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Multicriteria Decision Making Based on Generalized Maclaurin Symmetric Means with Multi-Hesitant Fuzzy Linguistic Information

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Abstract: In multicriteria decision making (MCDM), multi-hesitant fuzzy linguistic term sets (MHFLTSs) can eliminate the limitations of hesitant fuzzy linguistic term sets (HFLTSs) and hesitant fuzzy linguistic sets (HFLSs), and emphasize the importance of a repeated linguistic term (LT). Meanwhile, there is usually an interrelation between criteria. The Maclaurin symmetric mean (MSM) operator can capture the interrelationships among multi-input arguments. The purpose of this paper is to integrate MHFLTSs with MSM operators and to solve MCDM problems. Firstly, we develop the generalized MSM operator for MHFLTSs (MHFLGMSM), the generalized geometric MSM operator for MHFLTSs (MHFLGGMSM), the weighted generalized MSM operator for MHFLTSs (WMHFLGMSM) and the weighted generalized geometric MSM operator for MHFLTSs (WMHFLGGMSM), respectively. Then, we discuss their properties and some special cases. Further, we present a novel method to deal with MCDM problems with the MHFLTSs based on the proposed MSM operators. Finally, an illustrative example about how to select the best third-party logistics service provider is supplied to demonstrate the practicality and reliability of the proposed approaches in comparison with some existing approaches.

Keywords: multi-hesitant fuzzy linguistic term sets (MHFLTSs); maclaurin symmetric mean (MSM); multicriteria decision making (MCDM)

1. Introduction

Because human preferences are inherently vague, it is more suitable to deal with uncertainty and imprecise information by linguistic terms (LTs). For example, when evaluating customer satisfaction of a service or product, experts are inclined to select some LTs such as "poor", "good" or "excellent", etc., to give their evaluation of the criteria. Since Zadeh [1] presented the fuzzy linguistic technique in 1975, which utilized linguistic variables (LVs) to describe qualitative information, some methods based on LVs have been utilized in a number of areas, such as supply chain management [2], qualify function deploy [3], health care system [4], housing marker [5], risk evaluation [6], etc.

Generally, if the qualitative information is only represented by a single LT, it might sometimes not accurately reflect what it means. When the decision-making problems become more ambiguous, the experts might be hesitant among several LTs and need two or more LTs to express their preference. Based on hesitant fuzzy set (HFS), Rodríguez et al. [7,8] defined a hesitant fuzzy LT set (HFLTS), which is a set with ordered consecutive LTs. In real applications, experts can give several possible LTs instead of a single LT to evaluate the criteria. Although HFLTS improves the previous linguistic approaches, it is still difficult to express the hesitance preference of experts only by consecutive LTs.

In addition, there are some limitations in the calculation of HFLTS and this will be illustrated in Section 2 of this paper. Thus, hesitant fuzzy linguistic sets (HFLSs), as an extension of HFLTS, were produced, which allow for inconsecutive LTs. However, in practice, when multi-persons give the same LT in assessment information, this LT is handled as one time by default and the significance of repeated LTs is neglected. Since the evaluation values $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_2, x_3\}$ and $\{x_1, x_2, x_2, x_2, x_3\}$ are not equivalent to each other, the repeated occurrence of x_2 can convey importance or a certain special meaning. In addition, the LVs provided by experts generally fluctuate in evaluation period [9]. So, MHFLTS [10] was proposed based on multi-HFSs to solve the mentioned problems, and each multi-hesitant LT element (MHFLTE) can contain inconsecutive and repeated LTs. Based on the MHFLTS, one expert can assess an alternative under specific criteria by one or several arbitrary LTs, and the assessment values from the different experts, can be collected, and the frequency of a LT can be expressed in the assessment information [11]. In a word, MHFLTS is more energetic than the HFLTS and HFLS, and it can be applied to present the hesitance linguistic information by multi-repeated LTs in primary assessment information.

To date, many contributions have concentrated on the decision-making techniques based on HFLTSs, which are from three domains: (1) the theory of foundations, for instance, operational laws [7,8], comparative methods [12,13], distance and similarity measures [14], likelihood [15], outranking degree [16], consistency [17] or multiplicative consistency [18], correlation coefficient [19] and so on; (2) the extended multicriteria decision making (MCDM) approaches for HFLTS, such as TOPSIS [20], ELECTRE [21], VIKOR [22], TODIM [23], Entropy [24], GRA [25] and other methods, such as MACBETH [26], linear programming technique [27], Shapely [28], EDAS [29], Fuzzy Petri Net [30] and so on; (3) the MCDM techniques based on aggregation operators of HFLS. The MCDM methods based on aggregation operators can acquire the comprehensive values of alternatives by aggregating all attribute values, and then rank the alternatives. Obviously, they have more superiority than the traditional MCDM methods, and it is meaningful to research the aggregation operators and then to solve the MCDM problems.

As an important tool for information fusion, a great number of research achievements about aggregation operators of HFLS have been produced. Wang et al. [31] presented a 2-tuple linguistic aggregation operator for MHFLTEs; Wu [32] applied possibility distribution to develop the weighted average (WA) operator for HFLSs (HFLWA) and the ordered WA operator (OWA) for HFLSs (HFLOWA). Gou et al. [33] developed the hesitant fuzzy linguistic BM operator for HFLSs (HFLBM) and the weighted HFLBM operator (WHFLBM). Zhu et al. [34] investigated some linguistic hesitant fuzzy power aggregation (LHFPA) operators. Wang et al. [35] proposed the cloud WA (CWA) operator, cloud OWA (COWA) operator, and cloud hybrid arithmetic (CHA) operator for LVs. Liu et al. [36] investigated prioritized WA operator for HIFLSs (HIFLPWA) and prioritized weighted geometric operator for HIFLSs (HIFLPWG). However, these operators for HFLTSs do not pay attention to the interrelation among input arguments.

MSM was proposed by Maclaurin [37] and was further developed by Detemple and Robertson [38], which has distinct advantage, i.e., it can deal with the interrelationship among multi-inputs, however Bonferroni mean (BM) operator and Heronian mean (HM) operator can only consider the correlation between two arguments. Consequently, based on its main advantage of more flexible and robust in information fusion, the related research achievements of MSM operator have been quite fruitful. Liu and Zhang [39] introduced some MSM operator for single-valued trapezoidal neutrosophic numbers. Yu et al. [33] developed the MSM operator for HFLSs and the weighted MSM operator for HFLSs. Liu and Qin [40] extended the MSM operator for linguistic intuitionistic fuzzy numbers (LIFNs). Qin and Liu [41] investigated the dual MSM (DMSM) operator and extended the DMSM operator to uncertain LVs. Moreover, the MSM operators were also applied to handle the 2-tuple linguistic information [42] and single-valued neutrosophic linguistic information [43] and so on.

However, research about MHFLTSs and their application in MCDM is limited. Wang et al. [31] presented a generalized 2-tuple linguistic WA operator and a generalized 2-tuple linguistic OWA to deal

with MHFLTSs. Afterwards, a likelihood-based TODIM method based on the MHFLTSs for evaluation in logistics outsourcing was also proposed [11]. Then, Wang et al. [10] further developed the HM and prioritized operators to solve the MCDM problem with the MHFLTSs. Liu and Teng [44] proposed some normal neutrosophic number Heronian Mean operators for solving multiple attribute group decision making problems. However, because MHFLTSs can describe inconsecutive and repeated LTs and MSM aggregation operator can process the correlations among multi-inputs, to develop some MSM operators to deal with MHFLTSs is an important work. So, the aim of this paper is to establish some MSM operators for MHFLTSs and use them to solve the MCDM problem in which the attributes take the form of MHFLTSs.

This paper is organized as follows. Section 2 reviews and discusses some basic concepts and theories. Section 3 proposes some multi-hesitant fuzzy linguistic generalized MSM operators, including the generalized MSM operator for MHFLTSs (MHFLGMSM), the generalized geometric MSM operator for MHFLTSs (MHFLGGMSM), the weighted generalized MSM operator for MHFLTSs (WMHFLGMSM) and the weighted generalized geometric MSM operator for MHFLTSs (WMHFLGGMSM), and studies some properties and some particular examples of these operators. Section 4, outlines an MCDM approach based on the proposed aggregation operators. Section 5 gives a case to verify the availability of the presented methods. Section 6 presents a few conclusions.

2. Preliminaries

In this part, the definitions of HFLTSs, HFLSs and MHFLTEs are briefly reviewed and their corresponding operations are given. Subsequently, the linguistic scale function (LSF) and score function of MHFLTS were given.

2.1. HFLTSs

Assume that $S = \{s_i | i = 0, 1, 2, \cdots, 2g, g \in N\}$ is a LT set (LTs) with odd cardinality, where s_i is called a LV. Then the following requirements must be satisfied [2]:

(1) The set is ordered: $\alpha > \beta \Leftrightarrow s_\alpha > s_\beta$;
(2) There is a negation operator: $neg(s_\alpha) = s_{2g-\alpha}$.

Definition 1. *Let $S = \{s_i | i = 0, 1, 2, \cdots, 2g, g \in N\}$, then a HFLTS H_s is an ordered finite consecutive LTs of S [31].*

Definition 2. *Let H_s, H_s^1 and H_s^2 be any three HFLTSs on S [31]. Then we have*

(1) *The upper bound h_{s+} : $h_{s+} = \max\{h_\tau | h_\tau \in H_s\}$; and the lower bound h_{s-} of H_s: $h_{s-} = \min\{h_\tau | h_\tau \in H_s\}$ and $h_{s-} \leq h_{s+}$;*
(2) *The intersection between H_s^1 and H_s^2 : $H_s^1 \cap_s H_s^2 = \{h_\tau | h_\tau \in H_s^1 \text{ and } h_\tau \in H_s^2\}$;*
(3) *The union of H_s^1 and H_s^2 : $H_s^1 \cup_s H_s^2 = \{h_\tau | h_\tau \in H_s^1 \text{ or } h_\tau \in H_s^2\}$.*

Example 1. *Assume that $S = \{s_i | i = 0, 1, 2, \cdots, 8\}$, $H_s^1 = \{s_5, s_6\}$, $H_s^2 = \{s_3, s_4, s_5\}$ and $H_s^3 = \{s_1, s_2, s_3\}$ are three HFLTSs on S. Then, we get:*

(1) $h_{s+}^1 = \{s_6\}$ and $h_{s-}^1 = \{s_5\}$;
(2) $H_s^1 \cap_s H_s^2 = \{s_5\}$ and $H_s^1 \cup_s H_s^3 = \{s_1, s_2, s_3, s_5, s_6\}$.

Obviously, the union of H_s^1 and H_s^3 is not a HFLTS because the HFLTS must be consecutive LTs. So, the HFLTSs were extended to HFLSs.

2.2. HFLSs

In order to preserve the provided information, the LTs S is developed to a successive LTs $\overline{S} = \{s_k | k \in [0, 2g]\}$. If $s_k \in S$, then s_k is named the original LT; if not so, s_k is named the virtual LT which has no practical meaning and only it is used to operational process.

Definition 3. *Let X be a reference set and $\overline{S} = \{s_k | k \in [0, 2g]\}$. Then a HFLS H_{LS} on X can be defined in terms of a function H_{LS} that returns a subset of $\overline{S}$ [32].*

For simple and convenient, $h_{LS}(x) = h_{LS}$ is called a HFLE.

