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Analytic Solutions of Nonlinear Partial Differential Equations by the Power Index Method

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Abstract: An updated Power Index Method is presented for nonlinear differential equations (NLPDEs) with the aim of reducing them to solutions by algebraic equations. The Lie symmetry, translation invariance of independent variables, allows for traveling waves. In addition discrete symmetries, reflection, or 180° rotation symmetry, are possible. The method tests whether certain hyperbolic or Jacobian elliptic functions are analytic solutions. The method consists of eight steps. The first six steps are quickly applied; conditions for algebraic equations are more complicated. A few exceptions to the Power Index Method are discussed. The method realizes an aim of Sophus Lie to find analytic solutions of nonlinear differential equations.

Keywords: nonlinear; partial differential equations; symmetries; analytic solutions

1. Introduction

Nonlinear partial differential equations (NLPDEs) have been difficult to solve analytically. However, analytic solutions may serve as bench marks for numerical solutions where the computations can be tricky. There is no one method that works for all the cases of analytic solutions. The Inverse Scattering Method for solitons [1] is restricted to certain NLPDEs, but the inclusion of initial conditions is very advantageous. The Method of Characteristics [2] applies to quasi-linear partial differential equations (PDEs) that have first-order derivatives. Initial conditions can be applied too. For many NLPDEs these methods do not work. Lie symmetries [3–7] are employed to reduce the NLPDEs to nonlinear ordinary differential equations (NLODEs) that may then be reduced to quadradures if sufficient symmetries exist. Initial or boundary conditions complicate the application of Lie symmetries. Various non-classical symmetries, including hidden symmetries [8], broaden the class of NLPDEs that can be reduced. Sometimes the NLDEs can be integrated and first integrals are found. Certain NLPDEs are then reduced to quadratures. There are symbolic computer programs, such as Mathematica, which may solve the NLPDEs by various approaches.

If none of the above methods result in a solution, there are methods based on guessing possible functions [9,10] as solutions or by appending auxiliary equations. The functions elected as solutions include: hyperbolic functions, Jacobian elliptic functions, the logistic function, or the Weierstrass elliptic function. Various Expansion Methods such as F, G′/G [11,12] require auxiliary differential equations (ODEs). The Simplest Equation Method relies on linear ODEs [13]. The Tanh Method [14,15] and Sech Method have been applied with the homogeneous balance condition [16] that depends on balancing the most nonlinear terms in the NLPDEs. These methods have been applied to NLPDEs with the independent variables invariant under translations. As a consequence traveling waves are a common form of solutions.

2. Power Index Method

The updated Power Index Method for solution of NLPDEs is discussed here. An earlier version [17] has been modified to include several new criteria and the method is now presented as eight steps. The purpose is to reduce the NLPDEs to algebraic equations that may be solved. The intention is to present efficient and quick tests for determining if analytic solutions exist for the NLPDEs. The analysis is not exhaustive; other analytic solutions may exist. Those discussed below are for one real NLPDE but the method may be applied to complex NLPDEs, to several coupled NLPDEs and NLPDEs, with more than two independent variables.

The first step is to choose a NLPDE with two independent variables invariant under translations. By inspection if no independent variables appear explicitly, we see that the translation invariance holds. The second step is to assume traveling waves where the argument of the dependent variable u is a function of $\beta\overline{x}$ for β the wave number, $\overline{x} = x - ct + \alpha$ for x and t the independent variables, c the wave speed and α the phase angle. Once u is a function of $\overline{x}$, solving NLPDEs reduces to solving nonlinear ODEs. The third step is to assume that the variable u is given by a finite power series

$$u(\beta\overline{x}) = \sum_{l=0}^{p} a_l U^l(\beta\overline{x}) \tag{1}$$

where p is a positive integer, the a_l are expansion coefficients, and $U(\beta\overline{x})$ is a nonlinear function. The nonlinear function $U(\beta\overline{x})$ is one of the hyperbolic functions: tanh, sech, or the Jacobian elliptic functions: sn, cn, dn. There are other functions in these two classes, but they may be singular and are not studied here. Why do we need nonlinear functions? These nonlinear functions when differentiated produce a product of two functions in its class. This property is necessary to balance products of u and its derivatives by higher order derivatives. When the power series in Equation (1) is substituted in the NLPDE, another power series in $U(\beta\overline{x})$ is formed. This entails the application of function identities, such as $\tan h\prime(x) = 1 - \tan h(x)^2$, rather than $\sec h(x)^2$. It will be useful to remove some common factors of the NLPDE too.

