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Neutrosophic Cubic Power Muirhead Mean Operators with Uncertain Data for Multi-Attribute Decision-Making

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Received: 31 August 2018; Accepted: 25 September 2018; Published: 28 September 2018

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Abstract: The neutrosophic cubic set (NCS) is a hybrid structure, which consists of interval neutrosophic sets (INS) (associated with the undetermined part of information associated with entropy) and single-valued neutrosophic set (SVNS) (associated with the determined part of information). NCS is a better tool to handle complex decision-making (DM) problems with INS and SVNS. The main purpose of this article is to develop some new aggregation operators for cubic neutrosophic numbers (NCNs), which is a basic member of NCS. Taking the advantages of Muirhead mean (MM) operator and power average (PA) operator, the power Muirhead mean (PMM) operator is developed and is scrutinized under NC information. To manage the problems upstretched, some new NC aggregation operators, such as the NC power Muirhead mean (NCPMM) operator, weighted NC power Muirhead mean (WNCPMM) operator, NC power dual Muirhead mean (NCPMM) operator and weighted NC power dual Muirhead mean (WNCPDMM) operator are proposed and related properties of these proposed aggregation operators are conferred. The important advantage of the developed aggregation operator is that it can remove the effect of awkward data and it considers the interrelationship among aggregated values at the same time. Furthermore, a novel multi-attribute decision-making (MADM) method is established over the proposed new aggregation operators to confer the usefulness of these operators. Finally, a numerical example is given to show the effectiveness of the developed approach.

Keywords: NC power dual MM operator (NCPDMM) operator; NCPMM operator; MADM; MM operator; Neutrosophic cubic sets; PA operator

1. Introduction

One of the drawbacks of real MADM problems is expressing attribute values in fuzzy and indeterminate DM environments. Fuzzy sets (FSs) developed by Zadeh [1] emerged as a tool for describing and communicating uncertainties and vagueness. Since its beginning, FS has gained a significant focus from researchers all over the world who studied its practical and theoretical aspects. Several extensions of FSs have been developed, such as interval-valued FS (IVFS) [2], which explained the truth membership degree (TMD) on a closed interval value in the interval [0, 1], and intuitionistic FS (IFS) [3], which explained the TMD and falsity-membership degree (FMD). Therefore, IFS defines fuzziness and uncertainty more comprehensively than FS. However, neither FS nor IFS are capable to handle indeterminate and inconsistent information. For example, when we take a student opinion about the teaching skills of a professor with about 0.6 being the possibility that the teaching skills of the professor are good, 0.5 being the possibility that the teaching skills of the professor are bad and 0.3 is the possibility that he/she may not be sure about the teaching skills of the professor whether bad or good. To handle such type of information, Smarandache [4] added a new component "indeterminacy membership degree" (IMD) to the TMD and FMD, all being independent elements lying in $]0^-, 1^+[$. The resulting set is now familiarly known as neutrosophic set (NS). To use NS in practical and engineering problems, some scholars developed simplified forms of NS, such as SVNS [5], INS [6,7], simplified neutrosophic sets [8,9], multi-valued NS [10], Q-neutrosophic soft set [11], complex neutrosophic soft expert set [12] and others.

In the real world, sometimes it is difficult to express the TMD in some fuzzy problems completely by an exact value or interval value. Therefore, Jun et al. [13] developed the concept of cubic set (CS) by combining FS and IVFS. CS defined uncertainty and vagueness by an interval value and a fuzzy value concurrently. In recent years, some researchers established some extended forms of CS. Garg et al. [14,15] combined IFS and interval-valued intuitionistic FS (IVIFS) to form cubic IFS (CIFS), while Ali et al. [16] and Jun et al. [17] combined INS and SVNS to develop the cubic NS (CNS), consisting of internal and external NCSs. Jun et al. [18] further investigated P-union and P-intersection of NCS and discussed their related properties. Since then, various studies to solve MADM problems based on NCSs are developed. Zhang et al. [19] and Ye [20] developed some aggregation operators such as weighted averaging operators and weighted geometric operators on NCSs and applied these to MADM. Shi et al. [21], developed some aggregation operator for NCNs based on Dombi T-norm and T-conorm and applied these to MADM. To solve MADM problems under NC information, various similarity measures are developed for NCSs [22,23]. Pramanik et al. [24] introduced the NC-TODIM method to solve multiple-attribute group decision-making (MAGDM) problem.

Aggregation operator (AO) plays a dominant role in DM. Consequently, many scholars proposed different aggregation operators and their generalizations, such as Bonferroni mean (BM) operator [25,26], Heronian mean (HM) operator [27], Muirhead mean (MM) operator [28], Maclaurin symmetric mean (MSM) operator [29,30] and others. Certainly, different AOs have different functions. Some can remove the effect of awkward data given by prejudiced DMs, such as power average (PA) operator [31,32] developed by Yager [31] which can aggregate the input information by giving the weighted vector based on support degree among the input arguments. Some aggregation operators are capable to consider the interrelationship among two or more input arguments such as BM operator, HM operators, MSM operator and MM operator.

Due to the enhanced complexity in real decision-making problems, it is necessary to look over the following questions when selecting the best alternative. Firstly, the values of the attributes provided by the decision makers may be too low or too high, thus giving a negative impact on the final ranking results. The PA operator, however, permits the evaluated values to be mutually supported and enhanced. Therefore, we may use the PA operator to diminish such awful impact by designating distinct weights produced by the support measure. Secondly, the values of attributes are required to be dependent. Hence, the interrelationship among the values of the attributes should be examined. Some advantages of MM operator over BM and HM are discussed by Liu et al. [33,34]. Some existing aggregation operator such as the BM and MSM operators are special cases of the MM operator. The MM operator consists of the parameter vector, which enlarges the flexibility in the aggregation process. Recently, Li et al. [35] developed the concept of power Muirhead mean operator under Pythagorean fuzzy environment. From the existing literature, the PA operator and MM operator have not been yet combined to deal with NC information. To handle the issues raised, a few new aggregation operators will be proposed by incorporating both the PA and MM operators. These new aggregation operators are NC power MM operator (NCPMM), weighted NC power MM operator, NC power dual MM operator (NCPDMM) and weighted NC power dual MM (WNCPDMM) operator. Discussions on some basic properties and related cases with respect to the parameter vector will be dealt at length. The advantages of these proposed aggregation operators are to capture the interrelationship among input arguments by the MM operator, and simultaneously eliminate the effect of awkward data. Finally, a novel approach to solve MADM problems based on these proposed aggregation operators will be developed.

The rest of the article is organized as follows. In Section 2, some basic definitions and properties of NCSs, MM and PA operators are recalled. In Section 3, the PA and MM operators in the construction of new operators, namely NCPMM, WNCPMM, NCPDMM and WNCPDMM operators are incorporated followed by discussions on their related properties. In Section 4, a novel method to MADM is established based on the developed aggregation operators. In Section 5, a numerical example is illustrated to show the effectiveness of the proposed method to solve a MADM problem. In Section 6, a comparison with the existing methods is given followed by the conclusion.

2. Preliminaries

In this part, some basic concepts about SVNSs, INSs, NCSs, PA and MM operators are briefly overviewed.

2.1. The NCSs and Their Operations

Definition 1 ([4]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. A neutrosophic set N in Γ is defined as $N = \{\langle n; T_N(n), I_N(n), F_N(n)\rangle n \in \Gamma\}$ where $T_N(n), I_N(n)$ and $F_N(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively, such that $T; F; I : \Gamma \to]0^-, 1^+[$ and $0^- \leq T_N(n) + I_N(n) + F_N(n) \leq 3^+$.*

Smarandache [4] developed the concept of NS as a generalization of FS, IFS and IVIFS. To apply NS to real and engineering problems easily, its parameters should be specified. Hence, Wang et al. [5] provided the following definition.

Definition 2 ([5]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. A single-valued neutrosophic set S in Γ is defined as:*

$$S = \int_{\Gamma} \langle T_S(n), I_S(n), F_S(n)\rangle | n, n \in \Gamma \tag{1}$$

when Γ is continuous, and

$$S = \sum_{i=1}^{m} \langle T_S(n_i), I_S(n_i), F_S(n_i)\rangle | n_i, n_i \in \Gamma \tag{2}$$

when Γ is discrete, where $T_S(n), I_S(n)$ and $F_S(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively, such that $T; F; I : \Gamma \to [0,1]$ and $0 \leq T_S(n) + I_S(n) + F_S(n) \leq 3$.

Definition 3 ([6]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. An interval neutrosophic set A in Γ is defined as:*

$$A = \int_{\Gamma} \langle T_A(n), I_A(n), F_A(n)\rangle | n, n \in \Gamma \tag{3}$$

when Γ is continuous, and

$$A = \sum_{i=1}^{m} \langle T_A(n_i), I_A(n_i), F_A(n_i)\rangle | n_i, n_i \in \Gamma \tag{4}$$

when Γ is discrete, where $T_A(n), I_A(n)$ and $F_A(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively. For each element n in Γ, we have

$T_A(n) = \left[T_A^L(n), T_A^U(n)\right] \subseteq [0,1], I_A(n) = \left[I_A^L(n), I_A^U(n)\right] \subseteq [0,1]$, and $F_A(n) = \left[F_A^L(n), F_A^U(n)\right] \subseteq [0,1]$ such that

$0 \leq \sup T_A^U(n) + \sup I_A^U(n) + \sup F_A^U(n) \leq 3.$

Definition 4 ([16,17]). *Let Γ be a non-empty set. A neutrosophic cubic set (NCS) in Γ is a pair $Z = \langle A, \lambda \rangle$, where $A = \{\langle n, T_A(n), I_A(n), F_A(n) \rangle | n \in \Gamma\}$ is an interval neutrosophic set in Γ and $\lambda = \{\langle n, \lambda_T(n), \lambda_I(n), \lambda_F(n) \rangle | n \in \Gamma\}$ is a neutrosophic set in Γ.*

For simplicity, a basic element $\{n, \langle T(n), I(n), F(n) \rangle, \langle \lambda_T(n), \lambda_I(n), \lambda_F(n) \rangle\}$ in a NCS can be expressed by $z = (\langle T, I, F \rangle, \langle \lambda_T, \lambda_I, \lambda_F \rangle)$, which is called neutrosophic cubic number (NCN), where $T, I, F \subseteq [0,1]$ and $\lambda_T, \lambda_I, \lambda_F \in [0,1]$, satisfying $0 \leq T^U + I^U + F^U \leq 3$ and $0 \leq \lambda_T + \lambda_I + \lambda_F \leq 3$.