Definition 4. *Let $\overline{S} = \{s_k | k \in [0, 2g]\}$ be a LTs, h_{LS}, h_{LS}^1 and h_{LS}^2 be three HFLEs on $\overline{S}$ and $\lambda \in [0, 1]$, $I(s_k) = k$ [32]. Thus, the operations can be defined as follows:*

$$h_{LS}^1 \oplus h_{LS}^2 = \cup_{h_{\tau 1} \in h_{LS}^1, h_{\tau 2} \in h_{LS}^2} \{h_{\tau 1} \oplus h_{\tau 2}\} = \cup_{h_{\tau 1} \in h_{LS}^1, h_{\tau 2} \in h_{LS}^2} \left\{ s_{I(h_{\tau 1}) + I(h_{\tau 2})} \right\} \tag{1}$$

$$h_{LS}^1 \otimes h_{LS}^2 = \cup_{h_{\tau 1} \in h_{LS}^1, h_{\tau 2} \in h_{LS}^2} \{h_{\tau 1} \otimes h_{\tau 2}\} = \cup_{h_{\tau 1} \in h_{LS}^1, h_{\tau 2} \in h_{LS}^2} \left\{ s_{I(h_{\tau 1}) \times I(h_{\tau 2})} \right\} \tag{2}$$

$$\lambda h_{LS} = \cup_{h_\tau \in h_{LS}} \{\lambda h_\tau\} = \cup_{h_\tau \in h_{LS}} \left\{ s_{(\lambda I(h_\tau))} \right\} \tag{3}$$

$$(h_{LS})^\lambda = \cup_{h_\tau \in h_{LS}} \left\{ (h_\tau)^\lambda \right\} = \cup_{h_\tau \in h_{LS}} \left\{ s_{(I(h_\tau))^\lambda} \right\} \tag{4}$$

Example 2. *Suppose that $\overline{S} = \{s_k | k \in [0, 8]\}$ and h_{LS}^1, h_{LS}^2, h_{LS}^3 are three HFLEs on $\overline{S}$. $h_{LS}^1 = \{s_5, s_6\}$, $h_{LS}^2 = \{s_3, s_4, s_5\}$ and $h_{LS}^3 = \{s_1, s_2, s_3\}$. Then, $h_{LS}^1 \oplus h_{LS}^2 = \{s_8, s_9, s_{10}, s_{11}\}$, $h_{LS}^1 \otimes h_{LS}^3 = \{s_5, s_6, s_{10}, s_{12}, s_{15}, s_{18}\}$.*

The results obtained by Definition 4 have exceed the domain of $\overline{S}$. Many elements based on the subscript of LTs in $h_{LS}^1 \oplus h_{LS}^2$ and $h_{LS}^1 \otimes h_{LS}^3$ do not exist in $\overline{S}$.

Definition 5. *Let $\overline{S} = \{s_k | k \in [0, 2g]\}$ be a LTs and $h_{LS} = \{h_\tau | h_\tau \in \overline{S}\}$ be a HFLE on $\overline{S}$ [33]. The score function of h_{LS} is*

$$S(h_{LS}) = \frac{1}{\#h_{LS}} \sum_{h_\tau \in h_{LS}} \frac{I(h_\tau)}{2g} \tag{5}$$

where $\#h_{LS}$ indicates the count of elements in h_{LS}. h_{LS}^1, h_{LS}^2 are two arbitrary HFLEs, if $S(h_{LS}^1) > S(h_{LS}^2)$ then $h_{LS}^1 > h_{LS}^2$, and if $S(h_{LS}^1) = S(h_{LS}^2)$ then $h_{LS}^1 = h_{LS}^2$.

Example 3. *Assume that $\overline{S} = \{s_k | k \in [0, 8]\}$ and h_{LS}^1, h_{LS}^2 are two HFLEs on $\overline{S}$. $h_{LS}^1 = \{s_2, s_4, s_6\}$, $h_{LS}^2 = \{s_2, s_6\}$, $S(h_{LS}^1) = S(h_{LS}^2) = 0.5$, then $h_{LS}^1 = h_{LS}^2$. However, this comparison result is insufficient due to the existence of s_4. Therefore, it is essential to give some revisions.*

2.3. MHFLTS

Although the definitions of HFLTS and HFLS have been widely and consistently used, the frequency of one repeated LT is assigned to be one time and the operations are contradictory to the definitions. Thus, a further extended definition of them is proposed in this section.

Definition 6. *Suppose X is a reference set, a multi-HFS on X can be defined about a function h_E that returns a multi-subset of values in $[0, 1]$ [31].*

Definition 7. *Suppose X is a reference set and $\overline{S} = \{s_k | k \in [0, 2g]\}$ is a LTs. Then a MHFLTS H on X is defined about a function h that returns an ordered finite multi-subset of $\hat{S}$, denoted by [31]:*

$$H = \{\langle x, h(x) \rangle | x \in X\} \tag{6}$$

where *h(x)* indicates the possible membership degrees of the element $x \in X$ to the set X. For simple and convenient, *h(x) = h* is a MHFLTE and $\hat{H}$ is the set of all MHFLTEs. There is the strong possibility that the virtual LTs of h exist in $\overline{S}$ but not in S.

Essentially, MHFLTSs are a development of HFLTSs and HFLSs. If *h* is a set of successive LTs and no duplicate LT, then *h* is a HFLTSs. If *h* is a set of discrete LTs with no duplicate LT, then *h* is a HFLS. Therefore, all the operations and approaches on MHFLTSs can be applied to HFLTSs and HFLSs because they are both special cases of MHFLTSs.

Definition 8. *Let* $\overline{S} = \{s_k | k \in [0, 2g]\}$ *be a LTs. The LSF* $\vartheta : \overline{S} \to R^+$ *is a strictly monotone increasing and continuous function satisfying* $\vartheta(0) = 0$ *and* $\vartheta(2g) = 1$. *The inverse function of* ϑ *exists and is denoted by* $\vartheta^{-1}(\cdot)$. *Then* $\vartheta^{-1}(\delta) = h_\tau$, *where* $h_\tau \in \overline{S}$ *and* $\delta \in R^+$ *[41,42].*

There are five normal LSFs which are shown as below.

$$\vartheta(h_\tau) = \frac{I(h_\tau)}{2g} \quad I(h_\tau) \in [0, 2g] \tag{7}$$

$$\vartheta(h_\tau) = \left(\frac{I(h_\tau)}{2g}\right)^g \quad I(h_\tau) \in [0, 2g] \tag{8}$$

$$\vartheta(h_\tau) = \left(\frac{I(h_\tau)}{2g}\right)^{\frac{1}{g}} \quad I(h_\tau) \in [0, 2g] \tag{9}$$

$$\vartheta(h_\tau) = \begin{cases} \dfrac{a^g - a^{g-I(h_\tau)}}{2(a^g - 1)} & (I(h_\tau) \in [0, g]) \\ \dfrac{a^g + a^{I(h_\tau)-g} - 2}{2(a^g - 1)} & (I(h_\tau) \in [0, g]) \end{cases} \tag{10}$$

$$\vartheta(h_\tau) = \begin{cases} \dfrac{g^a - (g - I(h_\tau))^a}{2g^\alpha} & (I(h_\tau) \in [0, g]) \\ \dfrac{g^\beta + (I(h_\tau) - g)^\beta}{2g^\alpha} & (I(h_\tau) \in [0, g]) \end{cases}, \quad \text{where } \alpha, \beta \in (0, 1] \tag{11}$$

A great quantity of experiment research has proven the value of *a* in Equation (10) lies in the interval [1.36, 1.4]. Suppose $g = 4$, $a = 1.4$ and $\alpha = \beta = 0.8$, the character of Equations (7)–(11) can be shown in Figure 1.
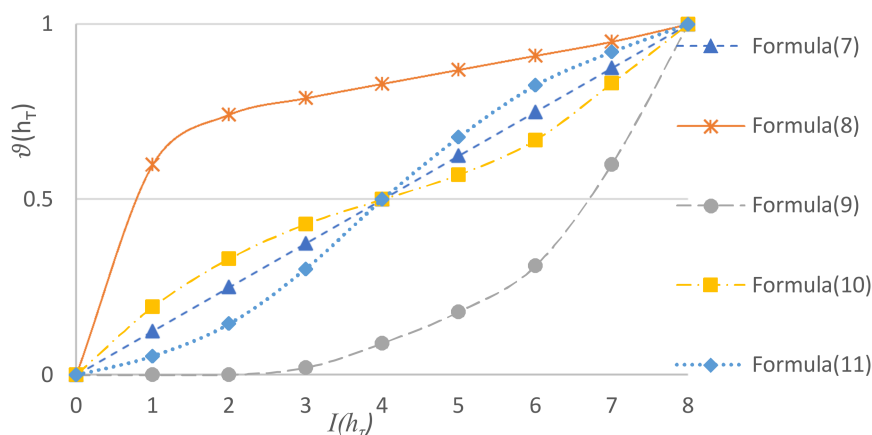

Figure 1. The graphical demonstration of Equations (7)–(11).

The Equation (10) is regarded as one representative among all Equations for $\vartheta^{-1}(\cdot)$, thus we choose it to deal with the MHFLTEs in this paper.

2.4. Basic Operations of MHFLTS

Definition 9. *Let h_1 and h_2 be any two MHFLTEs on $\overline{S}$. Then, we can give the following operations.*

$$h_1 \oplus h_2 = \left\{ \vartheta^{-1}(\vartheta(h_{\tau 1}) + \vartheta(h_{\tau 2})) \Big| h_{\tau 1} \in h_1, h_{\tau 2} \in h_2 \right\} \tag{12}$$

$$h_1 \otimes h_2 = \left\{ \vartheta^{-1}(\vartheta(h_{\tau 1}) + \vartheta(h_{\tau 2})) \Big| h_{\tau 1} \in h_1, h_{\tau 2} \in h_2 \right\} \tag{13}$$

$$\lambda h_1 = \left\{ \vartheta^{-1}(\lambda \vartheta(h_{\tau})) \Big| h_{\tau} \in h_1 \right\} \tag{14}$$

$$(h_1)^{\lambda} = \left\{ \vartheta^{-1}(\vartheta(h_{\tau}))^{\lambda} \Big| h_{\tau} \in h_1 \right\} \tag{15}$$

Definition 10. *Let h, h_1, h_2 be arbitrary MHFLTEs on $\overline{S}$. Then we give the score function of h as follows [31]:*

$$S(h) = \frac{1}{\#h} \sum_{h_{\tau} \in h} \vartheta(h_{\tau}) \tag{16}$$

and the variance function of h as follows:

$$V(h) = \frac{1}{\#h} \sum_{h_{\tau} \in h} (\vartheta(h_{\tau}) - S(h))^2 \tag{17}$$

where #h denotes the count of elements in h. Then the comparison rules for MHFLTEs can be re-defined as follows:

(1) *If $\vartheta(h_1^-) > \vartheta(h_1^+)$, then h_1 is strictly greater than h_2, denoted by $h_1 > h_2$;*
(2) *If $S(h_1) > S(h_2)$, then h_1 is greater than h_2, denoted by $h_1 > h_2$; If $S(h_1) = S(h_2)$, and $V(h_1) < V(h_2)$, then $h_1 > h_2$; If $S(h_1) = S(h_2)$, and $V(h_1) = V(h_2)$, then $h_1 = h_2$.*

2.5. MSM Operator

Definition 11. *Let $a_i(i = 1, 2, \cdots, n)$ be a group of positive real values and $m = 1, 2, \cdots, n$ [43]. If*

$$MSM^{(m)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_m \le i_n} \prod_{j=1}^{m} a_{i_j}}{C_n^m} \right)^{\frac{1}{m}} \tag{18}$$

Then $MSM^{(m)}$ is called the MSM operator where $(i_1, i_2, \cdots, i_m)$ traversal all the combination of $(1, 2, \cdots, n)$, C_n^m is the binominal coefficient.

