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The Cosine Measure of Single-Valued Neutrosophic Multisets for Multiple Attribute Decision-Making

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Received: 29 April 2018; Accepted: 10 May 2018; Published: 11 May 2018



Abstract: Based on the multiplicity evaluation in some real situations, this paper firstly introduces a single-valued neutrosophic multiset (SVNM) as a subclass of neutrosophic multiset (NM) to express the multiplicity information and the operational relations of SVNMs. Then, a cosine measure between SVNMs and weighted cosine measure between SVNMs are presented to measure the cosine degree between SVNMs, and their properties are investigated. Based on the weighted cosine measure of SVNMs, a multiple attribute decision-making method under a SVNM environment is proposed, in which the evaluated values of alternatives are taken in the form of SVNMs. The ranking order of all alternatives and the best one can be determined by the weighted cosine measure between every alternative and the ideal alternative. Finally, an actual application on the selecting problem illustrates the effectiveness and application of the proposed method.

Keywords: single valued neutrosophic set (SVNS); neutrosophic multiset (NM); single valued neutrosophic multiset (SVNM); cosine measure; multiple attribute decision-making

1. Introduction

In 1965, Zadeh [1] proposed the theory of fuzzy sets (FS), in which every fuzzy element is expressed by the membership degree $T(x)$ belonging to the scope of $[0, 1]$. While the fuzzy membership degree of $T(x)$ is difficult to be determined, or cannot be expressed by an exact real number, the practicability of FS is limited. In order to avoid the above situation, Turksen [2] extended a single-value membership to an interval-valued membership. Generally, when the membership degree $T(x)$ is determined, the non-membership degree can be calculated by $1 - T(x)$. Considering the role of the non-membership degree, Atanassov [3] put forward the intuitionistic fuzzy sets (IFS) and introduced the related theory of IFS. Since then, IFS has been widely used for solving the decision-making problems. Although the FS theory and IFS theory have been constantly extended and completed, they are not applicable to all the fuzzy problems. In 1998, Smarandache [4] added the uncertain degree to the IFS and put forward the theory of the neutrosophic set (NS), which is a general form of the FS and IFS. NS is composed of the neutrosophic components of truth, indeterminacy, and falsity denoted by T, I, F , respectively. Since then, many forms of the neutrosophic set were proposed as extensions of the neutrosophic set. Wang and Smarandache [5,6] introduced a single-valued neutrosophic set (SVNS) and an interval neutrosophic set (INS). Smarandache [7] and Smarandache and Ye [8] presented n -value and refined-single valued neutrosophic sets (R-SVNSs). Fan and Ye [9] presented a refined-interval neutrosophic set (R-INS). Ye [10] presented a dynamic single-valued neutrosophic multiset (DSVM), and so on.

Now, more researches have been done on the NS theory by experts and scholars. Ye [11,12] proposed the correlation coefficient and the weighted coefficient correlation of SVN and proved that cosine similarity is a special case of the SVN correlation coefficient. Broumi and Smarandache [13] proposed three vector similarity methods to simplify the similarity of SVN, including Jaccard similarity, Dice similarity, and cosine similarity. Majumdar and Samanta [14] gave the similarity formula of SVN. Broumi and Smarandache [15] gave the correlation coefficient of IN. Based on the Hamming and Euclidean distances, Ye [16] defined the similarity of IN. For the operation rules of NS, Smarandache, Ye, and Chi [4,16,17] gave different operation rules, respectively, where they all have certain rationality and applicability.

Recently, Smarandache [18] introduced the neutrosophic multiset and the neutrosophic multiset algebraic structures, in which one or more elements are repeated for some times, keeping the same or different neutrosophic components. Its concept is different from the concept of single-valued neutrosophic multiset in [10,19]. Until now, there are few studies and applications of neutrosophic multisets (NM) in science and engineering fields, so we introduce a single valued neutrosophic multiset (SVNM) as a subclass of the neutrosophic multiset (NM) to express the multiplicity information and propose a decision-making method based on the weighted cosine measures of SVNMs, and then provide a decision-making example to show its application under SVNM environments.

The remaining sections of this article are organized as follows. Section 2 describes some basic concepts of SVN, NM, and the cosine measure of SVN. Section 3 presents a SVNM and its basic operational relations. Section 4 proposes a cosine measure between SVNMs and a weighted cosine measure between SVNMs and investigates their properties. Section 5 establishes a multiple attribute decision-making method using the weighted cosine measure of SVNMs under SVNM environment. Section 6 presents an actual example to demonstrate the application of the proposed methods under SVNM environment. Section 7 gives a conclusion and further research.

2. Some Concepts of SVN and NM

Definition 1 [5]. Let X be a space of points (objects), with a generic element x in X . A SVN R in X can be characterized by a truth-membership function $T_R(x)$, an indeterminacy-membership function $I_R(x)$, and a falsity-membership function $F_R(x)$, where $T_R(x), I_R(x), F_R(x) \in [0, 1]$ for each point x in X . Then, a SVN R can be expressed by the following form:

$$R = \{ \langle x, T_R(x), I_R(x), F_R(x) \rangle | x \in X \}.$$

Thus, the SVN R satisfies the condition $0 \leq T_R(x) + I_R(x) + F_R(x) \leq 3$.

For two SVN M and N , the relations of them are defined as follows [5]:

- (1) $M \subseteq N$ if and only if $T_M(x) \leq T_N(x)$, $I_M(x) \geq I_N(x)$, $F_M(x) \geq F_N(x)$ for any x in X ;
- (2) $M = N$ if and only if $M \subseteq N$ and $N \subseteq M$;
- (3) $M^c = \{ \langle x, F_M(x), 1 - I_M(x), T_M(x) \rangle | x \in X \}$.

For writing convenience, an element called single-valued neutrosophic number (SVNN) in the SVN R can be denoted by $R = \langle T_R(x), I_R(x), F_R(x) \rangle$ for any x in X . For two SVNNs M and N , the operational relations of them can be defined as follows [5]:

- (1) $M \cup N = \langle \max(T_M(x), T_N(x)), \min(I_M(x), I_N(x)), \min(F_M(x), F_N(x)) \rangle$ for any x in X ;
- (2) $M \cap N = \langle \min(T_M(x), T_N(x)), \max(I_M(x), I_N(x)), \max(F_M(x), F_N(x)) \rangle$ for any x in X .