The fourth step is the homogenous balance condition. This balance condition was apparently first used in the Tanh Method [13,14]. We discuss the homogenous balance first for tanh functions. Any order derivative of $\tan h(\beta\overline{x})$ can be expressed in terms of $\tan h(\beta\overline{x})$. Therefore, one chooses the term in the NLPDE with the largest power index and equates it to another term with the largest power index in the NLPDE. Next, we introduce the power index P as the criterion in the homogenous balance condition. The homogenous condition is found by substituting $u = U(\beta\overline{x})^p$ in the NLPDE. Then, P is defined as

$$P = np + d \tag{2}$$

where n is the number of products of u and the derivatives of u. The power p is the highest power of $U(\beta\overline{x})$ in the power series Equation (1) and is a positive integer. If p is determined to be negative or a fraction, the dependent variable is changed and a new replacement power p calculated. The d is the net number of derivatives in a term in the NLPDE. For the Tanh Method each derivative adds a product of one extra $\tan h(\beta\overline{x})$ that appears in the most nonlinear term if function identities are used. As an example, consider the terms $u^2 u_x$ and u_{xxx}. The power index for the first term is $P_1 = 3p + 1$ and the power index for the second term is $P_2 = p + 3$. Equating P_1 and P_2 we find that $p = 1$.

The fifth step is to assess whether all the net number of derivatives d are even, odd, or mixed even and odd. The effect of the derivatives differs for tanh functions and the other functions. The homogenous balance condition has been used for the hyperbolic secant and Jacobian elliptic functions; its use is similar to that for the $\tan h(\beta\overline{x})$ function if all the d are even numbers. If all d are odd numbers, then a common factor for the $\sec h(\beta\overline{x})$ function is $\sec h(\beta\overline{x}) \tan h(\beta\overline{x})$ and for the $\mathrm{sn}(\beta\overline{x})$ function the common factor is $\mathrm{cn}(\beta\overline{x})\mathrm{dn}(\beta\overline{x})$ after all the identities are used. The effective d for products of only a $\sec \mathrm{h}(\beta\overline{x})$ function or only a $\mathrm{sn}(\beta\overline{x})$ function would be $d - 1$ and $d - 2$, respectively. However, the power p would be the same as

$$p = \frac{P_2 - P_1}{n_1 - n_2} = \frac{d_2 - d_1}{n_1 - n_2} \tag{3}$$

for $\tan h(\beta\overline{x})$ functions as the number of net derivatives are subtracted as seen in Equation (3).

If the net derivative d values are mixed numbers in the NLPDE, only $\tan h(\beta\overline{x})$ functions are usually possible solutions. There are unusual exceptions if the NLPDE can be split into two NLPDEs, as were done in fluid mechanics [18]. To split the NLPDE the terms with even and odd values of d are set equal to zero separately. An additional requirement is that same expression for $u(\beta\overline{x})$ in Equation (1) is used in the separate parts of the NLPDE where the values for the power p are the same. The split of a NLPDE into two separate NLPDEs has been reported with a solution of Equation (6) in [19] with a solution that is proportional to $\sec h^2(\beta\overline{x})$. Actually, the solution for the original NLPDE Equation (6) can be found by $\tan h(\beta\overline{x})$ functions. It is possible to formulate a NLPDE that can be solved only by splitting.

The sixth step is to check the discrete symmetries, reflection, or 180° rotation symmetry of $u(\beta\overline{x})$. These symmetries have been rarely applied with Lie symmetries. Assume $u(\beta\overline{x})$ is an even (odd) function of its argument. Then, check each term in the NLPDE as to whether it is even or odd. If the NLPDE is invariant in form, then $u(\beta\overline{x})$ is an even (odd) function. This attribute can reduce the number of terms in the power series Equation (1).

The preceding six steps can be checked quickly unless there is an exception to step five. The last two steps for the number of algebraic equations are more complicated. The seventh step determines the number N_e of algebraic equations. The algebraic equations are found from the NLPDE power series in the function $U(\beta\overline{x})$. To find the power series Equation (1) is substituted into the NLPDE and after evaluation of derivatives, employing function identities, and dividing of common factors a power series in $U(\beta\overline{x})$ is found. This operation may be performed by symbolic computing or by hand. The coefficients of each power of $U(\beta\overline{x})$ in the NLPDE set equal to zero constitute the algebraic equations.

