\partial}{\partial y}\Pi_1\phi + R_2\frac{\partial}{\partial x}\Pi_2\phi = 0 \\ (\frac{c}{K_1-K_2} + M)\nabla^2\psi_2 - R_1\frac{\partial}{\partial x}\Pi_2\phi + R_2\frac{\partial}{\partial y}\Pi_1\phi = 0 \end{cases} \tag{35}$$

in which $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \Pi_1 = 3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \Pi_2 = 3\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}$, c is given by above. By now, the numbers of equations and unknown functions have been reduced to 3.

We introduce a new unknown function $G(x, y)$ such as

$$\begin{cases} \phi = c_1\nabla^2\nabla^2 G, \\ \psi_1 = -(\frac{1}{R_1}\frac{\partial}{\partial y}\Pi_1 + R_2\frac{\partial}{\partial x}\Pi_2)\nabla^2 G \\ \psi_2 = (R_1\frac{\partial}{\partial x}\Pi_2 G - R_2\frac{\partial}{\partial y}\Pi_1)\nabla^2 G \end{cases} \tag{36}$$

If we let

$$\nabla^2\nabla^2\nabla^2\nabla^2 G = 0, \tag{37}$$

then Equation (35) is satisfied, in which

$$c_1 = \frac{c}{K_1 - K_2} + M \tag{38}$$

The general solution of Equation (38) can be expressed as

$$G = 2\mathrm{Re}\left[g_1(z) + \bar{z}g_2(z) + \frac{1}{2}\bar{z}^2 g_3(z) + \frac{1}{6}\bar{z}^3 g_4(z)\right] \tag{39}$$

in which $g_i(z)(i = 1, 2, 3, 4)$ are analytic functions of complex variable $z = x + iy = re^{i\theta}$.

2.3. The Elasticity of Three-Dimensional Quasicrystals

There is no periodicity in three directions of elasticity of three dimensional quasicrystals. Therefore, field variables including the phonon field u_i and the phase field w_i exist simultaneously in these three directions. Besides icosahedral quasicrystals, three dimensional quasicrystals also include cubic quasicrystals. Due to their tediousness, we only discuss the elasticity of three-dimensional icosahedral quasicrystals here. The elasticity of cubic quasicrystals is neglected.

The equations of deformation geometry are

$$\begin{cases} \varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\ \varepsilon_{yz} = \frac{1}{2}(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}), \varepsilon_{zx} = \frac{1}{2}(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}), \varepsilon_{xy} = \frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) \\ w_{xx} = \frac{\partial w_x}{\partial x}, w_{yy} = \frac{\partial w_y}{\partial y}, w_{zz} = \frac{\partial w_z}{\partial z} \\ w_{yz} = \frac{\partial w_y}{\partial z}, w_{zx} = \frac{\partial w_z}{\partial x}, w_{xy} = \frac{\partial w_x}{\partial y} \\ w_{zy} = \frac{\partial w_z}{\partial y}, w_{xz} = \frac{\partial w_x}{\partial z}, w_{yx} = \frac{\partial w_y}{\partial x} \end{cases} \tag{40}$$

which are similar in form to those given in previous, but here u_i and w_i have six components, and ε_{ij} and w_{ij} have 15 components in total.
The elastic constants of phonon field are very simple, merely two, and can be expressed by *Lame′* coefficient, that is

$$C_{ijhl} = \lambda\delta_{ij}\delta_{hl} + \mu(\delta_{ih}\delta_{jl}), i, j, h, l = 1, 2, 3 \tag{41}$$

The elastic constants of the phases are

$$\begin{cases} K_{1111} = K_{2222} = K_{1212} = K_{2121} \equiv K_1 \\ K_{1331} = K_{3111} = K_{1113} = K_{1311} = K_{2213} = K_{1322} = K_{2312} = K_{1223} = -K_{2231} \\ = -K_{3122} = -K_{2321} = -K_{2123} = -K_{1232} = -K_{3212} = -K_{3221} = -K_{2132} \equiv K_2 \\ K_{3333} = K_1 + K_2 \\ K_{2323} = K_{3131} = K_{3232} = K_{1313} = K_1 - K_2 \end{cases} \tag{42}$$

The phonon–phason coupling elastic constant is only one, and mark as R.
Based on the matrix expression of the generalized Hooke's law $\begin{bmatrix} \sigma_{ij} \\ H_{ij} \end{bmatrix} = \begin{bmatrix} [C] & [R] \\ [R]^{\mathrm{T}} & [K] \end{bmatrix} \begin{bmatrix} \varepsilon_{ij} \\ w_{ij} \end{bmatrix}$, $\begin{bmatrix} \sigma_{ij} \\ H_{ij} \end{bmatrix} = \begin{bmatrix} \sigma_{ij} & H_{ij} \end{bmatrix}^{\mathrm{T}}$, $\begin{bmatrix} \varepsilon_{ij} \\ w_{ij} \end{bmatrix} = \begin{bmatrix} \varepsilon_{ij} & w_{ij} \end{bmatrix}^{\mathrm{T}}$, we know the explicit relationship between stresses and strains as below:

$$\begin{cases} \sigma_{xx} = \lambda\theta + 2\mu\varepsilon_{xx} + R(w_{xx} + w_{yy} + w_{zz} + w_{xz}), \\ \sigma_{yy} = \lambda\theta + 2\mu\varepsilon_{yy} - R(w_{xx} + w_{yy} - w_{zz} + w_{xz}), \\ \sigma_{zz} = \lambda\theta + 2\mu\varepsilon_{yy} - 2Rw_{zz}, \\ \sigma_{yz} = 2\mu\varepsilon_{yz} + R(w_{zy} - w_{xy} - w_{yx}) = \sigma_{zy}, \\ \sigma_{zx} = 2\mu\varepsilon_{zx} + R(w_{xx} - w_{yy} - w_{zx}) = \sigma_{xz}, \\ \sigma_{xy} = 2\mu\varepsilon_{xy} + R(w_{yx} - w_{yz} - w_{xy}) = \sigma_{yx}, \\ H_{xx} = R(\varepsilon_{xx} - \varepsilon_{yy} + 2\varepsilon_{zx}) + K_1 w_{xx} + K_2(w_{zx} + w_{xz}), \\ H_{yy} = R(\varepsilon_{xx} - \varepsilon_{yy} - 2\varepsilon_{zx}) + K_1 w_{yy} + K_2(w_{xz} - w_{zx}), \\ H_{zz} = R(\varepsilon_{xx} + \varepsilon_{yy} - 2\varepsilon_{zz}) + (K_1 + K_2)w_{zz}, \\ H_{yz} = -2R\varepsilon_{xy} + (K_1 - K_2)w_{yz} + K_2(w_{xy} - w_{yx}), \\ H_{zx} = 2R\varepsilon_{zx} + (K_1 - K_2)w_{zx} + K_2(w_{xx} - w_{yy}), \\ H_{xy} = -2R(\varepsilon_{yz} + \varepsilon_{xy}) + K_1 w_{xy} + K_2(w_{yz} - w_{zy}), \\ H_{zy} = 2R\varepsilon_{yz} + (K_1 - K_2)w_{zy} - K_2(w_{xy} + w_{yx}), \\ H_{xz} = R(\varepsilon_{xx} - \varepsilon_{yy}) + K_2(w_{xx} + w_{yy}) + (K_1 - K_2)w_{xz}, \\ H_{yx} = 2R(\varepsilon_{xy} - \varepsilon_{yx}) + K_1 w_{yx} - K_2(w_{yz} + w_{zy}), \end{cases} \tag{43}$$