For two SVNNs M and N , the operational rules of them can be defined as follows [5]:

$$M + N = \langle T_M(x) + T_N(x) - T_M(x)T_N(x), I_M(x)I_N(x), F_M(x)F_N(x) \rangle \text{ for any } x \text{ in } X; \quad (1)$$

$$M \times N = \langle T_M(x)T_N(x), I_M(x) + I_N(x) - I_M(x)I_N(x), F_M(x) + F_N(x) - F_M(x)F_N(x) \rangle > \text{ for any } x \text{ in } X; \quad (2)$$

$$\varphi M = \langle 1 - (1 - T_M(x))^\varphi, (I_M(x))^\varphi, (F_M(x))^\varphi \rangle \text{ for } \varphi > 0 \text{ and any } x \text{ in } X; \quad (3)$$

$$M^\varphi = \langle (T_M(x))^\varphi, (1 - I_M(x))^\varphi, (1 - F_M(x))^\varphi \rangle, \text{ for } \varphi > 0 \text{ and any } x \text{ in } X. \quad (4)$$

Definition 2 [20]. Let $X = \{x_1, x_2, \dots, x_n\}$ be a space of points (objects), L and M be two SVNss. The cosine measure between L and M is defined as follows:

$$\rho(L, M) = \frac{1}{n} \sum_{i=1}^n \cos\left\{\frac{\pi}{6}(|T_L(x_i) - T_M(x_i)| + |I_L(x_i) - I_M(x_i)| + |F_L(x_i) - F_M(x_i)|)\right\}. \quad (5)$$

Obviously, the cosine measure between L and M satisfies the following properties [20]:

- ① $0 \leq \rho(L, M) \leq 1$;
- ② $\rho(L, M) = 1$ if and only if $L = M$;
- ③ $\rho(L, M) = \rho(M, L)$.

Definition 3 [18]. Let X be a space of points (objects), and a neutrosophic multiset is repeated by one or more elements with the same or different neutrosophic components.

For example, $M = \{(m_1, \langle 0.7, 0.2, 0.1 \rangle), (m_2, \langle 0.6, 0.4, 0.1 \rangle), (m_3, \langle 0.8, 0.3, 0.2 \rangle)\}$ is a neutrosophic set rather than a neutrosophic multiset; while $K = \{(k_1, \langle 0.7, 0.2, 0.1 \rangle), (k_1, \langle 0.7, 0.2, 0.1 \rangle), (k_1, \langle 0.7, 0.2, 0.1 \rangle), (k_2, \langle 0.6, 0.4, 0.1 \rangle)\}$ is a neutrosophic multiset, where the element k_1 is repeated. Then, we can say that the element k_1 has neutrosophic multiplicity 3 with the same neutrosophic components.

Meanwhile, $L = \{(l_1, \langle 0.7, 0.2, 0.1 \rangle), (l_1, \langle 0.6, 0.3, 0.1 \rangle), (l_1, \langle 0.8, 0.1, 0.1 \rangle), (l_2, \langle 0.6, 0.4, 0.1 \rangle)\}$ is also a neutrosophic multiset since the element l_1 is repeated, and then we can say that the element l_1 has neutrosophic multiplicity 3 with different neutrosophic components.

If the element l_1 is repeated times with the same neutrosophic components, we say l_1 has multiplicity. If the element l_1 is repeated times with different neutrosophic components, we say l_1 has the neutrosophic multiplicity (nm). The nm function can be defined as follows:

$nm: X \rightarrow N = \{1, 2, 3, \dots, \infty\}$ for any $r \in R$

$nm(r) = \{(p_1, \langle T_1, I_1, F_1 \rangle), (p_2, \langle T_2, I_2, F_2 \rangle), \dots, (p_i, \langle T_i, I_i, F_i \rangle), \dots\}$,

which means that r is repeated by p_1 times with the neutrosophic components $\langle T_1, I_1, F_1 \rangle$; r is repeated by p_2 times with the neutrosophic components $\langle T_2, I_2, F_2 \rangle$; ...; r is repeated by p_i times with the neutrosophic components $\langle T_i, I_i, F_i \rangle$; and so on. $p_1, p_2, \dots, p_i, \dots \in N$, and $\langle T_j, I_j, F_j \rangle \neq \langle T_k, I_k, F_k \rangle$, for $j \neq k$ and $j, k \in N$. Then a neutrosophic multiset R can be written as:

$$(R, nm(r)) \text{ or } \{(r, nm(r), \text{ for } r \in R)\}. \quad (6)$$

Now, with respect to the previous neutrosophic multisets K, L , we compute the neutrosophic multiplicity function:

$nm_K: K \rightarrow N$;

$nm_K(k_1) = \{(3, \langle 0.7, 0.2, 0.1 \rangle)\}$;

$nm_K(k_2) = \{(1, \langle 0.6, 0.4, 0.1 \rangle)\}$;

$nm_L: L \rightarrow N$;

$nm_L(l_1) = \{(1, \langle 0.7, 0.2, 0.1 \rangle), (1, \langle 0.6, 0.3, 0.1 \rangle), (1, \langle 0.8, 0.1, 0.1 \rangle)\}$;

$nm_L(l_2) = \{(1, \langle 0.6, 0.4, 0.1 \rangle)\}$.

