symmetry

MDPI

Article

New Multigranulation Neutrosophic Rough Set with Applications

Chunxin Bo [1], Xiaohong Zhang [2,*], Songtao Shao [1] and Florentin Smarandache [3]

1 College of Information Engineering, Shanghai Maritime University, Shanghai 201306, China; 201640311001@stu.shmtu.edu.cn (C.B.); 201740310005@stu.shmtu.edu.cn (S.S.)
2 Department of Mathematics, Shaanxi University of Science & Technology, Xi'an 710021, China
3 Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu
* Correspondence: zhangxiaohong@sust.edu.cn or zhangxh@shmtu.edu.cn

Received: 3 September 2018; Accepted: 6 October 2018; Published: 2 November 2018

check for updates

Abstract: After the neutrosophic set (NS) was proposed, NS was used in many uncertainty problems. The single-valued neutrosophic set (SVNS) is a special case of NS that can be used to solve real-word problems. This paper mainly studies multigranulation neutrosophic rough sets (MNRSs) and their applications in multi-attribute group decision-making. Firstly, the existing definition of neutrosophic rough set (we call it type-I neutrosophic rough set (NRS_I) in this paper) is analyzed, and then the definition of type-II neutrosophic rough set (NRS_{II}), which is similar to NRS_I, is given and its properties are studied. Secondly, a type-III neutrosophic rough set (NRS_{III}) is proposed and its differences from NRS_I and NRS_{II} are provided. Thirdly, single granulation NRSs are extended to multigranulation NRSs, and the type-I multigranulation neutrosophic rough set ($MNRS_I$) is studied. The type-II multigranulation neutrosophic rough set ($MNRS_{II}$) and type-III multigranulation neutrosophic rough set ($MNRS_{III}$) are proposed and their different properties are outlined. We found that the three kinds of MNRS*s* generate tcorresponding NRSs when all the NRs are the same. Finally, $MNRS_{III}$ in two universes is proposed and an algorithm for decision-making based on $MNRS_{III}$ is provided. A car ranking example is studied to explain the application of the proposed model.

Keywords: inclusion relation; neutrosophic rough set; multi-attribute group decision-making (MAGDM); multigranulation neutrosophic rough set (MNRS); two universes

1. Introduction

Many theories have been applied to solve problems with imprecision and uncertainty. Fuzzy set (FS) theories [1–3] use the degree of membership to solve the fuzziness. Rough set (RS) theories [4–7] deal with uncertainty by lower and upper approximation (LUA). Soft set theories [8–10] deal with uncertainty by using a parametrized set. However, all these theories have their own restrictions. Smarandache proposed the concept of the neutrosophic set (NS) [11], which was a generalization of the intuitionistic fuzzy set (IFS). To address real-world uncertainty problems, Wang et al. proposed the single-valued neutrosophic set (SVNS) [12]. Many theories about neutrosophic sets were studied and extended single-valued neutrosophic set [13–15]. Zhang et al. [16] analyzed two kinds of inclusion relations of the NS and then proposed the type-3 inclusion relation of NS. The combinations of the FS and RS are popular and produce many interesting results [17]. Broumi and Smarandache [18] combined the RS and NS, then produced a rough NS and studied its qualities. Yang et al. [19] combined the SVNS and RS, then produced the SVNRS (single-valued neutrosophic rough set) and studied its qualities.

From the view point of granular computing, the RS uses upper and lower approximations to solve uncertainty problems, shown by single granularity. However, with the complexity of real-word problems, we often encounter multiple relationship concepts. Qian and Liang [20] proposed a multigranularity rough set (MGRS). Many scholars have generalized MGRS and acquired some interesting consequences [21–26]. Zhang et al. [27] proposed non-dual MGRSs and investigated their qualities.

Few articles have been published about the combination of NSs and multigranulation rough sets. In this paper, we study three kinds of neutrosophic rough sets (NRSs) and multigranulation neutrosophic rough sets (MNRSs) that are based on three kinds of inclusion relationships of NS and corresponding union and intersection relationships [11,12,16]. Their different properties are discussed. We found that MNRSs degenerate to corresponding NRSs when the NRs are the same. Yang et al. [19] defined the NRS_I and considered its properties. Bo et al. [28] proposed $MNRS_I$ and discussed its properties. In this paper, we study NRS_{II} and $MNRS_{II}$. We also study NRS_{III} and $MNRS_{III}$, which are based on a type-3 inclusion relationship and corresponding union and intersection relationships. Finally, we use $MNRS_{III}$ on two universes to solve multi-attribute group decision-making (MAGDM) problems.

The structure of this article is as follows: In Section 2, some basic notions and operations of NRS_I and NRS_{II} are introduced. In Section 3, the definition of NRS_{III} is proposed and its qualities are investigated, and the differences between NRS_I, NRS_{II}, and NRS_{III} are illustrated using an example. In Section 4, $MNRS_I$ and $MNRS_{II}$ are discussed. In Section 5, $MNRS_{III}$ is proposed and its differences from $MNRS_I$ and $MNRS_{II}$ are studied. In Section 6, $MNRS_{III}$ on two universes is proposed and an application to solve the MAGDM problem is outlined. Finally, Section 7 provides our conclusions and outlook.

2. Preliminary

In this chapter, we look back at several basic concepts of type-I NRS, then propose the definition and properties of type-II NRS.

Definition 1. *[12] A single valued neutrosophic set A in X is denoted by:*

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\}, \tag{1}$$

where TA(x), IA(x), FA(x) ∈ [0, 1] for each point x in X and satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

*For convenience, "*SVNS*" is abbreviated to "*NS*" later. Here, NS(X) denotes the set of all* SVNS *in X.*

Definition 2. *[29] A neutrosophic relation* (NR) *is a neutrosophic fuzzy subset of* $X \times Y$, *that is,* $\forall x \in X$, $y \in Y$,

$$R(x, y) = (T_R, I_R, F_R), \tag{2}$$

where T_R: $X \times Y \to [0, 1]$, I_R: $X \times Y \to [0, 1]$, *and* F_R: $X \times Y \to [0, 1]$ *and satisfies* $0 \le T_R + I_R + F_R \le 3$. $NR(X \times Y)$ *denotes all the* NRs *in* $X \times Y$.

Definition 3. *[19] Suppose (U, R) is a neutrosophic approximation space* (NAS)*.* $\forall A \in NS(U)$, *the* LUA *of A, denoted by* $\underline{R}(A)$ *and* $\overline{R}(A)$, *is defined as:* $\forall x \in U$,

$$\underline{R}(A) = \underset{y \in U}{\cap} (R^c(x, y) \cup A(y)),\ \overline{R}(A) = \underset{y \in U}{\cup} (R(x, y) \cap A(y)). \tag{3}$$

The pair $\left(\underline{R}(A), \overline{R}(A)\right)$ *is called the* SVNRS *of A. In this paper, we called it type-I neutrosophic rough set* (NRS_I)*. Because the definition of* NRS_I *is based on the type-1 operator of* NS*, the definition can be written as:*

$$\underline{NRS_I}(A) = \underset{y \in U}{\cap_1} (R^c(x, y) \cup_1 A(y)),\ \overline{NRS_I}(A) = \underset{y \in U}{\cup_1} (R(x, y) \cap_1 A(y)). \tag{4}$$

Proposition 1. [19] *Suppose (U, R) is an* NAS. $\forall A, B \in NS(U)$, *we have:*

(1) *If* $A \subseteq_1 B$, *then* $\underline{NRS_I}(A) \subseteq_1 \underline{NRS_I}(B)$ *and* $\overline{NRS_I}(A) \subseteq_1 \overline{NRS_I}(B)$.
(2) $\underline{NRS_I}(A \cap_1 B) = \underline{NRS_I}(A) \cap_1 \underline{NRS_I}(B)$, $\overline{NRS_I}(A \cup_1 B) = \overline{NRS_I}(A) \cup_1 \overline{NRS_I}(B)$.
(3) $\underline{NRS_I}(A) \cup_1 \underline{NRS_I}(B) \subseteq_1 \underline{NRS_I}(A \cup_1 B)$, $\overline{NRS_I}(A \cap_1 B) \subseteq_1 \overline{NRS_I}(A) \cap_1 \overline{NRS_I}(B)$.