where $\theta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$.
The equilibrium equations are as follows:

$$\begin{cases} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0, & \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0, \\ \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0, & \frac{\partial H_{xx}}{\partial x} + \frac{\partial H_{xy}}{\partial y} + \frac{\partial H_{xz}}{\partial z} = 0, \\ \frac{\partial H_{yx}}{\partial x} + \frac{\partial H_{yy}}{\partial y} + \frac{\partial H_{yz}}{\partial z} = 0, & \frac{\partial H_{zx}}{\partial x} + \frac{\partial H_{zy}}{\partial y} + \frac{\partial H_{zz}}{\partial z} = 0. \end{cases} \tag{44}$$

Based on Equation (44), we can get the deformation compatibility equations

$$\begin{cases} \frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}, \frac{\partial\varepsilon_{yz}}{\partial x} = \frac{\partial\varepsilon_{zx}}{\partial y} \\ \frac{\partial w_{xy}}{\partial x} = \frac{\partial w_{xx}}{\partial y}, \frac{\partial w_{yy}}{\partial x} = \frac{\partial w_{yx}}{\partial y}, \frac{\partial w_{zy}}{\partial x} = \frac{\partial w_{zx}}{\partial y} \end{cases} \tag{45}$$

If we introduce five stress potential functions $\varphi_1(x,y)$, $\varphi_2(x,y)$, $\psi_1(x,y)$, $\psi_2(x,y)$, $\psi_3(x,y)$, which are satisfied with

$$\begin{cases} \sigma_{xx} = \frac{\partial^2\phi_1}{\partial y^2}, \sigma_{xy} = -\frac{\partial^2\phi_1}{\partial x\partial y}, \sigma_{yy} = \frac{\partial^2\phi_1}{\partial x^2} \\ \sigma_{zx} = \frac{\partial\phi_2}{\partial y}, \sigma_{zy} = -\frac{\partial\phi_2}{\partial x} \\ H_{xx} = \frac{\partial\psi_1}{\partial y}, H_{xy} = -\frac{\partial\psi_1}{\partial x}, H_{yx} = \frac{\partial\psi_2}{\partial y} \\ H_{yy} = -\frac{\partial\psi_2}{\partial x}, H_{zx} = \frac{\partial\psi_3}{\partial y}, H_{zy} = -\frac{\partial\psi_3}{\partial x} \end{cases} \tag{46}$$

and

$$\begin{cases} \phi_1 = c_2c_3R\frac{\partial}{\partial y}(2\frac{\partial^2}{\partial x^2}\Pi_2 - \Lambda^2\Pi_1)\nabla^2\nabla^2 G \\ \phi_2 = -c_3c_4\nabla^2\nabla^2\nabla^2\nabla^2 G \\ \psi_1 = c_1c_2R\frac{\partial^2}{\partial y^2}(2\frac{\partial^2}{\partial x^2}\Pi_1\Pi_2 - \Lambda^2\Pi_1^2)\nabla^2 G + c_2c_4\Lambda^2\nabla^2\nabla^2\nabla^2\nabla^2 G \\ \psi_2 = c_1c_2R\frac{\partial^2}{\partial x\partial y}(2\frac{\partial^2}{\partial x^2}\Pi_2^2 - \Lambda^2\Pi_1\Pi_2)\nabla^2 G + 2c_2c_4\frac{\partial^2}{\partial x\partial y}\nabla^2\nabla^2\nabla^2\nabla^2 G \\ \psi_3 = -\frac{1}{R}K_2c_3c_4\nabla^2\nabla^2\nabla^2\nabla^2\nabla^2 G, \end{cases} \tag{47}$$

then the equilibrium equations and the deformation compatibility equations will be satisfied automatically. After derivation, the final control equation is

$$\nabla^2\nabla^2\nabla^2\nabla^2\nabla^2\nabla^2 G = 0 \tag{48}$$

(the approximation of $\frac{R^2}{K_1\mu} << 1$), in which

$$\begin{aligned} c_1 &= \frac{R(2K_2-K_1)(\mu K_1+\mu K_2-3R^2)}{2(\mu K_1-2R^2)}, c_2 = \frac{1}{R}K_2(\mu K_2 - R^2) - R(2K_2 - K_1) \\ c_3 &= \mu(K_1 - K_2) - R^2 - \frac{(\mu K_2-R^2)^2}{\mu K_1-2R^2}, c_4 = c_1R + \frac{1}{2}c_3(K_1 + \frac{\mu K_1-2R^2}{\lambda+\mu}) \end{aligned} \tag{49a}$$

$$\Pi_1 = 3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \Pi_2 = 3\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \Lambda^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \tag{49b}$$

In summary, the icosahedral quasicrystals have six displacement components, 15 strain components, and 15 stress components, i.e., 36 unknown functions in total. It is the same as the number of fundamental equations. From the mathematical point of view, it can be solved under appropriate boundary conditions. Similarly, the problem may be decomposed into plane and anti-plane problems. In terms of the anti-plane problem, we will illustrate it with an example.

3. Some Examples

3.1. Complex Variable Theory for Elasticity of Quasicrystals with Defects

As we all know, it is very difficult to solve plane problems of the materials with defects in the physical plane (z-plane). For some complicated configuration, we have two common conformal mappings, and one is

$$z = \omega(\zeta) = R(\zeta + \sum_{k=0}^{n} d_k\zeta^{-k}) \sum_{k=0}^{n}|d_k| \leq 1 \tag{50}$$

to transform the exterior of the material with defects in the z-plane into the exterior of the unit circle in the ζ-plane. Through the transformation, we can have

$$\ln z = \ln\left[R\zeta(1 + \sum_{k=0}^{n} d_k\zeta^{-(k+1)})\right] = \ln R + \ln\zeta + \ln(1 + \sum_{k=0}^{n} d_k\zeta^{-(k+1)})$$

Because we have $|\zeta| > 1$ outside the unit, we can obtain $\sum\limits_{k=0}^{n}\left|\frac{d_k}{\zeta^{k+1}}\right| < 1$ and

$$\ln(1+\sum_{k=0}^{n} d_k\zeta^{-(k+1)}) = (\sum_{k=0}^{n} d_k\zeta^{-(k+1)}) + \frac{1}{2}(\sum_{k=0}^{n} d_k\zeta^{-(k+1)})^2 + \cdots\cdots = f(\zeta),$$

in which $f(\zeta)$ is analytic outside the unit circle. For the same reason, we have the formula $\frac{a_1}{z} = \frac{a_1}{R\zeta(1+\sum\limits_{k=0}^{n} d_k\zeta^{-(k+1)})} = \frac{a_1}{R\zeta}(1-\sum\limits_{k=0}^{n} d_k\zeta^{-(k+1)} - \cdots\cdots)$ and so on.