3. Single Valued Neutrosophic Multiset

Definition 4. Let X be a space of points (objects) with a generic element x in X and $N = \{1, 2, 3, \dots, \infty\}$. A SVN R in X can be defined as follows:

$R = \{x, ((p_{R1}, \langle T_{R1}(x), I_{R1}(x), F_{R1}(x) \rangle), (p_{R2}, \langle T_{R2}(x), I_{R2}(x), F_{R2}(x) \rangle), \dots, (p_{Rj}, \langle T_{Rj}(x), I_{Rj}(x), F_{Rj}(x) \rangle)) | x \in X\}$, where $T_{Rk}(x), I_{Rk}(x), F_{Rk}(x)$ express the truth-membership function, the indeterminacy-membership function, and the falsity-membership function, respectively. $T_{R1}(x), T_{R2}(x), \dots, T_{Rk}(x) \in [0, 1]$, $I_{R1}(x), I_{R2}(x), \dots, I_{Rk}(x) \in [0, 1]$, $F_{R1}(x), F_{R2}(x), \dots, F_{Rk}(x) \in [0, 1]$ and $0 \leq T_{Rk}(x) + I_{Rk}(x) + F_{Rk}(x) \leq 3$, for $k = 1, 2, \dots, j, j \in N, p_{R1}, p_{R2}, \dots, p_{Rj} \in N$ and $p_{R1} + p_{R2} + \dots + p_{Rj} \geq 2$.

For convenience, a SVN R can be denoted by the following simplified form:

$$R = \{x, (p_{Rk}, \langle T_{Rk}(x), I_{Rk}(x), F_{Rk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j.$$

For example, with a universal set $X = \{x_1, x_2\}$, a SVN R is given as:

$$R = \{(x_1, (2, \langle 0.6, 0.2, 0.1 \rangle), (1, \langle 0.8, 0.2, 0.2 \rangle)), (x_2, (1, \langle 0.7, 0.3, 0.1 \rangle), (2, \langle 0.7, 0.2, 0.3 \rangle))\}.$$

Then

$$\begin{aligned} nm_R(x_1) &= \{(2, \langle 0.6, 0.2, 0.1 \rangle), (1, \langle 0.8, 0.2, 0.2 \rangle)\}; \\ nm_R(x_2) &= \{(1, \langle 0.7, 0.3, 0.1 \rangle), (2, \langle 0.7, 0.2, 0.3 \rangle)\}. \end{aligned}$$

Definition 5. Let X be a space of points (objects) with a generic element x in X , M and L be two SVNMs,

$$M = \{x, (p_{Mk}, \langle T_{Mk}(x), I_{Mk}(x), F_{Mk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j,$$

$$L = \{x, (p_{Lk}, \langle T_{Lk}(x), I_{Lk}(x), F_{Lk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j,$$

Then the relations of them are given as follows:

- ① $M = L$, if and only if $p_{Mk} = p_{Lk}, T_{Mk}(x) = T_{Lk}(x), I_{Mk}(x) = I_{Lk}(x), F_{Mk}(x) = F_{Lk}(x)$, for $k = 1, 2, \dots, j$;
- ② $M \cup L = \{x, ((p_{Mk} \vee p_{Lk}), \langle T_{Mk}(x) \vee T_{Lk}(x), I_{Mk}(x) \wedge I_{Lk}(x), F_{Mk}(x) \wedge F_{Lk}(x) \rangle) | x \in X\}$, for $k = 1, 2, \dots, j$;
- ③ $M \cap L = \{x, ((p_{Mk} \wedge p_{Lk}), \langle T_{Mk}(x) \wedge T_{Lk}(x), I_{Mk}(x) \vee I_{Lk}(x), F_{Mk}(x) \vee F_{Lk}(x) \rangle) | x \in X\}$, for $k = 1, 2, \dots, j$.

For convenience, we can use $r = ((p_{r1}, \langle T_{r1}(x), I_{r1}(x), F_{r1}(x) \rangle), (p_{r2}, \langle T_{r2}(x), I_{r2}(x), F_{r2}(x) \rangle), \dots, (p_{rj}, \langle T_{rj}(x), I_{rj}(x), F_{rj}(x) \rangle))$ to express a basic element in a SVN R and call r a single valued neutrosophic multiset element (SVNME).

For example, with a universal set $X = \{x_1, x_2\}$, then two SVNMs M and L are given as:

$$M = \{(x_1, (2, \langle 0.6, 0.2, 0.1 \rangle), (1, \langle 0.4, 0.1, 0.2 \rangle)), (x_2, (1, \langle 0.7, 0.3, 0.1 \rangle))\};$$

$$L = \{(x_1, (1, \langle 0.6, 0.2, 0.1 \rangle), (1, \langle 0.8, 0.2, 0.1 \rangle)), (x_2, (1, \langle 0.9, 0.3, 0.1 \rangle))\};$$

$$M \cup L = \{(x_1, (2, \langle 0.6, 0.2, 0.1 \rangle), (1, \langle 0.4, 0.1, 0.2 \rangle), (1, \langle 0.8, 0.2, 0.1 \rangle)), (x_2, (1, \langle 0.7, 0.3, 0.1 \rangle), (1, \langle 0.9, 0.3, 0.1 \rangle))\}$$

$$M \cap L = \{x_1, (1, \langle 0.6, 0.2, 0.1 \rangle)\}.$$

Definition 6. Let X be a space of points (objects) with a generic element x in X and M be a SVN, we can change a SVN M into a SVN \tilde{M} by using the operational rules of SVN.

$$M = \{x, (p_{Mk}, \langle T_{Mk}(x), I_{Mk}(x), F_{Mk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j.$$

Then

$$\tilde{M} = \{ \langle x, 1 - \prod_{k=1}^j (1 - T_{Mk}(x))^{p_{Mk}}, \prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}}, \prod_{k=1}^j (F_{Mk}(x))^{p_{Mk}} \rangle | x \in X \}. \quad (7)$$

Proof. Set m_1, m_2, \dots, m_j are basic elements in M .

When $k = 1$, we can get

$$m_1 = (\langle T_{m1}(x), I_{m1}(x), F_{m1}(x) \rangle, \langle T_{m1}(x), I_{m1}(x), F_{m1}(x) \rangle, \dots, \langle T_{m1}(x), I_{m1}(x), F_{m1}(x) \rangle),$$

which has neutrosophic multiplicity p_{m1} .