According to the NRS_I, we can get the definition and properties of NRS_II, which is based on the type-2 operator of NS.

Definition 4. *Suppose (U, R) is an* NAS. $\forall A \in NS(U)$, *the type-II* LUA *of A, is defined as:*

$$\underline{NRS_{II}}(A) = \underset{y \in U}{\cap_2} (R^c(x,y) \cup_2 A(y)),\ \overline{NRS_{II}}(A) = \underset{y \in U}{\cup_2} (R(x,y) \cap_2 A(y)) \tag{5}$$

The pair $\left(\underline{NRS_{II}}(A), \overline{NRS_{II}}(A)\right)$ *is called* NRS_II *of A.*

Proposition 2. *Suppose (U, R) is an* NAS. $\forall A, B \in NS(U)$, *we have:*

(1) *If* $A \subseteq_2 B$, *then* $\underline{NRS_{II}}(A) \subseteq_2 \underline{NRS_{II}}(B)$, $\overline{NRS_{II}}(A) \subseteq_2 \overline{NRS_{II}}(B)$.
(2) $\underline{NRS_{II}}(A \cap_2 B) = \underline{NRS_{II}}(A) \cap_2 \underline{NRS_{II}}(B)$, $\overline{NRS_{II}}(A \cup_2 B) = \overline{NRS_{II}}(A) \cup_2 \overline{NRS_{II}}(B)$.
(3) $\underline{NRS_{II}}(A) \cup_2 \underline{NRS_{II}}(B) \subseteq_2 \underline{NRS_{II}}(A \cup_2 B)$, $\overline{NRS_{II}}(A \cap_2 B) \subseteq_2 \overline{NRS_{II}}(A) \cap_2 \overline{NRS_{II}}(B)$.

Definition 5. [22] *Suppose A, B are two* NSs, *then the Hamming distance between A and B is defined as:*

$$d_N(A,B) = \sum_{i=1}^{n} \{|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|\}. \tag{6}$$

3. Type-III NRS

In this chapter, we introduce a new NRS, type-III NRS (NRS_III). We provide the differences between the three kinds of NRSs. The properties of NRS_III are also given.

Definition 6. *Suppose (U, R) is an* NAS. $\forall A \in NS(U)$, *the type-III* LUA *of A, is defined as:*

$$\underline{NRS_{III}}(A) = \underset{y \in U}{\cap_3} (R^c(x,y) \cup_3 A(y)), \overline{NRS_{III}}(A) = \underset{y \in U}{\cup_3} (R(x,y) \cap_3 A(y)).$$

The pair $\left(\underline{NRS_{III}}(A), \overline{NRS_{III}}(A)\right)$ *is called* NRS_III *of A.*

Proposition 3. *Suppose (U, R) is an* NAS. $\forall A, B \in NS(U)$, *we have:*

(1) *If* $A \subseteq_3 B$, *then* $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$, $\overline{NRS_{III}}(A) \subseteq_3 \overline{NRS_{III}}(B)$.
(2) $\underline{NRS_{III}}(A \cap_3 B) \subseteq_3 \underline{NRS_{III}}(A) \cap_3 \underline{NRS_{III}}(B)$, $\underline{NRS_{III}}(A) \cup_3 \underline{NRS_{III}}(B) \subseteq_3 \underline{NRS_{III}}(A \cup_3 B)$.
(3) $\overline{NRS_{III}}(A \cap_3 B) \subseteq_3 \overline{NRS_{III}}(A) \cap_3 \overline{NRS_{III}}(B)$, $\overline{NRS_{III}}(A) \cup_3 \overline{NRS_{III}}(B) \subseteq_3 \overline{NRS_{III}}(A \cup_3 B)$.

Proof. (1) Assume $A \subseteq_3 B$,

Case 1: If $T_A(x) < T_B(x)$, $F_A(x) \geq F_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y \in U}{\wedge}[F_R(x,y) \vee T_A(y)] \leq \underset{y \in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y \in U}{\vee}[T_R(x,y) \wedge F_A(y)] \geq \underset{y \in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x).$$

Hence,

$$\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B).$$

Case 2: If $T_A(x) = T_B(x)$, $F_A(x) > F_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_A(y)] = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_A(y)] \geq \underset{y\in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x).$$

Hence,

$$\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B).$$

Case 3: suppose $T_A(x) = T_B(x)$, $F_A(x) = F_B(x)$ and $I_A(x) \leq I_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_A(y)] = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_A(y)] = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x)$$

$$I_{\underline{NRS_{III}}(A)}(x) = \begin{cases} I_A(y_j), & R^c(x,y_j) \subseteq_3 A(y_j) \subseteq_3 A(y_k), y_k, y_j \in U \\ I_{R^c}(x,y_j), & A(y_j) \subseteq_3 R^c(x,y_j) \\ 1, & else \end{cases}$$

$$I_{\underline{MNRS_{III}{}^o}(B)}(x) = \begin{cases} I_B(y_j), & R_i{}^c(x,y_j) \subseteq_3 B(y_j) \subseteq_3 B(y_k), y_k, y_j \in U \\ I_{R_i{}^c}(x,y_j), & B(y_j) \subseteq_3 R_i{}^c(x,y_j) \\ 1, & else \end{cases}.$$

Hence, $I_{\underline{NRS_{III}}(A)}(x) \leq I_{\underline{NRS_{III}}(B)}(x)$. So $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$.
Summing up the above, if $A \subseteq_3 B$, then $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$.
Similarly, we can get $\overline{NRS_{III}}(A) \subseteq_3 \overline{NRS_{III}}(B)$.
(2) According the Definition 6, we have:

$$\begin{aligned} \underline{NRS_{III}}(A \cap_3 B) &= \underset{y\in U}{\cap_3}[R^c(x,y) \cup_3 (A \cap_3 B)(y)] \\ &\subseteq_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 A(y))\right] \cap_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 B(y))\right] \\ &= \underline{NRS_{III}}(A) \cap_3 \underline{NRS_{III}}(B). \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{NRS_{III}}(A) \cup_3 \underline{NRS_{III}}(B) &= \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 A(y))\right] \cup_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 B(y))\right] \\ &\subseteq_3 \underset{y\in U}{\cap_3}[R^c(x,y) \cup_3 (A \cup_3 B)(y)] \\ &= \underline{NRS_{III}}(A \cup_3 B). \end{aligned}$$

(3) The proof is similar to that of Case 2. □

Example 1. *Define* NAS *(U, R), where U = {x_1, x_2} and R is given in Table 1.*

Table 1. A neutrosophic relation R.