The other conformal mapping is

$$z = \omega(\zeta) = R(\frac{1}{\zeta} + \sum_{k=0}^{n} C_k\zeta^k)\sum_{k=0}^{n}|C_k| \leq 1 \tag{51}$$

to transform the exterior of the material with defects in the z-plane into the interior of the unit circle in the ζ-plane.

3.2. An Extended Dugdale model for Anti-Plane Quasicrystals

We deduce the boundaries: it is assumed that there is a penetrating elliptical orifice along the periodic direction z in one-dimensional hexagonal quasicrystals. At this moment, it belongs to plane elasticity problem in the periodic plane $x-y$. In the periodic oxy plane, we suppose elliptic orifice equation is $L : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in which a and b are the long and short axis respectively in Figure 1. Considering that the quasicrystal is not subjected to external forces at infinity, it is subjected to the shearing force τ along the quasi-periodic direction on the surface of the elliptical hole. We can express the boundary by formulas as follows:

$$\begin{cases} z \to \infty, \sigma_{yz} = \sigma_{zx} = H_{zy} = H_{zx} = 0 \\ z = t \in L, \sigma_{yz} = -\tau, H_{zy} = -\tau_1 \end{cases} \tag{52}$$

Draw the conformal mapping

$$z = w(\zeta) = A(m\zeta + \frac{1}{\zeta}),\ A = \frac{a+b}{2}, m = \frac{a-b}{a+b} \tag{53}$$

into the problem to transform the region containing ellipse at the z-plane onto the interior of the unit circle at the ζ-plane. Obviously, $\zeta = \sigma = e^{i\theta}$ on the unit circle r. Signing $\phi_1(w(\zeta)) = \phi(\zeta), \psi_1(w(\zeta)) = \psi(\zeta)$, we deduce

$$\phi_1'(z) = \frac{\phi'(\zeta)}{w'(\zeta)}, \psi_1'(z) = \frac{\psi'(\zeta)}{w'(\zeta)} \tag{54}$$

Colligating Equations (52)–(54), we can get

$$\begin{cases} \varphi(\zeta) = \frac{K_2\tau - R_3\tau_1}{C_{44}K_2 - R_3^2} 2iAm\zeta \\ \psi(\zeta) = -\frac{C_{44}\tau_1 - R_3\tau}{C_{44}K_2 - R_3^2} 2iAm\zeta \end{cases} \tag{55}$$

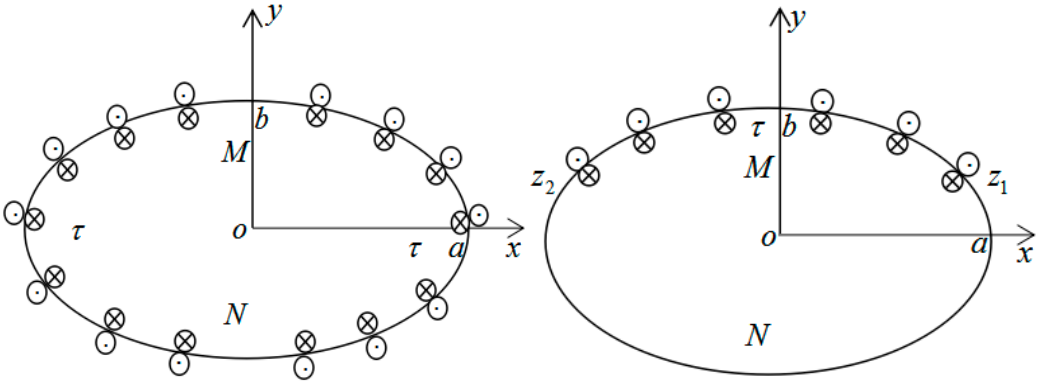

Figure 1. An elliptic notch in an anti-plane quasicrystal.

In a similar manner, we can also calculate the solution of following boundary conditions

$$\begin{cases} z \to \infty, \sigma_{yz} = \sigma_{zx} = H_{zy} = H_{zx} = 0 \\ z \in z_1\overset{\frown}{m}z_2, \sigma_{yz} = -\tau, H_{zy} = 0 \\ z \in z_1\overset{\frown}{n}z_2, \sigma_{yz} = 0, H_{zy} = 0. \end{cases} \tag{56}$$

The solution is

$$\begin{cases} \varphi(\zeta) = \frac{AK_2\tau}{\pi(C_{44}K_2-R_3^2)}[-\frac{4}{\zeta}\ln\sigma_2 + (m\zeta+\frac{1}{\zeta})\ln(\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta}\frac{\overline{\sigma_2}+\zeta}{\sigma_2+\zeta}) + (m\sigma_2+\frac{1}{\sigma_2})\ln\frac{\sigma_2+\zeta}{\sigma_2-\zeta} \\ +(m\overline{\sigma_2}+\frac{1}{\overline{\sigma_2}})\ln(\frac{\overline{\sigma_2}-\zeta}{\overline{\sigma_2}+\zeta})] \\ \psi(\zeta) = -\frac{AR_3\tau}{\pi(C_{44}K_2-R_3^2)}[-\frac{4}{\zeta}\ln\sigma_2 + (m\zeta+\frac{1}{\zeta})\ln(\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta}\frac{\overline{\sigma_2}+\zeta}{\sigma_2+\zeta}) + (m\sigma_2+\frac{1}{\sigma_2})\ln\frac{\sigma_2+\zeta}{\sigma_2-\zeta} \\ +(m\overline{\sigma_2}+\frac{1}{\overline{\sigma_2}})\ln(\frac{\overline{\sigma_2}-\zeta}{\overline{\sigma_2}+\zeta})] \end{cases} \tag{57}$$

Now, we solve the problem of the Dugdale model of one-dimensional hexagonal anti-plane quasicrystals. The boundary conditions are as follows:

$$\begin{cases} z \to \infty : \sigma_{yz}(z) = \tau^{(\infty)}, \sigma_{xz} = H_{xz} = H_{zx} = H_{yz} = H_{zy} = 0 \\ y = 0, |x| < a : \sigma_{xz} = \sigma_{yz} = H_{xz} = H_{yz} = H_{zx} \\ y = 0, a < |x| < a+R : \sigma_{yz} = \tau, \sigma_{xz} = H_{xz} = H_{yz} = H_{zx} = H_{zy} = 0 \end{cases} \tag{58}$$