According to the operational rules of SVNMs, we can get:

$$+ m_1 = 1 - (1 - T_{m1}(x))^{p_{m1}}, (I_{m1}(x))^{p_{m1}}, (F_{m1}(x))^{p_{m1}}.$$

As the same reason, when $k = 2$, we can get

$$+ m_2 = 1 - (1 - T_{m2}(x))^{p_{m2}}, (I_{m1}(x))^{p_{m2}}, (F_{m2}(x))^{p_{m2}}.$$

Then

$$\begin{aligned} m_1 + m_2 &= 1 - (1 - T_{m1}(x))^{p_{m1}} (1 - T_{m2}(x))^{p_{m2}}, (I_{m1}(x))^{p_{m1}} (I_{m1}(x))^{p_{m2}}, (F_{m2}(x))^{p_{m1}} (F_{m2}(x))^{p_{m2}} \\ &= 1 - \prod_{k=1}^2 (1 - T_{mk}(x))^{p_{mk}}, \prod_{k=1}^2 (I_{mk}(x))^{p_{mk}}, \prod_{k=1}^2 (F_{mk}(x))^{p_{mk}}; \end{aligned}$$

Suppose when $k = i$, the Equation (7) is established, then we can get:

$$m_1 + m_2 + \dots + m_i = 1 - \prod_{k=1}^i (1 - T_{mk}(x))^{p_{mk}}, \prod_{k=1}^i (I_{mk}(x))^{p_{mk}}, \prod_{k=1}^i (F_{mk}(x))^{p_{mk}};$$

Then

$$\begin{aligned} m_1 + m_2 + \dots + m_i + m_{i+1} &= 1 - \prod_{k=1}^i (1 - T_{mk}(x))^{p_{mk}} + 1 - (1 - T_{m(i+1)}(x))^{p_{m(i+1)}} \\ &\quad - \left(1 - \prod_{k=1}^i (1 - T_{mk}(x))^{p_{mk}} \right) \left(1 - (1 - T_{m(i+1)}(x))^{p_{m(i+1)}} \right), \\ &\quad \left(\prod_{k=1}^i (I_{mk}(x))^{p_{mk}} \right) (I_{m(i+1)}(x))^{p_{m(i+1)}}, \\ &\quad \left(\prod_{k=1}^i (F_{mk}(x))^{p_{mk}} \right) (F_{m(i+1)}(x))^{p_{m(i+1)}} \\ &= 1 - \prod_{k=1}^{i+1} (1 - T_{mk}(x))^{p_{mk}}, \prod_{k=1}^{i+1} (I_{mk}(x))^{p_{mk}}, \prod_{k=1}^{i+1} (F_{mk}(x))^{p_{mk}}. \end{aligned}$$

To sum up, when $k = i + 1$, Equation (7) is true, and then according to the mathematical induction, we can get that the aggregation result is also true. \square

Definition 7. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse, and M and N be two SVNMs, and then the operational rules of SVNMs are defined as follows:

$$M = \{x, (p_{Mk}, \langle T_{Mk}(x), I_{Mk}(x), F_{Mk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j;$$

$$N = \{x, (p_{Nk}, \langle T_{Nk}(x), I_{Nk}(x), F_{Nk}(x) \rangle) | x \in X\}, \text{ for } k = 1, 2, \dots, j;$$

$$\begin{aligned}
M \oplus N &= \{ \langle x, 1 \\
&\quad - \prod_{k=1}^j (1 - T_{Mk}(x))^{p_{Mk}} \prod_{k=1}^j (1 \\
&\quad - T_{Nk}(x))^{p_{Nk}}, \prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}} \prod_{k=1}^j (I_{Nk}(x))^{p_{Nk}}, \prod_{k=1}^j (F_{Mk}(x))^{p_{Mk}} \prod_{k=1}^j (F_{Nk}(x))^{p_{Nk}} \rangle | x \in X \} \\
M \otimes N &= \{ \langle x, \left(1 - \prod_{k=1}^j (1 - T_{Mk}(x))^{p_{Mk}} \right) \left(1 - \prod_{k=1}^j (1 - T_{Nk}(x))^{p_{Nk}} \right), \prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}} \\
&\quad + \prod_{k=1}^j (I_{Nk}(x))^{p_{Nk}} - \prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}} \prod_{k=1}^j (I_{Nk}(x))^{p_{Nk}}, \prod_{k=1}^j (F_{Mk}(x))^{p_{Mk}} \\
&\quad + \prod_{k=1}^j (F_{Nk}(x))^{p_{Nk}} - \prod_{k=1}^j (F_{Mk}(x))^{p_{Mk}} \prod_{k=1}^j (F_{Nk}(x))^{p_{Nk}} \rangle | x \in X \} \\
\varphi M &= \{ \langle x, \left(1 - \left(\prod_{k=1}^j (1 - T_{Mk}(x))^{p_{Mk}} \right)^\varphi \right), \left(\prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}} \right)^\varphi, \left(\prod_{k=1}^j (F_{Nk}(x))^{p_{Nk}} \right)^\varphi \rangle | x \in X \} \\
M^\varphi &= \{ \langle x, \left(1 - \prod_{k=1}^j (1 - T_{Mk}(x))^{p_{Mk}} \right)^\varphi, 1 - \left(1 - \prod_{k=1}^j (I_{Mk}(x))^{p_{Mk}} \right)^\varphi, 1 \\
&\quad - \left(1 - \prod_{k=1}^j (F_{Nk}(x))^{p_{Nk}} \right)^\varphi \rangle | x \in X \}
\end{aligned}$$

4. Cosine Measures of Single-Value Neutrosophic Multisets

Cosine measures are usually used in science and engineering applications. In this section, we propose a cosine measure of SVNMs and a weighted cosine measure of SVNMs.