Definition 5 ([20]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ and $z_2 = (\langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle, \langle \lambda_{T_2}, \lambda_{I_2}, \lambda_{F_2} \rangle)$ be any two NCNs and $\xi > 0$. Then the operational laws for NCNs defined by Ye [20] are as follows:*

$$\begin{aligned}(1)\ z_1 \oplus z_2 = \ & (\langle [T_1^L + T_2^L - T_1^L T_2^L,\ T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U] \rangle, \\ & \langle \lambda_{T_1} + \lambda_{T_2} - \lambda_{T_1} \lambda_{T_2}, \lambda_{I_1} \lambda_{I_2}, \lambda_{F_2} \lambda_{F_2} \rangle);\end{aligned} \tag{5}$$

$$\begin{aligned}(2)\ z_1 \otimes z_2 = \ & (\langle [T_1^L T_2^L,\ T_1^U T_2^U], [I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U] \rangle, \\ & \langle \lambda_{T_1} \lambda_{T_2}, \lambda_{I_1} + \lambda_{I_2} - \lambda_{I_1} \lambda_{I_2}, \lambda_{F_2} + \lambda_{F_2} - \lambda_{F_2} \lambda_{F_2} \rangle);\end{aligned} \tag{6}$$

$$(3)\ \xi z_1 = \left(\left\langle \left[1 - (1 - (T_1^L))^{\xi}, 1 - (1 - (T_1^U))^{\xi}\right], \left[(I_1^L)^{\xi}, (I_1^U)^{\xi}\right], \left[(F_1^L)^{\xi}, (F_1^U)^{\xi}\right] \right\rangle, \left\langle 1 - (1 - \lambda_{T_1})^{\xi}, (\lambda_{I_1})^{\xi}, (\lambda_{F_1})^{\xi} \right\rangle\right); \tag{7}$$

$$(4)\ z_1^{\xi} = \left\langle \left[(T_1^L)^{\xi}, 1 - (T_1^U)^{\xi}\right], \left[1 - (1 - I_1^L)^{\xi}, 1 - (1 - I_1^U)^{\xi}\right], \left[1 - (1 - F_1^L)^{\xi}, 1 - (1 - F_1^U)^{\xi}\right] \right\rangle, \left\langle (\lambda_{T_1})^{\xi}, 1 - (1 - \lambda_{I_1})^{\xi}, 1 - (1 - \lambda_{F_1})^{\xi} \right\rangle. \tag{8}$$

Definition 6 ([21]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ be an NCN. Then, the score, accuracy, and certainty functions of NCN are defined as follows:*

$$\widehat{S}(z_1) = \frac{4 + T_1^L - I_1^L - F_1^L + T_1^U - I_1^U - F_1^U + \lambda_{T_1} + 2 - \lambda_{I_1} - \lambda_{F_1}}{9}; \tag{9}$$

$$\widehat{A}(z_1) = \frac{T_1^L - I_1^L + T_1^U - I_1^U + \lambda_{T_1} - \lambda_{F_1}}{3} \text{ and } \hat{C}(z_1) = \frac{T_1^L + T_1^U + \lambda_{T_1}}{3}. \tag{10}$$

Theorem 1 ([21]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ and $z_2 = (\langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle, \langle \lambda_{T_2}, \lambda_{I_2}, \lambda_{F_2} \rangle)$. Then the comparison rules for NCNs can be defined as follows:*

(i) *If $\hat{S}(z_1) > \hat{S}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(ii) *If $\hat{S}(z_1) = \hat{S}(z_2)$, and $\hat{A}(z_1) > \hat{A}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(iii) *If $\hat{S}(z_1) = \hat{S}(z_2)$, $\hat{A}(z_1) = \hat{A}(z_2)$, and $\hat{C}(z_1) > \hat{C}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(iv) *If $\hat{S}(z_1) = \hat{S}(z_2)$, $\hat{A}(z_1) = \hat{A}(z_2)$, and $\hat{C}(z_1) = \hat{C}(z_2)$, then z_1 is equal to z_2, and is denoted by $z_1 = z_2$.*

2.2. Power Average (PA) Operator

The PA operator was first introduced by Yager [31] for classical number. The dominant edge of PA operator is its capacity to diminish the inadequate effect of unreasonably too high and too low arguments on the inconclusive results.

Definition 7 ([31]). *Let $\Re_g (g = 1, 2, \ldots, a)$ be a group of classical numbers. The PA operator is then represented as follows:*

$$PA(\Re_1, \Re_2, \ldots, \Re_a) = \sum_{g=1}^{a} \left(\frac{(1 + T(\Re_g))}{\sum_{x=1}^{a} (1 + T(\Re_x))} \Re_g \right) \tag{11}$$

where, $T(\Re_z) = \sum_{\substack{x=1 \\ g \neq x}}^{a} Supp(\Re_g, \Re_x)$ and $Supp(\Re_z, \Re_x)$ is the support degree for $\Re_g$ and $\Re_x$. The support degree must satisfy the following axioms:

(1) $Supp(\Re_g, \Re_x) \in [0, 1]$;
(2) $Supp(\Re_g, \Re_x) = Supp(\Re_x, \Re_g)$;
(3) *If $\overline{\overline{D}}(\Re_g, \Re_x) < \overline{\overline{D}}(\Re_l, \Re_m)$, then $Supp(\Re_g, \Re_x) > Supp(\Re_l, \Re_m)$, where $\overline{\overline{D}}(\Re_g, \Re_x)$ is the distance measure among $\Re_g$ and $\Re_x$.*

2.3. Muirhead Mean (MM) Operator

The MM operator was first introduced by Muirhead [28] for classical numbers. MM operator has the advantage of considering the interrelationship among all aggregated arguments.

Definition 8 ([28]). *Let $\Re_g (g = 1, 2, \ldots, a)$ be a group of classical numbers and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the MM operator is described as:*

$$MM^{Q}(\Re_1, \Re_2, \ldots, \Re_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \Re_{\theta(g)}^{q_g} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \tag{12}$$

where, S_a is the group of permutation of $(1, 2, \ldots, a)$ and $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$.

Now we can give some special cases with respect to the parameter vector Q of the MM operator, which are shown as follows:

(1) If $Q = (1, 0, 0, \ldots, 0)$, then the MM operator degenerates to the following form:

$$MM^{(1,0,\ldots,0)}(\Re_1, \Re_2, \ldots, \Re_a) = \frac{1}{a} \sum_{g=1}^{a} \Re_g. \tag{13}$$

That is, the MM operator degenerates into arithmetic averaging operator.

(2) If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a} \right)$, then the MM operator degenerates to the following form:

$$MM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(\Re_1, \Re_2, \ldots, \Re_a) = \prod_{g=1}^{a} \Re_g^{\frac{1}{a}}. \tag{14}$$

That is, the MM operator degenerates into geometric averaging operator.

(3) If $Q = (1, 1, 0, \ldots, 0)$, then the MM operator degenerates to the following form:

$$MM^{(1,1,0,\ldots,0)}(\Re_1, \Re_2, \ldots, \Re_a) = \left(\frac{1}{a(a+1)} \sum_{\substack{g,x=1 \\ g \neq x}}^{a} \Re_g \Re_x \right)^{\frac{1}{2}}. \tag{15}$$

That is, the MM operator degenerates into BM operator.

(4) If $Q = \left(\overbrace{1,1,\ldots,1}^{c},\overbrace{0,\ldots,0}^{a-c}\right)$, then the MM operator degenerates to the following form:

$$MM^{(\overbrace{1,1,\ldots,1}^{d},\overbrace{0,\ldots,0}^{a-d})}(\Re_1,\Re_2,\ldots,\Re_a) = \left(\frac{\sum\limits_{1\le x_1<x_2<\ldots<x_d\le a}\prod\limits_{y=1}^{d}\Re_{g_y}}{C_a^d}\right)^{\frac{1}{d}}. \tag{16}$$

That is, the MM operator degenerates into MSM operator.

3. Some Power Muirhead Mean Operator for NCNs

In this part, we first give the definitions of PMM operator and propose the concept of power dual Muirhead mean (PDMM) operator. Then, we extended both the aggregation operator to NCN environment.

Definition 9 ([35]). *Let $\Re_g(g=1,2,\ldots,a)$ be a group of classical numbers and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. Then, the PMM operator is explained as,*

$$PMM^Q(\Re_1,\Re_2,\ldots,\Re_a) = \left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(\frac{a\left(1+T\left(\Re_{\theta(g)}\right)\right)}{\sum\limits_{x=1}^{a}(1+T(\Re_x))}\Re_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \tag{17}$$

where, $T(\Re_g) = \sum\limits_{x=1,x\neq g}^{a} Supp(\Re_g,\Re_x)$ and $Supp(\Re_g,\Re_x)$ is the support degree for $\Re_g$ and $\Re_x$, satisfying the above conditions.

Definition 10. *Let $\Re_g(g=1,2,\ldots,a)$ be a group of classical numbers and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. Then, the PDMM operator is described as,*

$$PDMM^Q(\Re_1,\Re_2,\ldots,\Re_a) = \frac{1}{\sum\limits_{g=1}^{a}q_g}\left(\sum_{\theta\in S_a}\prod_{g=1}^{a}q_g\Re_{\theta(g)}^{\frac{a(1+T(\Re_{\theta(g)}))}{\sum_{x=1}^{a}(1+T(\Re_x))}}\right)^{\frac{1}{a!}} \tag{18}$$

where, $T(\Re_g) = \sum\limits_{x=1x\neq 1}^{a} Supp(\Re_g,\Re_x)$ and $Supp(\Re_g,\Re_x)$ is the support degree for $\Re_g$ and $\Re_x$, satisfying the above conditions.