This problem can be transformed into two simple questions:

$$\begin{cases} z \to \infty : \sigma_{yz}(z) = \tau^{(\infty)}, \sigma_{xz} = H_{xz} = H_{zx} = H_{yz} = H_{zy} = 0 \\ y = 0, |x| < a+R : \sigma_{yz} = \sigma_{xz} = H_{xz} = H_{yz} = H_{zx} = H_{zy} = 0 \end{cases} \tag{59a}$$

and

$$\begin{cases} z \to \infty : \sigma_{yz}(z) = \sigma_{xz} = H_{xz} = H_{zx} = H_{yz} = H_{zy} = 0 \\ y = 0, |x| < a : \sigma_{xz} = \sigma_{yz} = H_{xz} = H_{yz} = H_{zx} \\ y = 0, a < |x| < a+R : \sigma_{yz} = \tau, \sigma_{xz} = H_{xz} = H_{yz} = H_{zx} = H_{zy} = 0 \end{cases} \tag{59b}$$

The conformal mapping is

$$z = w(\zeta) = \frac{a+R}{2}(\zeta + \frac{1}{\zeta}) \tag{60}$$

to transform the region containing ellipse at the z-plane onto the interior or the exterior of the unit circle at the ζ-plane. The solution of problem I has been given. To highlight problem, the only thing to do is replace τ by σ^{∞}, A by $\frac{a+R}{2}$, and ζ of the formula by $\frac{1}{\zeta}$, $\tau_1 = 0, m = 1$. Thus, there are

$$\begin{cases} \varphi(\zeta) = \frac{(a+R)K_2\tau^{(\infty)}i}{C_{44}K_2-R_3^2}\frac{1}{\zeta} \\ \psi(\zeta) = -\frac{(a+R)R_3\tau^{(\infty)}i}{C_{44}K_2-R_3^2}\frac{1}{\zeta} \end{cases} \tag{61}$$

The solution of problem II has also been given, only to substitute $-\sigma_s$ for τ in (57), $a+R$ for a over there. Consequently,

$$\begin{cases} \varphi(\zeta) = -\frac{(a+R)K_2\tau}{2\pi(C_{44}K_2-R_3^2)}[-\frac{4i\phi_2}{\zeta} + \frac{2z}{a+R}\ln\frac{\sigma_2^2-\zeta^2}{\overline{\sigma}_2^2-\zeta^2} + (\overline{\sigma}_2^2+\sigma_2)\ln(\frac{\sigma_2+\zeta}{\sigma_2-\zeta}\frac{\overline{\sigma}_2-\zeta}{\overline{\sigma}_2+\zeta})] \\ \psi(\zeta) = \frac{(a+R)R_3\tau}{2\pi(C_{44}K_2-R_3^2)}[-\frac{4i\phi_2}{\zeta} + \frac{2z}{a+R}\ln\frac{\sigma_2^2-\zeta^2}{\overline{\sigma}_2^2-\zeta^2} + (\overline{\sigma}_2+\sigma_2)\ln(\frac{\sigma_2+\zeta}{\sigma_2-\zeta}\frac{\overline{\sigma}_2-\zeta}{\overline{\sigma}_2+\zeta})] \end{cases} \tag{62}$$

Superposition of the above two types, that is, the total complex potential. Because the stress is limited where $\zeta = 0$, the first item of Equation (29) must be zero. It is obtained as follows

$$\phi_2 = -\frac{\pi\tau^{(\infty)}}{2\tau} \tag{63}$$

Likewise, $\sigma_2 = e^{i\varphi_2}$ represents one of a value of ζ on the unit circle r, it corresponds to $z = a = (a+R)\cos\varphi_2$, hence it can get

$$R = a[\sec(\frac{\pi\tau^{(\infty)}}{2\tau}) - 1] \tag{64}$$

Based on the theory of fracture, the displacement of the curve L is

$$(u_z)_L = \mathrm{Re}[\varphi_1(z)]_L = \mathrm{Re}[\varphi(\zeta)]_r \tag{65}$$

Among which, $z = x+iy$, L indicate the crack surface, r the unit circle, $\phi(\zeta) = \phi_1(w(\zeta)) = \phi_1(z)$, $\zeta = \sigma = e^{i\varphi}$ on the unit circle, and there are

$$\begin{cases} \frac{\sigma_2^2-\zeta^2}{\overline{\sigma}_2^2-\zeta^2} = \frac{e^{2i\phi_2}-e^{2i\phi}}{e^{-2i\phi_2}-e^{2i\phi}} = \frac{\sin(\phi-\phi_2)}{\sin(\phi+\phi_2)}e^{2i\phi_2} \\ \frac{\sigma_2+\zeta}{\sigma_2-\zeta}\cdot\frac{\overline{\sigma}_2-\zeta}{\overline{\sigma}_2+\zeta} = \frac{e^{i\phi_2}+e^{i\phi}}{e^{i\phi_2}-e^{i\phi}}\cdot\frac{e^{-i\phi_2}-e^{i\phi}}{e^{-i\phi_2}+e^{i\phi}} = \frac{\sin\phi+\sin\phi_2}{\sin\phi-\sin\phi_2} \end{cases} \tag{66}$$

If we let

$$\delta = u_z(x,0^+) - u_z(x,0^-) = 2u_z(x,0) \tag{67}$$

denote the crack opening displacement. Deduce

$$\delta_t = \lim_{x\to a} 2u_z(x,0) = \lim_{\phi\to\phi_2} 2u_z(x,0) \tag{68}$$

Let $\varphi = \varphi_2 + \Delta\varphi$, we have

$$\sin(\phi-\phi_2) = \sin\Delta\phi \approx \Delta\phi, \sin\phi - \sin\phi_2 \approx \Delta\phi\cos\phi_2 \tag{69}$$

and can get the result

$$\delta_t = \frac{-4K_2\tau a}{\pi(C_{44}K_2-R_3^2)}[\ln\sec(\frac{\pi\tau^{(\infty)}}{2\tau})] \tag{70}$$

The problem of the Dugdale model of three-dimensional anti-plane quasicrystals is similar to the one of one-dimensional hexagonal anti-plane quasicrystals. After a series of calculating, in case of duplication, we show the conclusions directly

$$\delta_t = \frac{-2(K_1 - K_2)\tau a}{\pi[R^2 - \mu(K_1 - K_2)]}[\ln\sec(\frac{\pi\tau^{(\infty)}}{2\tau})] \tag{71}$$

3.3. An Extended Dugdale Model for Plane Problem of Two-Dimensional Quasicrystals

Secondly, we consider one example of plane problem of two-dimensional quasicrystals. There is a Griffith crack with the length of $2(l + b)$ along the z axis in the generalized cohesive force model of quasicrystals (see Figure 2.).