Definition 8. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse, M and N be two SVNMs,

$$M = \{x_i, \langle p_{M1}, \langle T_{M1}(x_i), I_{M1}(x_i), F_{M1}(x_i) \rangle \rangle, \langle p_{M2}, \langle T_{M2}(x_i), I_{M2}(x_i), F_{M2}(x_i) \rangle \rangle, \dots, \langle p_{Mj}, \langle T_{Mj}(x_i), I_{Mj}(x_i), F_{Mj}(x_i) \rangle \rangle \mid x_i \in X\},$$

$$N = \{x_i, \langle p_{N1}, \langle T_{N1}(x_i), I_{N1}(x_i), F_{N1}(x_i) \rangle \rangle, \langle p_{N2}, \langle T_{N2}(x_i), I_{N2}(x_i), F_{N2}(x_i) \rangle \rangle, \dots, \langle p_{Nj}, \langle T_{Nj}(x_i), I_{Nj}(x_i), F_{Nj}(x_i) \rangle \rangle \mid x_i \in X\}$$

Then, a cosine measure between two SVNMs M and N is defined as follows:

$$\begin{aligned}
\rho(M, N) &= \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{6} \left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| \right. \\
&\quad + \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| \\
&\quad \left. + \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| \right\}
\end{aligned} \quad (8)$$

Theorem 1. The cosine measure $\rho(M, N)$ between two SVNMs M and N satisfies the following properties:

- ① $\rho(M, N) = \rho(N, M)$;
- ② $0 \leq \rho(M, N) \leq 1$;
- ③ $\rho(M, N) = 1$, if and only if $M = N$.

Proof. ①: For $\left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| = \left| \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} - \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} \right| + \left| \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} - \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} \right| + \left| \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} - \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} \right|$, so we can get $\rho(M, N) = \rho(N, M)$.

- ②: For $0 \leq T_{Mk}(x_i) \leq 1$, $0 \leq I_{Mk}(x_i) \leq 1$, $0 \leq F_{Mk}(x_i) \leq 1$, $0 \leq T_{Nk}(x_i) \leq 1$, $0 \leq I_{Nk}(x_i) \leq 1$, $0 \leq F_{Nk}(x_i) \leq 1$;

Then, we can get

$$0 \leq 1 - T_{Mk}(x_i) \leq 1 \text{ and } 0 \leq 1 - T_{Nk}(x_i) \leq 1;$$

$$0 \leq \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} \leq 1 \text{ and } 0 \leq \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \leq 1;$$

So,

$$0 \leq \left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| \leq 1.$$

For the same reason, we can get

$$\begin{aligned} 0 &\leq \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| \leq 1 \text{ and } 0 \\ &\leq \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| \leq 1 \end{aligned}$$

Above all, we can get $0 \leq \left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| \leq 3$ and $0 \leq \sum_{i=1}^n \cos \left\{ \frac{\pi}{6} \left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| + \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| \right\} \leq 1$;

$$\begin{aligned} \rho(M, N) &= \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{6} \left| \prod_{k=1}^j (1 - T_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_i))^{p_{Nk}} \right| \right. \\ &\quad \left. + \left| \prod_{k=1}^j (I_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_i))^{p_{Nk}} \right| \right. \\ &\quad \left. + \left| \prod_{k=1}^j (F_{Mk}(x_i))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_i))^{p_{Nk}} \right| \right\} \\ &= \frac{1}{n} \left(\cos \left\{ \frac{\pi}{6} \left(\left| \prod_{k=1}^j (1 - T_{Mk}(x_1))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_1))^{p_{Nk}} \right| \right. \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (I_{Mk}(x_1))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_1))^{p_{Nk}} \right| \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (F_{Mk}(x_1))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_1))^{p_{Nk}} \right| \right) \right\} \\ &\quad + \cos \left\{ \frac{\pi}{6} \left(\left| \prod_{k=1}^j (1 - T_{Mk}(x_2))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_2))^{p_{Nk}} \right| \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (I_{Mk}(x_2))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_2))^{p_{Nk}} \right| \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (F_{Mk}(x_2))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_2))^{p_{Nk}} \right| \right) \right\} + \dots \\ &\quad + \cos \left\{ \frac{\pi}{6} \left(\left| \prod_{k=1}^j (1 - T_{Mk}(x_n))^{p_{Mk}} - \prod_{k=1}^j (1 - T_{Nk}(x_n))^{p_{Nk}} \right| \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (I_{Mk}(x_n))^{p_{Mk}} - \prod_{k=1}^j (I_{Nk}(x_n))^{p_{Nk}} \right| \right. \right. \\ &\quad \left. \left. + \left| \prod_{k=1}^j (F_{Mk}(x_n))^{p_{Mk}} - \prod_{k=1}^j (F_{Nk}(x_n))^{p_{Nk}} \right| \right) \right\} \end{aligned}$$

Let $\cos\left\{\frac{\pi}{6}\left(\left|\prod_{k=1}^j(1-T_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(1-T_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(I_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(I_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(F_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(F_{Nk}(x_i))^{p_{Nk}}\right|\right)\right\}=a_i$ ($i=1,2,\dots,n$), then $\rho(M,N)=\frac{1}{n}\{a_1+a_2+\dots+a_n\}$.

According to $0\leq\cos\left\{\frac{\pi}{6}\left(\left|\prod_{k=1}^j(1-T_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(1-T_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(I_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(I_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(F_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(F_{Nk}(x_i))^{p_{Nk}}\right|\right)\right\}=a_i\leq 1$, we can obtain $0\leq a_1+a_2+\dots+a_n\leq n$ and $0\leq\frac{1}{n}(a_1+a_2+\dots+a_n)\leq 1$, so we can get $0\leq\rho(M,N)\leq 1$.

③: If $M=N$ then $T_{Mk}(x_i)=T_{Nk}(x_i)$, $I_{Mk}(x_i)=I_{Nk}(x_i)$, and $F_{Mk}(x_i)=F_{Nk}(x_i)$ for any $x_i\in X$ and $i=1,2,\dots,n$, so we can get $\rho(M,N)=1$, if and only if $M=N$.

Now, we consider different weights for each element x_i ($i=1,2,\dots,n$) in X . Then, let $w=(w_1,w_2,\dots,w_n)^T$ be the weight vector of each element x_i ($i=1,2,\dots,n$) with $w_i\in[0,1]$, and $\sum_{i=1}^n w_i=1$. Hence, we further extend the cosine measure of Equation (8) to the following weighted cosine measure of SVNMs:

$$\rho_w(M,N)=\sum_{i=1}^n w_i \cos\left\{\frac{\pi}{6}\left(\left|\prod_{k=1}^j(1-T_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(1-T_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(I_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(I_{Nk}(x_i))^{p_{Nk}}\right|+\left|\prod_{k=1}^j(F_{Mk}(x_i))^{p_{Mk}}-\prod_{k=1}^j(F_{Nk}(x_i))^{p_{Nk}}\right|\right)\right\} \quad (9)$$

□

Theorem 2. The cosine measure $\rho_w(M,N)$ between two SVNMs M and N satisfies the following properties:

- ① $\rho_w(M,N)=\rho_w(N,M)$;
- ② $0\leq\rho_w(M,N)\leq 1$;
- ③ $\rho_w(M,N)=1$, if and only if $M=N$.