It is clear that the $MSM^{(m)}$ has the following properties:

(1) Idempotency. $MSM^{(m)}(a, a, \cdots, a) = a$;
(2) Monotonicity. $MSM^{(m)}(a_1, a_2, \cdots, a_n) \le MSM^{(m)}(b_1, b_2, \cdots, b_n)$, if $a_i < b_i$ for all i;
(3) Boundedness. $\min\{a_1, a_2, \cdots, a_n\} \le MSM^{(m)}(a_1, a_2, \cdots, a_n) \le \max\{a_1, a_2, \cdots, a_n\}$.

Especially, if $m = 1$, then the $MSM^{(1)}$ reduces to the WA operator as follows:

$$MSM^{(1)}(a_1, a_2, \cdots, a_n) = \frac{\sum_{1 \le i_1 \le i_n} a_i}{n} \tag{19}$$

If $m = 2$, then the $MSM^{(2)}$ reduces to a BM operator ($p = q = 1$), as follows:

$$MSM^{(2)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 < i_2 \leq i_n} \prod_{j=1}^2 a_{i_j}}{C_n^2} \right)^{\frac{1}{2}} = \left(\frac{2\sum_{1 \leq i_1 < i_2 \leq i_n} a_{i_1} a_{i_2}}{n(n-1)} \right)^{\frac{1}{2}}$$

$$= \left(\frac{\sum_{i,j=1, i \neq j}^n a_{i_1} a_{i_2}}{n(n-1)} \right)^{\frac{1}{2}} = BM^{1,1}(a_1, a_2, \cdots, a_n) \tag{20}$$

If $m = 3$, then the $MSM^{(3)}$ reduces to a generalized BM operator ($p = q = u = 1$) as follows:

$$MSM^{(3)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 < i_2 < i_3 \leq i_n} \prod_{j=1}^3 a_{i_j}}{C_n^3} \right)^{\frac{1}{3}} = \left(\frac{6\sum_{1 \leq i_1 < i_2 \leq i_n} a_{i_1} a_{i_2} a_{i_3}}{n(n-1)(n-2)} \right)^{\frac{1}{3}}$$

$$= \left(\frac{\sum_{i,j,r=1, i \neq j \neq r}^n a_{i_1} a_{i_2} a_{i_3}}{n(n-1)(n-2)} \right)^{\frac{1}{3}} = GBM^{1,1,1}(a_1, a_2, \cdots, a_n) \tag{21}$$

Definition 12. *Let $a_i (i = 1, 2, \cdots, n)$ be a set of positive real values and $m = 1, 2, \cdots, n$ [43]. If*

$$GMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m a_{i_j}^{p_j}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \tag{22}$$

Then $GMSM^{(m,p_1,p_2,\cdots,p_m)}$ is called the generalized MSM (GMSM) operator, where $p_1, p_2, \cdots, p_m \geq 0$, $(i_1, i_2, \cdots, i_m)$ traversal all the combination of $(1, 2, \cdots, n)$, C_n^m is the binominal coefficient.

The $GMSM^{(m,p_1,p_2,\cdots,p_m)}$ has the following properties:

Property 1.

(1) *Idempotency. $GMSM^{(m,p_1,p_2,\cdots,p_m)}(a, a, \cdots, a) = a$;*

(2) *Monotonicity.* $GMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq GMSM^{(m,p_1,p_2,\cdots,p_m)}(b_1, b_2, \cdots, b_n)$, *if $a_i \leq b_i$ for all i;*

(3) *Boundedness. $\min\{a_1, a_2, \cdots, a_n\} \leq GMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq \max\{a_1, a_2, \cdots, a_n\}$.*

Proof.

(1)
$$GMSM^{(m,p_1,p_2,\cdots,p_m)}(a, a, \cdots, a) = \left(\frac{\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m a^{p_j}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} = \left(\frac{\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} a^{p_1 + p_2 + \cdots + p_m}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}}$$
$$= \left(\frac{C_n^m a^{p_1 + p_2 + \cdots + p_m}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} = a$$

(2) Assume that m-tuple $(i_1, i_2, \cdots, i_m)$ is given randomly, and $p_1, p_2, \cdots, p_m \geq 0$. if $0 \leq a_i \leq b_i$ for all i, then $\prod_{j=1}^m a_{i_j}^{p_j} \leq \prod_{j=1}^m b_{i_j}^{p_j}$ and $\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m a_{i_j}^{p_j} \leq \sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m b_{i_j}^{p_j}$, therefore,

$$\left(\frac{\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m a_{i_j}^{p_j}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \leq \left(\frac{\sum_{1 \leq i_1 < \cdots < i_m \leq i_n} \prod_{j=1}^m b_{i_j}^{p_j}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}}$$

(3) Let $a^- = \min\{a_1, a_2, \cdots, a_n\}$ and $a^+ = \max\{a_1, a_2, \cdots, a_n\}$. According to the property of idempotency, $\min\{a_1, a_2, \cdots, a_n\} = a^- = GMSM^{(m,p_1,p_2,\cdots,p_m)}(a^-, a^-, \cdots, a^-)$. According to the property of monotonicity, when $a^- \leq a_i$ for all i, we have $GMSM^{(m,p_1,p_2,\cdots,p_m)}(a^-, a^-, \cdots, a^-) \leq GMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n)$. Similarly, we also have $GMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq GMSM^{(m,p_1,p_2,\cdots,p_m)}(a^+, a^+, \cdots, a^+)$.

Finally, $MSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq MSM^{(m,p_1,p_2,\cdots,p_m)}(b_1, b_2, \cdots, b_n)$. □

Especially, if $m = 1$, then the $GMSM^{(m,p_1,p_2)}(a_1, a_2, \cdots, a_n)$ reduces to the generalized WA operator.

$$GMSM^{(1,p)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 \leq i_n} a_i^p}{n} \right)^{\frac{1}{p}} \tag{23}$$

If $m = 2$, then the $GMSM^{(m,p_1,p_2)}(a_1, a_2, \cdots, a_n)$ reduces to the BM operator as follows:

$$GMSM^{(2,p_1,p_2)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 < i_2 \leq in} a_{i_1}^{p_1} a_{i_2}^{p_2}}{C_n^2} \right)^{\frac{1}{p_1+p_2}} = \left(\frac{2\sum_{1 \leq i < j \leq n} a_i^{p_1} a_j^{p_2}}{n(n-1)} \right)^{\frac{1}{p_1+p_2}}$$

$$= \left(\frac{\sum_{i,j=1, i \neq j}^n a_i^{p_1} a_j^{p_2}}{n(n-1)} \right)^{\frac{1}{p_1+p_2}} = BM^{p_1,p_2}(a_1, a_2, \cdots, a_n) \tag{24}$$

If $m = 3$, then the $GMSM^{(m,p_1,p_2,p_3)}(a_1, a_2, \cdots, a_n)$ reduces to the GBM operator as follows:

$$GMSM^{(3,p_1,p_2,p_3)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 < i_2 < i_3 \leq in} \prod_{j=1}^3 a_{i_j}^{p_j}}{C_n^3} \right)^{\frac{1}{p_1+p_2+p_3}} = \left(\frac{6\sum_{1 \leq i_1 < i_2 \leq in} a_{i_1}^{p_1} a_{i_2}^{p_2} a_{i_3}^{p_3}}{n(n-1)(n-2)} \right)^{\frac{1}{p_1+p_2}}$$

$$= \left(\frac{\sum_{i,j,r=1, i \neq j \neq r}^n a_i^{p_1} a_j^{p_2} a_r^{p_3}}{n(n-1)(n-2)} \right)^{\frac{1}{p_1+p_2+p_3}} = GBM^{p_1,p_2,p_3}(a_1, a_2, \cdots, a_n) \tag{25}$$

Definition 13. *Let $a_i (i = 1, 2, \cdots, n)$ be a group of positive real values and $m = 1, 2, \cdots, n$ [43]. If*

$$GGMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) = \frac{1}{p_1 + p_2 + \cdots + p_m} \left(\prod_{1 \leq i_1 < \cdots < i_m \leq i_n} \sum_{j=1}^m p_j a_{i_j} \right)^{\frac{1}{C_n^m}} \tag{26}$$

Then $GGMSM^{(m,p_1,p_2,\cdots,p_m)}$ is called the generalized geometric MSM ($GGMSM$), where $p_1, p_2, \cdots, p_m \geq 0$, $(i_1, i_2, \cdots, i_m)$ traversal all the combination of $(1, 2, \cdots, n)$, C_n^m is the binominal coefficient.

The $GGMSM^{(m,p_1,p_2,\cdots,p_m)}$ has the following properties:

Property 2.

(1) *Idempotency.* $GGMSM^{(m,p_1,p_2,\cdots,p_m)}(a, a, \cdots, a) = a$;

(2) *Monotonicity.* $GGMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq GGMSM^{(m,p_1,p_2,\cdots,p_m)}(b_1, b_2, \cdots, b_n)$, *if $0 \leq a_i \leq b_i$ for all i;*

(3) *Boundedness.* $min\{a_1, a_2, \cdots, a_n\} \leq GGMSM^{(m,p_1,p_2,\cdots,p_m)}(a_1, a_2, \cdots, a_n) \leq max\{a_1, a_2, \cdots, a_n\}$.

Property 2 is similar to the Property 1, the proof is omitted here.

Especially, if $m = 1$, then the $GGMSM^{(m,p_1,p_2,\cdots,p_m)}$ reduces to the generalized geometric operator

$$GGMSM^{(1,p)}(a_1, a_2, \cdots, a_n) = \frac{1}{p} \left(\prod_{i=1}^n p a_i \right)^{\frac{1}{n}} \tag{27}$$

If $m = 2$, then the $GGMSM^{(m,p_1,p_2,\cdots,p_m)}$ reduces to the geometric BM operator as follows:

$$GGMSM^{(2,p_1,p_2)}(a_1, a_2, \cdots, a_n) = \frac{1}{p_1+p_2} \left(\prod_{1 \leq i_1 < i_2 \leq i_n} (p_1 a_{i_1} + p_2 a_{i_2}) \right)^{\frac{1}{C_n^2}}$$

$$= \frac{1}{p_1+p_2} \left(\prod_{i,j=1, i \neq j}^n (p_1 a_i + p_2 a_j) \right)^{\frac{1}{C_n^2}} = GBM^{p_1,p_2}(a_1, a_2, \cdots, a_n) \tag{28}$$

If $m = 3$, then the $GGMSM^{(3,p_1,p_2,p_3)}(a_1, a_2, \cdots, a_n)$ reduces to the generalized geometric BM operator as follows:

$$
\begin{aligned}
GGMSM^{(3,p_1,p_2,p_3)}(a_1, a_2, \cdots, a_n) &= \frac{1}{p_1+p_2+p_3}\left(\prod_{1 \le i_1 < i_2 < i_3 \le i_n}\left(p_1 a_{i_1} + p_2 a_{i_2} + p_3 a_{i_3}\right)\right)^{\frac{1}{C_n^3}} \\
&= \frac{1}{p_1+p_2+p_3}\left(\prod_{i,j,r=1,i\ne j\ne r}^{n}\left(p_1 a_i + p_2 a_j + p_3 a_r\right)\right)^{\frac{1}{C_n^3}} = GGBM^{p_1,p_2,p_3}(a_1, a_2, \cdots, a_n)
\end{aligned}
\tag{29}
$$

3. Some Multi-Hesitant Fuzzy Linguistic MSM Operators

In this section, we will extend GMSM operator and GGMSM operator to process MHFLTSs and develop the MHFLGMSM operator, MHFLGGMSM operator, WMHFLGMSM operator and WMHFLGGMSM operator.

