symmetry

MDPI

Article

Symmetries of Differential Equations in Cosmology

Michael Tsamparlis [1,*,†] and Andronikos Paliathanasis [2,3,4,†]

1 Faculty of Physics, Department of Astronomy-Astrophysics-Mechanics, University of Athens, Panepistemiopolis, 157 83 Athens, Greece
2 Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia 5090000, Chile; anpaliat@phys.uoa.gr
3 Department of Mathematics and Natural Sciences, Core Curriculum Program, Prince Mohammad Bin Fahd University, Al Khobar 31952, Saudi Arabia
4 Institute of Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, South Africa
* Correspondence: mtsampa@phys.uoa.gr
† These authors contributed equally to this work.

Received: 15 May 2018; Accepted: 15 June 2018; Published: 21 June 2018

Abstract: The purpose of the current article is to present a brief albeit accurate presentation of the main tools used in the study of symmetries of Lagrange equations for holonomic systems and subsequently to show how these tools are applied in the major models of modern cosmology in order to derive exact solutions and deal with the problem of dark matter/energy. The key role in this approach are the first integrals of the field equations. We start with the Lie point symmetries and the first integrals defined by them, that is, the Hojman integrals. Subsequently, we discuss the Noether point symmetries and the well-known method for deriving the Noether integrals. By means of the Inverse Noether Theorem, we show that, to every Hojman quadratic first integral, it is possible to associate a Noether symmetry whose Noether integral is the original Hojman integral. It is emphasized that the point transformation generating this Noether symmetry need not coincide with the point transformation defining the Lie symmetry which produces the Hojman integral. We discuss the close connection between the Lie point and the Noether point symmetries with the collineations of the metric defined by the kinetic energy of the Lagrangian. In particular, the generators of Noether point symmetries are elements of the homothetic algebra of that metric. The key point in the current study of cosmological models is the introduction of the mini superspace, which is the space that is defined by the physical variables of the model, which is not the spacetime where the model evolves. The metric in the mini superspace is found from the kinematic part of the Lagrangian and we call it the kinetic metric. The rest part of the Lagrangian is the effective potential. We consider coordinate transformations of the original mini superspace metric in order to bring it to a form where we know its collineations, that is, the Killing vectors, the homothetic vector, etc. Then, we write the field equations of the cosmological model and we use the connection of these equations with the collineations of the mini superspace metric to compute the first integrals and subsequently to obtain analytic solutions for various allowable potentials and finally draw conclusions about the problem of dark energy. We consider the ΛCDM cosmological model, the scalar field cosmology, the Brans–Dicke cosmology, the $f(R)$ gravity, the two scalar fields cosmology with interacting scalar fields and the Galilean cosmology. In each case, we present the relevant results in the form of tables for easy reference. Finally, we discuss briefly the higher order symmetries (the contact symmetries) and show how they are applied in the cases of scalar field cosmology and in the $f(R)$ gravity.

Keywords: Lie symmetries; Noether symmetries; dynamical systems; integrability; conservation laws; invariants; dark energy; modified theories of gravity; cosmology

1. Introduction

In order to understand properly the role of symmetries in Cosmology, we have to make a short detour into General Relativity and the relativistic models in general. General Relativity associates the gravitational field with the geometry of spacetime as this is specified by the metric of the Riemannian structure. Concerning the matter, this is described in terms of various dynamical fields which are related to the mater via Einstein equations $G_{ab} = T_{ab}$. Einstein equations are not equations, in the sense that they equate known quantities in terms of unknown ones, that is there is no point to look for a "solution" of them in this form. These equations are rather generators of equations which result after one introduces certain assumptions according to the model required. These assumptions are of two kinds: (a) geometric assumptions; and (b) other non-geometric assumptions among the physical fields which we call equations of state. The first specifies the metric to a certain degree and are called collineations [1] (or "symmetries" which is in common use and is possible to create confusion in our discussion of symmetries in Cosmology). The collineations are the familiar Killing vectors (KV), the conformal Killing vectors (CKV), etc. The collineations affect the Einstein tensor, which is expressed in terms of the metric. For example, for a KV X, one has $L_X g_{ab} = L_X G_{ab} = 0$.