The maximum number of terms in the NLPDE power series is $P_{max} + 1$, where P_{max} is the power index of the most nonlinear terms in the NLPDE and the 1 counts the constant term if present. Then, the number of algebraic equations is given by

$$N_e \leq \; P_{max} + 1. \tag{4}$$

Now, N_e is frequently smaller than the maximum value in the Equation (4); some examples are noted here. For example, if all of the terms in the NLPDE have derivatives, then there is no constant term in the NLPDE power series and the 1 on the right side of Equation (4) is gone. Common factors are likely when Equation (1) is substituted into the NLPDE and dividing these out reduces the N_e. In addition, when all of the net derivatives d are odd, the common factors if $U(\beta\overline{x})$ is a sech function, contain a tanh function and the common factors if $U(\beta\overline{x})$ is a Jacobian elliptic function contain a product of the other Jacobian elliptic functions. If the NLPDE has even or odd symmetry, then there are fewer terms in the NLPDE power series except at times for P_{max}= 1 for even symmetry.

The eighth step determines the number of parameters N_p in the algebraic equations. Parameters in the original NLPDE power series consist of the expansion coefficients a_l in Equation (1), β, and c and any other constants in the NLPDE, but the effective parameters are those that appear in the algebraic equations. These effective parameters may be compound, that is as the sum or product of the original parameters and hence may be fewer in number. In the examples discussed here that may reduce the value of N_p. Again, the algebraic equations are determined by equating the coefficients of each power of $U(\beta\overline{x})$ in the NLPDE series expansion to zero. For a solution of the algebraic equations

$$N_p > N_e \tag{5}$$

which is a sufficient condition. Some algebraic equations are redundant or consist of an identity. Then, the actual number of algebraic equation is fewer. Equation (5) may be valid for Jacobian functions but not for hyperbolic functions until the actual algebraic equations are written down. Those Jacobian elliptic function solutions should reduce to solutions of hyperbolic functions when the modulus approaches one. An example is discussed in Section 3.3.

3. Examples

The examples of NLPDEs are all invariant under translations in the independent variables. Therefore, traveling waves are chosen for all of the examples. We start with determination of the power p from homogeneous balance.

3.1. KdV and mKdV Equations

Consider the KdV and mKdV equations. These equations with traveling waves may be integrated twice and reduced to quadratures, but that is not possible for some NLPDEs. The KdV equation was originally derived for nonlinear fluid flow in a channel. We have

$$u_t + u^l u_x = \gamma u_{xxx} \tag{6}$$

for γ a constant and where $l = 1$ for the KdV equation and $l = 2$ for the mKdV equation. Equating the maximum power index P for the second and third terms, we find $p = 2$ for the KdV equation and $p = 1$ for the mKdV equation. All of the net derivatives are odd for both equations. Both the KdV and mKdV equations may have solutions in the two hyperbolic functions and the three Jacobian elliptic functions. The reflection and/or 180° rotation symmetry is next checked. The $u(\beta\overline{x})$ for the KdV equation has reflection symmetry; the $u(\beta\overline{x})$ for the mKdV equation has both a reflection symmetry and 180° rotation symmetry. Then, $u(\beta\overline{x})$ for the KdV equation is an even function where we consider the hyperbolic tangent function or

$$u(\beta\overline{x}) = a_0 + a_2 \tan h^2(\beta\overline{x}) \tag{7}$$

The number N_e of algebraic equations for the KdV equation equals 2. How do we arrive at that number? The number of terms in the NLPDE power series, P_{max}, is $p + 3$, but since $u(\beta\overline{x})$ is an even function, the highest power of the nonlinear function in the power series for the NLPDE is four. Then the common factors when all of the terms have at least one derivative is two terms or $N_e = 2$. The effective parameters are $a_0 - c,\ a_2, \beta^2\gamma$, where some are compound. Then, $N_p = 3$. As $N_p > N_e$, a solution of the algebraic equations exists where $a_0 = c - 8\beta^2\gamma, a_2 = 12\beta^2\gamma$.