The proof of Theorem 2 is similar to that of the Theorem 1, so we omitted it here.

5. Cosine Measure of SVNMs for Multiple Attribute Decision-Making

In this section, we use the weighted cosine measure of SVNMs to deal with the multiple attribute decision-making problems with SVNMs information.

Let $G=\{g_1,g_2,\dots,g_m\}$ as a set of alternatives and $X=\{x_1,x_2,\dots,x_n\}$ as a set of attributes, then they can be established in a decision-making problem. However, sometimes x_i ($i=1,2,\dots,n$) may have multiplicity, and then we can use the form of a SVNMs to represent the evaluation value.

Let $g_r=\{x_i,(p_{gr1},\langle T_{gr1}(x_i),I_{gr1}(x_i),F_{gr1}(x_i)\rangle),(p_{gr2},\langle T_{gr2}(x_i),I_{gr2}(x_i),F_{gr2}(x_i)\rangle),\dots,(p_{grj},\langle T_{grj}(x_i),I_{grj}(x_i),F_{grj}(x_i)\rangle)|x_i\in X\}$, for $r=1,2,\dots,m$ and $i=1,2,\dots,n$. Then we can establish the SVNMs decision matrix D , which is shown in Table 1.

Table 1. The single-valued neutrosophic multiset (SVNM) decision matrix D .

	x_1	...
g_1	$x_1, (p_{g_1 1}, \langle T_{g_1 1}(x_1), I_{g_1 1}(x_1), F_{g_1 1}(x_1) \rangle), \dots, (p_{g_1 j}, \langle T_{g_1 j}(x_1), I_{g_1 j}(x_1), F_{g_1 j}(x_1) \rangle)$...
g_2	$x_1, (p_{g_2 1}, \langle T_{g_2 1}(x_1), I_{g_2 1}(x_1), F_{g_2 1}(x_1) \rangle), \dots, (p_{g_2 j}, \langle T_{g_2 j}(x_1), I_{g_2 j}(x_1), F_{g_2 j}(x_1) \rangle)$...
...
g_m	$x_1, (p_{g_m 1}, \langle T_{g_m 1}(x_1), I_{g_m 1}(x_1), F_{g_m 1}(x_1) \rangle), \dots, (p_{g_m j}, \langle T_{g_m j}(x_1), I_{g_m j}(x_1), F_{g_m j}(x_1) \rangle)$...
	x_n	
	$x_n, (p_{g_1 1}, \langle T_{g_1 1}(x_n), I_{g_1 1}(x_n), F_{g_1 1}(x_n) \rangle), \dots, (p_{g_1 j}, \langle T_{g_1 j}(x_n), I_{g_1 j}(x_n), F_{g_1 j}(x_n) \rangle)$	
	$x_n, (p_{g_2 1}, \langle T_{g_2 1}(x_n), I_{g_2 1}(x_n), F_{g_2 1}(x_n) \rangle), \dots, (p_{g_2 j}, \langle T_{g_2 j}(x_n), I_{g_2 j}(x_n), F_{g_2 j}(x_n) \rangle)$	
	...	
	$x_n, (p_{g_m 1}, \langle T_{g_m 1}(x_n), I_{g_m 1}(x_n), F_{g_m 1}(x_n) \rangle), \dots, (p_{g_m j}, \langle T_{g_m j}(x_n), I_{g_m j}(x_n), F_{g_m j}(x_n) \rangle)$	

Step 1: By using Equation (7), we change the SVNM decision matrix D into SVNS decision matrix \tilde{D} , which is shown in Table 2.

Table 2. The single-valued neutrosophic set (SVNS) decision matrix \tilde{D} .

	x_1	...
\tilde{g}_1	$\langle x_1, 1 - \prod_{k=1}^j (1 - T_{g_1 k}(x_1))^{p_{g_1 k}}, \prod_{k=1}^j (I_{g_1 k}(x_1))^{p_{g_1 k}}, \prod_{k=1}^j (F_{g_1 k}(x_1))^{p_{g_1 k}} \rangle$...
\tilde{g}_2	$\langle x_1, 1 - \prod_{k=1}^j (1 - T_{g_2 k}(x_1))^{p_{g_2 k}}, \prod_{k=1}^j (I_{g_2 k}(x_1))^{p_{g_2 k}}, \prod_{k=1}^j (F_{g_2 k}(x_1))^{p_{g_2 k}} \rangle$...
...
\tilde{g}_m	$\langle x_1, 1 - \prod_{k=1}^j (1 - T_{g_m k}(x_1))^{p_{g_m k}}, \prod_{k=1}^j (I_{g_m k}(x_1))^{p_{g_m k}}, \prod_{k=1}^j (F_{g_m k}(x_1))^{p_{g_m k}} \rangle$...
	x_n	
	$\langle x_n, 1 - \prod_{k=1}^j (1 - T_{g_1 k}(x_n))^{p_{g_1 k}}, \prod_{k=1}^j (I_{g_1 k}(x_n))^{p_{g_1 k}}, \prod_{k=1}^j (F_{g_1 k}(x_n))^{p_{g_1 k}} \rangle$	
	$\langle x_n, 1 - \prod_{k=1}^j (1 - T_{g_2 k}(x_n))^{p_{g_2 k}}, \prod_{k=1}^j (I_{g_2 k}(x_n))^{p_{g_2 k}}, \prod_{k=1}^j (F_{g_2 k}(x_n))^{p_{g_2 k}} \rangle$	
	...	
	$\langle x_n, 1 - \prod_{k=1}^j (1 - T_{g_m k}(x_n))^{p_{g_m k}}, \prod_{k=1}^j (I_{g_m k}(x_n))^{p_{g_m k}}, \prod_{k=1}^j (F_{g_m k}(x_n))^{p_{g_m k}} \rangle$	