Through Einstein field equations collineations pass over to T_{ab} and restrict its possible forms, therefore the types of matter that can be described in the given geometry-model. For example, if one considers the symmetries of the Friedmann–Robertson–Walker (FRW) model, which we shall discussed in the following, then the collineations restrict the T_{ab} to be of the form

$$T_{ab} = \rho u_a u_b + p h_{ab}, \tag{1}$$

where ρ, p are two dynamical scalar fields the density and the isotropic pressure, u^a are the comoving observers u^a $(u^a u_a = -1)$ and $h_{ab} = g_{ab} + u_a u_b$ is the spatial projection operator. An equation of state is a relation between the dynamical variables ρ, p.

Once one specifies the metric by the considered collineations (and perhaps some additional requirements of geometric nature) of the model and consequently the dynamical fields in the energy momentum tensor, then Einstein equations provide a set of differential equations which describe the defined relativistic model. What remains is the solution of these equations and the consequent determination of the physics of the model. At this point, one introduces the equations of state which simplify further the resulting field equations.

When one has the final form of the field equations considers a second use of the concept of "symmetry" which is the main objective of the current work. Let us refer briefly some history.

In the late 19th century, Sophus Lie, in a series of works [2–4] with the title "Theory of transformation groups" introduced a new method for the solution for differential equations via the concept of "symmetry". In particular, Lie defined the concept of symmetry of a differential equation by the requirement of the point transformation to leave invariant the set of solution curves of the equation, that is, under the action of the transformation, a point form one solution curve is mapped to a point on another solution curve. Subsequently, Lie introduced a simple algebraic algorithm for the determination of these types of symmetries and consequently on the solution of differential equations. Since then, symmetries of differential equations is one of the main methods which is used for the determination of solutions for differential equations. Some important works which established the importance of symmetries in the scientific society are those of Ovsiannikov [5], Bluman and Kumei [6], Ibragimov [7], Olver [8], Crampin [9], Kalotas [10] and many others; for instance, see [11–20].

As it is well known, an important fact in the solution of a differential equation are the first integrals. Inspired by the work of Lie in the early years of the 20th century, Emmy Noether required another definition of symmetry, which is known as Noether symmetry. This symmetry concerns Lagrangian dynamical systems and it is defined by the requirement that the action integral under the action of the point transformation changes up to a total derivative so that the Lagrange equation(s) remain the same [21]. She established a connection between a Noether symmetry and the existence of a first integral which she expressed by a simple mathematical formula. In addition, Noether's work except for its simplicity had a second novelty by alloying the continuous transformation to depend also on the derivatives of the dependent function, which was the first generalization of the context of symmetry from point transformations to higher-order transformations. Since then, symmetries play an important role in various theories of physics, from analytical mechanics [22], to particle physics [23,24], and gravitational physics [1,25].

In the following, we shall present briefly the approach of Lie and Noether symmetries and will show how the symmetries of differential equations are related to the collineations of the metric in the mini superspace. We shall develop an algorithm which indicates how one should work in order to get the analytical solution of a cosmological model. The various examples will demonstrate the application of this algorithm.

In conclusion, by symmetries in Cosmology, we mean the work of Lie and Noether applied to the solution of the field equations of a given cosmological model scenario.