R	x_1	x_2
x_1	(0.4, 0.6, 0.7)	(0.2, 0.2, 0.9)
x_2	(0.7, 0.1, 0.4)	(0.8, 0.8, 0.6)

Suppose A is an NS and $A = \{(x_1, 0.8, 0.2, 0.1), (x_2, 0.4, 0.9, 0.5)\}$. Then, by Definitions 3, 4 and 6, we can get:

$$\begin{array}{ll}
\underline{NRS_I}(A)(x_1) = (0.8, 0.8, 0.2), & \underline{NRS_I}(A)(x_2) = (0.6, 0.2, 0.5), \\
\overline{NRS_I}(A)(x_1) = (0.4, 0.6, 0.7), & \overline{NRS_I}(A)(x_2) = (0.7, 0.2, 0.4), \\
\underline{NRS_{II}}(A)(x_1) = (0.8, 0.4, 0.2), & \underline{NRS_{II}}(A)(x_2) = (0.6, 0.9, 0.5), \\
\overline{NRS_{II}}(A)(x_1) = (0.4, 0.2, 0.7), & \overline{NRS_{II}}(A)(x_2) = (0.7, 0.8, 0.4), \\
\underline{NRS_{III}}(A)(x_1) = (0.8, 1, 0.2), & \underline{NRS_{III}}(A)(x_2) = (0.6, 0, 0.5), \\
\overline{NRS_{III}}(A)(x_1) = (0.4, 0.6, 0.7), & \overline{NRS_{III}}(A)(x_2) = (0.7, 0.1, 0.4).
\end{array}$$

4. Type-I and Type-II MNRS

We have proposed a kind of multigranulation neutrosophic rough set [30] (we called it type-I multigranulation neutrosophic rough set in this paper). $\mathrm{MNRS_I}$ is based on a type-1 operator of NRs. In this chapter, we define the type-II multigranulation neutrosophic rough set ($\mathrm{MNRS_{II}}$), which is based on a type-2 operator of NRs.

Definition 7. *[28] Suppose U is a non-empty finite universe, and R_i ($1 \le i \le m$) is a binary* NR *on U. We call the tuple ordered set (U, R_i) the multigranulation neutrosophic approximation space* (MNAS).

Definition 8. *[28] Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-I optimistic* LUA *of A, represented by $\underline{MNRS_I}^o(A)$ and $\overline{MNRS_I}^o(A)$, is defined as:*

$$\underline{MNRS_I}^o(A)(x) = \overset{m}{\underset{i=1}{\cup_1}} \left(\underset{y \in U}{\cap_1} (R_i{}^c(x,y) \cup_1 A(y)) \right)$$

$$\overline{MNRS_I}^o(A)(x) = \overset{m}{\underset{i=1}{\cap_1}} \left(\underset{y \in U}{\cup_1} (R_i(x,y) \cap_1 A(y)) \right).$$

Then, A is named a definable NS *when $\underline{MNRS_I}^o(A) = \overline{MNRS_I}^o(A)$. Alternatively, we name the pair $\left(\underline{MNRS_I}^o(A), \overline{MNRS_I}^o(A)\right)$ an optimistic* $\mathrm{MNRS_I}$.

Definition 9. *[30] Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-I pessimistic* LUA *of A, represented by $\underline{MNRS_I}^p(A)$ and $\overline{MNRS_I}^p(A)$, is defined as:*

$$\underline{MNRS_I}^p(A)(x) = \overset{m}{\underset{i=1}{\cap_1}} \left(\underset{y \in U}{\cap_1} (R_i{}^c(x,y) \cup_1 A(y)) \right)$$

$$\overline{MNRS_I}^p(A)(x) = \overset{m}{\underset{i=1}{\cup_1}} \left(\underset{y \in U}{\cup_1} (R_i(x,y) \cap_1 A(y)) \right).$$

Similarly, A is named a definable NS *when $\underline{MNRS_I}^p(A) = \overline{MNRS_I}^p(A)$. Alternatively, we name the pair $\left(\underline{MNRS_I}^p(A), \overline{MNRS_I}^p(A)\right)$ a pessimistic* $\mathrm{MNRS_I}$.

Definition 10. *Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-II optimistic* LUA *of A, represented by $\underline{MNRS_{II}}^o(A)$ and $\overline{MNRS_{II}}^o(A)$, is defined as:*

$$\underline{MNRS_{II}}^o(A)(x) = \overset{m}{\underset{i=1}{\cup_2}} \left(\underset{y \in U}{\cap_2} (R_i{}^c(x,y) \cup_2 A(y)) \right)$$

$$\overline{MNRS_{II}}^o(A)(x) = \overset{m}{\underset{i=1}{\cap_2}} \left(\underset{y \in U}{\cup_2} (R_i(x,y) \cap_2 A(y)) \right).$$

Then, A is named a definable NS *when* $\underline{MNRS_{II}}^{o}(A) = \overline{MNRS_{II}}^{o}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{II}}^{o}(A), \overline{MNRS_{II}}^{o}(A)\right)$ *an optimistic* $\mathrm{MNRS_{II}}$.

Definition 11. *Suppose* (U, R_i) *is an* MNAS. $\forall A \in NS(U)$, *the type-II pessimistic* LUA *of* A, *represented by* $\underline{MNRS_{II}}^{p}(A)$ *and* $\overline{MNRS_{II}}^{p}(A)$, *is defined as:*

$$\underline{MNRS_{II}}^{p}(A)(x) = \mathop{\cap_2}_{i=1}^{m}\left(\mathop{\cap_2}_{y\in U}(R_i{}^{c}(x,y)\cup_2 A(y))\right)$$

$$\overline{MNRS_{II}}^{p}(A)(x) = \mathop{\cup_2}_{i=1}^{m}\left(\mathop{\cup_2}_{y\in U}(R_i(x,y)\cap_2 A(y))\right).$$

Similarly, A is named a definable NS *when* $\underline{MNRS_{II}}^{p}(A) = \overline{MNRS_{II}}^{p}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{II}}^{p}(A), \overline{MNRS_{II}}^{p}(A)\right)$ *a pessimistic* $\mathrm{MNRS_{II}}$.

Proposition 4. *Suppose* (U, R_i) *is an* MNAS. $\forall A, B \in NS(U)$, *then:*

(1) $\underline{MNRS_{II}}^{o}(A) = \sim \overline{MNRS_{II}}^{o}(\sim A)$, $\underline{MNRS_{II}}^{p}(A) = \sim \overline{MNRS_{II}}^{p}(\sim A)$.
(2) $\overline{MNRS_{II}}^{o}(A) = \sim \underline{MNRS_{II}}^{o}(\sim A)$, $\overline{MNRS_{II}}^{p}(A) = \sim \underline{MNRS_{II}}^{p}(\sim A)$.
(3) $\underline{MNRS_{II}}^{o}(A\cap_2 B) = \underline{MNRS_{II}}^{o}(A)\cap_2 \underline{MNRS_{II}}^{o}(B)$, $\underline{MNRS_{II}}^{p}(A\cap_2 B) = \underline{MNRS_{II}}^{p}(A)\cap_2 \underline{MNRS_{II}}^{p}(B)$.
(4) $\overline{MNRS_{II}}^{o}(A\cup_2 B) = \overline{MNRS_{II}}^{o}(A)\cup_2 \overline{MNRS_{II}}^{o}(B)$, $\overline{MNRS_{II}}^{p}(A\cup_2 B) = \overline{MNRS_{II}}^{p}(A)\cup_2 \overline{MNRS_{II}}^{p}(B)$.
(5) $A\subseteq_2 B \Rightarrow \underline{MNRS_{II}}^{o}(A)\subseteq_2 \underline{MNRS_{II}}^{o}(B), \underline{MNRS_{II}}^{p}(A)\subseteq_2 \underline{MNRS_{II}}^{p}(B)$.
(6) $A\subseteq_2 B \Rightarrow \overline{MNRS_{II}}^{o}(A)\subseteq_2 \overline{MNRS_{II}}^{o}(B), \overline{MNRS_{II}}^{p}(A)\subseteq_2 \overline{MNRS_{II}}^{p}(B)$.
(7) $\underline{MNRS_{II}}^{o}(A)\cup_2 \underline{MNRS_{II}}^{o}(B) \subseteq_2 \underline{MNRS_{II}}^{o}(A\cup_2 B)$, $\underline{MNRS_{II}}^{p}(A)\cup_2 \underline{MNRS_{II}}^{p}(B) \subseteq_2 \underline{MNRS_{II}}^{p}(A\cup_2 B)$.
(8) $\overline{MNRS_{II}}^{o}(A\cap_2 B) \subseteq_2 \overline{MNRS_{II}}^{o}(A)\cap_2 \overline{MNRS_{II}}^{o}(B)$, $\overline{MNRS_{II}}^{p}(A\cap_2 B) \subseteq_2 \overline{MNRS_{II}}^{p}(A)\cap_2 \overline{MNRS_{II}}^{p}(B)$.