The solutions for the mKdV equation are both odd and even functions

$$u(\beta\overline{x}) = \mathrm{a}_1\ \tan\mathrm{h}(\beta\overline{x}) \text{ or } u(\beta\overline{x}) = b_0 +\ b_1 \sec\mathrm{h}(\beta\overline{x}) \tag{8}$$

where $a_1,\ b_0,\ b_1$ are constants. Now, $N_e = 2$ for the odd function where P_{max} is 4, but the common factors reduce it to 2. Then $N_e = 3$ for the even function where again P_{max} is 4 which gives a maximum of five terms in the power series in the NLPDE as $N_e = P_{max} + 1$. Once the common factors are divided out there remain three terms in the power series. The effective parameters are $a_1^2, c, \beta^2\gamma$ for the odd function and $b_0^2,\ b_0b_2,\ b_2^2$, c, $\beta^2\gamma$ for the even function. owever However, $b_0^2,\ b_0b_2,\ b_2^2$ are not all independent. Then, $N_p\ = 3$ for the odd function and $N_p\ = 4$ for the even function. The relation $N_p > N_e$, is obeyed for both cases where $a_1 = \pm\beta\sqrt{6\gamma}$, $c = 2\beta^2\gamma$ for the odd function and $b_0 = 0,\ b_1 = \pm i\beta\sqrt{6\gamma}, c = -\beta^2\gamma$ for the even function.

3.2. Burgers Equation

Next consider Burgers equation [20], a model equation for shock waves,

$$u_t + uu_x = \gamma u_{xx} \tag{9}$$

for γ a constant. The homogenous balance gives $p = 1$. As the net values of d are both odd and even numbers, only the nonlinear function $\tan\mathrm{h}(\beta\overline{x})$ is a possible solution. A split into two NLPDEs does not hold. There is no reflection or rotation symmetry; consequently, the solution can have even and odd terms. Now, $P_{max} + 1 = 4$, but the number of equations in the NLPDE is N_e = 2 when the common factors are divided out. The original number of terms was 4, because the solution has both odd and even terms. The effective parameters are $a_0 - c,\ a_1,\ \beta\gamma$ or the number of parameters $N_P = 3$. Then,

$$u(\beta\overline{x}) = \ a_0 + a_1 \tan h(\beta\overline{x}) \tag{10}$$

where $a_0 = c,\ a_1 = -2\beta\gamma$.

3.3. Blasius Equation

An example of a NLPDE that has no solution in terms of the five nonlinear functions is the Blasius equation [4,21] for laminar flow of a fluid past a plate

$$u_{xxx} + \gamma u u_{xx} = 0 \tag{11}$$

where γ is a constant. This NLODE has two Lie symmetries, but cannot be reduced to quadratures, as has been known. The application of homogenous balance gives power $p = 1$. There are only two terms in the NLPDE so that lesser nonlinear terms in the expansion of Equation (11) by Equation (1) are not balanced. The net derivatives d are mixed even and odd numbers that means only a $\tan h(\beta\overline{x})$ is possible. As there is the 180° rotation symmetry, $u(\beta\overline{x})$ is an odd function. The number of equations in the NLPDE, $N_e = 2$, as there are four terms originally in the NLPDE power series, but common terms are divided out. The effective parameters are $a_1,\ \beta$ or $N_p = 2$. Because $N_p = 2$, no solution in the hyperbolic or Jacobian functions exists. Equation (12)

$$u_x = \ \frac{1}{3}\, u^2 \tag{12}$$

from [21] becomes the Blasius equation when differentiated twice. The solution of Equation (12) is

$$u = \ \pm\frac{3}{x + C} \tag{13}$$

with C a constant. Then, Equation (13) is a solution of the Blasius equation but it may be singular.

3.4. Kaup-Boussinesq Equations

The Kaup-Boussinesq like equations [22] are coupled NLPDEs. They are

$$h_t + (uh)_x + \frac{1}{4}\, u_{xxx} = 0 \tag{14a}$$

$$u_t + u u_x + h_x = 0. \tag{14b}$$

where we assume the functions $h(\beta\overline{x})$ and $u(\beta\overline{x})$ have the dependence indicated. For homogeneous balance we use a power series for $h(\beta\overline{x})$

$$h(\beta\overline{x}) = \textstyle\sum_{l=0}^{q} a_l H(\beta\overline{x})^l \tag{15}$$

and Equation (1) for $u(\beta\overline{x})$.