3.1. The Neutrosophic Cubic Power Muirhead Mean (NCPMM) Operator

In this subsection, we extend the PMM operator to neutrosophic cubic environment and discuss some basic properties, and special cases of these developed aggregation operators with respect to the parameter Q.

Definition 11. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. If,*

$$NCPMM^Q(z_1, z_2, \ldots, z_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \left(\frac{a\left(1 + T\left(z_{\theta(g)}\right)\right)}{\sum\limits_{x=1}^{a} (1 + T(z_x))} z_{\theta(g)} \right)^{q_g} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \tag{19}$$

then, we call $NCPMM^Q$ the neutrosophic cubic power Muirhead mean operator, where S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and $T(z_x) = \sum\limits_{x=1, x \neq g}^{a} Supp(z_g, z_x)$, $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;
(2) $Supp(z_g, z_x) = Supp(z_x, z_z)$;
(3) *If $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, then $Supp(z_g, z_x) > Supp(z_u, z_v)$, where $\overline{\overline{D}}(z_g, z_x)$ is the distance among z_g and z_x.*

To write Equation (20) in a simple form, we can specify it as:

$$\Theta_g = \frac{(1 + T(z_g))}{\sum\limits_{x=1}^{a} (1 + T(z_x))}. \tag{20}$$

For suitability, we can call $(\Theta_1, \Theta_2, \ldots, \Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0, 1]$ and $\sum\limits_{g=1}^{a} \Theta_g = 1$. From the use of Equation (20), Equation (19) can be expressed as:

$$NCPMM^Q(z_1, z_2, \ldots, z_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \left(a\Theta_g z_{\theta(g)} \right)^{q_g} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}. \tag{21}$$

Based on the operational rules given in Definition 3 for NCNs, and Definition 11, we can have the following Theorem 2.

Theorem 2. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (21) is still an NCN and,*

$$\begin{gathered} NCPMM^Q(z_1, z_2, \ldots, z_a) = \\ \left(\left\langle \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (T^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (T^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (I^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \right. \right. \right. \\ \left. 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (I^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (F^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (F^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \Bigg\rangle \\ \left. \left\langle \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_I)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_F)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right). \end{gathered} \tag{22}$$

Proof. According to the operational laws for NCNs, we have

$$a\Theta_g z_{\theta(g)} = \left(\left\langle \left[1 - \left(1 - \left(T^L \right)_{\theta(g)} \right)^{a\Theta_g}, 1 - \left(1 - \left(T^U \right)_{\theta(g)} \right)^{a\Theta_g} \right], \left[\left(I^L \right)_{\theta(g)}^{a\Theta_g}, \left(I^U \right)_{\theta(g)}^{a\Theta_g} \right], \left[\left(F^L \right)_{\theta(g)}^{a\Theta_g}, \left(F^U \right)_{\theta(g)}^{a\Theta_g} \right] \right\rangle, \left\langle 1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g}, (\lambda_I)_{\theta(g)}^{a\Theta_g}, (\lambda_F)_{\theta(g)}^{a\Theta_g} \right\rangle \right).$$

Therefore,

$$\begin{aligned} \left(a\Theta_g z_{\theta(g)} \right)^{q_g} = & \left(\left\langle \left[\left(1 - \left(1 - \left(T^L \right)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g}, \left(1 - \left(1 - \left(T^U \right)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right], \left[, 1 - \left(1 - \left(I^L \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} 1 - \left(1 - (I)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right], \left[1 - \left(1 - \left(F^L \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - \left(F^U \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right] \right\rangle, \right. \\ & \left. \left\langle \left(1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - (\lambda_I)_{\theta(g)}^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - (\lambda_F)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right\rangle \right). \end{aligned}$$

Therefore,

$$\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \Bigg(\Bigg\langle \Bigg[\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}, \prod_{g=1}^{a}\Big(1-\big(1-(T^U)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg], \Bigg[1-\prod_{g=1}^{a}\Big(1-(I^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}, 1-\prod_{g=1}^{a}\Big(1-(I^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg], \Bigg[1-\prod_{g=1}^{a}\Big(1-(F^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}, 1-\prod_{g=1}^{a}\Big(1-(F^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg]\Bigg\rangle, \Bigg\langle \prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}, 1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}, 1-\prod_{g=1}^{a}\Big(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg\rangle\Bigg),$$

and

$$\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \Bigg\langle \Bigg[1-\prod_{\theta\in S_a}\Bigg(\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg), 1-\prod_{\theta\in S_a}\Bigg(\prod_{g=1}^{a}\Big(1-\big(1-(T^U)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg], \Bigg[\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg), \prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg]\Bigg), \Bigg[\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg), \prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg]\Bigg\rangle, \Bigg\langle 1-\prod_{\theta\in S_a}\Bigg(\prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg), \prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg), \prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_f)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg\rangle\Bigg).$$

Furthermore,

$$\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \Bigg(\Bigg\langle \Bigg[1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}, 1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^U)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg], \Bigg[\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}, \Bigg\langle \Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg], \Bigg[\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}, \Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg]\Bigg\rangle, \Bigg\langle 1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}, \Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}, \Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg\rangle\Bigg).$$

Hence,

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g \Gamma_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} = \Bigg(\Bigg\langle \Bigg[\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg]\Bigg\rangle, \Bigg\langle \Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg),$$

$$NCPMM^{Q}(z_1, z_2, \ldots, z_a) =$$
$$\Bigg(\Bigg\langle \Bigg[\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(T^U)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(I^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(F^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg]\Bigg\rangle, \Bigg\langle \Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg).$$

This is the required proof of Theorem 2. □

In the above equations, we calculate the PWV Θ, after calculating the support degree $Supp(z_g, z_x)$. First, we determined the $Supp(z_g, z_x)$ using

$$Supp(z_g, z_x) = 1 - \overline{\overline{D}}(z_g, z_x), \tag{23}$$

where,

$$\overline{\overline{D}}(z_g, z_x) = \sqrt{\begin{array}{l}\frac{1}{9}\Big(\left(T_g^L - T_x^L\right)^2 + \left(T_g^U - T_x^U\right)^2 + \left(I_g^L - I_x^L\right)^2 + \left(I_g^U - I_x^U\right)^2 + \left(F_g^L - F_x^L\right)^2 + \left(F_g^U - F_x^U\right)^2 \\ \left(\lambda_{T_g} - \lambda_{T_x}\right)^2 + \left(\lambda_{I_g} - \lambda_{I_x}\right)^2 + \left(\lambda_{F_g} - \lambda_{F_x}\right)^2\Big)\end{array}}. \tag{24}$$

Therefore, we use the equation

$$T(z_g) = \sum_{g=1, g\neq x}^{a} Supp(z_g, z_x) \tag{25}$$

to obtain the values of $T(z_g)$ $(g = 1, 2, \ldots, a)$. Then using Equation (20) we can get the PWV.

Theorem 3. *(Idempotency) Let $z_g(g=1,2,\ldots,a)$ be a group of NCNs, and $z_g=z$, for all $g=1,2,\ldots,a$. Then,*

$$NCPMM^{Q}(z_1,z_2,\ldots,z_a)=CN. \tag{26}$$

Proof. As $z_g=z$ for all $g=1,2,\ldots,a$, we have $Supp(z_g,z_x)=1$ for all $g,x=1,2,\ldots,a$. Therefore, we can get $\Theta_g=\frac{1}{a}$ for all g. Moreover,

$$\begin{aligned}
&CNPMM^{Q}(z_1,z_2,\ldots,z_a)=CNPMM^{Q}(z,z,\ldots,z)\\
&=\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(1-T^{L})^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(1-T^{L})^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-I^{La\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.\\
&\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-I^{Ua\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-F^{La\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-F^{Ua\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,\\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(1-\lambda_T)^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\lambda_I^{a\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\ 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\lambda_F^{a\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right),\\
&=\left(\left\langle\left[\left(1-\left(\left(1-(1-(1-T^{L}))^{\sum_{g=1}^{a}q_g}\right)^{a!}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\left(1-(1-(1-T^{U}))^{\sum_{g=1}^{a}q_g}\right)^{a!}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\left(1-(1-I^{L})^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.\\
&\left.1-\left(1-\left(1-(1-I^{U})^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left.\left[1-\left(1-\left(1-(1-F^{L})^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\left(1-(1-F^{U})^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,\\
&\left.\left\langle\left(1-\left(\left(1-(1-(1-\lambda_T))^{\sum_{g=1}^{a}q_g}\right)^{a!}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\left(1-(1-\lambda_I)^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\left(1-(1-\lambda_F)^{\sum_{g=1}^{a}q_g}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right),\\
&=\left(\left\langle\left[T^{L},T^{U}\right],\left[I^{L},I^{U}\right],\left[F^{L},F^{U}\right]\right\rangle,\left\langle\lambda_T,\lambda_I,\lambda_F\right\rangle\right)=z.
\end{aligned}$$

This is the required proof of Theorem 3. □

Theorem 4. *(Boundedness) Let $z_g(g=1,2,\ldots,a)$ be a group of NCNs. Where*

$$\begin{aligned}
\overset{-}{z}&=\min(z_1,z_2,\ldots,z_a)=\left(\left\langle\left[T^{-L},T^{-U}\right],\left[I^{+L},I^{+U}\right],\left[F^{+L},F^{+U}\right]\right\rangle,\left\langle{\lambda_T}^{-},{\lambda_I}^{+},{\lambda_F}^{+}\right\rangle\right),\ \text{and}\\
\overset{+}{z}&=\max(z_1,z_2,\ldots,z_a)=\left(\left\langle\left[T^{+L},T^{+U}\right],\left[I^{-L},I^{-U}\right],\left[F^{-L},F^{-U}\right]\right\rangle,\left\langle{\lambda_T}^{+},{\lambda_I}^{-},{\lambda_F}^{-}\right\rangle\right).
\end{aligned}$$