Proof. Equations (1), (2), (5), and (6) are obviously according to Definitions 10 and 11. Next, we will prove Equations (3), (4), (7), and (8).

(3) By Definition 10,

$$\begin{aligned}
\underline{MNRS_{II}}^{o}(A\cap_2 B)(x) &= \mathop{\cup_2}_{i=1}^{m}\left(\mathop{\cap_2}_{y\in U}(R_i{}^{c}(x,y)\cup_2 (A\cap_2 B)(y))\right)\\
&= \mathop{\cup_2}_{i=1}^{n}\left(\mathop{\cap_2}_{y\in U}((R_i{}^{c}(x,y)\cup_2 A(y))\cap(R_i{}^{c}(x,y)\cup_2 B(y)))\right)\\
&= \left(\mathop{\cup_2}_{i=1}^{n}\left(\mathop{\cap_2}_{y\in U}(R_i{}^{c}(x,y)\cup_2 A(y))\right)\right)\cap_2\left(\mathop{\cup_2}_{i=1}^{n}\left(\mathop{\cap_2}_{y\in U}(R_i{}^{c}(x,y)\cup_2 B(y))\right)\right)\\
&= \underline{MNRS_{II}}^{o}A(x)\cap_2 \underline{MNRS_{II}}^{o}B(y).
\end{aligned}$$

Similarly, from Definition 11, we can get the following:

$$\underline{MNRS_{II}}^{p}(A\cap_2 B) = \underline{MNRS_{II}}^{p}(A)\cap_2 \underline{MNRS_{II}}^{p}(B).$$

(4) The proof is similar to that of Equation (3).

(7) By Definition 10, we can get:

$$\begin{aligned}
T_{\underline{MNRS_{II}}^{o}(A\cup_2 B)}(x) &= \overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}\{\max[F_{R_i}(x,y),(\max(T_A(y),T_B(y)))]\}\\
&= \overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}\{\max[(\max(F_{R_i}(x,y),T_A(y))),(\max(F_{R_i}(x,y),T_B(y)))]\}\\
&\geq \max\left\{\left[\overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}(\max(F_{R_i}(x,y),T_A(y)))\right],\left[\overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}(\max(F_{R_i}(x,y),T_B(y)))\right]\right\}\\
&= \max\left(T_{\underline{MNRS_{II}}^{o}(A)}(x),T_{\underline{MNRS_{II}}^{o}(B)}(x)\right).
\end{aligned}$$

$$\begin{aligned}
I_{\underline{MNRS_{II}}^{o}(A\cup_2 B)}(x) &= \overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}\{\max[(1-I_{R_i}(x,y)),(\max(I_A(y),I_B(y)))]\}\\
&= \overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}\{\max[(\max((1-I_{R_i}(x,y)),I_A(y))),(\max((1-I_{R_i}(x,y)),I_B(y)))]\}\\
&\geq \max\left\{\left[\overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}(\max((1-I_{R_i}(x,y)),I_A(y)))\right],\left[\overset{m}{\underset{i=1}{\max}}\underset{y\in U}{\min}(\max((1-I_{R_i}(x,y)),I_B(y)))\right]\right\}\\
&= \max\left(I_{\underline{MNRS_{II}}^{o}(A)}(x),I_{\underline{MNRS_{II}}^{o}(B)}(x)\right).
\end{aligned}$$

$$\begin{aligned}
F_{\underline{MNRS_{II}}^{o}(A\cup_2 B)}(x) &= \overset{m}{\underset{i=1}{\min}}\underset{y\in U}{\max}\{\min[T_{R_i}(x,y),(\min(F_A(y),F_B(y)))]\}\\
&= \overset{m}{\underset{i=1}{\min}}\underset{y\in U}{\max}\{\min[\min(T_{R_i}(x,y),F_A(y))],[\min(T_{R_i}(x,y),F_B(y))]\}\\
&\leq \min\left\{\left[\overset{m}{\underset{i=1}{\min}}\underset{y\in U}{\max}(\min(T_{R_i}(x,y),F_A(y)))\right],\left[\overset{m}{\underset{i=1}{\min}}\underset{y\in U}{\max}(\min(T_{R_i}(x,y),F_B(y)))\right]\right\}\\
&= \min\left(F_{\underline{MNRS_{II}}^{o}(A)}(x),F_{\underline{MNRS_{II}}^{o}(B)}(x)\right).
\end{aligned}$$

Hence, $\underline{MNRS_{II}}^{o}(A)\cup_2\underline{MNRS_{II}}^{o}(B)\subseteq_2\underline{MNRS_{II}}^{o}(A\cup_2 B)$.

Additionally, according to Definition 11, we can get $\underline{MNRS_{II}}^{p}(A)\cup_2\underline{MNRS_{II}}^{p}(B)\subseteq_2\underline{MNRS_{II}}^{p}(A\cup_2 B)$.

(8) The proof is similar to that of Equation (7). □

Remark 1. *Note that if the* NRs *are the same one, then the optimistic (pessimistic)* $\mathrm{MNRS_{II}}$ *degenerates into* $\mathrm{NRS_{II}}$ *in Section* 2.

5. Type-III MNRS

In this chapter, $\mathrm{MNRS_{III}}$, which is based on a type-3 inclusion relation and corresponding union and intersection relations, is proposed and their characterizations are provided.

Definition 12. *Suppose (U, R_i) is an* MNAS. *∀A ∈ NS(U), the type-III optimistic* LUA *of A, represented by* $\underline{MNRS_{III}}^{o}(A)$ *and* $\overline{MNRS_{III}}^{o}(A)$*, is defined as:*

$$\underline{MNRS_{III}}^{o}(A)(x)=\overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i^{c}(x,y)\cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^{o}(A)(x)=\overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y)\cap_3 A(y))\right).$$

Then, A is named a definable NS *when* $\underline{MNRS_{III}}^{o}(A)=\overline{MNRS_{III}}^{o}(A)$*. Alternatively, we name the pair* $\left(\underline{MNRS_{III}}^{o}(A),\overline{MNRS_{III}}^{o}(A)\right)$ *an optimistic* $\mathrm{MNRS_{III}}$.

Definition 13. *Suppose (U, R_i) is an* MNAS. *∀A ∈ NS(U), the type-III pessimistic* LUA *of A, represented by* $\underline{MNRS_{III}}^{p}(A)$ *and* $\overline{MNRS_{III}}^{p}(A)$*, is defined as:*

$$\underline{MNRS_{III}}^{p}(A)(x)=\overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cap_3}(R_i^{c}(x,y)\cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cup_3} \left(R_i(x,y) \cap_3 A(y) \right) \right).$$

Similarly, A is named a definable NS *when* $\underline{MNRS_{III}}^{p}(A) = \overline{MNRS_{III}}^{p}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{III}}^{p}(A), \overline{MNRS_{III}}^{p}(A) \right)$ *a pessimistic* MNRS_{III}.

