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Symmetric Identities for Fubini Polynomials

Taekyun Kim [1,2], Dae San Kim [3], Gwan-Woo Jang [2] and Jongkyum Kwon [4,*]

1 Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China; tkkim@kw.ac.kr or kwangwoonmath@hanmail.net
2 Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea; gwjang@kw.ac.kr
3 Department of Mathematics, Sogang University, Seoul 121-742, Korea; dskim@sogang.ac.kr
4 Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo 52828, Korea
* Correspondence: mathkjk26@gnu.ac.kr

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Abstract: We represent the generating function of w-torsion Fubini polynomials by means of a fermionic p-adic integral on $\mathbb{Z}_p$. Then we investigate a quotient of such p-adic integrals on $\mathbb{Z}_p$, representing generating functions of three w-torsion Fubini polynomials and derive some new symmetric identities for the w-torsion Fubini and two variable w-torsion Fubini polynomials.

Keywords: Fubini polynomials; w-torsion Fubini polynomials; fermionic p-adic integrals; symmetric identities

1. Introduction and Preliminaries

In recent years, various p-adic integrals on $\mathbb{Z}_p$ have been used in order to find many interesting symmetric identities related to some special polynomials and numbers. The relevant p-adic integrals are the Volkenborn, fermionic, q-Volkenborn, and q-fermionic integrals of which the last three were discovered by the first author T. Kim (see [1–3]). They have been used by a good number of researchers in various contexts and especially in unfolding new interesting symmetric identities. This verifies the usefulness of such p-adic integrals. Moreover, we can expect that people will find some further applications of these p-adic integrals in the years to come. The present paper is an effort in this direction. Assume that p is any fixed odd prime number. Throughout our discussion, we will use the standard notations $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Assume that $f(x)$ is a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral of $f(x)$ on $\mathbb{Z}_p$ was introduced by Kim (see [2]) as

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1}$$

where $\mu_{-1}(x+p^N\mathbb{Z}_p) = (-1)^x$.

We can easily deduce from (1) that (see [2,3])

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{2}$$

By invoking (2), we easily get (see [2,4])

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{3}$$

where $E_n(x)$ are the usual Euler polynomials.

As is known, the two variable Fubini polynomials are defined by means of the following (see [5,6])

$$\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} = \frac{1}{1-y(e^t-1)}e^{xt}. \tag{4}$$

When $x = 0$, $F_n(y) = F_n(0,y)$, $(n \geq 0)$, are called Fubini polynomials. Further, if $y = 1$, then $Ob_n = F_n(0,1)$ are the ordered Bell numbers (also called Frobenius numbers). They first appeared in Cayley's work on a combinatorial counting problem in 1859 and have many different combinatorial interpretations. For example, the ordered Bell numbers count the possible outcomes of a multi-candidate election. From (3) and (4), we note that $F_n(x,-1/2) = E_n(x)$, $(n \geq 0)$. By (4), we easily get (see [6]),

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \ (n \geq 0), \tag{5}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind.

For $w \in \mathbb{N}$, we define the two variable w-torsion Fubini polynomials given by

$$\frac{1}{1-y^w(e^t-1)^w}e^{xt} = \sum_{n=0}^{\infty} F_{n,w}(x,y)\frac{t^n}{n!}. \tag{6}$$

In particular, for $x = 0$, $F_{n,w}(y) = F_{n,w}(0,y)$ are called the w-torsion Fubini polynomials. It is obvious that $F_{n,1}(x,y) = F_n(x,y)$.

We represent the generating function of w-torsion Fubini polynomials by means of a fermionic p-adic integral on $\mathbb{Z}_p$. Then we investigate a quotient of such p-adic integrals on $\mathbb{Z}_p$, representing generating functions of three w-torsion Fubini polynomials and derive some new symmetric identities for the w-torsion Fubini and two variable w-torsion Fubini polynomials. Recently, a number of researchers have studied symmetric identities for some special polynomials. The reader may refer to [7–11] as an introduction to this active area of research. Some symmetric identities for q-special polynomials and numbers were treated in [12–15], including q-Bernoulli, q-Euler, and q-Genocchi numbers and polynomials. While some identities of symmetry for degenerate special polynomials were discussed in the more recent papers [6,16,17]. Finally, interested readers may want to have a glance at [18,19] as general references on polynomials.