3.1. MHFLGMSM Operator

Definition 14. *Let $h_i(i = 1, 2, \cdots, n)$ be a group of MHFLTESs on $\overline{S}$. A MHFLGMSM operator is a mapping $\hat{H}^n \to \hat{H}$ defined as follows:*

$$
MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n) = \left(\frac{\oplus_{1 \le \tau_{i_1} < \cdots < \tau_{i_m} < n}\left(\otimes_{j=1}^{m} h_{\tau_{i_j}}^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}
\tag{30}
$$

where $m = 1, 2, \cdots n$ and $\hat{H}$ is the set of all MHFLTESs.

Based on the operations of the MHFLTESs described in Section 2, we can get the following theorem.

Theorem 1. *Let $h_i(i = 1, 2, \cdots, n)$ be a set of MHFLTESs on $\overline{S}$, the value aggregated by the MHFLGMSM operator from (30) is still a MHFLTSs and*

$$
MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n) = \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n}\left(\vartheta^{-1}\left(\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^{m}\left(\vartheta\left(h_{\tau_{i_j}^k}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}\right)\right)
\tag{31}
$$

where $m = 1, 2, \cdots, n$ and $k = 1, 2, \ldots, C_n^m$. $h_{\tau_{i_j}^k}$ represents the i_jth element in kth union of each permutation which consists of one element from each $h_i(i = 1, 2, \cdots, n)$. Because the form of $MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}$ involves selecting an element from each $h_i(i = 1, 2, \cdots, n)$, each h_i must mutual calculation and obviously the MHFLGMSM operator will be used $\prod(\#h_i, i = 1, 2, \cdots, n)$ times, where $\#h_i$ denotes the number of elements in h_i. The final aggregated result consists of $\prod(\#h_i, i = 1, 2, \cdots, n)$ elements because of each aggregated result becomes an element which based on the operational laws of MHFLTSs.

Property 3. *Let $h_i(i = 1, 2, \cdots, n)$ be a set of MHFLTESs on $\overline{S}$. The value aggregated by $MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}$ operator has the following desirable properties:*

(1) *Idempotency. If the $h_i = h$, then*

$$
S\left(MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h, h, \cdots, h)\right) = S(h)
$$

(2) *Monotonicity. If $(h_1, h_2, \cdots, h_n)$ and $(h'_1, h'_2, \cdots, h'_n)$ are two sets of MHFLTESs on $\overline{S}$ and $h_i^+ < h_i'^-$, then for $i = 1, 2, \cdots, n$,*

$$MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1,h_2,\cdots,h_n) < MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1',h_2',\cdots,h_n')$$

(3) *Boundedness. If $h_i^{\min}$ is obtained by replacing the minimum of h_i for each element of h_i, h_i^{max} is obtained by replacing the maximum of h_i for each element of h_i, then*

$$MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1^{\min},h_2^{\min},\cdots,h_n^{\min}\right) \leq MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1,h_2,\cdots,h_n)$$
$$\leq MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1^{\max},h_2^{\max},\cdots,h_n^{\max}\right)$$

(4) *Commutativity. Let $\left(h_1',h_2',\cdots,h_n'\right)$ be any permutation of $(h_1,h_2,\cdots,h_n)$, then*

$$MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1',h_2',\cdots,h_n'\right) = MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1,h_2,\cdots,h_n)$$

Proof

(1) Since the MHFLTSs $h_i = h$, then

$$S\left(MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h,h,\cdots,h)\right) = S\left(\cup_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\left(\vartheta^{-1}\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}\right)\right)$$
$$= S\left(\cup_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\left(\vartheta^{-1}\left(\vartheta\left(h_{\tau_i}\right)\right)\right)\right) = \frac{1}{(\#h)^n}\sum_{k=1}^{(\#h)^n}\vartheta\left(h_{\tau_i}\right) = S(h).$$

(2) If $(h_1,h_2,\cdots,h_n)$ and $\left(h_1',h_2',\cdots,h_n'\right)$ are two sets of MHFLTESs on $\overline{S}$ and $h_i^+ < h_i'^-$, then for $i = 1,2,\cdots,n$

$$S\left(MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1,h_2,\cdots,h_n)\right)$$
$$= S\left(\cup_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\left(\vartheta^{-1}\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}\right)\right)$$
$$= \frac{1}{\#h_1\times\#h_2\times\cdots\times\#h_n}\sum_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\vartheta\left(\vartheta^{-1}\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}\right)$$
$$= \frac{1}{\#h_1\times\#h_2\times\cdots\times\#h_n}\sum_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}$$

The $h_i^+ < h_i'^-$, and for $i = 1,2,\cdots,n$ states the minimum element of $h_i'^-$ is more than the maximum element of h_i^+, then $\vartheta\left(h_i^+\right) < \vartheta\left(h_i'^-\right)$. In addition, MHFLGMSM operator can satisfy the property of monotonicity, so

$$\left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}} < \left(\frac{\sum_{k=1}^{C_n^m}\left(\prod_{j=1}^m\left(\vartheta\left(h_{\tau_i}'\right)\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\cdots+p_m}}$$

Finally,

$$MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1,h_2,\cdots,h_n) < MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1',h_2',\cdots,h_n'\right)$$

(3) If $h_i^{\min}$ is obtained by replacing the minimum of h_i for each element of h_i and $h_i^{\max}$ is obtained by replacing the maximum of h_i for each element of h_i, then $h_{\tau_i}^{\min} \leq h_{\tau_i} \leq h_{\tau_i}^{\max}$ and $\vartheta\left(h_{\tau_i}^{\min}\right) \leq \vartheta(h_{\tau_i}) \leq \vartheta\left(h_{\tau_i}^{\max}\right)$ for $h_{\tau_i}^{\min} \in h_i^{\min}, h_{\tau_i} \in h_i, h_{\tau_i}^{\max} \in h_i^{\max}$. In addition, the MHFLGMSM operator can satisfy the property of monotonicity, so

$$
\left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i}^{\min} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \leq \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}}
$$

$$
\leq \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i}^{\max} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}}
$$

Then

$$
S \left(\cup_{h_{\tau_1}^{\min} \in h_1^{\min}, h_{\tau_2}^{\min} \in h_2^{\min}, \cdots, h_{\tau_n}^{\min} \in h_n^{\min}} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i}^{\min} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right)
$$

$$
\leq S \left(\cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right)
$$

$$
\leq S \left(\cup_{h_{\tau_1}^{\max} \in h_1^{\max}, h_{\tau_2}^{\max} \in h_2^{\max}, \cdots, h_{\tau_n}^{\max} \in h_n^{\max}} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_i}^{\max} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right)
$$

Finally,

$$
MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)} \left(h_1^{\min}, h_2^{\min}, \cdots, h_n^{\min} \right) \leq MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)} \left(h_1, h_2, \cdots, h_n \right)
$$
$$
\leq MHFLGMSM^{(m,p_1,p_2,\cdots,p_n)} \left(h_1^{\max}, h_2^{\max}, \cdots, h_n^{\max} \right).
$$

(4)　Because $\left(h_1', h_2', \cdots, h_n' \right)$ is a permutation of $\left(h_1, h_2, \cdots, h_n \right)$, so $\prod(\# h_i, i = 1, 2, \cdots, n) = \prod(\# h_i', i = 1, 2, \cdots, n)$. Therefore

$$
\left(\frac{\oplus_{1 \leq \tau_{i_1} < \cdots < \tau_{i_m} < n} \left(\otimes_{j=1}^m h_{\tau_{i_j}}^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} = \left(\frac{\oplus_{1 \leq \tau_{i_1} < \cdots < \tau_{i_m} < n} \left(\otimes_{j=1}^m \left(h_{\tau_{i_j}}' \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}}
$$

where $\exists h_{\tau_i} \in h_i$ for $h_{\tau_i}' \in h_i'$. So

$$
MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)} \left(h_1, h_2, \cdots, h_n \right)
$$

$$
= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_{i_j}^k} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right)
$$

$$
= \cup_{h_{\tau_1}' \in h_1', h_{\tau_2}' \in h_2', \cdots, h_{\tau_n}' \in h_n'} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^m \left(\vartheta \left(h_{\tau_{i_j}^k}' \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right)
$$

$$
= MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)} \left(h_1', h_2', \cdots, h_n' \right).
$$

The proof of Theorem 2 is completed now. □

Next, by taking into account diverse values of the parameter *m*, some particular cases of the $MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}$ can be discussed as follows:

(1)　When $m = 1$, then

$$MHFLGMSM^{(1,p_1)}(h_1, h_2, \cdots, h_n) = \left(\frac{\oplus_{i=1}^{n} h_i^{p_1}}{n} \right)^{\frac{1}{p_1}} = \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{\sum_{k=1}^{C_n^m} \left(\vartheta \left(h_{\tau_i^k} \right) \right)^{p_1}}{n} \right)^{\frac{1}{p_1}} \right) \quad (32)$$

(2)　When $m = 2$, then

$$MHFLGMSM^{(2,p_1,p_2)}(h_1, h_2, \cdots, h_n) = \left(\frac{\oplus_{i=1,j=1,i \neq j}^{n} h_i^{p_1} h_j^{p_2}}{n(n-1)} \right)^{\frac{1}{p_1+p_2}}$$

$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{\sum_{i=1,j=1,i \neq j}^{n} \left(\vartheta \left(h_{\tau_i^k} \right) \right)^{p_1} \left(\vartheta \left(h_{\tau_j^k} \right) \right)^{p_2}}{n(n-1)} \right)^{\frac{1}{p_1+p_2}} \right) \quad (33)$$

(3)　When $m = 3$, then

$$MHFLGMSM^{(3,p_1,p_2,p_3)}(h_1, h_2, \cdots, h_n) = \left(\frac{\oplus_{i=1,j=1,r=1,i \neq j \neq r}^{n} h_i^{p_1} h_j^{p_2} h_r^{p_3}}{n(n-1)(n-2)} \right)^{\frac{1}{p_1+p_2+p_3}}$$

$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{\sum_{i=1,j=1,r=1,i \neq j \neq r}^{n} \left(\vartheta \left(h_{\tau_i^k} \right) \right)^{p_1} \left(\vartheta \left(h_{\tau_j^k} \right) \right)^{p_2} \left(\vartheta \left(h_{\tau_r^k} \right) \right)^{p_3}}{n(n-1)(n-2)} \right)^{\frac{1}{p_1+p_2+p_3}} \right)$$

3.2. MHFLGGMSM Operator

Definition 15. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a set of MHFLTESs on $\overline{S}$. A MHFLGGMSM operator is a mapping $\hat{H}^n \to \hat{H}$ defined as follows:*

$$MHFLGGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n) = \frac{1}{p_1 + p_2 + \cdots + p_m} \left(\otimes_{1 \leq \tau_{i_1} < \cdots < \tau_{i_m} < n} \left(\oplus_{j=1}^{m} p_j h_{i_j} \right) \right)^{\frac{1}{C_n^m}} \quad (35)$$

where $m = 1, 2, \cdots, n$ and $\hat{H}$ is the set including all MHFLTSs.