Proposition 5. *Suppose* (U, R_i) *is an* MNAS. $\forall A, B \in NS(U)$, *then:*

(1) $\underline{MNRS_{III}}^{o}(A) = \sim \overline{MNRS_{III}}^{o}(\sim A)$, $\underline{MNRS_{III}}^{p}(A) = \sim \overline{MNRS_{III}}^{p}(\sim A)$.
(2) $\overline{MNRS_{III}}^{o}(A) = \sim \underline{MNRS_{III}}^{o}(\sim A)$, $\overline{MNRS_{III}}^{p}(A) = \sim \underline{MNRS_{III}}^{p}(\sim A)$.
(3) $A \subseteq_3 B \Rightarrow \underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B), \underline{MNRS_{III}}^{p}(A) \subseteq_3 \underline{MNRS_{III}}^{p}(B)$.
(4) $A \subseteq_3 B \Rightarrow \overline{MNRS_{III}}^{o}(A) \subseteq_3 \overline{MNRS_{III}}^{o}(B), \overline{MNRS_{III}}^{p}(A) \subseteq_3 \overline{MNRS_{III}}^{p}(B)$.
(5) $\underline{MNRS_{III}}^{o}(A \cap_3 B) \subseteq_3 \underline{MNRS_{III}}^{o}(A) \cap_3 \underline{MNRS_{III}}^{o}(B)$, $\underline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \underline{MNRS_{III}}^{p}(A) \cap_3 \underline{MNRS_{III}}^{p}(B)$.
(6) $\overline{MNRS_{III}}^{o}(A) \cup_3 \overline{MNRS_{III}}^{o}(B) \subseteq_3 \overline{MNRS_{III}}^{o}(A \cup_3 B)$, $\overline{MNRS_{III}}^{p}(A) \cup_3 \overline{MNRS_{III}}^{p}(B) \subseteq_3 \overline{MNRS_{III}}^{p}(A \cup_3 B)$.
(7) $\underline{MNRS_{III}}^{o}(A) \cup_3 \underline{MNRS_{III}}^{o}(B) \subseteq_3 \underline{MNRS_{III}}^{o}(A \cup_3 B)$, $\underline{MNRS_{III}}^{p}(A) \cup_3 \underline{MNRS_{III}}^{p}(B) \subseteq_3 \underline{MNRS_{III}}^{p}(A \cup_3 B)$.
(8) $\overline{MNRS_{III}}^{o}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{o}(A) \cap_3 \overline{MNRS_{III}}^{o}(B)$, $\overline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{p}(A) \cap_3 \overline{MNRS_{III}}^{p}(B)$.

Proof. Equations (1) and (2) can be directly derived from Definitions 12 and 13. We only provide the proof of Equations (3)–(8).

(3) Suppose $A \subseteq_3 B$, then:

Case 1: If $T_A(x) < T_B(x)$, $F_A(x) \geq F_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y) \right] \leq \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y) \right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y) \right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y) \right] = F_{\underline{MNRS_{III}}^{o}(B)}(x).$$

Hence, $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.

Case 2: If $T_A(x) = T_B(x)$, $F_A(x) > F_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y) \right] = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y) \right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y) \right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y) \right] = F_{\underline{MNRS_{III}}^{o}(B)}(x).$$

Hence, $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.

Case 3: suppose $T_A(x) = T_B(x)$, $F_A(x) = F_B(x)$ and $I_A(x) \leq I_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y) \right] = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y) \right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y) \right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y) \right] = F_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$I_{\underline{MNRS_{III}}^{o}(A)}(x) = \begin{cases} I_A(y_j), {R_i}^c(x,y_j) \subseteq_3 A(y_j) \subseteq_3 A(y_k), y_k, y_j \in U \\ I_{{R_i}^c}(x,y_j), A(y_j) \subseteq_3 {R_i}^c(x,y_j) \\ 0, else \end{cases}$$

$$I_{\underline{MNRS_{III}}^{o}(B)}(x)=\begin{cases} I_B(y_j), R_i{}^c(x,y_j)\ \subseteq_3 B(y_j)\subseteq_3 B(y_k), y_k, y_j\in U\\ I_{R_i{}^c}(x,y_j), B(y_j)\subseteq_3 R_i{}^c(x,y_j)\\ 0, else\end{cases}.$$

Hence, $I_{\underline{MNRS_{III}}^{o}(A)}(x)\leq I_{\underline{MNRS_{III}}^{o}(B)}(x)$. So, $\underline{MNRS_{III}}^{o}(A)\subseteq_3 \underline{MNRS_{III}}^{o}(B)$.
Summing up the above, if $A\subseteq_3 B$, then $\underline{MNRS_{III}}^{o}(A)\subseteq_3 \underline{MNRS_{III}}^{o}(B)$.
Similarly, we can get $\underline{MNRS_{III}}^{p}(A)\subseteq_3 \underline{MNRS_{III}}^{p}(B)$.
(4) The proof is similar to that of Equation (3).
(5) From Definition 12, we have:

$$\begin{aligned}\underline{MNRS_{III}}^{o}(A\cap_3 B) &= \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3(A(y)\cap_3 B(y)))\right)\\ &\subseteq_3 \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}((R_i{}^c(x,y)\cup_3 A(y))\cap_3(R_i{}^c(x,y)\cup_3 B(y)))\right)\\ &\subseteq_3 \left(\overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 A(y))\right)\right)\cap_3\left(\overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 B(y))\right)\right)\\ &= \underline{MNRS_{III}}^{o}(A)\cap_3 \underline{MNRS_{III}}^{o}(B).\end{aligned}$$

Similarly, from Definition 13, we can get $\underline{MNRS_{III}}^{p}(A\cap_3 B)\subseteq_3 \underline{MNRS_{III}}^{p}(A)\cap_3 \underline{MNRS_{III}}^{p}(B)$.
(6) From Definition 12, we have:

$$\begin{aligned}\overline{MNRS_{III}}^{o}(A)\cup_3\overline{MNRS_{III}}^{o}(B) &= \left(\overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y)\cap_3 A(y))\right)\right)\cup_3\left(\overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y)\cap_3 B(y))\right)\right)\\ &\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}((R_i(x,y)\cap_3 A(y))\cup_3(R_i(x,y)\cap_3 B(y)))\right)\\ &\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y)\cap_3(A(y)\cup_3 B(y)))\right)\\ &= \overline{MNRS_{III}}^{o}(A\cup_3 B).\end{aligned}$$

Similarly, from Definition 13, we can get $\overline{MNRS_{III}}^{p}(A\cup_3 B)=\overline{MNRS_{III}}^{p}(A)\cup_3\overline{MNRS_{III}}^{p}(B)$.
(7) From Definition 12, we have:

$$\begin{aligned}\underline{MNRS_{III}}^{o}(A\cup_3 B) &= \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3(A\cup_3 B)(y))\right)\\ &= \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3(A(y)\cup_3 B(y)))\right)\\ &\supseteq_3 \overset{m}{\underset{i=1}{\cup_3}}\left(\left(\left[\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 A(y))\right]\cup_3\left[\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 B(y))\right]\right)\right)\\ &= \left(\overset{m}{\underset{i=1}{\cup_3}}\left[\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 A(y))\right]\right)\cup_3\left(\overset{m}{\underset{i=1}{\cup_3}}\left[\underset{y\in U}{\cap_3}(R_i{}^c(x,y)\cup_3 B(y))\right]\right)\\ &= \underline{MNRS_{III}}^{o}(A)\cup_3\underline{MNRS_{III}}^{o}(B).\end{aligned}$$

Hence, $\underline{MNRS_{III}}^{o}(A)\cup_3\underline{MNRS_{III}}^{o}(B)\subseteq_3\underline{MNRS_{III}}^{o}(A\cup_3 B)$.
Additionally, from Definition 13, we can get $\underline{MNRS_{III}}^{p}(A)\cup_3\underline{MNRS_{III}}^{p}(B)\subseteq_3 \underline{MNRS_{III}}^{p}(A\cup_3 B)$.