Step 2: Setting $T_{g^*}(x_i)$ is the maximum truth value in each column x_i of the decision matrix \tilde{D} , $I_{g^*}(x_i)$ and $F_{g^*}(x_i)$ are the minimum indeterminate and falsity values in each column x_i of the decision matrix \tilde{D} , respectively, the ideal solution can be determined as x_i^* .

$$x_i^* = \langle T_{g^*}(x_i), I_{g^*}(x_i), F_{g^*}(x_i) \rangle, \text{ for } i = 1, 2, \dots, n.$$

So, we can get the ideal alternative $g^* = \{x_1^*, x_2^*, \dots, x_n^*\}$.

Step 3: When the weight vector of attributes for the different importance of each attribute $x_i (i = 1, 2, \dots, n)$ is given by $w = (w_1, w_2, \dots, w_n)^T$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then we utilize the weighted cosine measure to deal with multiple attribute decision-making problems with SVNM information. The weighted cosine measure between an alternative $\tilde{g}_r (r = 1, 2, \dots, m)$ and the ideal alternative g^* can be calculated by using the following formula:

$$\begin{aligned} \rho_w(\tilde{g}_r, g^*) &= \rho_w(\tilde{g}_r, g^*) \\ &= \sum_{i=1}^n w_i \cos \left\{ \frac{\pi}{6} \left(\left| T_{\tilde{g}_r}(x_i) - T_{g^*}(x_i) \right| + \left| I_{\tilde{g}_r}(x_i) - I_{g^*}(x_i) \right| \right. \right. \\ &\quad \left. \left. + \left| F_{\tilde{g}_r}(x_i) - F_{g^*}(x_i) \right| \right) \right\}. \end{aligned} \quad (10)$$

Step 4: According to the values of $\rho_w(\tilde{g}_r, g^*)$ for $r = 1, 2, \dots, m$, we rank the alternatives and select the best one.

Step 5: End.

The formalization of the steps is illustrated in Figure 1.

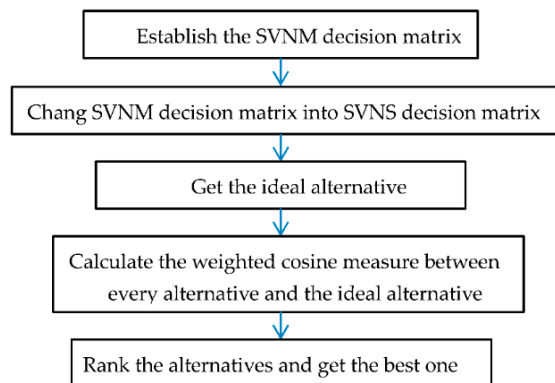


Figure 1. Flowchart of the decision steps.

6. Numerical Example and Comparative Analysis

6.1. Numerical Example

Now, we utilize a practical example for the decision-making problem adapted from the literature [21] to demonstrate the applications of the proposed method under a SVN environment. Now, one customer wants to buy a car, he selects four types of cars and evaluates them according to four attributes. Then, we build a decision model. There are four possible alternatives (g_1, g_2, g_3, g_4) to be considered. The decision should be taken according to four attributes: fuel economy (x_1), price (x_2), comfort (x_3), and safety (x_4). The weight vector of these four attributes is given by $w = (0.5, 0.25, 0.125, 0.125)^T$. Then, the customer tests the four cars on the road with less obstacles and on the road with more obstacles, respectively, and after testing, some attributes may have two different evaluated values or the same value. So, the customer evaluates the four cars (alternatives) under the four attributes by the form of SVNMs.

Step 1: Establish the SVN decision matrix D provided by the customer, which is given as the following SVN decision matrix D in Table 3.

Table 3. The SVN decision matrix D .

	x_1	x_2	x_3	x_4
g_1	$(1, \langle 0.5, 0.7, 0.2 \rangle), (1, \langle 0.7, 0.3, 0.6 \rangle)$	$1, \langle 0.4, 0.4, 0.5 \rangle$	$(1, \langle 0.7, 0.7, 0.5 \rangle), (1, \langle 0.8, 0.7, 0.6 \rangle)$	$(1, \langle 0.1, 0.5, 0.7 \rangle), (1, \langle 0.5, 0.2, 0.8 \rangle)$
g_2	$(1, \langle 0.9, 0.7, 0.5 \rangle), (1, \langle 0.7, 0.7, 0.1 \rangle)$	$1, \langle 0.7, 0.6, 0.8 \rangle$	$2, \langle 0.9, 0.4, 0.6 \rangle$	$(1, \langle 0.5, 0.2, 0.7 \rangle), (1, \langle 0.5, 0.1, 0.9 \rangle)$
g_3	$(1, \langle 0.3, 0.4, 0.2 \rangle), (1, \langle 0.6, 0.3, 0.7 \rangle)$	$1, \langle 0.2, 0.2, 0.2 \rangle$	$(1, \langle 0.9, 0.5, 0.5 \rangle), (1, \langle 0.6, 0.5, 0.2 \rangle)$	$(1, \langle 0.7, 0.5, 0.3 \rangle), (1, \langle 0.4, 0.2, 0.2 \rangle)$
g_4	$(1, \langle 0.9, 0.7, 0.2 \rangle), (1, \langle 0.8, 0.6, 0.1 \rangle)$	$1, \langle 0.3, 0.5, 0.2 \rangle$	$(1, \langle 0.5, 0.4, 0.5 \rangle), (1, \langle 0.1, 0.7, 0.2 \rangle)$	$2, \langle 0.4, 0.2, 0.8 \rangle$

Step 2: By using Equation (7), we change the SVN decision matrix D into SVNS decision matrix \tilde{D} , which is shown in Table 4.