　　Based on the operations of MHFLTSs described in Section 2, we can get the following theorem.

Theorem 2. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a set of MHFLTESs on $\overline{S}$, the value aggregated by MHFLGGMSM operator from (35) is still a MHFLTSs and*

$$MHFLGGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n)$$

$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\frac{1}{p_1+p_2+\cdots+p_m} \left(\prod_{k=1}^{C_n^m} \left(\sum_{j=1}^{m} p_j \vartheta \left(h_{\tau_{i_j}^k} \right) \right) \right)^{\frac{1}{C_n^m}} \right) \right) \quad (36)$$

where $m = 1, 2, \cdots, n$ and $k = 1, 2, \ldots, C_n^m$. $h_{\tau_{i_j}^k}$ represents the i_jth element in kth union of each permutation which consists of one element from every $h_i(i = 1, 2, \cdots, n)$. Because the form of MHFLGGMSM operator involves selecting an element from every $h_i(i = 1, 2, \cdots, n)$, each h_i must mutual calculation, and obviously the MHFLGGMSM operator will be used $\prod(\#h_i, i = 1, 2, \cdots, n)$ times, where $\#h_i$ indicates the count of elements in h_i. The final aggregated result consists of $\prod(\#h_i, i = 1, 2, \cdots, n)$ elements because of each aggregated result becomes an element which is based on the operation law of MHFLTSs.

Property 4. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a group of the MHFLTESs on $\overline{S}$. The value aggregated by MHFLGGMSM operator has the following desirable properties:*

(1)   *Idempotency. If the $h_i = h$, then*

$$S\left(MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h,h,\cdots,h)\right) = S(h)$$

(2)   *Monotonicity. If $(h_1, h_2, \cdots, h_n)$ and $(h_1', h_2', \cdots, h_n')$ are two collections of MHFLTSs on $\overline{S}$ and $h_i^+ < h_i'^-$, then for $i = 1, 2, \cdots, n$.*

$$MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1, h_2, \cdots, h_n) < MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1', h_2', \cdots, h_n').$$

(3)   *Boundedness. If $h_i^{\min}$ is obtained by replacing the minimum of h_i for each element of h_i, h_i^{max} is obtained by replacing the maximum of h_i for each element of h_i, then*

$$MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1^{\min}, h_2^{\min}, \cdots, h_n^{\min}\right) \leq MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1, h_2, \cdots, h_n)$$
$$\leq MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}\left(h_1^{\max}, h_2^{\max}, \cdots, h_n^{\max}\right).$$

(4)   *Commutativity.   Let $(h_1', h_2', \cdots, h_n')$ be any permutation of $(h_1, h_2, \cdots, h_n)$, then $MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1', h_2', \cdots, h_n') = MHFLGGMSM^{(m,p_1,p_2,\cdots,p_n)}(h_1, h_2, \cdots, h_n).$*

The proof of property 4 is similar to Property 3, therefore, it is omitted here.

Next, by taking into account diverse values of the parameter *m*, some particular cases of the MHFLGGMSM operator can be discussed as below:

(1)   When *m* = 1, then

$$MHFLGGMSM^{(1,p_1)}(h_1, h_2, \cdots, h_n) = \frac{1}{p_1}\left(\otimes_{i=1}^n (p_1 h_i)\right)^{\frac{1}{n}}$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n}\left(\vartheta^{-1}\left(\frac{1}{p_1}\left(\prod_{k=1}^n \left(p_1 \vartheta\left(h_{\tau_i^k}\right)\right)\right)^{\frac{1}{n}}\right)\right) \tag{37}$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n}\left(\vartheta^{-1}\left(\prod_{k=1}^n \vartheta\left(h_{\tau_i^k}\right)\right)^{\frac{1}{n}}\right)$$

(2)   When *m* = 2, then

$$MHFLGGMSM^{(2,p_1,p_2)}(h_1, h_2, \cdots, h_n) = \frac{1}{p_1+p_2}\left(\otimes_{i=1,j=1,i\neq j}^n (p_1 h_i + p_2 h_j)\right)^{\frac{1}{n(n-1)}}$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n}\left(\vartheta^{-1}\left(\frac{1}{p_1+p_2}\left(\prod_{i=1,j=1,i\neq j}^n \left(p_1 \vartheta(h_{\tau_i}) + p_2 \vartheta\left(h_{\tau_j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)\right) \tag{38}$$

(3)   When *m* = 3, then

$$MHFLGGMSM^{(3,p_1,p_2,p_3)}(h_1, h_2, \cdots, h_n)$$
$$= \frac{1}{p_1+p_2+p_3}\left(\otimes_{i=1,j=1,r=1,i\neq j\neq r}^n (p_1 h_i + p_2 h_j + p_3 h_r)\right)^{\frac{1}{n(n-1)(n-2)}} \tag{39}$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n}\left(\vartheta^{-1}\left(\frac{1}{p_1+p_2+p_3}\left(\prod_{i=1,j=1,r=1,i\neq j\neq r}^n \left(p_1 \vartheta(h_{\tau_i}) + p_2 \vartheta\left(h_{\tau_j}\right) + p_3 \vartheta(h_{\tau_r})\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)\right)$$

3.3. WMHFLGMSM Operator

Under many practical situations, each attribute has different importance. In this section, taking the weight of the attributes into account, we propose WMHFLGMSM operator which can be defined as below:

Definition 16. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a set of MHFLTESs on $\overline{S}$, and $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector, which satisfies $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0 (i = 1, 2, \cdots, n)$. Each w_i denotes the importance degree of h_i. The WMHFLGMSM operator: $\hat{H}^n \to \hat{H}$ is:*

$$WMHFLMGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n) = \left(\frac{\oplus_{1 \leq \tau_{i_1} < \cdots < \tau_{i_m} < n} \left(\otimes_{j=1}^{m} \left(nw_{i_j} \otimes h_{\tau_{i_j}} \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \quad (40)$$

where $m = 1, 2, \cdots, n$ and $\hat{H}$ is the set containing all MHFLTSs.

By the calculation laws for *MHFLTSs* depicted earlier, we can get the following theorem.

Theorem 3. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a set of MHFLTESs on $\overline{S}$, and $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector, which satisfies $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0 (i = 1, 2, \cdots, n)$. Each w_i denotes the importance degree of h_i. Then, the overall value aggregated by WMHFLGMSM operator from (40) is still a MHFLTSs and*

$$WMHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n)$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^{m} \left(nw_{i_j^k} \left(\vartheta \left(h_{\tau_{i_j}^k} \right) \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right) \quad (41)$$

where $m = 1, 2, \cdots, n$ and $k = 1, 2, \ldots, C_n^m$. $h_{\tau_{i_j}^k}$ represents the i_jth element in kth union of each permutation which consist of one element from each $h_i(i = 1, 2, \cdots, n)$.

Property 5. *(Reducibility). Let $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T$. Then $WMHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n)$ $= MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n)$.*

Proof

$$WMHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n)$$
$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^{m} \left(n \cdot \frac{1}{n} \left(\vartheta \left(h_{\tau_{i_j}^k} \right) \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right)$$

$$= \cup_{h_{\tau_1} \in h_1, h_{\tau_2} \in h_2, \cdots, h_{\tau_n} \in h_n} \left(\vartheta^{-1} \left(\left(\frac{\sum_{k=1}^{C_n^m} \left(\prod_{j=1}^{m} \left(\vartheta \left(h_{\tau_{i_j}^k} \right) \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_m}} \right) \right)$$

$$= MHFLGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1, h_2, \cdots, h_n). \quad \square$$

3.4. WMHFLGGMSM Operator

In this section, taking the weight into account, we propose WMHFLGGMSM operator which can be defined as follows:

Definition 17. *Suppose $h_i(i = 1, 2, \cdots, n)$ is a set of MHFLTESs on $\overline{S}$, and $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector, which satisfies $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0 (i = 1, 2, \cdots, n)$. w_i indicates the weight of h_i. The WMHFLGGMSM operator: $\hat{H}^n \to \hat{H}$ is:*

$$WMHFLGGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1,h_2,\cdots,h_n)$$

$$= \frac{1}{p_1+p_2+\cdots+p_m}\left(\otimes_{1\leq\tau_{i_1}<\cdots<\tau_{i_m}<n}\left(\oplus_{j=1}^m\left(p_jh_{i_j}\right)^{nw_{i_j}}\right)\right)^{\frac{1}{C_n^m}} \quad (42)$$

where $m = 1,2,\cdots,n$ and $\hat{H}$ is the set including all MHFLTSs.

By the operations of MHFLTSs depicted in Section 2, we can get the theorem as follows.

Theorem 4. *Suppose $h_i(i = 1,2,\cdots,n)$ is a set of MHFLTESs on $\overline{S}$, and $w = (w_1,w_2,\cdots,w_n)^T$ is the weight vector, which satisfies $\sum_{i=1}^n w_i = 1$ and $w_i > 0 (i = 1,2,\cdots,n)$. w_i indicates the weight of h_i. Then, the overall value aggregated by WMHFLGGMSM operator from (42) is still a MHFLTSs and*

$$WMHFLGGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1,h_2,\cdots,h_n)$$

$$= \cup_{h_{\tau_1}\in h_1,h_{\tau_2}\in h_2,\cdots,h_{\tau_n}\in h_n}\left(\vartheta^{-1}\left(\frac{1}{p_1+p_2+\cdots+p_m}\left(\prod_{k=1}^{C_n^m}\left(\sum_{j=1}^m\left(p_j\vartheta\left(h_{\tau_{i_j}^k}\right)\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^m}}\right)\right)\right) \quad (43)$$

where $m = 1,2,\cdots,n$ and $k = 1,\ 2,\ \ldots\ ,C_n^m$. $h_{\tau_{i_j}^k}$ represents the i_jth element in kth union of each permutation which consist of one element from each $h_i(i = 1,2,\cdots,n)$.

Property 6. *(Reducibility)*
Let $w = \left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$. Then

$$WMHFLGGMSM^{(m,p_1,p_2,\cdots,p_m)}(h_1,h_2,\cdots,h_n).$$

The proof of property 6 is similar to Property 5, therefore, it is omitted here.

4. A MCDM Approach with MHFLTs

In this section, based on the defined aggregation operators, a MCDM approach is developed to process the criteria information with the MHFLTs, which is also valid and feasible for the HFLTS or HFLEs.