(8) From Definition 12, we have:

$$\begin{aligned}
\overline{MNRS_{III}}^{o}(A\cap_3 B) &= \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}\left(R_i(x,y)\cap_3(A\cap_3 B)(y)\right)\right)\\
&= \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}\left(R_i(x,y)\cap_3(A(y)\cap_3 B(y))\right)\right)\\
&\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}}\left(\left[\underset{y\in U}{\cup_2}\left(R_i(x,y)\cap_3 A(y)\right)\right]\cap_3\left[\underset{y\in U}{\cup_3}\left(R_i(x,y)\cap_3 B(y)\right)\right]\right)\\
&= \left(\overset{m}{\underset{i=1}{\cap_3}}\left[\underset{y\in U}{\cup_3}\left(R_i(x,y)\cap_3 A(y)\right)\right]\right)\cap_3\left(\overset{m}{\underset{i=1}{\cap_3}}\left[\underset{y\in U}{\cup_3}\left(R_i(x,y)\cap_3 B(y)\right)\right]\right)\\
&= \overline{MNRS_{III}}^{o}(A)\cap_3\overline{MNRS_{III}}^{o}(B).
\end{aligned}$$

Hence, $\overline{MNRS_{III}}^{o}(A\cap_3 B)\subseteq_3\overline{MNRS_{III}}^{o}(A)\cap_3\overline{MNRS_{III}}^{o}(B)$.
Similarly, from Definition 13, we can get $\overline{MNRS_{III}}^{p}(A\cap_3 B)\subseteq_3\overline{MNRS_{III}}^{p}(A)\cap_3\overline{MNRS_{III}}^{p}(B)$.
□

Remark 2. *Note that if the* NRs *are the same one, then the optimistic (pessimistic)* $\mathrm{MNRS_{III}}$ *degenerates into* $\mathrm{NRS_{III}}$ *in Section 3.*

6. Type-III MNRS in Two Universes with Its Applications

In this chapter, we propose the concept of $\mathrm{MNRS_{III}}$ in two universes and use it to deal with the MAGDM problem.

Definition 14. *[28] Suppose U, V are two non-empty finite universes, and $R_i \in NS(U\times V)$ $(1\le i\le m)$ is a binary* NR. *We call (U, V, R_i) the* MNAS *in two universes.*

Definition 15. *Suppose (U, V, R_i) is an* MNAS *in two universes. $\forall A\in NS(V)$ and $x\in U$, the type-III optimistic* LUA *of A in (U, V, R_i), represented by $\underline{MNRS_{III}}^{o}(A)$ and $\overline{MNRS_{III}}^{o}(A)$, is defined as:*

$$\underline{MNRS_{III}}^{o}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in V}{\cap_3}\left(R_i{}^{c}(x,y)\cup_3 A(y)\right)\right)$$

$$\overline{MNRS_{III}}^{o}(A)(x) = \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in V}{\cup_3}\left(R_i(x,y)\cap_3 A(y)\right)\right).$$

Then, A is named a definable NS *in two universes when $\underline{MNRS_{III}}^{o}(A)=\overline{MNRS_{III}}^{o}(A)$. Alternatively, we name the pair $\left(\underline{MNRS_{III}}^{o}(A),\overline{MNRS_{III}}^{o}(A)\right)$ an optimistic* $\mathrm{MNRS_{III}}$ *in two universes.*

Definition 16. *Suppose (U, V, R_i) is an* MNAS *in two universes. $\forall A\in NS(V)$ and $x\in U$, the type-III pessimistic* LUA *of A in (U, V, R_i), denoted by $\underline{MNRS_{III}}^{p}(A)$ and $\overline{MNRS_{III}}^{p}(A)$, is defined as follows:*

$$\underline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in V}{\cap_3}\left(R_i{}^{c}(x,y)\cup_3 A(y)\right)\right)$$

$$\overline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in V}{\cup_3}\left(R_i(x,y)\cap_3 A(y)\right)\right).$$

Similarly, A is named a definable NS *when $\underline{MNRS_{III}}^{p}(A)=\overline{MNRS_{III}}^{p}(A)$. Alternatively, we name the pair $\left(\underline{MNRS_{III}}^{p}(A),\overline{MNRS_{III}}^{p}(A)\right)$ a pessimistic* $\mathrm{MNRS_{III}}$ *in two universes.*

Remark 3. *Note that if the two domains are the same, then the optimistic (pessimistic)* $\mathrm{MNRS}_{\mathrm{III}}$ *in two universes degenerates into the optimistic (pessimistic)* $\mathrm{MNRS}_{\mathrm{III}}$ *in a single universe in Section 5.*

The MAGDM problem is becoming more and more generally present in our daily life. MAGDM means to select or rank all the feasible alternatives in various criterions. There are many ways to solve the MAGDM problem, but we use MNRS to solve it in this paper. Next, we give the basic description of the considered MAGDM problem.

For the car-ranking question, suppose $U = \{x_1, x_2, \ldots, x_n\}$ is the decision set and $V = \{y_1, y_2, \ldots, y_m\}$ is the criteria set in which x_1 represents "very popular", x_2 represents "popular", x_3 represents "less popular", ..., x_n represents "not popular", y_1 represents the vehicle type", y_2 represents the size of the space, y_3 represents the ride height, y_4 represents quality, and ..., y_m represents length of durability. Then, l selection experts make evaluations about the criteria sets according to their own experiences. Here, the evaluations were shown by NRs. Next, we calculate the degree of popularity for a given car. Therefore, we need to use MGNRS to solve the above problem. For the MAGDM problem under a multigranulation neutrosophic environment, the optimistic lower approximation can be regarded as an optimistic risk decision, and the optimistic upper approximation can be regarded as an optimistic conservative decision. Additionally, the pessimistic lower approximation can be regarded as a pessimistic risk decision and the pessimistic upper approximation can be regarded as a pessimistic conservative decision. According to the distance of neutrosophic sets, we define the difference function $d_N(A, B)(x_i) = (1/3)(|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|)$. We used the difference function to represent the distance of optimistic (pessimistic) upper and lower approximation. The smaller the value of the distance is, the better the alternative x_i is, because the risk decision and the conservative decision are close. By comparing the distance value, all alternatives can be ranked and we can choose the optimal alternative. In this paper, we only used three kinds of optimistic upper and lower approximation to decision-making.

Next, we show the process of the above car-ranking question based on MGNRSs over two universes. Let $R_l \in NR(U \times V)$ be NRs from U to V, where $\forall (x_i, y_j) \in U \times V$, $R_l(x_i, y_j)$ denotes the degree of popularity for criteria set y_j $(y_j \in V)$. R_l can be obtained according to experts' experience. Given a car A, according to the unconventional questionnaire (suppose there are three options—"like", "not like", and "neutral" to choose for each of the criteria sets, and everyone can choose one or more options), then we can get the popularity of every criterion as described by an NS A in the universe V according to the questionnaire. By use of the following Algorithm 1, we can determine the degree of popularity of the given car A.

Algorithm 1 Decision algorithm

Input Multigranulation neutrosophic decision information systems $(U, V, \mathbf{R})$.

Output The degree of popularity of the given car.

Step 1 Computing three kinds of optimistic multigranulation LUA $\underline{MNRS_I}^o(A)$, $\overline{MNRS_I}^o(A)$, $\underline{MNRS_{II}}^o(A)$, $\overline{MNRS_{II}}^o(A)$, $\underline{MNRS_{III}}^o(A)$, $\overline{MNRS_{III}}^o(A)$.

Step 2 Calculate $d(\underline{MNRS_I}^o(x_i), \overline{MNRS_I}^o(x_i))$, $d(\underline{MNRS_{II}}^o(x_i), \overline{MNRS_{II}}^o(x_i))$ and $d(\underline{MNRS_{III}}^o(x_i), \overline{MNRS_{III}}^o(x_i))$.