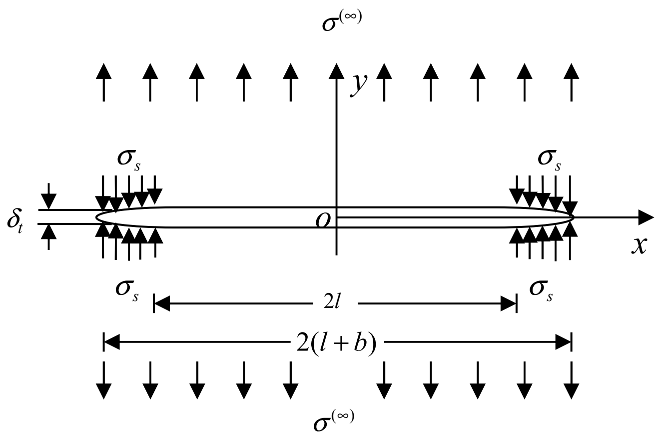

Figure 2. Generalized cohesive force model of quasicrystals.

We have assumed already that the stress distribution within the plastic zone is known. Thus, the generalized cohesive force model of quasicrystals can be expressed as follows:

$$\begin{cases} \sigma_{yy} = \sigma^{(\infty)},\ H_{yy} = 0,\ \sigma_{xx} = \sigma_{xy} = 0,\ H_{xx} = H_{xy} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0,\ H_{yy} = H_{xy} = 0 & y = 0,\ |x| < l \\ \sigma_{yy} = \sigma_s,\ \sigma_{xy} = 0,\ H_{yy} = 0,\ H_{xy} = 0 & y = 0,\ l < |x| < l+b \end{cases} \tag{72}$$

This problem can be reduced into a superposition of the following two problems:

$$\begin{cases} \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0,\ H_{xx} = H_{xy} = H_{yy} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0, H_{xy} = H_{yy} = 0 & y = 0,\ |x| < l \\ \sigma_{yy} = \sigma_s,\ \sigma_{xy} = 0,\ H_{xy} = H_{yy} = 0 & l < |x| < l+b \end{cases} \tag{73a}$$

and

$$\begin{cases} \sigma_{yy} = \sigma^{(\infty)},\ \sigma_{xx} = \sigma_{xy} = 0, H_{xx} = H_{xy} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0,\ H_{yy} = H_{xy} = 0 & y = 0,\ |x| < l+b \end{cases} \tag{73b}$$

Thus, the nonlinear problem is reduced into an "equivalent" elasticity problem. However, the calculation cannot be completed in the z-plane due to the complicacy of the evaluation, and we use the conformal mapping

$$z = \omega(\zeta) = \frac{(l+b)}{2}(\zeta + \frac{1}{\zeta}) \tag{74}$$

to transform the exterior of the crack in the z-plane into the exterior or interior of the unit circle in the ζ-plane. It is worth noting that $\widetilde{h_2}(\zeta)$ and $h_2(\zeta)$ will no longer be referred in the following text. It is clear that the boundary condition can be written as follow:

$$f_4(\zeta) + \overline{f_3(\zeta)} + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{f_4'(\zeta)} = \frac{i}{32c_1}\int (T_x + iT_y)ds \tag{75}$$

in which $f_4(\zeta)$ denotes $h_4(\zeta)$ or $\widetilde{h_4}(\zeta)$; $f_3(\zeta)$ represents $h_3(\zeta)$ or $\widetilde{h_3}(\zeta)$; and T_x and T_y are generalized surface tractions in the x-direction and y-direction, respectively. In above analysis process, the phason filed can be discussed similarly, so it is omitted here.

For the problem in Equation (73a), we have the solution

$$\begin{cases} h_4(\zeta) = \frac{1}{32c_1} \cdot \frac{\sigma_s(l+b)\varphi_2}{\pi} \cdot \frac{1}{\zeta} - \frac{1}{32c_1} \cdot \frac{\sigma_s}{2\pi i}\left[z(\ln\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta} + \ln\frac{\sigma_2+\zeta}{\overline{\sigma_2}+\zeta}) - l\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}\right], \\ h_3(\zeta) = \frac{1}{32c_1} \cdot \frac{\sigma_s(l+b)\varphi_2}{\pi} \cdot \frac{2\zeta}{\zeta^2-1} - \frac{1}{32c_1} \cdot \frac{\sigma_s l}{2\pi i}\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}, \end{cases} \tag{76}$$

and for the problem in Equation (73b), the solution is

$$\begin{cases} \widetilde{h_4}(\zeta) = -\frac{1}{32c_1}\frac{\sigma^{(\infty)}}{2}(l+b)\frac{1}{\zeta} \\ \widetilde{h_3}(\zeta) = -\frac{\sigma^{(\infty)}}{32c_1}(l+b)\left[\frac{\zeta}{(\zeta^2-1)}\right] \end{cases} \tag{77}$$

in which

$$\sigma_2 = e^{i\varphi_2}, l = (l+b)\cos\varphi_2 \tag{78}$$

and we must point out that $\sigma_2 = e^{i\varphi_2}$ corresponds to $z = l$ in the z-plane.

Thus, we have the complex potentials of the generalized cohesive force model of quasicrystals as follows:

$$\begin{aligned} &\widehat{h_4}(\zeta) = h_4(\zeta) + \widetilde{h_4}(\zeta) = \\ &\frac{(l+b)}{32c_1} \cdot \left(\frac{\sigma_s\varphi_2}{\pi} - \frac{\sigma^{(\infty)}}{2}\right)\frac{1}{\zeta} - \frac{1}{32c_1} \cdot \frac{\sigma_s}{2\pi i}\left[z(\ln\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta} + \ln\frac{\zeta+\sigma_2}{\zeta+\overline{\sigma_2}}) - l\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}\right], \\ &\widehat{h_3}(\zeta) = h_3(\zeta) + \widetilde{h_4}(\zeta) = \frac{(l+b)}{32c_1} \cdot \left(\frac{2\sigma_s\varphi_2}{\pi} - \sigma^{(\infty)}\right)\frac{2\zeta}{\zeta^2-1} - \frac{1}{32c_1} \cdot \frac{\sigma_s l}{2\pi i}\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}. \end{aligned} \tag{79}$$

Because the stresses are finite values in the tip of plastic zone, we have

$$\frac{(l+b)}{32c_1} \cdot \left(\frac{\sigma_s\varphi_2}{\pi} - \frac{\sigma^{(\infty)}}{2}\right)\frac{1}{\zeta} = 0 \tag{80}$$

and we obtain

$$\varphi_2 = \frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s} \tag{81}$$

At the same time, noting that $l = (l+b)\cos\varphi_2$, one can have the size of plastic zone

$$b = l\left[\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) - 1\right] \tag{82}$$

Now, we will calculate the crack tip opening displacement δ_t. According to the solutions in Equations (76) and (79) and the displacement formula, after a lengthy calculation, we have

$$u_y(x,0) = (128c_1c_2 - 64c_3)\mathrm{Im}(\widehat{h_4}(\zeta)) \tag{83}$$

By substituting Equation (79) into Equation (83) and noting Equation (81), one can obtain

$$u_y(x,0) = \frac{(4c_1c_2 - 2c_3)}{c_1} \cdot \frac{\sigma_s(l+b)}{2\pi} \cdot \left[\cos\varphi \ln\frac{\sin(\varphi_2 - \varphi)}{\sin(\varphi_2 + \varphi)} - \cos\varphi_2 \ln\frac{(\sin\varphi_2 - \sin\varphi)}{(\sin\varphi_2 + \sin\varphi)}\right] \tag{84}$$