Considering a MCDM problem as follows. There are d alternatives, denoted by $T = \{t_1,t_2,\cdots,t_d\}$ and n criteria, denoted by $R = \{r_1,r_2,\cdots,r_n\}$, which weight vector is $w = (w_1,w_2,\cdots,w_n)^T$ satisfying $w_i > 0$, $\sum_{i=1}^n w_i = 1(i = 1,2,\cdots,n)$. The assessments are performed by l experts $E = \{e_1,e_2,\cdots,e_l\}$ and the result of alternative t_i given by e_k under r_j is denoted by a_{ij}^k based on the LTs $S = \{s_i|i = 0,1,2,\cdots,2g,g\in N\}$. b_{ij} is a MHFLTE that contains inconsecutive and repetitive LTs by combing with the result a_{ij}^k from l experts, then we obtain the decision matrix $B = (b_{ij})_{d\times n}$, and the goal is to rank the alternatives.

The presented method for this MCDM problem is shown as follows:

Step 1. Normalize the MHFLTE matrix $B = (b_{ij})_{d\times n}$. Only for the cost criterion r_j, b_{ij} is normalized by using the negation operator.

Step 2. Integrate the assessment value of every alternative under n criteria and obtain the comprehensive assessment results b_i for $t_i(i = 1,2,\cdots,d)$.

Step 3. Calculate $S(b_i)$ and $V(b_i)$ for alternative t_i.

Step 4. Ranking all alternatives based on $S(b_i)$ and $V(b_i)$.

5. An Illustrative Example

In this part, we cited an example (adapted from [10]) to display the flexibility of the proposed methods. In order to get a fast development, the car manufacturer company will select a logistics service provider to concentrate on core competencies. After thorough investigation, five possible

logistic providers will take into account, denoted by $T = \{t_1, t_2, \cdots, t_5\}$. The criteria, denoted by $R = \{r_1, r_2, r_3, r_4\}$, are shown as follows: r_1: cost (such as the total cost of logistics operation); r_2: relationship (such as shared risks and cooperation rewards); r_3: service (such as breadth, specialization, variety); r_4: quality (such as management and improvement). The weight vector of the criteria is $w = (0.40, 0.26, 0.17, 0.17)^T$. The evaluation is carry out by three experts, denoted by $E = \{e_1, e_2, e_3\}$. $S = \{s_0 = Extremely\ Poor\ (EP), s_1 = Very\ Poor\ (VP), s_2 = Poor\ (P), s_3 = Slightly\ Poor\ (SP), s_4 = Fair\ (F), s_5 = Slightly\ Good\ (SG), s_6 = Good\ (G), s_7 = Very\ Good\ (VG), s_8 = Extremely\ Good\ (EG)\}$ be a LTs. The evaluation value of each criteria b_{ij} from three experts are represented in the form of MHFLTEs as demonstrated in Table 1.

Table 1. Linguistic evaluation result in the form of multi-hesitant linguistic term elements (MHFLTEs).

Item	r_1	r_2	r_3	r_4
t_1	{F, SG, VG}	{SG, SG, EG}	{G, G, EG}	{VP, SP, F}
t_2	{SG, G, G}	{G, EG, EG}	{VP, SP, SP }	{SP, G, G}
t_3	{P, SP, SP}	{G, G, G}	{P, SP, SG}	{F, F, F}
t_4	{VG, VG, VG}	{SP, SG, SG}	{G, EG, EG}	{VP, P, SG}
t_5	{G, EG, EG}	{F, SG, SG}	{SP, SG, EG}	{VP, P, P}

The evaluation value in the form of MHFLTSs are transformed into numerical data using Equation (6). If $g = 4$ and $a = 1.4$, then $\vartheta(s_0) = 0$, $\vartheta(s_1) = 0.1931$, $\vartheta(s_2) = 0.3311$, $\vartheta(s_3) = 0.4296$, $\vartheta(s_4) = 0.5$, $\vartheta(s_5) = 0.5704$, $\vartheta(s_6) = 0.6689$, $\vartheta(s_7) = 0.8069$, $\vartheta(s_8) = 1$.

In order to obtain the best alternative, we adopt the approach described in Section 4 to solve the MCDM problems.

5.1. Procedure of Decision Making Based on WMHFLGMSM Operator

Step 1. Get the normalized decision matrix $B = (b_{ij})_{n \times m}$.

Because the four criteria $R = \{r_1, r_2, r_3, r_4\}$ are all benefit type in this example, thus, this step can be omitted. The decision matrix is shown below:

$$B = \begin{bmatrix} \{s_4, s_5, s_7\} & \{s_5, s_5, s_8\} & \{s_6, s_6, s_8\} & \{s_1, s_3, s_4\} \\ \{s_5, s_6, s_6\} & \{s_6, s_8, s_8\} & \{s_1, s_3, s_3\} & \{s_3, s_6, s_6\} \\ \{s_2, s_3, s_3\} & \{s_6, s_6, s_6\} & \{s_2, s_3, s_5\} & \{s_4, s_4, s_4\} \\ \{s_7, s_7, s_7\} & \{s_3, s_5, s_5\} & \{s_6, s_8, s_8\} & \{s_1, s_2, s_5\} \\ \{s_6, s_8, s_8\} & \{s_4, s_5, s_5\} & \{s_3, s_5, s_8\} & \{s_1, s_2, s_2\} \end{bmatrix}$$

Step 2. Integrate the evaluation value of every alternative for four criteria and obtain the comprehensive assessment results b_i for $t_i (i = 1, 2, \cdots, 5)$ by the WMHFGMSM operator (suppose $m = 2$, $p_1 = p_2 = 1$).

Step 3. Calculate $S(b_i)$ for alternative $t_i (i = 1, 2, \cdots, 5)$, and get

$$S(b_1) = 0.5970, S(b_2) = 0.5987, S(b_3) = 0.4820, S(b_4) = 0.6329, S(b_5) = 0.5838.$$

Step 4. Rank alternatives based on $S(b_i)$, and get

$$t_4 > t_2 > t_1 > t_5 > t_3$$

5.2. Procedure of Decision Making Based on WMHFLGGMSM Operator

Step 1. Same as the above step 1.

Step 2. Integrate the evaluation value of every alternative for four criteria and obtain the comprehensive assessment results b_i for $t_i(i = 1, 2, \cdots, 5)$ by the WMHFLGGMSM operator (suppose $m = 2$, $p_1 = p_2 = 1$).

Step 3. Calculate $S(b_i)$ for alternative $t_i(i = 1, 2, \cdots, 5)$.

$$S(b_1) = 0.5919, S(b_2) = 0.5921, S(b_3) = 0.5109, S(b_4) = 0.6312, S(b_5) = 0.5895.$$

Step 4. Rank alternatives based on $S(b_i)$, and get

$$t_4 > t_2 > t_1 > t_5 > t_3$$

From the above two cases, we can find that the methods based on WMHFLGMSM operator and based on WMHFLGGMSM operator get the same ranking result, and the optimal alternative is all t_4. The WMHFLGMSM operator has the dominant advantage of stressing on the influence of the entire and general data, which permits forceful supplementary among the attribute values, while the WMHFLGGMSM operator has the dominant advantage of stressing on the counterpoise and the coordination among the attribute values.

5.3. Analysis the Effect of the Parameters m, p_1, p_2, ... , p_m

In order to observe the effects of the parameters $m, p_1, p_2, \cdots, p_m$ on this illustrate example, we assign distinct parameter values $m, p_1, p_2, \cdots, p_m$ to obtain the ranking results, and the results are demonstrated in Tables 2–5.

Table 2. Ranking results when $m = 2$ by weighted generalized MSM operator for MHFLTSs (WMHFLGMSM) operator.

p_1	p_2	Score Function $S(b_i)$	Ranking
1	0	$S_1 = 0.8507, S_2 = 0.8497, S_3 = 0.5996, S_4 = 0.9710, S_5 = 0.9769$	$t_5 > t_4 > t_1 > t_2 > t_3$
0	1	$S_1 = 0.4057, S_2 = 0.4194, S_3 = 0.3865, S_4 = 0.3946, S_5 = 0.3209$	$t_2 > t_1 > t_4 > t_3 > t_5$
1	2	$S_1 = 0.5978, S_2 = 0.6047, S_3 = 0.5436, S_4 = 0.5866, S_5 = 0.4691$	$t_2 > t_1 > t_4 > t_3 > t_5$
1	3	$S_1 = 0.5961, S_2 = 0.6371, S_3 = 0.4797, S_4 = 0.5892, S_5 = 0.5371$	$t_2 > t_1 > t_4 > t_3 > t_5$
1	4	$S_1 = 0.6102, S_2 = 0.6720, S_3 = 0.4980, S_4 = 0.5881, S_5 = 0.5394$	$t_2 > t_1 > t_4 > t_3 > t_5$
1	5	$S_1 = 0.6241, S_2 = 0.7026, S_3 = 0.5175, S_4 = 0.5894, S_5 = 0.5442$	$t_2 > t_1 > t_4 > t_3 > t_5$
2	1	$S_1 = 0.6952, S_2 = 0.7033, S_3 = 0.5277, S_4 = 0.7809, S_5 = 0.7526$	$t_4 > t_5 > t_2 > t_1 > t_3$
3	1	$S_1 = 0.7580, S_2 = 0.7672, S_3 = 0.5558, S_4 = 0.8852, S_5 = 0.8705$	$t_4 > t_5 > t_2 > t_1 > t_3$
4	1	$S_1 = 0.8024, S_2 = 0.8102, S_3 = 0.5751, S_4 = 0.8915, S_5 = 0.8758$	$t_4 > t_5 > t_2 > t_1 > t_3$
5	1	$S = 0.8356, S_2 = 0.8411, S_3 = 0.5894, S_4 = 0.9106, S_5 = 0.9054$	$t_4 > t_5 > t_2 > t_1 > t_3$
0.5	0.5	$S_1 = 0.5232, S = 0.5295, S_3 = 0.4702, S_4 = 0.5542, S_5 = 0.5010.$	$t_4 > t_2 > t_1 > t_5 > t_3$
1	1	$S_1 = 0.5970, S_2 = 0.5987, S_3 = 0.4820, S_4 = 0.6329, S_5 = 0.5838$	$t_4 > t_2 > t_1 > t_5 > t_3$
2	2	$S_1 = 0.6619, S_2 = 0.6827, S_3 = 0.5054, S_4 = 0.7118, S_5 = 0.6732$	$t_4 > t_2 > t_1 > t_5 > t_3$
3	3	$S_1 = 0.7010, S_2 = 0.7398, S_3 = 0.5273, S_4 = 0.7575, S_5 = 0.7266$	$t_4 > t_2 > t_1 > t_5 > t_3$
4	4	$S_1 = 0.7285, S_2 = 0.7811, S_3 = 0.5464, S_4 = 0.7879, S_5 = 0.7621$	$t_4 > t_2 > t_1 > t_5 > t_3$
5	5	$S_1 = 0.7872, S_2 = 0.8094, S_3 = 0.5623, S_4 = 0.8112, S_5 = 0.7489$	$t_4 > t_2 > t_1 > t_5 > t_3$

Note: S_i is abbreviation of score value $S(b_i)$.