Step 3 The best choice is to select x_h (which means that the most welcome degree is x_h) if $d(\underline{MNRS}^o(x_h), \overline{MNRS}^o(x_h)) = \min_{i \in \{1,2,\cdots,n\}} d(\underline{MNRS}^o(x_i), \overline{MNRS}^o(x_i))$.

Step 4 If h has two or more values, then each x_h will be the best choice. In this case, the car may have two or more popularities and each x_k will be regarded as the most possible popularity; otherwise, we use other methods to make a decision.

Next, we use an example to explain the algorithm.

Let $U = \{x_1, x_2, x_3, x_4\}$ be the decision set, in which x_1 denotes "very popular", x_2 denotes "popular", x_3 denotes "less popular", and x_4 denotes "not popular". Let $V = \{y_1, y_2, y_3, y_4, y_5\}$ be criteria sets, in which y_1 denotes the vehicle type, y_2 denotes the size of the space, y_3 denotes the ride height, y_4 denotes quality, and y_5 denotes length of durability.

Suppose that R_1, R_2, and R_3 are given by three invited experts. They provide their evaluations for all criteria y_j with respect to decision set elements x_i. The evaluation R_1, R_2, and R_3 are NRs between attribute set V and decision evaluation set U., that is., there are $R_1, R_2, R_3 \in NR(U \times V)$.

Suppose three experts present their judgment (the neutrosophic relation R_1, R_2, and R_3) for the attribute and decision sets in Tables 2–4:

Table 2. Neutrosophic relation R_1.

R_1	y_1	y_2	y_3	y_4	y_5
x_1	(0.8, 0.6, 0.5)	(0.2, 0.3, 0.9)	(0, 0, 1)	(0.7, 0.5, 0.6)	(0, 0, 1)
x_2	(0.6, 0.4, 0.6)	(0.9, 0.3, 0.4)	(1, 0, 0)	(0, 0, 1)	(0.3, 0.6, 0.7)
x_3	(0.2, 0.5, 0.9)	(0.6, 0.7, 0.5)	(0.8, 0.7, 0.8)	(0, 0, 1)	(1, 0, 0)
x_4	(0.6, 0.4, 0.7)	(0, 0, 1)	(0, 0, 1)	(0.9, 0.8, 0.1)	(0, 0, 1)

Table 3. Neutrosophic relation R_2.

R_2	y_1	y_2	y_3	y_4	y_5
x_1	(0.9, 0.3, 0.6)	(0, 0, 1)	(0, 0, 1)	(0.5, 0.6, 0.5)	(0.2, 0.3, 0.9)
x_2	(0.3, 0.7, 0.8)	(0.7, 0.5, 0.6)	(0.9, 0.1, 0.1)	(0, 0, 1)	(0.4, 0.5, 0.8)
x_3	(0.1, 0.6, 0.8)	(0.3, 0.6, 0.5)	(0.7, 0.3, 0.6)	(0, 0, 1)	(1, 0, 0)
x_4	(0.7, 0.5, 0.6)	(0, 0, 1)	(0, 0, 1)	(1, 0, 0)	(0, 0, 1)

Table 4. Neutrosophic relation R_3.

R_3	y_1	y_2	y_3	y_4	y_5
x_1	(0.6, 0.9, 0.4)	(0.1, 0.1, 0.8)	(0.1, 0, 0.9)	(0.8, 0.4, 0.8)	(0, 0, 1)
x_2	(0.5, 0.6, 0.6)	(0.6, 0.2, 0.7)	(1, 0, 0)	(0, 0, 1)	(0, 0, 1)
x_3	(0.1, 0.4, 0.7)	(0.2, 0.2, 0.7)	(0.5, 0.7, 0.6)	(0, 0, 1)	(0.9, 0.1, 0.2)
x_4	(0.6, 0.3, 0.4)	(0, 0, 1)	(0, 0, 1)	(0.7, 0.5, 0.4)	(0, 0, 1)

Suppose A is a car and each criterion in V is as follows:

$$A = \{(y_1, 0.9, 0.2, 0.2), (y_2, 0.2, 0.7, 0.8), (y_3, 0, 1, 0.3), (y_4, 0.7, 0.6, 0.3), (y_5, 0.1, 0.8, 0.9)\}.$$

Then, we can calculate the three kinds of optimistic LUAs of A as follow:

$$\begin{gathered}
\underline{MNRS_I}^o(A)(x_1) = (0.8, 1, 0.3),\ \underline{MNRS_I}^o(A)(x_2) = (0.1, 0.9, 0.6),\\
\underline{MNRS_I}^o(A)(x_3) = (0.2, 0.8, 0.9),\ \underline{MNRS_I}^o(A)(x_4) = (0.7, 1, 0.3),\\
\overline{MNRS_I}^o(A)(x_1) = (0.7, 0.6, 0.5),\ \overline{MNRS_I}^o(A)(x_2) = (0.3, 0.6, 0.3),\\
\overline{MNRS_I}^o(A)(x_3) = (0.2, 0.6, 0.8),\ \overline{MNRS_I}^o(A)(x_4) = (0.7, 0.5, 0.4),\\
\underline{MNRS_{II}}^o(A)(x_1) = (0.8, 0.6, 0.3),\ \underline{MNRS_{II}}^o(A)(x_2) = (0.1, 0.6, 0.6),\\
\underline{MNRS_{II}}^o(A)(x_3) = (0.2, 0.6, 0.9),\ \underline{MNRS_{II}}^o(A)(x_4) = (0.7, 0.6, 0.3),\\
\overline{MNRS_{II}}^o(A)(x_1) = (0.7, 0.4, 0.5),\ \overline{MNRS_{II}}^o(A)(x_2) = (0.3, 0.2, 0.3),\\
\overline{MNRS_{II}}^o(A)(x_3) = (0.2, 0.6, 0.8),\ \overline{MNRS_{II}}^o(A)(x_4) = (0.7, 0.2, 0.4),\\
\underline{MNRS_{III}}^o(A)(x_1) = (0.8, 0, 0.3),\ \underline{MNRS_{III}}^o(A)(x_2) = (0.1, 0, 0.6),\\
\underline{MNRS_{III}}^o(A)(x_3) = (0.2, 0.9, 0.9),\ \underline{MNRS_{III}}^o(A)(x_4) = (0.7, 0.6, 0.3),\\
\overline{MNRS_{III}}^o(A)(x_1) = (0.7, 1, 0.5),\ \overline{MNRS_{III}}^o(A)(x_2) = (0.3, 0, 0.3),\\
\overline{MNRS_{III}}^o(A)(x_3) = (0.2, 0.7, 0.8)\ \overline{MNRS_{III}}^o(A)(x_4) = (0.7, 0.5, 0.4).
\end{gathered}$$

Therefore, we can get:

$$d(\underline{MNRS_I}^o(x_1), \overline{MNRS_I}^o(x_1)) = 0.7/3, d(\underline{MNRS_I}^o(x_2), \overline{MNRS_I}^o(x_2)) = 0.8/3,$$
$$d(\underline{MNRS_I}^o(x_3), \overline{MNRS_I}^o(x_3)) = 0.1, d(\underline{MNRS_I}^o(x_4), \overline{MNRS_I}^o(x_4)) = 0.2,$$
$$d(\underline{MNRS_{II}}^o(x_1), \overline{MNRS_{II}}^o(x_1)) = 0.5/3, d(\underline{MNRS_{II}}^o(x_2), \overline{MNRS_{II}}^o(x_2)) = 0.3,$$
$$d(\underline{MNRS_{II}}^o(x_3), \overline{MNRS_{II}}^o(x_3)) = 0.1/3, d(\underline{MNRS_{II}}^o(x_4), \overline{MNRS_{II}}^o(x_4)) = 0.5/3,$$
$$d(\underline{MNRS_{III}}^o(x_1), \overline{MNRS_{III}}^o(x_1)) = 1.3/3, d(\underline{MNRS_{III}}^o(x_2), \overline{MNRS_{III}}^o(x_2)) = 0.5/3,$$
$$d(\underline{MNRS_{III}}^o(x_3), \overline{MNRS_{III}}^o(x_3)) = 0.1, d(\underline{MNRS_{III}}^o(x_4), \overline{MNRS_{III}}^o(x_4)) = 0.2/3.$$

Thus, for the type-I and type-II MNRS, the optimistic best choice is to select x_3, that is, this car is less popular; for the type-III MNRS, the optimistic best choice is to select x_4, that is, this car is not popular.