2. Point Transformations

On a manifold M with coordinates (t, q^a), one defines the jet space $J^m(M)$ of order m over M to be a manifold with coordinates $t, q^a, \frac{dq^a}{dt}, ..., \frac{d^m q^a}{dt^m}$. Let $X = \xi\left(t, q, ..., q^{[m]}\right)\frac{\partial}{\partial t} + \eta^{a[A]}\left(t, q, ..., q^{[m]}\right)\frac{\partial^{[A]}}{\partial q^{a[A]}}$ where $A = 1, ..., m$ be a vector field on $J^m(M)$ which generates the infinitesimal point transformation

$$\begin{aligned} t' &= t + \varepsilon\xi + O^2\left(\varepsilon^2\right) + \cdots, \\ q^{a'} &= q^a + \varepsilon\eta^{a[1]} + O^2\left(\varepsilon^2\right) + \cdots, \\ &\;\;... \\ q^{a'[m]} &= q^{a[m]} + \varepsilon\eta^{a[m]} + O\left(\varepsilon^2\right) + \cdots. \end{aligned} \tag{2}$$

Well behaved infinitesimal point transformations form a group under the operation of composition of transformations. If the infinitesimal point transformation depends on many parameters, that is,

$$q'^i = q'^i\left(q^i, q^{i(m)}, \mathbf{E}\right), \tag{3}$$

where $\mathbf{E} = \varepsilon^\beta\partial_\beta$, is a vector field in $\mathbb{R}^\kappa$, $\beta = 1...\kappa$ with the same properties as in the case of the one parameter infinitesimal point transformations, then the point transformation is called a multi-parameter point transformation. These transformations are generated by $\kappa-$vector fields which form a (finite or infinite dimensional) Lie algebra. That is, if the vector fields X, Y are generators of a multi-parameter point transformation, so is the commutator $Z = [X, Y]$.

2.1. Prolongation of Point Transformations

A differential equation $H\left(t, q, q', ..., q^{(m)}\right) = 0$ where $q^i(t)$ and $q^{i(m)} = \frac{d^m q^i}{dt^m}$ may be considered as a function on the jet space $J^m(M)$. In order to study the effect of a point transformation in the base manifold $M(t, q)$ to the differential equation, one has to prolong the transformation to the space $J^m(M)$. To do that, we consider in $J^m(M)$ the induced point transformation

$$\begin{gathered} \bar{t} = t + \varepsilon\xi\,,\; \bar{y} = x + \varepsilon\eta, \\ \bar{q}^{i(1)} = q^{i(1)} + \varepsilon X^{[1]}\,,\; ...., \\ ... \\ \bar{q}^{i(n)} = q^{i(n)} + \varepsilon X^{[n]}, \end{gathered}$$

where $X^{[k]}$ $k = 1, 2, ..., m$ are the components of a vector field

$$W = \xi\partial_x + \eta\partial_y + X^{[1]}\partial_{y^{(1)}} + ... + X^{[m]}\partial_{y^{[m]}}, \tag{4}$$

in $J^m(M)$ called the lift of the vector field $X = \xi(t,q)\frac{\partial}{\partial x^i} + \eta^i(t,q)\frac{\partial}{\partial q^i}$ of M.

There are many ways to lift a vector field from the base manifold to a vector field in the bundle space $J^m(M)$ depending on the geometric properties one wants to preserve.

One type of lift, the complete lift or prolongation, is defined by the requirement that the tangent to the vector field X in M goes over to the tangent of the vector field W in $J^m(M)$ at the corresponding lifted point. Equivalently, one may define the prolongation by the requirement that under the action of the point transformation the variation of the variables equals the difference of the derivatives before and after the action of the one parameter transformation. For example, for $\eta^{[1]}$, we have

$$\eta^{[1]} \equiv \lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon}\left(\bar{q}^{(1)} - q^{(1)}\right)\right] = \frac{d\eta}{dt} - q^{i(1)}\frac{d\xi}{dq^i}. \tag{5}$$

For the $k-th$ prolongation, $\eta^{[k]}$ follows from the recursive formula:

$$\eta^{[k]}(t,q^i,q^{i(1)},...,q^{i(m)}) = \frac{d\eta^{k-1}}{dt} - q^{i(k)}\frac{d\xi}{dq^i} = \frac{d^k}{dq^{ik}}\left(\eta - q^{i(1)}\xi\right) + q^{i(k+1)}\xi. \tag{6}$$

Two important observations for the prolongation $\eta^{[n]}$ are, (a) $\eta^{[n]}$ is linear in $q^{i(n)}$, and (b) $\eta^{[n]}$ is a polynomial in the derivatives $q^{i(1)},...,q^{i(n)}$ whose coefficients are linear homogeneous in the functions $\xi(x,y), \eta(x,y)$ up to nth order partial derivatives.