Thus, we have

$$\delta_t = \lim_{x\to l} 2u_y(x,0) = \lim_{\phi\to\phi_2} 2u_y(x,0) = \frac{(8c_1c_2 - 4c_3)\sigma_s l}{c_1\pi} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) \tag{85}$$

When we assume that and $R_2 = 0$ in all equations, then δ_t will become the corresponding solution of point group 10 mm quasicrystals

$$\delta_t = \frac{2\sigma_s l}{\pi}\left[\frac{1}{L+M} + \frac{K_1}{MK_1 - R^2}\right] \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) \tag{86}$$

Let the phonon–phason coupling constant $R = o(K_1)$, $K_1 \to 0$, then one will have the crack tip opening displacement for conventional crystals such as

$$\delta_t = \frac{2\sigma_s l}{\pi}\frac{(L+2M)}{(L+M)M} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) \tag{87}$$

in which $L = C_{12}$ and $M = C_{66}$. If the crystals are of isotropic solid, there are $L = \lambda$, and $M = \mu$, which are Lame constants, i.e.,

$$\delta_t = \begin{cases} \frac{(1+\kappa)\sigma_s l}{\pi\mu} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) = \frac{4(1-\nu)\sigma_s l}{\pi\mu} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}), & \text{plane strain state} \\ \frac{(1+\kappa')\sigma_s l}{\pi\mu} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) = \frac{4\sigma_s l}{(1+\nu)\pi\mu} \ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}), & \text{plane stress state} \end{cases} \tag{88}$$

in which $\kappa = 3 - 4\nu$ is for plane strain, $\kappa' = \frac{3-\nu}{1+\nu}$ is for plane stress, and ν is not only the Poisson ratio of the isotropic solid, but is also exactly reduced into the well-known classic Dugdale solution in nonlinear fracture mechanics of conventional structural (engineering) materials.

3.4. An Elastic Problem of Plane Problem of Two-Dimensional Quasicrystals with Elliptical Hole with Double Cracks

Based on Equation (36), it leads to

$$\begin{cases} \sigma_{xx} = -32c_1\mathrm{Re}(\Omega(z) - 2g4'''(z)) \\ \sigma_{yy} = 32c_1\mathrm{Re}(\Omega(z) + 2g4'''(z)) \\ \sigma_{xy} = \sigma_{yx} = 32c_1\mathrm{Im}\Omega(z) \end{cases} \tag{89}$$

in which

$$\Omega(\mathrm{z}) = \mathrm{g_3}^{(4)}(z) + \bar{\mathrm{z}}\mathrm{g_4}^{(4)}(\mathrm{z})$$

in addition, we replace $g_3'''(z)$ and $g_4''(z)$ by $h_3(z)$ and $h_4(z)$, and we can get

$$\sigma_{xx} + \sigma_{yy} = 128c_1\mathrm{Re}g_4'''(z) = 128c_1\mathrm{Re}h_4'(z) \tag{90a}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 64\left(g_3^{(\mathrm{IV})}(z) + \bar{z}g_4^{(\mathrm{IV})}(z)\right) = 64\left(h_3'(z) + \bar{z}h_4''(z)\right) \tag{90b}$$

According to Reference [23], there are

$$h_4(z) = d_1(X + iY)\ln z + Bz + h_4^0(z) \tag{91a}$$

$$h_3(z) = d_2(X - iY)\ln z + \left(B' + iC'\right)z + h_3^0(z) \tag{91b}$$

in which

$$d_1 = \frac{1}{64c_1\pi\times(32(4c_1c_2-c_3-c_1c_4)+1)}$$
$$d_2 = -\frac{4c_1c_2-c_3-c_1c_4}{2c_1\pi\times(32(4c_1c_2-c_3-c_1c_4)+1)}$$

$$c_2 = \frac{(L+M)(K_1+K_2)+c}{4c(L+M)}, c_3 = \frac{R_1^2+R_2^2}{c}, c_4 = \frac{K_1+K_2}{c}$$

Thereby, we have

$$h_4^*(\zeta) + \overline{h_3^*(0)} + \frac{1}{2\pi i}\int_r \frac{w(\sigma)}{\overline{w'(\sigma)}}\frac{\overline{h_4^{*\prime}(\sigma)}}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_r \frac{f_0(\sigma)}{\sigma-\zeta}d\sigma \tag{92}$$

$$h_3^*(\zeta) + \overline{h_4^*(0)} + \frac{1}{2\pi i}\int_r \frac{\overline{w(\sigma)}}{w'(\sigma)}\frac{h_4^{*\prime}(\sigma)}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_r \frac{\overline{f_0(\sigma)}}{\sigma-\zeta}d\sigma \tag{93}$$

$$\begin{aligned} f_0(\sigma) \quad &= \tfrac{i}{32c_1}\int \overline{X}+i\overline{Y}ds - (d_1-d_2)(X+iY)\ln\sigma - \tfrac{w(\sigma)}{w'(\sigma)}d_1(X-iY)\sigma \\ &-2Bw(\sigma) - (B'-iC')\overline{w(\sigma)} \end{aligned} \tag{94}$$

in which $\overline{X}$ and $\overline{Y}$ indicate the surface force components of the surface force $\overline{X}$ and $\overline{Y}$, respectively; and B and $B' - iC'$ are decided by primary stress.

Supposing that elliptic hole with two cracks is in an infinite plane, the major semi-axis is a, the minor semi-axis is b, the crack length is $c - a$, we may have the elliptical hole center as the origin, and the crack where the straight line coordinate system is established for the X axis (Figure 3).

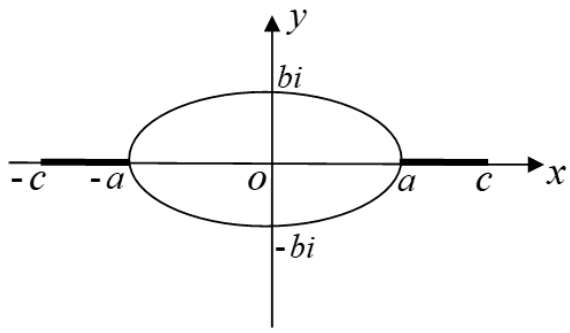

Figure 3. A model of two dimensional quasicrystals with an elliptical hole with double cracks.