2. Symmetric Identities for w-torsion Fubini and Two Variable w-torsion Fubini Polynomials

From (2), we note that

$$\int_{\mathbb{Z}_p} (-1)^x (y(e^t-1))^x d\mu_{-1}(x) = \frac{2}{1-y(e^t-1)} = 2\sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \tag{7}$$

and

$$e^{xt}\int_{\mathbb{Z}_p} (-1)^z (y(e^t-1))^z d\mu_{-1}(z) = \frac{2}{1-y(e^t-1)}e^{xt} = 2\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}. \tag{8}$$

From (7) and (8), we note that

$$\begin{aligned}\left(\sum_{l=0}^{\infty} x^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} 2F_m(y)\frac{t^m}{m!}\right) &= e^{xt}\int_{\mathbb{Z}_p}(-1)^z(y(e^t-1))^z d\mu_{-1}(z) \\ &= \sum_{n=0}^{\infty} 2F_n(x,y)\frac{t^n}{n!}.\end{aligned} \tag{9}$$

Thus, by (9), we easily get

$$\sum_{l=0}^{n}\binom{n}{l}x^l F_{n-l}(y) = F_n(x,y),\ (n \geq 0). \tag{10}$$

Now, we observe that

$$\begin{aligned}\frac{1-y^k(e^t-1)^k}{1-y(e^t-1)} &= \sum_{i=0}^{k-1} y^i(e^t-1)^i = \sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}y^i e^{lt} \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}y^i l^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k-1} y^i \Delta^i 0^n\right)\frac{t^n}{n!},\end{aligned} \tag{11}$$

where $\Delta f(x) = f(x+1) - f(x)$.

For $w \in \mathbb{N}$, the w-torsion Fubini polynomials are represented by means of the following fermionic p-adic integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p}(-y^w(e^t-1)^w)^x d\mu_{-1}(x) = \frac{2}{1-y^w(e^t-1)^w} = \sum_{n=0}^{\infty} 2F_{n,w}(y)\frac{t^n}{n!}, \tag{12}$$

From (7) and (12), we have

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y(e^t-1))^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1}(e^t-1)^{w_1}}{1-y(e^t-1)} = \sum_{i=0}^{w_1-1} y^i(e^t-1)^i \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{w_1-1} y^i\Delta^i 0^n\right)\frac{t^n}{n!},\ (w_1 \in \mathbb{N}).\end{aligned} \tag{13}$$

For $w_1, w_2 \in \mathbb{N}$, we let

$$I = \frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^{x_1} d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^{x_2} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}. \tag{14}$$

Here it is important to observe that (14) has the built-in symmetry. Namely, it is invariant under the interchange of w_1 and w_2.

Then, by (14), we get

$$I = \left(\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)\right) \times \left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right). \tag{15}$$

First, we observe that

$$
\begin{aligned}
\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_2}(e^t-1)^{w_2}} = \sum_{i=0}^{w_1-1} y^{w_2 i}(e^t-1)^{w_2 i} \\
&= \sum_{i=0}^{w_1-1} y^{w_2 i} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} (-1)^{w_2 i-l} e^{lt} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_1-1} y^{w_2 i} \Delta^{w_2 i} 0^n \right) \frac{t^n}{n!}.
\end{aligned} \tag{16}
$$

From (15) and (16), we can derive the following equation.

$$
\begin{aligned}
I &= \left(\int_{\mathbb{Z}_p} (-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\sum_{m=0}^{\infty} 2F_{m,w_1}(y) \frac{t^m}{m!} \right) \times \left(\sum_{k=0}^{\infty} (\sum_{i=0}^{w_1-1} y^{w_2 i} \Delta^{w_2 i} 0^k \right) \frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \left(2 \sum_{k=0}^{n} \sum_{i=0}^{w_1-1} y^{w_2 i} \Delta^{w_2 i} 0^k F_{n-k,w_1}(y) \binom{n}{k} \right) \frac{t^n}{n!}.
\end{aligned} \tag{17}
$$

Interchanging the roles of w_1 and w_2, by (14), we get

$$
I = \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} \right). \tag{18}
$$

We note that

$$
\begin{aligned}
\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_1}(e^t-1)^{w_1}} = \sum_{i=0}^{w_2-1} y^{w_1 i}(e^t-1)^{w_1 i} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_2-1} y^{w_1 i} \Delta^{w_1 i} 0^n \right) \frac{t^n}{n!}.
\end{aligned} \tag{19}
$$

Thus, by (18) and (19), we get

$$
\begin{aligned}
I &= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\sum_{m=0}^{\infty} 2F_{m,w_2}(y) \frac{t^m}{m!} \right) \times \left(\sum_{k=0}^{\infty} (\sum_{i=0}^{w_2-1} y^{w_1 i} \Delta^{w_1 i} 0^k \right) \frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \left(2 \sum_{k=0}^{n} \sum_{i=0}^{w_2-1} y^{w_1 i} \Delta^{w_1 i} 0^k F_{n-k,w_2}(y) \binom{n}{k} \right) \frac{t^n}{n!}.
\end{aligned} \tag{20}
$$

The following theorem is now obtained by Equations (17) and (20).