Table 4. The SVNS decision matrix \tilde{D} .

	x_1	x_2	x_3	x_4
\tilde{g}_1	$\langle 0.85, 0.21, 0.12 \rangle$	$\langle 0.4, 0.4, 0.5 \rangle$	$\langle 0.94, 0.49, 0.3 \rangle$	$\langle 0.55, 0.1, 0.56 \rangle$
\tilde{g}_2	$\langle 0.97, 0.49, 0.05 \rangle$	$\langle 0.7, 0.6, 0.8 \rangle$	$\langle 0.99, 0.16, 0.36 \rangle$	$\langle 0.75, 0.02, 0.63 \rangle$
\tilde{g}_3	$\langle 0.72, 0.12, 0.14 \rangle$	$\langle 0.2, 0.2, 0.2 \rangle$	$\langle 0.96, 0.25, 0.1 \rangle$	$\langle 0.82, 0.1, 0.06 \rangle$
\tilde{g}_4	$\langle 0.98, 0.42, 0.02 \rangle$	$\langle 0.3, 0.5, 0.2 \rangle$	$\langle 0.55, 0.28, 0.1 \rangle$	$\langle 0.64, 0.04, 0.64 \rangle$

Step 3: According to the decision matrix \tilde{D} , we can get the ideal alternative g^* :

$$g^* = \{ \langle 0.98, 0.12, 0.02 \rangle \langle 0.7, 0.2, 0.2 \rangle \langle 0.99, 0.16, 0.1 \rangle \langle 0.82, 0.02, 0.06 \rangle \}. \quad (11)$$

Step 4: By applying the Equation (10), we can obtain the values of the weighted cosine measure between each alternative and the ideal alternative g^* as follows:

$$\begin{aligned} \rho_w(g_1, g^*) &= 0.9535, \rho_w(g_2, g^*) = 0.9511, \\ \rho_w(g_3, g^*) &= 0.9813 \text{ and } \rho_w(g_4, g^*) = 0.9616. \end{aligned} \quad (12)$$

Step 5: According to the above values of weighted cosine measure, we can rank the four alternatives: $g_3 \succ g_4 \succ g_1 \succ g_2$. Therefore, the alternative g_3 is the best choice.

This example clearly indicates that the proposed decision-making method based on the weighted cosine measure of SVNMs is relatively simple and easy for dealing with multiple attribute decision-making problems under SVN environment.

6.2. Comparative Analysis

In what follows, we compare the proposed method for SVN with other existing related methods for SVN; all the results are shown in Table 5.

Table 5. The ranking orders by utilizing four different methods.

Method	Result	Ranking Order	The Best Alternative
Method 1 based on correlation coefficient in [11]	$\rho_w(g_1, g^*) = 0.9053,$ $\rho_w(g_2, g^*) = 0.9017,$ $\rho_w(g_3, g^*) = 0.9516,$ $\rho_w(g_4, g^*) = 0.8816.$	$g_3 \succ g_1 \succ g_2 \succ g_4$	g_3
Method 2 based on similarity in [16]	$\rho_w(g_1, g^*) = 0.8204,$ $\rho_w(g_2, g^*) = 0.8108,$ $\rho_w(g_3, g^*) = 0.8867,$ $\rho_w(g_4, g^*) = 0.8358.$	$g_3 \succ g_4 \succ g_2 \succ g_1$	g_3
Method 3 based on similarity in [16]	$\rho_w(g_1, g^*) = 0.7898,$ $\rho_w(g_2, g^*) = 0.7121,$ $\rho_w(g_3, g^*) = 0.8125,$ $\rho_w(g_4, g^*) = 0.7553.$	$g_3 \succ g_1 \succ g_4 \succ g_2$	g_3
The proposed method	$\rho_w(g_1, g^*) = 0.9535,$ $\rho_w(g_2, g^*) = 0.9511,$ $\rho_w(g_3, g^*) = 0.9813,$ $\rho_w(g_4, g^*) = 0.9616.$	$g_3 \succ g_4 \succ g_1 \succ g_2$	g_3

From Table 5, these four methods have the same best alternative g_3 . Many methods such as similarity measure, correlation coefficient, and cosine measure can all be used in SVN to handle the multiple attribute decision-making problems and can get the similar results.

The proposed decision-making method can express and handle the multiplicity evaluated data given by decision makers or experts, while various existing neutrosophic decision-making methods cannot deal with these problems.

7. Conclusions

Based on the multiplicity evaluation in some real situations, this paper introduced a SVN as a subclass of NM to express the multiplicity information and the operational relations of SVN. The SVN is expressed by its one or more elements, which may have multiplicity. Therefore, SVN has the desirable advantages and characteristics of expressing and handling the multiplicity problems, while existing neutrosophic sets cannot deal with them.

Then, we proposed the cosine measure of SVNMs and weighted cosine measure of SVNMs and investigated their properties. Based on the weighted cosine measure of SVNMs, the multiple attribute decision-making methods under SVNm environments was proposed, in which the evaluated values were taken the form of SVNMEs. Through the weighed cosine measure between each alternative and the ideal alternative, one can determine the ranking order of all alternatives and can select the best one. Finally, a practical example adapted from the literature [21] about buying cars was presented to demonstrate the effectiveness and practicality of the proposed method in this paper. According to the ranking orders, we can find that the ranking result with weighted cosine measures is agreement with the ranking results in literature [21]. Then, the proposed method is suitable for actual applications in multiple attribute decision-making problems with single-value neutrosophic multiplicity information.

In the future, we shall extend SVNMs to interval neutrosophic multisets and develop the application of interval neutrosophic multisets for handling the decision-making methods or other domains.

Author Contributions: C.F. originally proposed the LNNNWBM and LNNNWGBM operators and investigated their properties; J.Y. and E.F. provided the calculation and comparative analysis; all authors wrote the paper together.

Funding: This research was funded by the National Natural Science Foundation of China grant number [61703280] and Science and Technology Planning Project of Shaoxing City of China grant number [2017B70056].

Acknowledgments: This work was supported by the National Natural Science Foundation of China under grant Nos. 61703280, and Science and Technology Planning Project of Shaoxing City of China (No. 2017B70056).

Conflicts of Interest: The author declares no conflict of interest.

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