Then,

$$m\leq NCPMM^{Q}(z_1,z_2,\ldots,z_a)\leq n \tag{27}$$

where,

$$\begin{aligned}
m=&\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{-}{T}{}^{L}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{-}{T}{}^{U}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{+}{I}{}^{La\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.\\
&\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{+}{I}{}^{Ua\Theta_z}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{+}{F}{}^{La\Theta_z}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{+}{F}{}^{Ua\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,\\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\ 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).
\end{aligned}$$

and

$$\begin{aligned}
n=&\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{+}{T}{}^{L}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{+}{T}{}^{U}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{-}{I}{}^{La\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.\\
&\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{-}{I}{}^{Ua\Theta_z}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{-}{F}{}^{La\Theta_z}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\overset{-}{F}{}^{Ua\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,\\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\ 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).
\end{aligned}$$

Proof.

$$
\begin{aligned}
a\Theta_g z_{\theta(g)} &= \left(\left\langle\left[1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}, 1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right], \left[(I^L)^{a\Theta_g}_{\theta(g)}, (I^U)^{a\Theta_g}_{\theta(g)}\right], \left[(F^L)^{a\Theta_g}_{\theta(g)}, (F^U)^{a\Theta_g}_{\theta(g)}\right]\right\rangle, \left\langle 1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}, (\lambda_I)^{a\Theta_g}_{\theta(g)}, (\lambda_F)^{a\Theta_g}_{\theta(g)}\right\rangle\right) \\
&\geq \left(\left\langle\left[1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}, 1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right], \left[\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}, \left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right], \left[\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}, \left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right]\right\rangle, \left\langle 1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}, \left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}, \left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right\rangle\right).
\end{aligned}
$$

and

$$
\begin{aligned}
\left(a\Theta_g z_{\theta(g)}\right)^{q_g} &= \left(\left\langle\left[\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[, 1-\left(1-(I^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g} 1-\left(1-(I^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right],\right.\right. \\
&\left.\left.\left[1-\left(1-(F^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g} 1-\left(1-(F^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right]\right\rangle, \left\langle\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-(\lambda_I)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\left(1-(\lambda_F)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right\rangle\right) \\
&\geq \left(\left\langle\left[\left(1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[, 1-\left(1-\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g} 1-\left(1-\left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right],\right.\right. \\
&\left.\left.\left[1-\left(1-\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g} 1-\left(1-\left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right]\right\rangle, \left\langle\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right\rangle\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-(I^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(I^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right],\right.\right. \\
&\left.\left.\left[1-\prod_{g=1}^{a}\left(1-(F^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(F^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right]\right\rangle, \left\langle\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(\lambda_I)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(\lambda_F)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right\rangle\right) \geq \\
&\left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right],\right.\right. \\
&\left.\left.\left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right]\right\rangle, \left\langle\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right\rangle\right).
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right], \left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right],\right.\right. \\
&\left.\left.\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right]\right\rangle, \left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right\rangle\right) \\
&\geq \left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right], \left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right],\right.\right. \\
&\left.\left.\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right]\right\rangle, \left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right\rangle\right).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\right.\right. \\
&\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right], \left.\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle, \\
&\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-(\lambda_F)^{a\Theta_z}_{\theta(z)}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right) \\
&\geq \left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\right.\right. \\
&\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right], \left.\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle, \\
&\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_z}_{\theta(z)}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_z \Gamma_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a} q_z}} = \left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right],\right.\right. \\
&\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right]\right\rangle, \\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right\rangle\right) \\
&\geq \left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}},\right.\right.\right. \\
&\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right]\right\rangle, \\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right\rangle\right).
\end{aligned}
$$

This implies that $m \leq NCPMM^{Q}(z_1, z_2, \ldots, z_a)$. $\square$

In a similar way we can show that $NCPMM^{Q}(z_1, z_2, \ldots, z_a) \leq n$. Hence, $m \leq NCPMM^{Q}(z_1, z_2, \ldots, z_a) \leq n$.

The NCPMM operator does not have the property of monotonicity.

One of the leading advantages of NCPMM is its capacity to represent the interrelationship among NCNs. Furthermore, the NCPMM operator is more flexible in aggregation process due to parameter vector. Now, we discuss some special cases of NCPMM operators by assigning different values to the parameter vector.

Case 1. If $Q = (1, 0, \ldots, 0)$, then the NCPMM operator degenerates into the following form:

$$NCPMM^{(1,0,\ldots,0)}(z_1, z_2, \ldots, z_a) = \left(\sum_{g=1}^{a} \frac{\left(1 + T(z_g)\right)}{\sum_{x=1}^{a} \left(1 + T(z_x)\right)} z_g \right). \tag{28}$$

This is the NC power averaging operator.

Case 2. If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)$, then the NCPMM operator degenerates into the following form:

$$NCPMM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(z_1, z_2, \ldots, z_a) = \prod_{g=1}^{a} z_g^{\frac{\left(1+T(z_{\theta(g)})\right)}{\sum_{x=1}^{a}(1+T(z_x))}}. \tag{29}$$

This is the NC power geometric operator.

Case 3. If $Q = (1, 1, \ldots, 0)$, then the NCPMM operator degenerates into the following form:

$$\begin{aligned}
&NCPMM^{(1,1,0,\ldots,0)} = (z_1, z_2, \ldots, z_a) = \\
&\Bigg(\Bigg\langle \Bigg[\Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - T_g^L\right)^{\Theta_g}\right)\left(1 - \left(1 - T_x^L\right)^{\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}}, \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - T_g^U\right)^{\Theta_g}\right)\left(1 - \left(1 - T_x^U\right)^{\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}} \Bigg], \Bigg[1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - I_g^{L\Theta_g}\right)\left(1 - I_x^{L\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}}, \\
&1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - I_g^{U\Theta_g}\right)\left(1 - I_x^{U\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}} \Bigg], \Bigg[1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - F_g^{L\Theta_g}\right)\left(1 - F_x^{L\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}}, 1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - F_g^{U\Theta_g}\right)\left(1 - F_x^{U\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}} \Bigg], \\
&\Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - (\lambda_T)_g\right)^{\Theta_z}\right)\left(1 - \left(1 - T_x\right)^{\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}}, 1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (\lambda_I)_g^{\Theta_g}\right)\left(1 - (\lambda_T)_x^{\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}}, 1 - \Bigg(1 - \Bigg(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (\lambda_F)_z^{\Theta_z}\right)\left(1 - (\lambda_F)_x^{\Theta_x}\right)\right) \Bigg)^{\frac{1}{a^2-a}} \Bigg)^{\frac{1}{2}} \Bigg\rangle \Bigg).
\end{aligned} \tag{30}$$

This is the NC power Bonferroni mean operator $(p = q = 1)$.

Case 4. If $Q = \left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)$, then the NCPMM operator degenerates into the following form:

$$
\begin{aligned}
&NCPMM^{\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)}(z_1,z_2,\ldots,z_a) = \\
&\left\langle \left[\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-T^L_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, \left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-T^U_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right], \left[1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-I_{gx}^{L\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\right. \right. \\
&\left.\left. 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-I_{gx}^{U\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right], \left[1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-F_{gx}^{L\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-F_{gx}^{U\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right]\right\rangle, \\
&\left\langle \left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_T)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_I)_{gx}^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_F)_{gx}^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right\rangle.
\end{aligned}
\tag{31}
$$

This is the NC power Maclaurin symmetric mean operator.

3.2. Weighted Neutrosophic Cubic Power Muirhead Mean (WNCPMM) Operator

The NCPMM operator does not consider the weight of the aggregated NCNs. In this subsection, we develop the WNCPMM operator, which has the capacity of taking the weights of NCNs.

Definition 12. *Let $z_g(g = 1,2,\ldots,a)$ be a group of NCNs and $Q = (q_1, q_2,\ldots,q_a) \in R^a$ be a vector of parameters. If,*

$$
WNCPMM^{Q}(z_1,z_2,\ldots,z_a) = \left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(\frac{a\Xi_{\vartheta(g)}\Theta_{\theta(g)}}{\sum\limits_{x=1}^{a}\Xi_x\Theta_x} z_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum\limits_{g=1}^{a} q_g}}
\tag{32}
$$

then, we $WNCPMM^{Q}$ the weighted neutrosophic cubic power Muirhead mean operator, where $\Xi = (\Xi_1, \Xi_2,\ldots,\Xi_a)^T$ is the weight vector of $z_g(g = 1,2,\ldots,a)$ such that $\Xi_z \in [0,1]$, $\sum\limits_{z=1}^{a}\Xi_z = 1$, S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and Θ_g is power weight vector (PWV) satisfying $\Theta_g = \frac{(1+T(z_g))}{\sum\limits_{g=1}^{a}(1+T(z_g))}$, $\sum\limits_{g=1}^{a}\Theta_g = 1$, $T(z_x) = \sum\limits_{x=1,x\neq g}^{a} Supp(z_g,z_x)$, $Supp(z_g,z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g,z_x) \in [0,1]$;

(2) $Supp(z_g,z_x) = Supp(z_x,z_g)$;

(3) *If $\overline{\overline{D}}(z_g,z_x) < \overline{\overline{D}}(z_u,z_v)$, then $Supp(z_g,z_x) > Supp(z_u,z_v)$, where $\overline{\overline{D}}(z_g,z_x)$ is distance among z_g and z_x.*

From Definition 12, we have the following Theorem 5.

Theorem 5. *Let $z_g(g = 1,2,\ldots,a)$ be a group of NCNs and $Q = (q_1,q_2,\ldots,q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (32) is still an NCN and*

$$
\begin{aligned}
&WNCPMM^{Q}(z_1,z_2,\ldots,z_a) = \\
&\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right. \\
&\left.\left. 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle, \\
&\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).
\end{aligned}
\tag{33}
$$

Proof. Proof of Theorem 5 is same as Theorem 2. □

3.3. The Neutrosophic Cubic Power Dual Muirhead Mean (NCPDMM) Operator

In this subsection, we develop the NCPDMM operator and discuss some related properties.