Table 3. Ranking results when $m = 2$ by WMHFLGGMSM operator.

p_1	p_2	Score Function $S(b_i)$	Ranking
1	0	$S_1 = 0.5990, S_2 = 0.4033, S_3 = 0.3773, S_4 = 0.6969, S_5 = 0.6990$	$t_4 > t_5 > t_1 > t_2 > t_3$
0	1	$S_1 = 0.4537, S_2 = 0.4475, S_3 = 0.6103, S_4 = 0.4372, S_5 = 0.3719$	$t_3 > t_1 > t_2 > t_4 > t_5$
1	2	$S_1 = 0.5191, S_2 = 0.5386, S_3 = 0.4945, S_4 = 0.5286, S_5 = 0.4816$	$t_2 > t_4 > t_1 > t_3 > t_5$
1	3	$S_1 = 0.4690, S_2 = 0.4819, S_3 = 0.4527, S_4 = 0.4799, S_5 = 0.4363$	$t_2 > t_4 > t_1 > t_3 > t_5$
1	4	$S_1 = 0.4327, S_2 = 0.4470, S_3 = 0.4288, S_4 = 0.4389, S_5 = 0.3965$	$t_2 > t_4 > t_1 > t_3 > t_5$
1	5	$S_1 = 0.4084, S_2 = 0.4199, S_3 = 0.4050, S_4 = 0.4092, S_5 = 0.3671$	$t_2 > t_4 > t_1 > t_3 > t_5$
2	1	$S_1 = 0.6741, S_2 = 0.6650, S_3 = 0.5121, S_4 = 0.7551, S_5 = 0.7264$	$t_4 > t_5 > t_1 > t_2 > t_3$
3	1	$S_1 = 0.7461, S_3 = 0.7292, S_3 = 0.5335, S_4 = 0.8521, S_5 = 0.8302$	$t_4 > t_5 > t_1 > t_2 > t_3$
4	1	$S_1 = 0.8075, S_2 = 0.7835, S_3 = 0.5587, S_4 = 0.8604, S_5 = 0.9429$	$t_4 > t_5 > t_1 > t_2 > t_3$
5	1	$S_1 = 0.8604, S_2 = 0.8299, S_3 = 0.5836, S_4 = 0.8960, S_5 = 0.8815$	$t_4 > t_5 > t_1 > t_2 > t_3$
0.5	0.5	$S_1 = 0.6010, S_2 = 0.6056, S_3 = 0.5523, S_4 = 0.6298, S_5 = 0.5787$	$t_4 > t_2 > t_1 > t_5 > t_3$
1	1	$S_1 = 0.5919, S_2 = 0.5921, S_3 = 0.5109, S_4 = 0.6312, S_5 = 0.5895$	$t_4 > t_2 > t_1 > t_5 > t_3$
2	2	$S_1 = 0.6063, S_2 = 0.6240, S_3 = 0.4903, S_4 = 0.6580, S_5 = 0.6016$	$t_4 > t_2 > t_1 > t_5 > t_3$
3	3	$S_1 = 0.6261, S_2 = 0.6554, S_3 = 0.4882, S_4 = 0.6857, S_5 = 0.6184$	$t_4 > t_2 > t_1 > t_5 > t_3$
4	4	$S_1 = 0.6454, S_2 = 0.6827, S_3 = 0.4911, S_4 = 0.7108, S_5 = 0.6354$	$t_4 > t_2 > t_1 > t_5 > t_3$
5	5	$S_1 = 0.6633, S_2 = 0.7068, S_3 = 0.4960, S_4 = 0.7334, S_5 = 0.6515$	$t_4 > t_2 > t_1 > t_5 > t_3$

Note: S_i is abbreviation of score value $S(b_i)$.

Table 4. Ranking results when $m = 3$ by WMHFLGMSM operator.

p_1	p_2	p_3	Score Function $S(b_i)$	Ranking
1	1	1	$S_1 = 0.4856, S_2 = 0.4956, S_3 = 0.4583, S_4 = 0.5140, S_5 = 0.4644$	$t_4 > t_2 > t_1 > t_5 > t_3$
1	0	0	$S_1 = 0.9580, S_2 = 0.9464, S_3 = 0.6501, S_4 = 0.9609, S_5 = 0.9804$	$t_5 > t_4 > t_2 > t_1 > t_3$
0	1	0	$S_1 = 0.6360, S_2 = 0.5600, S_3 = 0.4987, S_4 = 0.5747, S_5 = 0.5111$	$t_1 > t_4 > t_2 > t_5 > t_3$
0	0	1	$S_1 = 0.2905, S_2 = 0.3492, S_3 = 0.3304, S_4 = 0.3054, S_5 = 0.2259$	$t_2 > t_3 > t_4 > t_1 > t_5$
1	1	2	$S_1 = 0.5251, S_2 = 0.4998, S_3 = 0.4302, S_4 = 0.5567, S_5 = 0.4791$	$t_4 > t_1 > t_2 > t_5 > t_3$
1	1	3	$S_1 = 0.5119, S_2 = 0.4736, S_3 = 0.4088, S_4 = 0.5513, S_5 = 0.4618$	$t_4 > t_1 > t_2 > t_5 > t_3$
1	1	4	$S_1 = 0.5069, S_2 = 0.4581, S_3 = 0.3958, S_4 = 0.5527, S_5 = 0.4564$	$t_4 > t_1 > t_2 > t_5 > t_3$
1	1	5	$S_1 = 0.5056, S_2 = 0.4480, S_3 = 0.3872, S_4 = 0.5561, S_5 = 0.4514$	$t_4 > t_1 > t_2 > t_5 > t_3$
1	2	1	$S_1 = 0.5885, S_2 = 0.6040, S_3 = 0.4933, S_4 = 0.5765, S_5 = 0.5302$	$t_2 > t_1 > t_4 > t_5 > t_3$
1	3	1	$S_1 = 0.6106, S_2 = 0.6506, S_3 = 0.5181, S_4 = 0.5739, S_5 = 0.5356$	$t_2 > t_1 > t_4 > t_5 > t_3$
1	4	1	$S_1 = 0.6278, S_2 = 0.6857, S_3 = 0.5398, S_4 = 0.5732, S_5 = 0.5406$	$t_2 > t_1 > t_4 > t_5 > t_3$
1	5	1	$S_1 = 0.6415, S_2 = 0.7165, S_3 = 0.5579, S_4 = 0.5734, S_5 = 0.5450$	$t_2 > t_1 > t_4 > t_5 > t_3$
2	1	1	$S_1 = 0.6449, S_2 = 0.6395, S_3 = 0.5089, S_4 = 0.7142, S_5 = 0.6661$	$t_4 > t_5 > t_1 > t_2 > t_3$
3	1	1	$S_1 = 0.7052, S_2 = 0.7004, S_3 = 0.5341, S_4 = 0.8110, S_5 = 0.7721$	$t_4 > t_5 > t_1 > t_2 > t_3$
4	1	1	$S_1 = 0.7498, S_2 = 0.7447, S_3 = 0.5521, S_4 = 0.8844, S_5 = 0.8534$	$t_4 > t_5 > t_1 > t_2 > t_3$
5	1	1	$S_1 = 0.7841, S_2 = 0.7783, S_3 = 0.5658, S_4 = 0.9416, S_5 = 0.9173$	$t_4 > t_5 > t_1 > t_2 > t_3$

Note: S_i is abbreviation of score value $S(b_i)$.

Table 5. Ranking results when $m = 3$ by WMHFLGGMSM operator.

p_1	p_2	p_3	Score Function $S(b_i)$	Ranking
1	1	1	$S_1 = 0.6215, S_2 = 0.6304, S_3 = 0.5656, S_4 = 0.6505, S_5 = 0.5933$	$t_4 > t_2 > t_1 > t_5 > t_3$
1	0	0	$S_1 = 0.5398, S_2 = 0.5622, S_3 = 0.2976, S_4 = 0.7049, S_5 = 0.7455$	$t_5 > t_4 > t_2 > t_1 > t_3$
0	1	0	$S_1 = 0.7604, S_2 = 0.4687, S_3 = 0.6121, S_4 = 0.6823, S_5 = 0.6272$	$t_1 > t_4 > t_5 > t_3 > t_2$
0	0	1	$S = 0.4309, S = 0.4387, S_3 = 0.6095, S_4 = 0.4303, S_5 = 0.3510$	$t_3 > t_2 > t_1 > t_4 > t_5$
1	1	2	$S = 0.5274, S_2 = 0.5573, S_3 = 0.4807, S_4 = 0.5451, S_5 = 0.5126$	$t_2 > t_4 > t_3 > t_1 > t_5$
1	1	3	$S_1 = 0.4703, S_2 = 0.4968, S_3 = 0.4468, S_4 = 0.4974, S_5 = 0.4501$	$t_2 > t_4 > t_3 > t_1 > t_5$
1	1	4	$S_1 = 0.4276, S_2 = 0.4589, S_3 = 0.4188, S_4 = 0.4483, S_5 = 0.4046$	$t_2 > t_4 > t_3 > t_1 > t_5$
1	1	5	$S_1 = 0.3946, S_2 = 0.4286, S_3 = 0.3957, S_4 = 0.4126, S_5 = 0.3699$	$t_2 > t_4 > t_3 > t_1 > t_5$
1	2	1	$S_1 = 0.6105, S_2 = 0.5878, S_3 = 0.5184, S_4 = 0.6209, S_5 = 0.5792$	$t_1 > t_4 > t_2 > t_5 > t_3$
1	3	1	$S_1 = 0.6045, S_2 = 0.5676, S_3 = 0.5116, S_4 = 0.5987, S = 0.5579$	$t_1 > t_4 > t_2 > t_5 > t_3$
1	4	1	$S_1 = 0.5962, S_2 = 0.5489, S_3 = 0.5032, S_4 = 0.5802, S_5 = 0.5402$	$t_1 > t_4 > t_2 > t_5 > t_3$
1	5	1	$S_1 = 0.5876, S_2 = 0.5322, S_3 = 0.4947, S_4 = 0.5647, S_5 = 0.5255$	$t_1 > t_4 > t_2 > t_5 > t_3$
2	1	1	$S_1 = 0.6931, S_2 = 0.6984, S_3 = 0.5173, S_4 = 0.7903, S_5 = 0.8062$	$t_5 > t_4 > t_2 > t_1 > t_3$
3	1	1	$S_1 = 0.7968, S_2 = 0.8078, S_3 = 0.5444, S_4 = 0.8108, S_5 = 0.8273$	$t_5 > t_4 > t_2 > t_1 > t_3$
4	1	1	$S_1 = 0.90422, S_2 = 0.9215, S_3 = 0.5935, S_4 = 0.9266, S_5 = 0.9329$	$t_5 > t_4 > t_2 > t_1 > t_3$
5	1	1	$S_1 = 0.9275, S_2 = 0.9297, S_3 = 0.6308, S_4 = 0.9439, S_5 = 0.9527$	$t_5 > t_4 > t_2 > t_1 > t_3$

Note: S_i is abbreviation of score value $S(b_i)$.

As we can see from Tables 2–5, the ranking results may be different for the distinct parameter values.

(1)   If we keep the balance of parameter assignment (i.e., $p_1 = p_2 = \cdots = p_m$), the ranking results of alternatives are identical in the condition of distinct parameter values. If the balance of parameter assignment is broken, the ranking results of alternatives will start to be changed, but as the value of parameter gap becomes lager and lager or the value of one parameter is much larger than other parameters, the ranking results will not be changed. So, the risk preference of experts plays an important role in real MCDM.