Then, we have the two relations: $p + q + 1 = p + 3$ and $2p + 1 = q + 1$. Then, the powers are $p = 1,\ q = 2$. As the net derivatives in both Equations (14a) and (14b) are all odd, we find that the hyperbolic and Jacobian elliptic functions are possible solutions. In addition, both $h(\beta\overline{x})$ and $u(\beta\overline{x})$ are even functions of their arguments, so that there is reflection symmetry. Then, we propose that

$$h(\beta\overline{x}) = b_0 \ + \ b_2 \sec h^2(\beta\overline{x}),\ \ u(\beta\overline{x}) = a_0 + \ a_1 \sec h(\beta\overline{x}) \tag{16}$$

Why is there no $b_1 \sec h(\beta\overline{x})$ in (16)? If we replace $b_2 \sec h^2(\beta\overline{x})$ by $b_2\left(1-\tan h^2(\beta\overline{x})\right)$, then we see that the term $b_1 \sec h(\beta\overline{x})$ will not appear. The number of algebraic equations $N_e = 3$ for Equation (14a) and $N_e = 2$ for Equation (14b) power series when the common factors are divided out. The original parameters are $a_{0,}a_{1,}\ b_{0,}\ b_{1,}c, \beta$. Essential parameters for Equation (14a) are $a_{0,}\ b_0,\ b_1,\ c,\ \beta^2$ and essential parameters for Equation (14b) are $a_0,\ a_1^2,\ b_{1,}\ c$. The common essential parameters are $a_{0,}\ a_1^2,\ b_0,\ b_1,\ c,\ \beta^2$. The resulting coefficients are

$$a_0 = c,\ \ a_1 = \pm i\beta,\ \ b_0 = -\frac{\beta^2}{4},\ \ b_2 = \frac{\beta^2}{2}. \tag{17}$$

3.5. Fermi-Pasta-Ulam Equation

The Fermi-Pasta-Ulam equation is an equation for the perturbations in the Fermi-Pasta-Ulam mass chain. The NLPDE [23] is

$$u_t + uu_x + \delta^2 u_{xxx} + 2\delta^2 u_x u_{xx} + \delta^2 uu_{xxx} + 0.4\delta^4 u_{xxxxx} - \mu u^2 u_x - 4\delta^2 \mu uu_x u_{xx} - \delta^2 \mu u_x^3 - \delta^2 \mu u^2 u_{xxx} = 0 \tag{18}$$

where μ and δ are constants. The solution of Equation (18) was found in [23] by assuming the logistic function that can be expressed in terms of the tanh function.

$$[1 - \mathrm{Exp}(-2\beta\overline{x})]^{-1} = 0.5[1 + \tan h(\beta\overline{x})] \tag{19}$$

The solution is

$$u(\beta\overline{x}) = a_0 + a_1 \tan h(\beta\overline{x}) \tag{20}$$

where $a_0 = \frac{1}{2\mu}$, $a_1 = \pm[(28\mu + 15)]/\left(30\mu^2\right)]^{1/2}$, $\beta = [(28\mu + 15)]/\left(96\mu\delta^2\right)]^{1/2}$.

We next analyze Equation (18) by the Power Index Method. Now, Equation (18) is invariant under translations in the independent variables, a classic Lie symmetry. Then, again we can assume traveling waves. The harmonic balance results in $p = 1$. The net derivatives are all odd; thus the hyperbolic and Jacobian elliptic functions are possible solutions. Consideration of reflection or 180° rotation symmetry of $u(\beta\overline{x})$ shows that it can be a mixed or even function of its argument. We choose to look at the mixed function for $u(\beta\overline{x})$, as we wish to demonstrate that the Power Index Method will give a result equivalent to that found in Equation (20) from [23] by the logistic equation. The number of algebraic equations from the NLPDE $N_e = 5$ because

$$P_{max} + 1 = p + 5 + 1 = 7 \tag{21}$$

but dividing by common factors reduces the number of algebraic equations to five. The number of original parameters $N_p = 5$ since they are $a_0,\ a_1, c, \mu, \beta\delta$. Since $N_p > N_e$ does not hold, how is there a solution? We look at the essential parameters in the five algebraic equations found from the coefficients of four powers of the $\tan h(\beta\overline{x})$ and a constant. The essential parameters are $\mu a_1^2,\ (\beta\delta)^2$ for the fourth power, $a_0\mu$ for the third power, $\mu, (\beta\delta)^2$ for the second power, an identity for the first power and $c,\ a_0, \mu, \mu a_1^2\ \mu a_0^2,\ (\beta\delta)^2, (\beta\delta)^4,\ (\beta\delta)^2 a_0^2,\ (\beta\delta)^2 a_1^2$ for the constant term. Therefore, there are only four algebraic equations. In addition, a solution with a $\tan h(\beta\overline{x})$ exists because it is the limit of the Jacobian elliptic function $\mathrm{sn}(\beta\overline{x})$ as the modulus $k \to 1$. The modulus is an additional parameter giving $N_p = 6$. The solution for $\mathrm{sn}(\beta\overline{x})$ is