Definition 13. *Let $z_g(g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. If,*

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = \frac{1}{\sum\limits_{g=1}^{a} q_g} \left(\prod_{\theta \in S_a} \sum_{g=1}^{a} \left(q_g z_{\theta(g)}^{\frac{a(1+T(z_{\theta(g)}))}{\sum_{x=1}^{a}(1+T(z_x))}} \right) \right)^{\frac{1}{a!}} \tag{34}$$

then, we call $NCPDMM^Q$ the neutrosophic cubic power dual Muirhead mean operator, where S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and $T(z_x) = \sum\limits_{x=1, x \neq g}^{a} Supp(z_g, z_x)$, $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;
(2) $Supp(z_g, z_x) = Supp(z_x, z_g)$;
(3) *If $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, then $Supp(z_g, z_x) > Supp(z_u, z_v)$, where $\overline{\overline{D}}(z_g, z_x)$ is distance among z_g and z_x.*

To write Equation (34) in a simple form, we can specify it as:

$$\Theta_g = \frac{(1 + T(z_g))}{\sum\limits_{x=1}^{a} (1 + T(z_x))}. \tag{35}$$

For suitability, we can call $(\Theta_1, \Theta_2, \ldots, \Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0, 1]$ and $\sum\limits_{g=1}^{a} \Theta_g = 1$. From, the use of Equation (35), Equation (34) can be expressed as,

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = \frac{1}{\sum\limits_{g=1}^{a} q_g} \left(\prod_{\theta \in S_a} \sum_{g=1}^{a} \left(q_g z_{\theta(g)}^{a\Theta_{\theta(g)}} \right) \right)^{\frac{1}{a!}}. \tag{36}$$

Theorem 6. *Let $z_g(g = 1, 2, \ldots, a)$ be a group of SVNNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (36) is still an NCN and,*

$$\begin{gathered} NCPDMM^Q(z_1, z_2, \ldots, z_a) = \\ \left(\left\langle \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (T^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (T^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (I^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right) ^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \right. \right. \right. \\ \left. \left. \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (I^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (F^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (F^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \right\rangle \\ \left. 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_T)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{z=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_I)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_F)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right). \end{gathered} \tag{37}$$

Proof. Proof of Theorem 6 is similar to that of Theorem 2. □

Theorem 7 (Idempotency)**.** *Let $z_g(g = 1, 2, \ldots, a)$ be a group of NCNs, and $z_g = z$, for all $g = 1, 2, \ldots, a$. Then,*

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = z. \tag{38}$$

Theorem 8 (Boundedness). *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs, $\overset{-}{z} = \min(z_1, z_2, \ldots, z_a) = \left(\left\langle \left[\overset{-}{T}^L, \overset{-}{T}^U\right], \left[\overset{+}{I}^L, \overset{+}{I}^U\right], \left[\overset{+}{F}^L, \overset{+}{F}^U\right] \right\rangle, \langle \lambda_T{}^-, \lambda_I{}^+, \lambda_F{}^+ \rangle\right)$, and $z^+ = \max(z_1, z_2, \ldots, z_a) = \left(\left\langle \left[\overset{+}{T}^L, \overset{+}{T}^U\right], \left[\overset{-}{I}^L, \overset{-}{I}^U\right], \left[\overset{-}{F}^L, \overset{-}{F}^U\right] \right\rangle, \langle \lambda_T{}^+, \lambda_I{}^-, \lambda_F{}^- \rangle\right)$.*

Then,

$$m \leq NCPDMM^Q(z_1, z_2, \ldots, z_a) \leq n. \tag{39}$$

where,

$$m = \left(\left\langle\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{T}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{T}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{I}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}},\right.\right.\right.$$
$$\left.\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{I}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left.\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{F}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{F}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right]\right\rangle,$$
$$\left.\left\langle 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a} q_z}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle\right),$$

and

$$n = \left(\left\langle\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{T}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{T}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{I}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}},\right.\right.\right.$$
$$\left.\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{I}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right], \left.\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{F}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{F}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}\right]\right\rangle,$$
$$\left.\left\langle 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a} q_z}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle\right).$$

Now we will discuss some special cases of NCPDMM operator with respect to the parameter vector Q.

Case 1. If $Q = (1, 0, \ldots, 0)$, then NCPDMM operators degenerate into the following form:

$$NCPDMM^{(1,0,\ldots,0)}(z_1, z_2, \ldots, z_a) = \left(\prod_{g=1}^{a} z_g^{\frac{(1+T(z_g))}{\sum_{x=1}^{a}(1+T(z_x))}}\right) \tag{40}$$

This is the NC power geometric averaging operator.

Case 2. If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)$, then NCPMM operators degenerate into the following form:

$$NCPDMM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(z_1, z_2, \ldots, z_a) = \sum_{g=1}^{a} \frac{(1+T(z_g))}{\sum_{x=1}^{a}(1+T(z_x))} z_g \tag{41}$$

This is NC power arithmetic averaging operator.

Case 3. If $Q=(1,1,0,\ldots,0)$, then NCPDMM operators degenerate into the following form:

$$
\begin{aligned}
&NCPDMM^{(1,1,0,\ldots,0)}(z_1,z_2,\ldots,z_a)=\Bigg(\Bigg\langle\Bigg[1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(T^L)_g^{\Theta_g}\right)\left(1-(T^L)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(T^U)_g^{\Theta_g}\right)\left(1-(T^U)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(I^L)_g\right)^{\Theta_g}\right)\left(1-\left(1-(I^L)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(I^U)_g\right)^{\Theta_g}\right)\left(1-\left(1-(I^U)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(F^L)_g\right)^{\Theta_g}\right)\left(1-\left(1-(F^L)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(F^U)_g\right)^{\Theta_g}\right)\left(1-\left(1-(F^U)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(\lambda_T)_g^{\Theta_g}\right)\left(1-(\lambda_T)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\lambda_I)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\lambda_I)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\\
&\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\lambda_F)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\lambda_F)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{42}
$$

This is the NC power geometric Bonferroni mean operator ($p=q=1$).

Case 4. If $Q=\left(\overbrace{1,1,\ldots,1}^{i}\overbrace{0,0,\ldots,0}^{z-i}\right)$, then the NCPDMM operator degenerates into the following form:

$$
\begin{aligned}
&NCPDMM^{(\overbrace{1,1,.,.,1}^{i}\overbrace{0,0,\ldots,0}^{z-i})}(z_1,z_2,\ldots,z_a)=\\
&\Bigg(\Bigg\langle\Bigg[1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(T^L)_{gx}^{\Theta_{zg}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(T^U)_{gx}^{\Theta_{zg}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg],\Bigg[\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(I^L)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\\
&\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(I^U)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg],\Bigg[\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(F^L)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(F^U)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg]\Bigg\rangle\\
&\Bigg\langle 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_T)_{gx}^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_I)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_F)_{gx}\right)^{\Theta_{gx}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{43}
$$

This is the NC power dual Maclaurin symmetric mean operator.

3.4. Weighted Neutrosophic Cubic Power Dual Muirhead Mean (WNCPDMM) Operator

The NCPDMM operator does not consider the weight of the aggregated NCNs. In this subsection, we develop the WNCPDMM operator, which has the capacity of taking the weights of NCNs.

Definition 14. *Let $z_g(g=1,2,\ldots,a)$ be a group of NCNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. If,*

$$
WNCPDMM^{Q}(z_1,z_2,\ldots,z_a)=\frac{1}{\sum_{g=1}^{a}q_g}\left(\prod_{\theta\in S_a}\sum_{g=1}^{a}\left(q_g z_{\theta(g)}^{\frac{a\Xi_{\vartheta(g)}\Theta_{\theta(g)}}{\sum_{x=1}^{a}\Xi_x\Theta_x}}\right)\right)^{\frac{1}{a!}}
\tag{44}
$$

then, we call $WNCPDMM^{Q}$ the weighted neutrosophic cubic power dual Muirhead mean operator, where $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_a)^T$ is the weight vector of $z_g (g = 1, 2, \ldots, a)$ such that $\Xi_g \in [0, 1]$, $\sum_{g=1}^{a} \Xi_g = 1$, S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and Θ_g is PVW satisfying $\Theta_g = \frac{(1+T(z_g))}{\sum_{g=1}^{a}(1+T(z_g))}$, $\sum_{g=1}^{a} \Theta_g = 1$, $T(z_x) = \sum_{x=1, x\neq g}^{a} Supp(z_g, z_x)$, and $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;

(2) $Supp(z_g, z_x) = Supp(z_x, z_g)$;

(3) *If* $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, *then* $Supp(z_g, z_x) > Supp(z_u, z_v)$, *where* $\overline{\overline{D}}(z_g, z_x)$ *is distance among* z_g *and* z_x.

From Definition 14, we have the following Theorem 9.

Theorem 9. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (44) is still an NCN and*

$$\begin{aligned}
&WNCPDMM^{Q}(z_1, z_2, \ldots, z_a) = \Bigg\langle \Bigg[1 - \left(1 - \prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \binom{L}{T}_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \binom{U}{T}_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \Bigg], \\
&\Bigg[\left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - \binom{L}{I}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - \binom{U}{I}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \Bigg], \\
&\Bigg[\left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - \binom{L}{F}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - \binom{U}{F}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \Bigg] \Bigg\rangle, \\
&1 - \left(1 - \prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - (\lambda_T)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - (\lambda_I)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \\
&, \left(1 - \left(\prod_{\theta\in S_a}\left(1 - \prod_{g=1}^{a}\left(1 - \left(1 - (\lambda_F)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \Bigg\rangle \Bigg).
\end{aligned} \tag{45}$$

Proof. Proof of Theorem 9 is similar to that of Theorem 2. □

4. The MADM Approach Based on WNCPMM Operator and WNCPDMM Operator

In this section, we give a novel method to MADM with NCNs, in which the attributes values gain the form of NCNs. For a MADM problem, let the series of alternatives is represented by $\hbar = \{\hbar_1, \hbar_2, \ldots, \hbar_a\}$, and the series of attributes is represented by $\lambdabar = \{\lambdabar_1, \lambdabar_2, \ldots, \lambdabar_b\}$. The weight vector of the attributes is denoted by $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_b)^T$ such that $\varpi_p \in [0, 1]$, $\sum_{p=1}^{b} \varpi_p = 1$. Assume that $z_{gh} = \left(\left\langle \left[T_{gh}^L, T_{gh}^U\right], \left[I_{gh}^L, I_{gh}^U\right], \left[F_{gh}^L, F_{gh}^U\right]\right\rangle, \langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \rangle\right)$ is the assessment values of the alternatives $\hbar_g$ on the attribute l_h, which is expressed by the form of NCN. Then, the main aim is to rank the alternatives. The following decision steps are to be followed.