The problem is solved using the conformal mappings

$$\begin{aligned} z = w(\zeta) = &\tfrac{a+b}{2}\tfrac{2(d^2+1)(1+\zeta^2)+(d^2-1)\sqrt{(1+\zeta)^4+2k(1-\zeta^2)^2+(1-\zeta)^4}}{8d\zeta} \\ &+\tfrac{a-b}{2}\tfrac{8d\zeta}{2(d^2+1)(1+\zeta^2)+(d^2-1)\sqrt{(1+\zeta)^4+2k(1-\zeta^2)^2+(1-\zeta)^4}} \end{aligned} \tag{95}$$

among which

$$k = \frac{d^4+6d^2+1}{(d^2-1)^2}, d = \frac{c+\sqrt{c^2-a^2+b^2}}{a+b} \tag{96}$$

This mapping maps the exterior of the crack in the z-plane into the exterior or interior of the unit circle in the ζ-plane.

The mapping maps the inner angle of the unit circle on the mathematical plane to the infinite plane of the elliptical hole with double cracks on the physical plane, and $w^{-1}(c) \to 1$, $w^{-1}(bi) \to -i$, $w^{-1}(-c) \to -1$, $w^{-1}(-bi) \to i$. Meanwhile, the shore point $(a^+, 0)$ is mapped to the point $\left(\frac{2d}{d^2+1}, -\frac{d^2-1}{d^2+1}\right)$, and the lower shore $(a^-, 0)$ is mapped to the point $\left(\frac{2d}{d^2+1}, \frac{d^2-1}{d^2+1}\right)$. In the fracture theory, defining the complex stress intensity factor is

$$K = K_{\mathrm{I}} - iK_{\mathrm{II}} \tag{97}$$

Besides, we consider $z - z_1 = r_1 e^{i\theta_1}$, where z_1 indicates one of the end of the crack tip field, namely, $z \to z_1$, there is

$$\sigma_{xx} + \sigma_{yy} = 2\mathrm{Re}\frac{K}{\sqrt{2\pi(z - z_1)}} \tag{98}$$

We know

$$\sigma_{xx} + \sigma_{yy} = 128c_1 \mathrm{Re} h_4'(z) \tag{99}$$

Thus,

$$K = 64\sqrt{2\pi}c_1\sqrt{z - z_1}h_4'(z) \tag{100}$$

Then, the stress intensity factor can be determined

$$K_{\mathrm{I}}^{(c,0)} - iK_{\mathrm{II}}^{(c,0)} = 64\sqrt{2\pi}c_1\sqrt{z - z_1}h_4'(z) \tag{101}$$

There are boundary conditions:

$$\sqrt{x^2 + y^2} \to \infty : \sigma_{xx} = \sigma_{xy} = 0, \sigma_{yy} = q, H_{xx} = H_{xy} = H_{yx} = H_{yy} = 0 \tag{102a}$$

$$(x, y) \in L : \begin{cases} \sigma_{xx}\cos(n, x) + \sigma_{xy}\cos(n, y) = 0 \\ \sigma_{yx}\cos(n, x) + \sigma_{yy}\cos(n, y) = 0 \\ H_{xx}\cos(n, x) + H_{xy}\cos(n, y) = 0 \\ H_{yx}\cos(n, x) + H_{yy}\cos(n, y) = 0 \end{cases} \tag{102b}$$

$$y = \pm 0, a < x < c : \sigma_{xy} = \sigma_{yy} = 0, H_{yx} = H_{yy} = 0 \tag{102c}$$

$$y = \pm 0, -c < x < -a : \sigma_{xy} = \sigma_{yy} = 0, H_{yx} = H_{yy} = 0 \tag{102d}$$

L means elliptical hole, $\cos(n, x)$ and $\cos(n, y)$ are on behalf of outside normal direction cosine of any point on the line L, respectively, and n is outward normal of the curve.

We take care of the following facts

$$B = \frac{q}{128c_1}, B' + iC' = \frac{q}{64c_1}, \overline{X} = \overline{Y} = X = Y = 0 \tag{103a}$$

and

$$h_4(\zeta) = \frac{q}{128c_1}w(\zeta) + h_4^*(\zeta), h_3(z) = \frac{q}{64c_1}w(\zeta) + h_3^*(\zeta) \tag{103b}$$

By the knowledge of the Cauchy integral formula and the analytic continuation, and paying attention to $|\sigma| = 1$, $w(\sigma) = \overline{w(\sigma)}$, we know

$$h_4^*(\zeta) + \overline{h_3^*(0)} + \frac{1}{2\pi i}\int_r \frac{w(\sigma)}{\overline{w'(\sigma)}} \frac{\overline{h_4^{*\prime}(\sigma)}}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i}\int_r \frac{f_0(\sigma)}{\sigma - \zeta} d\sigma \tag{104}$$

It can be

$$h_4^*(\zeta) = \frac{1}{2\pi i}\int_r \frac{f_0(\sigma)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i}\int_r \frac{-\frac{q}{64c_1}w(\sigma) - \frac{q}{64c_1}\overline{w(\sigma)}}{\sigma - \zeta} d\sigma = -\frac{q}{32c_1} \cdot \frac{1}{2\pi i}\int_r \frac{w(\sigma)}{\sigma - \zeta} d\sigma \tag{105}$$

In same manner, we have

$$h_3^*(\zeta) = -\frac{q}{32c_1} \cdot \frac{1}{2\pi i}\int_r \frac{w(\sigma)}{\sigma - \zeta} - \frac{\overline{w(\sigma)}}{w'(\sigma)} h_4^{*\prime}(\sigma) d\sigma \tag{106}$$

In view of the above facts, it is easy to obtain

$$h_4^{*\prime}(\zeta) = -\frac{q}{32c_1} \cdot \frac{1}{2\pi i}\int_r \frac{w'(\sigma)}{\sigma - \zeta} d\sigma \tag{107}$$

Meanwhile, we note the mapping in Equation (95) yields

$$\begin{gathered} w'(\zeta) = (1-\zeta^2)\left[(d^2+1) + \frac{(d^2-1)(1+k)(1+\zeta^2)}{\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}}\right] \\ \left\{-\frac{a+b}{8d\zeta^2} + \frac{8d(a-b)}{\left[2(d^2+1)(1+\zeta^2) + (d^2-1)\sqrt{(1+\zeta)^4 + 2k(1-\zeta^2)^2 + (1-\zeta)^4}\right]^2}\right\} \end{gathered} \tag{108}$$

It is obvious that $w'(\zeta)$ is analytical outside the unit circle and continuously to the boundary, thus we can get

$$\frac{1}{2\pi i}\int_r \frac{w'(\sigma)}{\sigma - \zeta} d\sigma = \frac{(a+b)(d^2+1)}{4d} \tag{109}$$

Therefore,

$$h_4'(\zeta) = \frac{q}{128c_1} w'(\zeta) - \frac{q}{128c_1}\frac{(a+b)(d^2+1)}{d} \tag{110}$$

Finally, we have the tress intensity factor

$$K_{\mathrm{I}}^{(c,0)} = \frac{q(a+b)\sqrt{\pi(d^4-1)}}{\sqrt{2d[(d^2-1)a + (d^2+1)b]}},\ K_{\mathrm{II}}^{(c,0)} = 0 \tag{111}$$

If we regard static stress intensity factor $K_{\mathrm{I}}^{\parallel,\mathrm{static}} = \sqrt{\pi a}q$ of mode I centric crack problems as the normalized stress intensity factor, then the dimensionless stress intensity factor can be introduced as $\frac{K^{\parallel}}{K^{\parallel,\mathrm{static}}}$. The comparison of the normalized stress intensity factor of various crack radio is depicted in Figure 4.