(2)   In Tables 2 and 3, if $p_1 = 0$ or $p_2 = 0$, the WMHFLGMSM operator and WMHFLGGMSM operator cannot consider the interrelation of the criteria. Moreover, the similar situation exists in Tables 4 and 5.

(3)   For the WMHFLGMSM operator, the greater value of m becomes, in other words, the more interrelationships of criteria we take into account, the smaller value of score function will get. Nevertheless, for the WMHFLGGMSM operator, the results are the opposite, the greater value of m becomes, in other words, the more interrelationships of criteria we take into account, the greater value of score function will get.

Therefore, experts should select the suitable parameter values in order to keep their risk preference at real MCDM environment. If expert has risk preference, he/she should assign the parameter as large as possible; if expert has risk aversion, he/she should assign the parameter as small as possible. Generally, we propose that experts assign the values of the parameters as $m = 2$, $p_1 = p_2 = 1$ in practical problems, which are not only simple for operations, but also take into account the interrelations for two parameters.

5.4. Comparison with the Other Methods

To verify the validity of the proposed approaches, we use the existing generalized weighted average operator for HFLSs (GHFLWA) [45], weighted BM operator for HFLSs (WHFLBM) [33]

and multi-HFLS weighted generalized HM (MHFL-WGHM) operator [10] respectively to solve an illustrative example from [40] which is listed as follows:

For the sake of further optimizing healthcare resource allocation, we need to select a general hospital to improve the traditional healthcare system and build a new comprehensive healthcare system. After thorough investigation, four general hospitals will take into account, denoted by $T = \{t_1, t_2, t_3, t_4\}$, namely SU hospital ($t_1$), FU hospital ($t_2$), UMC hospital ($t_3$), PLA hospital ($t_4$). Three main criteria, denoted by $R = \{r_1, r_2, r_3\}$, are shown as follows: service environment (r_1), diagnosis and treatment (r_2) and social resource allocation (r_3); The weight vector of the criteria is $w = (0.3, 0.2, 0.5)^T$. The linguistic evaluation result is listed in Table 6 and the ranking results of different methods are shown in Table 7.

Table 6. The Linguistic evaluation result for instance proposed by [40].

Item	r_1	r_2	r_3
t_1	$\{s_3, s_4\}$	$\{s_5\}$	$\{s_5\}$
t_2	$\{s_4\}$	$\{s_2, s_3\}$	$\{s_5\}$
t_3	$\{s_4\}$	$\{s_4, s_5\}$	$\{s_5, s_6\}$
t_4	$\{s_5\}$	$\{s_3, s_4\}$	$\{s_4\}$

Table 7. The comparisons of different methods for example proposed by [40].

Aggregation Operator	Parameter Value	Ranking Results
GHFLWA [45]	$\lambda = 1$	$t_3 > t_2 > t_4 > t_1$
$WHFLBM^{(p,q)}$ [33]	$p = q = 1$	$t_3 > t_4 > t_2 > t_1$
$MHFL\text{-}WGHM^{(p,q)}$ [10]	$p = q = 1$	$t_3 > t_4 > t_2 > t_1$
$WMHFLGMSM^{(1,p)}$ proposed in this paper	$m = 1, p = 1$	$t_3 > t_2 > t_4 > t_1$
$WMHFLGMSM^{(2,p_1,p_2)}$ proposed in this paper	$m = 2, p_1 = p_2 = 1$	$t_3 > t_4 > t_2 > t_1$
$WMHFLGMSM^{(3,p_1,p_2,p_3)}$ proposed in this paper	$m = 3, p_1 = p_2 = p_3 = 1$	$t_3 > t_4 > t_2 > t_1$

From Table 7, we can find that the approach based on the $WMHFLGMSM^{(1,p)}$ operator in this paper and the approach based on GHFLWA operator [45] have the same ranking result, i.e., $t_3 > t_2 > t_4 > t_1$, and the method based on $WMHFLGMSM^{(2,p_1,p_2)}$ operator proposed in this paper and the method based on WHFLBM operator and MHFL-WGHM operator have also the same ranking result, i.e., $t_3 > t_4 > t_2 > t_1$. Obviously, we can easily explain these results (1) because both of the proposed approaches based on the $WMHFLGMSM^{(1,p)}$ operator in this paper and the approach based on GHFLWA operator [45] don't consider the interrelationship of criteria, they produce the same ranking result; (2) while all of the proposed methods based on the $WMHFLGMSM^{(2,p_1,p_2)}$ operator and the approach based on WHFLBM operator [33] and MHFL-WGHM operator [10] can consider the interrelationship between two criteria, they get the same ranking result. So, these results can demonstrate that the presented approaches are rational and valid in HFLS.

In order to show the advantage of the novel method based on WMHFLGMSM operator in this paper, we continue to calculate the same example from [10] by comparing with some existing methods. We can get the final result demonstrated in Table 8.

From Table 8, we can find that the final result by the approach based on the MHFL-WGHM operator [10] is the same with using the WMHFLGMSM operator proposed in this paper, whereas the methods using GHFLWA operator [45] and WHFLBM operator [33] and the method using the WMHFLGMSM operator get different ranking results.

Table 8. The comparisons of different methods.

Aggregation Operator	Parameter Value	Ranking Results
GHFLWA [45]	$\lambda = 1$	$t_5 > t_4 > t_1 > t_2 > t_3$
WHFLBM$^{(p,q)}$ [33]	$m = 2, p_1 = p_2 = 1$	$t_1 > t_4 > t_5 > t_2 > t_3$
MHFL-WGHM$^{(p,q)}$ [10]	$p = q = 1$	$t_4 > t_2 > t_1 > t_5 > t_3$
WMHFLGMSM$^{(m,p_1,p_2,\cdots p_n)}$ prposed in this paper	$m = 2, p_1 = p_2 = 1$	$t_4 > t_2 > t_1 > t_5 > t_3$

In the following, we analyze the reason of producing these results as follows.

(1)   The final overall results from the GHFLWA operator [45] and the WHFLBM operator [33] are achieved under the condition that the repeated assessment values in Table 1 are eliminated. Obviously, this can lead to information loss, and make the ranking results unreasonable. From Table 8, we can also get this conclusion because they produced the different ranking results.

(2)   The approaches based on the MHFL-WGHM operator [10] and the WMHFLGMSM operator proposed in this paper produced the same ranking results because they can fully express the evaluation information, and all considered the interrelationship between two criteria. However, the proposed method based on the WMHFLGMSM operator in this paper can consider the interrelationship among any number of attributes by some parameters.

Obviously, the presented approach based on the WMHFLGMSM operator in this paper can get over the weakness of the approaches based on the GHFLWA operator [45] and the WHFLBM operator [33] because the HFLTSs cannot sufficiently convey the hesitance of experts, and WMHFLGMSM operator in this paper can take into the interrelation among any number of the criteria, which is more general than some existing methods. In summary, the presented approach based on the WMHFLGMSM operator not only considers the significance of repeated assessment values, but also takes into account the interrelationships among any criteria.

In the following, we give a detailed contrastive analysis for the different methods, and state these clearly in Table 9.

Table 9. The comparisons of the features for different methods.

Operators	Describe Fuzzy Information and Easier Parameters	Consider the Interrelationship of Two Arguments	Consider the Interrelationship of Any Multi-Input Arguments	Consider the Importance of Repeated Linguistic Values
GHFLWA [45]	Yes	No	No	No
WHFLBM$^{(p,q)}$ [33]	Yes	Yes	No	No
MHFL-WGHM$^{(p,q)}$ [10]	Yes	Yes	No	Yes
The operators prposed in this paper	Yes	Yes	Yes	Yes

From Table 9, we obtain the conclusion as follows:

(1)   Compared with the approach based on GHFLWA operator proposed by [45], we can note the weakness of the GHFLWA operator is that the input assessment values are independent and does not think about the interrelationships among input arguments. Under these circumstances, the GHFLWA operator proposed by [45] is a particular example of WMHFLGMSM operator when $m = 1, p = 1$. Our novel method takes into account the real decision environment in which there are some relationships among some criteria. In above illustrative example, the level of management and improvement will have an effect on the breadth, specialization, variety of logistic company, that is to say we should consider the interrelationships between r_3 and r_4. Therefore, our presented approach is more suitable for dealing with actual decision problems

than the presented approach based on GHFLWA operator proposed by [45]. Another weakness of the presented approach based on GHFLWA operator [45] is that it adopts HFLTSs which cannot sufficiently convey the hesitance of experts. At the same time, our presented approach can completely express the evaluation information.

(2) Compared with the approach based on the WHFLBM operator presented by [33], our new method can preserve the repetition of linguistic evaluation information and considers the interrelationship among more than two input arguments. However, the approach based on the WHFLBM operator only considers the interrelationship between two input arguments and the frequencies of repeated values are neglected. Therefore, as an extension of HFLSs and HFLTSs, the presented approach based on the WMHFLGMSM operator is more reasonable to aggregate the repeated linguistic information in practice. Moreover, we also noticed that the WMHFLGMSM operator with parameters will degrade into the WHFLBM operator proposed by [33] if $m = 2$, $p_1 = 1$, $p_2 = 1$. Therefore, the WMHFLGMSM operators have more generality and are more robust.

(3) Compared with the approach based on the MHFL-WGHM operator presented by [10], our presented novel method can take into account the correlation among multi-inputs. However, the MHFL-WGHM operator can only consider correlation between two inputs. In many real decision-making problems, the interrelationships among multi-input arguments must be considered. Thus, the presented approach is more general and wider, and it is more adequate to deal with MCDM problems.

By the contrast analysis above, we can show that the WMHFLGMSM operator we prosed in this paper is better than the WHFLBM operator proposed by [33], the MHFL-WGHM operator proposed by [10], and the GHFLWA operator proposed by [45] for aggregating the multi-hesitant fuzzy linguistic information.

6. Conclusions

A MHFLTS is an extension of HFLS and HFLTS, which can preserve the frequencies of repeated LTs. In addition, the MSM operator can take into account the correlativity among multi-inputs. In this paper, we investigate the MCDM problems with the information of the MHFLTSs. Firstly, we developed the MHFLGMSM operator, MHFLGGMSM operator, WMHFLGMSM operator, and WMHFLGGMSM operator, respectively. Then, their properties and particular cases of these operators are further studied. Based on these operators, we proposed a method to solve the MCDM problem with MHFLTEs. Finally, some examples are provided to contrast the presented approaches with the existing approaches, such the methods based on the WHFLBM operator [33], the MHFL-WGHM operator [10], the GHFLWA operator [45]. The comparison results demonstrate that the presented approaches outperform some existing approaches [10,33,45].

Fortunately, we maybe use these novel operators to solve new MCDM problems in practice, such as industrial site selection [46], intrusion detection [47], EEG signals analysis [48], consensus model [49], temporary disaster debris Management [50], etc. The limitation of proposed operators is that if we consider interrelationships among more than three criteria, more parameters will be given, which may bewilder experts, such that they perhaps cannot decide how to choose the values of parameters. In further research, it is necessary to apply these operators to real decision-making problems, such as evaluations on population resources and environment [51–54] or Chinese culture [55].

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