Step 1. Standardize the decision matrix. Generally, there are two types of attributes, one is of cost type and the other is of benefit type. We need to convert the cost type of attributes into benefit types of attributes by using the following formula:

$$\begin{aligned}
z_{gh} &= \left(\left\langle \left[T_{gh}^L, T_{gh}^U\right], \left[I_{gh}^L, I_{gh}^U\right], \left[F_{gh}^L, F_{gh}^U\right]\right\rangle, \left\langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle\right), \\
&= \begin{cases} \left(\left\langle \left[T_{gh}^L, T_{gh}^U\right], \left[I_{gh}^L, I_{gh}^U\right], \left[F_{gh}^L, F_{gh}^U\right]\right\rangle, \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle), & \text{for benefit attribute } \Gamma_h, \\ \left(\left\langle \left[F_{gh}^L, F_{gh}^U\right], \left[1 - I_{gh}^U, 1 - I_{gh}^L\right], \left[T_{gh}^L, T_{gh}^U\right]\right\rangle, \lambda_{T_{gh}}, 1 - \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle), & \text{for cost attribute } \Gamma_h. \end{cases}
\end{aligned} \tag{46}$$

Hence, the decision matrix $M = \left[z_{gh}\right]_{a\times b}$ can be transformed into normalized decision matrix $N = \left[\delta_{gh}\right]_{a\times b}$.

Step 2. Determine the supports $Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$ by,

$$Supp\left(\delta_{gh}, \delta_{gl}\right) = 1 - \overline{\overline{D}}\left(\delta_{gh}, \delta_{gh}\right) \tag{47}$$

where, $\overline{\overline{D}}\left(\delta_{gh}, \delta_{gh}\right)$ is the distance measure among two NCNs δ_{gh} and δ_{gl} defined in Equation (25).

Step 3. Determine $T\left(\delta_{gh}\right)$ by,

$$T\left(\delta_{gh}\right) = \sum_{\substack{l=1\\ l\neq h}}^{b} Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b) \tag{48}$$

Step 4. Determine the weights related with the NCN $\delta_{gh}(g = 1, 2, \ldots, a; h = 1, 2, \ldots, b)$ with the formula

$$\Psi_{gh} = \frac{b\varpi_h\left(1 + T\left(\delta_{gh}\right)\right)}{\sum_{d=1}^{b} \varpi_d\left(1 + T\left(\delta_{gh}\right)\right)}(g = 1, 2, \ldots, a; h, d = 1, 2, \ldots, b), \tag{49}$$

where, $T\left(\delta_{gh}\right) = \sum_{\substack{l=1\\ l\neq h}}^{b} Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$ is weighted support of NCN δ_{gh} by the other NCN $\delta_{gl}(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$.

Step 5. Use the WNCPMM or WNCPDMM operators

$$\delta_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right], \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPMM^Q\left(\delta_{g1}, \delta_{g2}, \ldots, \delta_{gb}\right) \tag{50}$$

or

$$\delta_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right], \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPDMM^Q\left(\delta_{g1}, \delta_{g2}, \ldots, \delta_{gb}\right) \tag{51}$$

to calculate the overall NCNs, $\delta_g(g = 1, 2, \ldots, a)$.

Step 6. Determine the score values of the collective NCNs $\delta_g(g = 1, 2, \ldots, a)$, using Definition 6.

Step 7. Rank all the alternatives according to their score values, and the select the best one using Theorem 1.

5. An Illustrative Example

To show the application of the developed MADM method, an illustrative example is embraced from [19,21] with NC information.

Example 1. *A passenger wants to travel and select the best vans (alternatives)* $\hbar_g(g = 1, 2, 3, 4)$ *among the possible four vans. The customer takes the following four attributes into account to evaluate the possible four alternatives: (1) the facility* $\bar{\lambda}_1$*; (2) saving rent* $\bar{\lambda}_2$*; (3) comfort* $\bar{\lambda}_3$*; (4) safety* $\bar{\lambda}_4$*. The importance degree of the attributes is expressed by* $\varpi = (0.5, 0.25, 0.125, 0.125)^T$*. Therefore, the following decision matrix* $M = \left[z_{gh}\right]_{4\times 4}$ *can be obtained in the form of NCNs shown in Table 1.*

Table 1. The decision matrix $M = \left[CN_{gh}\right]_{4\times 4}$.

	$ƛ_1$	$ƛ_2$	$ƛ_3$	$ƛ_4$
$\hbar_1$	$(\langle[0.2,0.5],[0.3,0.7],[0.1,\ 0.2]\rangle,\langle 0.9,0.7,0.2\rangle)$	$(\langle[0.2,0.4],[0.4,0.5],[0.2,0.5]\rangle,\langle 0.7,0.4,0.5\rangle)$	$(\langle[0.2,0.7],[0.4,0.9],[0.5,0.7]\rangle,\langle 0.7,0.7,0.5\rangle)$	$(\langle[0.1,0.6],[0.3,0.4],[0.5,0.8]\rangle,\langle 0.5,0.5,0.7\rangle)$
$\hbar_2$	$(\langle[0.3,0.9],[0.2,0.7],[0.3,0.5]\rangle,\langle 0.5,0.7,0.5\rangle)$	$(\langle[0.3,0.7],[0.6,0.8],[0.2,0.4]\rangle,\langle 0.7,0.6,0.8\rangle)$	$(\langle[0.3,0.9],[0.4,0.6],[0.6,0.8]\rangle,\langle 0.9,\ 0.4,0.6\rangle)$	$(\langle[0.2,0.5],[0.4,0.9],[0.5,0.8]\rangle,\langle 0.5,0.2,0.7\rangle)$
$\hbar_3$	$(\langle[0.3,0.4],[0.4,0.8],[0.2,0.6]\rangle,\langle 0.1,0.4,0.2\rangle)$	$(\langle[0.2,0.4],[0.2,0.3],[0.2,0.5]\rangle,\langle 0.2,0.2,0.2\rangle)$	$(\langle[0.4,0.7],[0.1,0.2],[0.4,0.5]\rangle,\langle 0.9,0.5,0.5\rangle)$	$(\langle[0.6,0.7],[0.3,0.6],[0.3,0.7]\rangle,\langle 0.7,0.5,0.3\rangle)$
$\hbar_4$	$(\langle[0.5,0.9],[0.1,0.8],[0.2,0.6]\rangle,\langle 0.4,0.6,0.2\rangle)$	$(\langle[0.4,0.6],[0.5,0.7],[0.1,0.2]\rangle,\langle 0.5,0.3,0.2\rangle)$	$(\langle[0.5,0.6],[0.2,0.4],[0.3,0.5]\rangle,\langle 0.5,0.4,0.5\rangle)$	$(\langle[0.3,0.7],[0.7,0.8],[0.6,0.7]\rangle,\langle 0.4,0.2,0.8\rangle)$

Then, we apply the WNCPMM operator or WNCPDMM operator to solve the MADM problem. Now, we use the WNCPMM operator for this decision-making problem as follows:

Step 1. Since all the attributes are the same, hence there is no need for conversion.

Step 2. Use Equation (47), to calculate the support degrees $Supp\left(z_{gh}, z_{gl}\right)(1,2,\ldots,4; h,l = 1,2,\ldots,4)$. We denote $Supp\left(z_{gh}, z_{gl}\right)$ by $Supp_{gh,gl}$.

$$Supp_{11,12} = Supp_{12,11} = 0.79452, Supp_{11,13} = Supp_{13,11} = 0.735425, Supp_{11,14} = Supp_{14,11} = 0.65359,$$
$$Supp_{12,13} = Supp_{13,12} = 0.771478, Supp_{12,14} = Supp_{14,12} = 0.805635, Supp_{13,14} = Supp_{14,13} = 0.786563;$$
$$Supp_{21,22} = Supp_{22,21} = 0.7972, Supp_{21,23} = Supp_{23,21} = 0.7667, Supp_{21,24} = Supp_{24,21} = 0.727155,$$
$$Supp_{22,23} = Supp_{23,22} = 0.750556, Supp_{22,24} = Supp_{24,22} = 0.750556, Supp_{23,24} = Supp_{23,24} = 0.76906,$$
$$Supp_{31,32} = Supp_{32,31} = 0.8, Supp_{31,33} = Supp_{33,31} = 0.614139, Supp_{31,34} = Supp_{34,31} = 0.735425,$$
$$Supp_{32,33} = Supp_{33,32} = 0.690879, Supp_{32,34} = Supp_{34,32} = 0.711325, Supp_{33,34} = Supp_{33,34} = 0.797241,$$
$$Supp_{41,42} = Supp_{42,41} = 0.7551, Supp_{41,43} = Supp_{43,41} = 0.783975, Supp_{41,44} = Supp_{44,41} = 0.645662,$$
$$Supp_{42,43} = Supp_{43,42} = 0.783975, Supp_{42,44} = Supp_{44,42} = 0.675107, Supp_{43,44} = Supp_{44,43} = 0.7152.$$

Step 3. Use Equation (48), to get $T\left(\delta_{gh}\right)(g,h = 1\ to\ 4)$. We denote $T\left(\delta_{gh}\right)$ by T_{gh}.