7. Conclusions

NRS and MNRS are extensions of the Pawlak rough set theory. In this paper, we analysed the NRS_I and NRS_{II}, we proposed model NRS_{III}, and used an example to outline the differences between the three kinds of NRS. We gave the definition of $MNRS_{III}$, which is based on the type-3 operator relation of NS, and considered their properties. Furthermore, we proposed $MNRS_{III}$ in two universes and we presented an algorithm of the MAGDM problem based on it.

In the future, we will be researching other types of fusions of MGRSs and NSs. We will also study the applications of concepts in this paper to some algebraic systems (for example, pseudo-BCI algebras, neutrosophic triplet groups, see [30,31]).

Author Contributions: X.Z. and C.B. initiated the research and wrote the paper, S.S. participated in some of the research work, and F.S. supervised the research work and provided helpful suggestions.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 61573240) and the Graduate Student Innovation Project of Shanghai Maritime University 2017ycx082.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Atanassov, K.T. Type-1 Fuzzy Sets and Intuitionistic Fuzzy Sets. *Algorithms* **2017**, *10*, 106. [CrossRef]
2. Bisht, K.; Joshi, D.K.; Kumar, S. Dual Hesitant Fuzzy Set-Based Intuitionistic Fuzzy Time Series Forecasting. In Proceedings of the International Conference on Recent Advancement in Computer, Communication and Computational Sciences, Ajmer, India, 2–3 September 2017; Springer: Singapore, 2018; pp. 317–329.
3. Kumar, K.; Garg, H. TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. *Comput. Appl. Math.* **2018**, *37*, 1319–1329. [CrossRef]
4. Jia, X.; Shang, L.; Zhou, B.; Yao, Y. Generalized attribute reduct in rough set theory. *Knowl. Based Syst.* **2016**, *91*, 204–218. [CrossRef]
5. Maji, P. Advances in Rough Set Based Hybrid Approaches for Medical Image Analysis. In Proceedings of the International Joint Conference, Olsztyn, Poland, 3–7 July 2017; Springer: Cham, Switzerland, 2017; pp. 25–33.
6. Yao, Y.; She, Y. Rough set models in multigranulation spaces. *Inf. Sci.* **2016**, *327*, 40–56. [CrossRef]
7. Zhan, J.; Liu, Q.; Herawan, T. A novel soft rough set: Soft rough hemirings and corresponding multicriteria group decision-making. *Appl. Soft Comput.* **2017**, *54*, 393–402. [CrossRef]
8. Ma, X.L.; Zhan, J.M.; Ali, M.I. A survey of decision making methods based on two classes of hybrid soft set models. *Artif. Intell. Rev.* **2018**, *49*, 511–529. [CrossRef]
9. Zhang, X.H.; Bo, C.X.; Smarandache, F.; Park, C. New operations of totally dependent-neutrosophic sets and totally dependent-neutrosophic soft sets. *Symmetry* **2018**, *10*. [CrossRef]
10. Zhang, X.; Park, C.; Wu, S. Soft set theoretical approach to pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2018**, *34*, 559–568. [CrossRef]
11. Smarandache, F. Neutrosophic set—A generialization of the intuitionistics fuzzy sets. *Int. J. Pure Appl. Math.* **2005**, *24*, 287–297.

12. Wang, H.B.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistructure* **2010**, *4*, 410–413.
13. Singh, P.K. Three-way fuzzy concept lattice representation using neutrosophic set. *Int. J. Mach. Learn. Cybern.* **2017**, *8*, 69–79. [CrossRef]
14. Peng, X.; Liu, C. Algorithms for neutrosophic soft decision making based on EDAS, new similarity measure and level soft set. *J. Intell. Fuzzy Syst.* **2017**, *32*, 955–968. [CrossRef]
15. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2459–2466.
16. Zhang, X.; Bo, C.X.; Smarandache, F.; Dai, J.H. New inclusion relation of neutrosophic sets with applications and related lattice structure. *Int. J. Mach. Learn. Cybern.* **2018**, *9*, 1753–1763. [CrossRef]
17. Dubois, D.; Prade, H. Rough fuzzy sets and fuzzy rough sets. *Inter. J. General Syst.* **1990**, *17*, 191–209. [CrossRef]
18. Broumi, S.; Smarandache, F.; Dhar, M. Rough neutrosophic sets. *Neut. Sets Syst.* **2014**, *3*, 62–67.
19. Yang, H.L.; Zhang, C.L.; Guo, Z.L.; Liu, Y.L.; Liao, X. A hybrid model of single valued neutrosophic sets and rough sets: Single valued neutrosophic rough set model. *Soft Comput.* **2017**, *21*, 6253–6267. [CrossRef]
20. Qian, Y.H.; Liang, J.Y.; Yao, Y.Y.; Dang, C.Y. MGRS: A multi-granulation rough set. *Inf. Sci.* **2010**, *180*, 949–970. [CrossRef]
21. Kumar, S.S.; Inbarani, H.H. Optimistic multi-granulation rough set based classification for medical diagnosis. *Procedia Comput. Sci.* **2015**, *47*, 374–382. [CrossRef]
22. Majumdar, P.; Samanta, S.K. On similarity and entropy of neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 1245–1252.
23. Kang, Y.; Wu, S.; Li, Y.; Liu, J.; Chen, B. A variable precision grey-based multi-granulation rough set model and attribute reduction. *Knowl. Based Syst.* **2018**, *148*, 131–145. [CrossRef]
24. Sun, B.Z.; Ma, W.M.; Qian, Y.H. Multigranulation fuzzy rough set over two universes and its application to decision making. *Knowl. Based Syst.* **2017**, *123*, 61–74. [CrossRef]
25. Pan, W.; She, K.; Wei, P. Multi-granulation fuzzy preference relation rough set for ordinal decision system. *Fuzzy Sets Syst.* **2017**, *312*, 87–108. [CrossRef]
26. Huang, B.; Guo, C.; Zhuang, Y.; Li, H.; Zhou, X. Intuitionistic fuzzy multi-granulation rough sets. *Inf. Sci.* **2014**, *277*, 299–320. [CrossRef]
27. Zhang, X.; Miao, D.; Liu, C.; Le, M. Constructive methods of rough approximation operators and multi-granulation rough sets. *Knowl. Based Syst.* **2016**, *91*, 114–125. [CrossRef]
28. Bo, C.X.; Zhang, X.; Shao, S.T.; Smarandache, F. Multi-granulation neutrosophic rough sets on a single domain and dual domains with applications. *Symmetry* **2018**. accepted. [CrossRef]
29. Yang, H.L.; Guo, Z.L.; She, Y.; Liao, X.W. On single valued neutrosophic relations. *J. Intell. Fuzzy Syst.* **2016**, *30*, 1045–1056. [CrossRef]
30. Zhang, X.H. Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2017**, *33*, 1767–1774. [CrossRef]
31. Zhang, X.H.; Smarandache, F.; Liang, X.L. Neutrosophic duplet semi-group and cancellable neutrosophic triplet groups. *Symmetry* **2017**, *9*. [CrossRef]