$$u(\beta\overline{x}) = b_0 + b_1 \mathrm{sn}(\beta\overline{x}) \tag{22}$$

where b_0 is the same as a_0 in (20) and we find

$$b_1 = \pm\left[(28\mu + 15)\,k^2\right]/\left(15\left(1+k^2\right)\mu^2\right)]^{1/2},\ \beta = [(28\mu + 15)]/\left(48\left(1+k^2\right)\mu\delta^2\right)]^{1/2}.$$

We note two aspects of the solutions in Equations (20) and (22) by the Power Index Method that are relevant. First, the solution in Equation (20) was not expected since the inequality in Equation (5) does not hold. This suggests caution if Jacobian elliptic functions are solutions. Second, the choice of the logistic function as a possible solution missed solutions. There was also a hyperbolic secant solution that was not reported here.

A solution of Equation (18) by an elliptic Weierstrass function was also reported in [23], where the form was guessed and the Equation (18) was integrated twice. The G′/G method can be used to find the same result in a more compact form. The extension of the Power Index Method to the G′/G Method has not been completed.

4. Results

4.1. Scope of Power Index Method

The updated Power Index Method as a test for determining the analytic solutions of nonlinear partial differential equations has been presented. The aim is to demonstrate whether the NLPDE can be reduced to algebraic equations that then can be solved. The method is limited to NLPDEs invariant under translation symmetries in the independent variables that then support traveling waves. The method has eight steps of which six steps are easy and quick to apply with a couple of exceptions. The dependent variable is expanded in a finite power series in one of the nonlinear functions that are the hyperbolic or Jacobian functions. Homogeneous balance is applied in order to determine the number of terms of the power series before additional constraints are imposed. Then, steps are introduced that may reduce the number of terms in the power series or possible functions. An important step is to determine the even, oddness, or mixed even and oddness of the net derivatives of the NLPDEs terms. Then, for the mixed case only hyperbolic tangent functions are possible solutions. The first five tests that include the previous restrictions were in the original Power Index Method although some details were not explicit and the exceptions were not discussed.

The updated version of the Power Index Method is more systematic and consists of eight steps described in detail. The sixth step considers 'discrete' symmetries of reflection and 180° rotation, which may reduce the number of terms in the algebraic equations. Several unusual cases are discussed. One is a split of the NLPDE into two parts with special conditions. Another case is the necessity to test Jacobian elliptic functions because in the limit as the modulus goes to one, a solution of the hyperbolic tangent function exists. The determination of a possible solution of the algebraic equations is more complicated. The last two steps estimate the number of algebraic equations and the number of parameters. The number of parameters must exceed the number of necessary algebraic equations in order to find a solution where some algebraic equations may be redundant or identities. Examples of various NLPDEs are analyzed by the eight steps.

4.2. New Results

The <u>fifth</u> step has been expanded to include justification for the power p for the hyperbolic secant function and the three Jacobian elliptic functions. The <u>sixth</u> step introduces discrete symmetries, where an even (odd) function $u\ (\beta\overline{x})$of its argument can reduce the number of terms in the power expansion for $u\ (\beta\overline{x})$. The <u>seventh</u> and <u>eight</u> steps consider the number of algebraic equations and the parameters in those equations. A sufficient condition is proposed that the number of parameters exceeds the number of algebraic equations.

4.3. Exceptions and Restrictions

The dependent variable is restricted to $u\ (\beta\overline{x})$ and must be expanded in a finite power series expansion of some nonlinear function of $\beta\overline{x}$. The nonlinear functions are restricted to two hyperbolic functions and three Jacobian elliptic functions. Homogeneous balance of the NLPDE holds. If these restrictions are not met, the Power Index Method fails at this point. Another restriction limits the nonlinear functions to tan h functions if the net derivatives are not all even numbers or odd numbers. A rare exception may be found by equation splitting. A restriction to keeping all of the terms in the power series expansion for $u\ (\beta\overline{x})$ occurs if neither of the discrete symmetries holds. The condition that the number of parameters exceeds the number of algebraic equations may fail if some algebraic equations are redundant or identities. This failure was found for the Fermi-Pasta-Ulam equation for the tanh function.

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