$$\mathrm{T}_{11} = 2.183534, \mathrm{T}_{12} = 2.371633, \mathrm{T}_{13} = 2.293466, \mathrm{T}_{14} = 2.245787;$$
$$\mathrm{T}_{21} = 2.291063, \mathrm{T}_{22} = 2.298354, \mathrm{T}_{23} = 2.286283, \mathrm{T}_{24} = 2.246771$$
$$\mathrm{T}_{31} = 2.149564, \mathrm{T}_{32} = 2.202204, \mathrm{T}_{33} = 2.102259, \mathrm{T}_{34} = 2.243991,$$
$$\mathrm{T}_{41} = 2.184688, \mathrm{T}_{42} = 2.214133, \mathrm{T}_{43} = 2.28315, \mathrm{T}_{44} = 2.035969.$$

Step 4. Use Equation (49), to obtain $\Psi_{gh}(g,h = 1,2,3,4)$.

$$\Psi_{11} = 1.957844, \Psi_{12} = 1.036761, \Psi_{13} = 0.506363, \Psi_{14} = 0.499032,$$
$$\Psi_{21} = 2.002623, \Psi_{22} = 1.00353, \Psi_{23} = 0.499929, \Psi_{24} = 0.493918,$$
$$\Psi_{31} = 1.987975, \Psi_{32} = 1.010601, \Psi_{33} = 0.489529, \Psi_{34} = 0.511894,$$

Step 5. Use the WNCPMM given in Equation (50),

$$z_g = \left(\left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right] \right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle\right) = WCNPMM^Q(z_{g1}, z_{g2}, \ldots, z_{g4})(g = 1,2,\ldots,4).$$

To get the overall NCNs $z_g(g = 1,2,\ldots,4)$. Assume that $Q = (1,1,1,1)$.

$$z_1 = (\langle[0.1399, 0.4650], [0.4421, 0.7027], [0.4691, 0.6847]\rangle, \langle 0.5483, 0.6368, 0.6029\rangle);$$
$$z_2 = (\langle[0.2238, 0.6021], [0.5236, 0.8162], [0.5122, 0.715]\rangle, \langle 0.5617, 0.5505, 0.7294\rangle);$$
$$z_3 = \langle[0.3002, 0.4736], [0.3232, 0.5782], [0.3881, 0.6445]\rangle, \langle 0.3255, 0.4952, 0.415668\rangle;$$
$$z_4 = \langle[0.3413, 0.5540], [0.5437, 0.7485], [0.4487, 0.5965]\rangle, \langle 0.3762, 0.4451, 0.5976\rangle.$$

Step 6. Using Definition 6, we calculate the score values of the collective NCNs $z_g(g = 1, 2, \ldots, a)$.

$$\widetilde{SC}(z_1) = 0.4022, \widetilde{SC}(z_2) = 0.393352, \widetilde{SC}(z_3) = 0.472717, \widetilde{SC}(z_4) = 0.4324.$$

Step 7. According to the score values, ranking order of the alternative is $\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2$.

Hence using Theorem 1, the best alternative is $\hbar_3$ and the worst is $\hbar_2$.

Similarly, by using WNCPDMM operator for this decision-making problem, we will have, the Steps 1 to 4 are similar to that of weighted neutrosophic cubic power Muirhead mean operator.

Step 5. Use the WNCPDMM given in Equation (51),

$$z_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right] \right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPDMM^Q(z_{g1}, z_{g2}, \ldots, z_{g4})(g = 1, 2, \ldots, 4).$$

To get the overall NCNs $z_g(g = 1, 2, \ldots, 4)$. Assume that, $Q = (1, 1, 1, 1)$.

$$\begin{aligned}
z_1 &= \langle [0.2569, 0.6239], [0.2929, 0.5112], [0.2375, 0.4571] \rangle, \langle 0.7682, 0.4666, 0.3905 \rangle; \\
z_2 &= \langle [0.3642, 0.8179], [0.3110, 0.6479], [0.3194, 0.5430] \rangle, \langle 0.7416, 0.3336, 0.5561 \rangle; \\
z_3 &= \langle [0.4935, 0.6438], [0.1794, 0.3224], [0.2248, 0.4812] \rangle, \langle 0.6502, 0.3206, 0.2330 \rangle; \\
z_4 &= \langle [0.4995, 0.7691], [0.2570, 0.5332], [0.2130, 0.3815] \rangle, \langle 0.5355, 0.2744, 0.3248 \rangle.
\end{aligned}$$

Step 6. Using Definition 6, we calculate the score values of the collective NCNs $z_g(g = 1, 2, \ldots, a)$.

$$\widetilde{SC}(z_1) = 0.5881, \widetilde{SC}(z_2) = 0.5782, \widetilde{SC}(z_3) = 0.6688, \widetilde{SC}(z_4) = 0.6467.$$

Step 7. According to the score values, ranking order of the alternative is $\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2$.

Hence using Theorem 1, the best alternatives is $\hbar_3$, while the worst is $\hbar_2$.

From the above obtained results, we can see that by using WNCPMM operator or WNCPDMM operator, the best alternative obtained is $\hbar_3$, while the worst is $\hbar_2$.

Effect of the Parameter Q on the Decision Result

In this subsection, different values to the parameter vector and the results obtained from these values are shown in Tables 2 and 3. From Tables 2 and 3, it can be seen that, when the value of the parameter vector Q is $(1, 0, 0, 0)$, that is, when the interrelationship among the attributes is not considered, then according to the score values the best alternative is $\hbar_4$ while the worst is $\hbar_2$. Similarly, when the value of the parameter vector Q is $(1, 1, 0, 0)$, that is, when WCNPMM operator and WNCPDMM operator degenerate into neutrosophic cubic power Bonferroni mean operator and neutrosophic cubic power geometric Bonferroni mean operator respectively, the best alternative is $\hbar_3$ and $\hbar_4$ while the worst for both cases is $\hbar_2$. When the value of the parameter vector Q is $(1, 1, 1, 0)$, the best alternative is $\hbar_3$ and the worst is $\hbar_2$. When the value of the parameter vector Q is $(1, 1, 1, 1)$, the best alternative is $\hbar_3$ and the worst is $\hbar_2$. Similarly, for other values of the parameter vector the score values and ranking order vary. Thus, one can select the value of the parameter vector according to the needs of the situations.

Table 2. Score values and ranking orders for different parameter values in WCNPMM operator.

Parameter Vector Q	Score Values	Ranking Orders
$Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5671, \widetilde{SC}(CN_2) = 0.5230,$ $\widetilde{SC}(CN_3) = 0.5593, \widetilde{SC}(CN_4) = 0.6031.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
$Q(1,1,0,0)$	$\widetilde{SC}(CN_1) = 0.4579, \widetilde{SC}(CN_2) = 0.4468,$ $\widetilde{SC}(CN_3) = 0.5092, \widetilde{SC}(CN_4) = 0.5027.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,0)$	$\widetilde{SC}(CN_1) = 0.4227, \widetilde{SC}(CN_2) = 0.4133,$ $\widetilde{SC}(CN_3) = 0.4866, \widetilde{SC}(CN_4) = 0.4607.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(0.5,0.5,0.5,0.5)$	$\widetilde{SC}(CN_1) = 0.3988, \widetilde{SC}(CN_2) = 0.3910,$ $\widetilde{SC}(CN_3) = 0.4708, \widetilde{SC}(CN_4) = 0.4306.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(5,0,0,0)$	$\widetilde{SC}(CN_1) = 0.6608, \widetilde{SC}(CN_2) = 0.6235,$ $\widetilde{SC}(CN_3) = 0.6313, \widetilde{SC}(CN_4) = 0.6854.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$

Table 3. Score values and ranking orders for different parameter values in weighted neutrosophic cubic power dual Muirhead mean operator.

Parameter Vector Q	Score Values	Ranking Orders
$Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5588, \widetilde{SC}(CN_2) = 0.5346,$ $\widetilde{SC}(CN_3) = 0.6040, \widetilde{SC}(CN_4) = 0.6081.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
$Q(1,1,0,0)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_4 > \hbar_3 > \hbar_1 > \hbar_2.$
$Q(1,1,1,0)$	$\widetilde{SC}(CN_1) = 0.5760, \widetilde{SC}(CN_2) = 0.5582,$ $\widetilde{SC}(CN_3) = 0.6478, \widetilde{SC}(CN_4) = 0.6276.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(0.5,0.5,0.5,0.5)$	$\widetilde{SC}(CN_1) = 0.5909, \widetilde{SC}(CN_2) = 0.5817,$ $\widetilde{SC}(CN_3) = 0.6741, \widetilde{SC}(CN_4) = 0.6488.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(5,0,0,0)$	$\widetilde{SC}(CN_1) = 0.4671, \widetilde{SC}(CN_2) = 0.4073,$ $\widetilde{SC}(CN_3) = 0.4022, \widetilde{SC}(CN_4) = 0.4559.$	$\hbar_1 > \hbar_4 > \hbar_2 > \hbar_3.$

6. Comparison with Existing Methods

To show the efficiency and advantages of the proposed method, we give a comparative analysis. We exploit some existing methods to solve the same example and examine the final results. We compare our method in this paper with the methods developed by Qin et al. [30] based on weighted IFMSM operator, and the one developed by Liu et al. [32]-based generalized INPWA operator. We extend the IFMSM operator method [30] for intuitionistic fuzzy information to neutrosophic cubic Maclaurin symmetric mean operator. We also extend the GINPWA operator [32] for interval neutrosophic information to generalized neutrosophic cubic power average operator.

The method developed by Qin et al. [30], is based on MSM operator, which can consider the interrelationship among the attribute values, but unable to remove the effect of awkward data. The MSM operator is a special case of the proposed aggregation operator. Also, the ranking result obtained using the method of Qin et al. [30], is different from the one obtained using the proposed method.

Similarly, the method developed by Liu et al. [32], is based on power weighted averaging operator, which can remove the effect of awkward data but cannot consider the interrelationship among the attributes values. From Table 4, it can be seen that the ranking result obtained using Liu et al. [32] is the same as the ranking order obtained from the proposed method, when $Q(1,0,0,0)$. That is, when the interrelationship between NCNs are not considered. This shows the validity of the proposed approach. The ranking order is different when $Q(1,1,1,1)$. That is, when the interrelationship among four attributes are considered, then the ranking order is different. The main reason behind the different ranking results is due to the existing aggregation operators, can only consider a single characteristic at a time while aggregating the NCNs, meaning that they can only either consider interrelationship among attributes or remove the effect of awkward data. Our proposed aggregation operator, however, can consider two characteristics at a time. It can consider the interrelationship among the attributes and remove the effect of awkward data. In fact, these existing aggregation operators can be regarded as special cases to our proposed aggregation operator. Hence, our proposed aggregation operator is more practical and flexible to be used in decision-making problems.