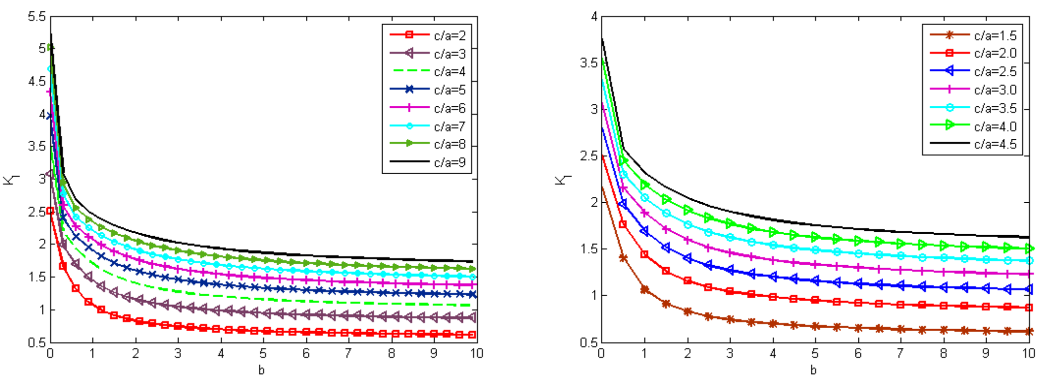

Figure 4. Variation and comparison of normalized stress intensity factor with crack radio $\frac{c}{a}$.

An Extended Dugdale Model for Plane Problems of Three-Dimensional Icosahedral Quasicrystal

The boundary conditions of the extended Dugdale model for plane problems of three-dimensional icosahedral quasicrystal can be expresses as follows:

$$\begin{cases} z \to \infty : \sigma_{yy}(z) = \sigma^{(\infty)}, \sigma_{xx} = \sigma_{xy} = H_{xx} = H_{xy} = H_{yy} = 0 \\ y = 0, |x| < a : \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = H_{xx} = H_{yy} = H_{xy} = H_{yx} \\ y = 0, |x| < a + R : \sigma_{yy} = \sigma_s, \sigma_{xx} = \sigma_{xy} = H_{xx} = H_{yy} = H_{xy} = H_{yx} = 0 \end{cases} \tag{112}$$

According to Reference [23], the general solution of Equation (48) is

$$G = \frac{1}{128}\mathrm{Re}[g_1(z) + \bar{z}g_2(z) + \bar{z}^2 g_3(z) + \bar{z}^3 g_4(z) + \bar{z}^4 g_5(z) + \bar{z}^5 g_6(z)].$$

Meanwhile, it yields

$$\begin{cases} \sigma_{xx} + \sigma_{yy} = 48c_2c_3R\mathrm{Im}\Gamma'(z) \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 8ic_2c_3R(12\psi'(z) - \Omega'(z)) \\ u_y + iu_x = -6c_3R(\frac{2c_2}{\lambda+\mu} + c_4)\overline{\Gamma(z)} - 2c_3c_7R\Omega(z) \\ \lambda = C_{12}, \mu = \frac{C_{11}-C_{12}}{2} \end{cases} \tag{113}$$

The boundary condition above can be written as follows:

$$3\overline{\Gamma}(\bar{z}) - \Omega(z) = 0 \tag{114}$$

In the above analytic process, the phason field has been omitted. We can calculate the solution by the complex variable function method, however, the formulation and calculations are lengthy and omitted; we only give the final outcome:

$$\Gamma(\zeta) = \frac{(a+R)}{12c_2c_3R}[\frac{\sigma_s\phi_2}{\pi} - \frac{\sigma^{(\infty)}}{2}]\frac{1}{\zeta} - \frac{1}{12c_2c_3R}\frac{\sigma_s}{2\pi}\left[z(\ln\frac{\sigma_2-\zeta}{\overline{\sigma}_2-\zeta} + \ln\frac{\sigma_2+\zeta}{\overline{\sigma}_2+\zeta}) - a\ln\frac{(\zeta-\sigma_2)}{(\zeta+\sigma_2)}\frac{(\zeta+\overline{\sigma}_2)}{(\zeta-\overline{\sigma}_2)}\right] \tag{115}$$

in which

$$z = w(\zeta) = \frac{a+R}{2}(\zeta + \frac{1}{\zeta})$$

Based on the same principle with the above problem, we have

$$\frac{(a+R)}{12c_2c_3R}[\frac{\sigma_s\varphi_2}{\pi} - \frac{\sigma^{(\infty)}}{2}]\frac{1}{\zeta} = 0. \tag{116}$$

Then,

$$\varphi_2 = \frac{\pi\sigma^{(\infty)}}{2\sigma_s}. \tag{117a}$$

$$R = a[\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) - 1]. \tag{117b}$$

$$u_y(x,0) = -12c_3R(\frac{c_2}{\lambda+\mu} + c_4)\mathrm{Re}(\Gamma(\zeta)). \tag{117c}$$

$$\delta_t = \mathrm{CTOD} = \lim_{x\to a} 2u_y(x,0) = \lim_{\varphi\to\varphi_2} 2u_y(x,0) = 2(\frac{1}{\lambda+\mu} + \frac{c_4}{c_2})\frac{\sigma_s a}{\pi}\ln\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}). \tag{117d}$$

$$\sec(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}) = 1 + \frac{1}{2}(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s})^2 + \cdots\cdots \tag{117e}$$

All parameters have been given above. All these solutions can be found in Reference [26]. We neglect the complicated and lengthy computation process.

4. Conclusions

The static elastic problems of quasicrystals are mathematically daunting. For the statics problem of quasicrystals, systematical and direct methods of mathematical physics are provided to solve the equations under appropriate boundary value conditions, and many analytical solutions are constructed. Based on previous theory static elastic problems of quasicrystals, this paper mainly presents for static elastic response of quasicrystals by using complex potential method. Two kinds of case are considered. One is the anti-plane problem of static elastic problems of the crack for quasicrystals, in which we mainly consider one-dimensional hexagonal quasicrystals and three-dimensional icosahedral quasicrystals. The results of analysis are obtained based on some regular specimen. The other is the plane problem of static elastic problems of the crack for two-dimensional quasicrystals, in which we mainly consider the quasicrystals with ten-fold symmetries. The phonon and phason elastic fundamental fields along with their coupling effect in crack analysis are explicitly presented in terms of analytical expressions. Through comparing the results of quasicrystals, this paper reveals that the influence of phason field and phonon–phason coupling, which occupy an important position in elastic deformation behavior of quasicrystals, should not be neglected.

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