Theorem 1. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $n \geq 0$, we have*

$$
\sum_{k=0}^{n} \sum_{i=0}^{w_1-1} \binom{n}{k} F_{n-k,w_1}(y) y^{w_2 i} \Delta^{w_2 i} 0^k = \sum_{k=0}^{n} \sum_{i=0}^{w_2-1} \binom{n}{k} F_{n-k,w_2}(y) y^{w_1 i} \Delta^{w_1 i} 0^k. \tag{21}
$$

Remark 1. *In particular, for $w_1 = 1$, we have*

$$F_n(y) = \sum_{k=0}^{n} \sum_{i=0}^{w_2-1} \binom{n}{k} F_{n-k,w_2}(y) y^i \Delta^i 0^k. \tag{22}$$

By expressing I in a different way, we have

$$\begin{aligned}
I &= \left(\int_{\mathbb{Z}_p} (-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1 w_2}(e^t-1)^{w_1 w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\int_{\mathbb{Z}_p} (-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x) \right) \times \left(\frac{1-y^{w_1 w_2}(e^t-1)^{w_1 w_2}}{1-y^{w_2}(e^t-1)^{w_2}} \right) \\
&= \left(\sum_{i=0}^{w_1-1} y^{w_2 i}(e^t-1)^{w_2 i} \right) \times \left(\frac{2}{1-y^{w_1}(e^t-1)^{w_1}} \right) \\
&= \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} y^{w_2 i}(-1)^l \frac{2}{1-y^{w_1}(e^t-1)^{w_1}} e^{(w_2 i - l)t} \\
&= 2 \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} y^{w_2 i}(-1)^l F_{n,w_1}(w_2 i - l, y) \right) \frac{t^n}{n!}.
\end{aligned} \tag{23}$$

Interchanging the roles of w_1 and w_2, by (14), we get

$$\begin{aligned}
I &= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1 w_2}(e^t-1)^{w_1 w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{1-y^{w_1 w_2}(e^t-1)^{w_1 w_2}}{1-y^{w_1}(e^t-1)^{w_1}} \right) \\
&= \left(\sum_{i=0}^{w_2-1} y^{w_1 i}(e^t-1)^{w_1 i} \right) \times \left(\frac{2}{1-y^{w_2}(e^t-1)^{w_2}} \right) \\
&= \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i} \binom{w_1 i}{l} (-1)^l \frac{2}{1-y^{w_2}(e^t-1)^{w_2}} e^{(w_1 i - l)t} \\
&= 2 \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i} \binom{w_1 i}{l} (-1)^l F_{n,w_2}(w_1 i - l, y) \right) \frac{t^n}{n!}.
\end{aligned} \tag{24}$$

Hence, by Equations (23) and (24), we obtain the following theorem.

Theorem 2. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1$ (mod 2), $w_2 \equiv 1$ (mod 2), $n \geq 0$, we have*

$$\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} y^{w_2 i} \binom{w_2 i}{l} (-1)^l F_{n,w_1}(w_2 i - l, y) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i} \binom{w_1 i}{l} (-1)^l F_{n,w_2}(w_1 i - l, y). \tag{25}$$

Remark 2. *Especially, if we take $w_1 = 1$, then by Theorem 2, we get*

$$F_n(y) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{i} \binom{i}{l} y^i (-1)^l F_{n,w_2}(i-l, y). \tag{26}$$

3. Conclusions

In this paper, we introduced w-torsion Fubini polynomials as a generalization of Fubini polynomials and expressed the generating function of w-torsion Fubini polynomials by means of a fermionic p-adic integral on $\mathbb{Z}_p$. Then we derived some new symmetric identities for the w-torsion Fubini and two variable w-torsion Fubini polynomials by investigating a quotient of such p-adic integrals on $\mathbb{Z}_p$, representing generating functions of three w-torsion Fubini polynomials. It seems that they are the first double symmetric identities on Fubini polynomials. As was done, for example in [4,20,21], we expect that this result can be extended to the case of triple symmetric identities. That is one of our next projects.

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