Table 4. Score values and ranking orders for different parameter values in WCNPDMM operator.

Aggregation Operator	Score Values	Ranking Orders
NCMSM operator [30]	$\widetilde{SC}(CN_1) = 0.6263, \widetilde{SC}(CN_2) = 0.6153,$ $\widetilde{SC}(CN_3) = 0.6355, \widetilde{SC}(CN_4) = 0.6373.$	$\hbar_4 > \hbar_3 > \hbar_1 > \hbar_2.$
GNCPWA operator [32]	$\widetilde{SC}(CN_1) = 0.5694, \widetilde{SC}(CN_2) = 0.5266,$ $\widetilde{SC}(CN_3) = 0.5646, \widetilde{SC}(CN_4) = 0.6054.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPMM operator $Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5671, \widetilde{SC}(CN_2) = 0.5230,$ $\widetilde{SC}(CN_3) = 0.5593, \widetilde{SC}(CN_4) = 0.6031.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPDMM operator $Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5588, \widetilde{SC}(CN_2) = 0.5346,$ $\widetilde{SC}(CN_3) = 0.6040, \widetilde{SC}(CN_4) = 0.6081.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPMM operator $Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
Proposed WNCPDMM operator $Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$

7. Conclusions

In this article, we incorporate both the PA operator and MM operator to form a few new aggregation operators to aggregate CNNs, such as the cubic neutrosophic power Muirhead mean (CNPMM) operator, WCNPMM operator, CNPDMM operator and WCNPDMM operator. We discussed several basic results and properties, along with a few special cases of the proposed aggregation operators. In other words, the developed aggregation operators do not only consider the interrelationship among the NCNs, but also remove the influence of too high or too low arguments in the final results. Based on these aggregation operators, a novel approach to MADM problem is developed. Finally, a numerical example is illustrated to show the effectiveness and practicality of the proposed approach.

Our main contribution is enhancing the neutrosophic cubic aggregation operator and its MADM method under neutrosophic cubic environment. In future, we will incorporate the PA operator with the MM operator under the intuitionistic fuzzy environment [3], interval neutrosophic environment [6] and multi-valued neutrosophic environment [10], to develop new operators such as IFPMM, IFPDMM, INPMM, INPDMM, multi-valued neutrosophic power Muirhead mean (NPMM) and multi-valued neutrosophic power dual Muirhead mean (NPDMM) operators along with their weighted forms. We will apply these to MAGDM, data mining, decision support, recommender system and pattern recognition.

Author Contributions: Q.K. drafted the initial manuscript, T.M. supervised, N.H. corrected the draft and sourced the funding. All authors approved the publication work.

Funding: The APC is funded by Universiti Kebangsaan Malaysia under the grant GUP-2017-105.

Acknowledgments: We, the authors of this project are deeply grateful for the valuable suggestions and comments provided by the respected reviewers to improve the quality and worth of this research. We are also indebted to Universiti Kebangsaan Malaysia for providing financial support and facilities for this research under the grant GUP-2017-105.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
INS	Interval neutrosophic set
INN	Interval neutrosophic number
MADM	Multiple-attribute decision-making
MAGDM	Multiple-attribute group decision-making
MM	Muirhead Mean
NS	Neutrosophic set
NC	Neutrosophic cubic
NCN	Neutrosophic cubic number
NCPMM	Neutrosophic cubic power Muirhead mean operator
NCPDMM	Neutrosophic cubic power dual Muirhead mean operator
PA	Power average operator
PWV	Power weight vector
SVNS	Single-valued neutrosophic set
SVNN	Single-valued neutrosophic number
WNCPMM	Weighted neutrosophic cubic power Muirhead mean
WNCPDMM	Weighted neutrosophic cubic power dual Muirhead mean operator

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Turksen, I.B. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets Syst.* **1986**, *20*, 191–210. [CrossRef]
3. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
4. Smarandache, F. *Neutrosophy: Neutrosophic Probability, Set, and Logic*; American Research Press: Rehoboth, IL, USA, 1998.
5. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. In Proceedings of the 8th Joint Conference on Information Sciences, Salt Lake, UT, USA, 21–26 July 2005; pp. 94–97.
6. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*; Hexis: Phoenix, AZ, USA, 2005.
7. Zhang, H.Y.; Wang, J.Q.; Chen, X.H. Interval neutrosophic sets and their application in multicriteria decision making problems. *Sci. World J.* **2014**. [CrossRef] [PubMed]
8. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2459–2466. [CrossRef]
9. Peng, J.J.; Wang, J.Q.; Wang, J.; Zhang, H.Y.; Chen, X.H. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. *Int. J. Syst. Sci.* **2016**, *47*, 2342–2358. [CrossRef]
10. Peng, J.J.; Wang, J.Q.; Wu, X.H.; Wang, J.; Chen, X.H. Multi-valued neutrosophic sets and power aggregation operators with their applications in multi-criteria group decision-making problems. *Int. J. Comput. Intell. Syst.* **2015**, *8*, 345–363. [CrossRef]
11. Abu Qamar, M.; Hassan, N. Q-Neutrosophic Soft Relation and Its Application in Decision Making. *Entropy* **2018**, *20*, 172. [CrossRef]
12. Al-Quran, A.; Hassan, N. The complex neutrosophic soft expert relation and its multiple attribute decision-making method. *Entropy* **2018**, *20*, 101. [CrossRef]
13. Jun, Y.B.; Kim, C.S.; Yang, K.O. Cubic sets. *Ann. Fuzzy Math. Inform.* **2012**, *4*, 83–98.
14. Kaur, G.; Garg, H. Multi-attribute decision-making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment. *Entropy* **2018**, *20*, 65. [CrossRef]
15. Kaur, G.; Harish, G. Cubic intuitionistic fuzzy aggregation operators. *Int. J. Uncertain. Quantif.* **2018**, *8*, 405–427. [CrossRef]

16. Ali, M.; Deli, I.; Smarandache, F. The theory of neutrosophic cubic sets and their applications in pattern recognition. *J. Intell. Fuzzy Syst.* **2018**, *30*, 1957–1963. [CrossRef]
17. Jun, Y.B.; Smarandache, F.; Kim, C.S. Neutrosophic cubic sets. *New Math. Nat. Comput.* **2017**, *13*, 41–54. [CrossRef]
18. Jun, Y.B.; Smarandache, F.; Kim, C.S. P-union and P-intersection of neutrosophic cubic sets. *Analele Universitatii Ovidius Constanta-Seria Matematica* **2017**, *25*, 99–115. [CrossRef]
19. Zhan, J.; Khan, M.; Gulistan, M.; Ali, A. Applications of neutrosophic cubic sets in multi-criteria decision-making. *Int. J. Uncertain. Quantif.* **2017**, *7*, 377–394. [CrossRef]
20. Ye, J. Operations and aggregation method of neutrosophic cubic numbers for multiple attribute decision-making. *Soft Comput.* **2018**, 1–10. [CrossRef]
21. Shi, L.; Ye, J. Dombi aggregation operators of neutrosophic cubic sets for multiple attribute decision-making. *Algorithms* **2018**, *11*, 29. [CrossRef]
22. Lu, Z.; Ye, J. Cosine measures of neutrosophic cubic sets for multiple attribute decision-making. *Symmetry* **2017**, *9*, 121. [CrossRef]
23. Pramanik, S.; Dalapati, S.; Alam, S.; Roy, T.K.; Smarandache, F. Neutrosophic cubic MCGDM method based on similarity measure. *Neutrosophic Sets Syst.* **2017**, *16*, 44–56.
24. Pramanik, S.; Dalapati, S.; Alam, S.; Roy, T.K. NC-TODIM-based MAGDM under a neutrosophic cubic set environment. *Information* **2017**, *8*, 149. [CrossRef]
25. Bonferroni, C. Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana* **1950**, *5*, 267–270.
26. Xu, Z.; Yager, R.R. Intuitionistic fuzzy Bonferroni means. *IEEE Trans. Syst. Man Cybern. Part B Cybern.* **2011**, *41*, 568–578. [CrossRef]
27. Sykora, S. Mathematical Means and Averages: Generalized Heronian Means. *Stan's Libr.* **2009**. [CrossRef]
28. Muirhead, R.F. Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters. *Proc. Edinb. Math. Soc.* **1902**, *21*, 144–162. [CrossRef]
29. Maclaurin, C. A second letter to Martin Folkes, Esq.; Concerning the roots of equations, with the demonstration of other rules of algebra. *Philos. Trans.* **1729**, *36*, 59–96.
30. Qin, J.; Liu, X. An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2177–2190. [CrossRef]
31. Yager, R.R. The power average operator. *IEEE Trans. Syst. Man Cybern. Part A Syst. Hum.* **2001**, *31*, 724–731. [CrossRef]
32. Liu, P.; Tang, G. Some power generalized aggregation operators based on the interval neutrosophic sets and their application to decision making. *J. Intell. Fuzzy Syst.* **2016**, *30*, 2517–2528. [CrossRef]
33. Liu, P.; Li, D. Some Muirhead mean operators for intuitionistic fuzzy numbers and their applications to group decision making. *PLoS ONE* **2017**, *12*, e0168767. [CrossRef] [PubMed]
34. Liu, P.; You, X. Interval neutrosophic Muirhead mean operators and their application in multiple attribute group decision-making. *Int. J. Uncertain. Quantif.* **2017**, *7*, 303–334. [CrossRef]
35. Li, L.; Zhang, R.; Wang, J.; Zhu, X.; Xing, Y. Pythagorean fuzzy power Muirhead mean operators with their application to multi-attribute decision making. *J. Intell. Fuzzy Syst.* **2018**. [CrossRef]