Concerning the general vector W on $J^m(M)$, one defines its components by the requirement

$$X^{[m]}(t,q^i,q^{i(1)},...,q^{i(m)}) = \eta^{[k]}(t,q^i,q^{i(1)},...,q^{i(m)}) + \phi^m,$$

where $\phi^m(t,q^i,q^{i(1)},...,q^{i(m)})$ are some functions on $J^m(M)$ that will be defined by additional requirements. For example, the complete lift (prolongation) is defined by the requirement $\phi^i = 0$.

In the case that we have a manifold with n independent variables $\{x^i : i = 1...,n\}$ and m dependent variables $\{u^A : A = 1...,m\}$, we consider the one parameter point transformation in the jet space $\{x^i, u^A\}$ of the form

$$\bar{x}^i = \Xi^i\left(x^i, u^A, \varepsilon\right) \ , \ \bar{u}^A = \Phi^A\left(x^i, u^A, \varepsilon\right).$$

The infinitesimal generator is

$$X = \xi^i\left(x^k, u^A\right)\partial_i + \eta^A\left(x^k, u^A\right)\partial_A, \tag{7}$$

where

$$\xi^i\left(x^k, u^A\right) = \frac{\partial \Xi^i\left(x^i, u^A, \varepsilon\right)}{\partial \varepsilon}|_{\varepsilon\to 0} \ , \ \eta^A\left(x^k, u^A\right) = \frac{\partial \Phi\left(x^i, u^A, \varepsilon\right)}{\partial \varepsilon}|_{\varepsilon \to 0}.$$

In a similar way, the prolongation vector is calculated to be

$$X^{[n]} = X + \eta_i^A \partial_{u_i} + ... + \eta^A_{ij..i_n}\partial_{u_{ij..i_n}},$$

where now

$$\eta_i^A = D_i \eta^A - u^A_{,j} D_i \xi^j, \tag{8}$$

$$\eta^A_{ij..i_n} = D_{i_n}\eta^A_{ij..i_{n-1}} - u_{ij..k} D_{i_n}\xi^k, \tag{9}$$

and the operator D_i is defined as follows:

$$D_i = \frac{\partial}{\partial x^i} + u_i^A \frac{\partial}{\partial u^A} + u^A_{ij}\frac{\partial}{\partial u^A_j} + ... + u^A_{ij..i_n}\frac{\partial}{\partial u^A_{jk..i_n}}. \tag{10}$$

In terms of the partial derivatives of the components $\xi^i\left(x^k, u^A\right)$, $\eta^A\left(x^k, u^A\right)$ of the generator (7), the first and the second extension of the symmetry vector are given by the relations

$$X^{[1]} = X + \left(\eta^A_{,i} + u^B_{,i}\eta^A_{,B} - \xi^j_{,i}u^A_{,j} - u^A_{,i}u^B_{,j}\xi^j_{,B}\right)\partial_{u^A_i}, \tag{11}$$

$$X^{[2]} = X^{[1]} + \left[\begin{array}{c} \eta^A_{,ij} + 2\eta^A_{,B(i}u^B_{,j)} - \xi^k_{,ij}u^A_{,k} + \eta^A_{,BC}u^B_{,i}u^C_{,j} - 2\xi^k_{,(i|B|}u^B_{j)}u^A_{,k} + \\ -\xi^k_{,BC}u^B_{,i}u^A_{,j}u^A_{,k} + \eta^A_{,B}u^B_{,ij} - 2\xi^k_{,(j}u^A_{,i)k} + -\xi^k_{,B}\left(u^A_{,k}u^B_{,ij} + 2u^B_{(,j}u^A_{,i)k}\right) \end{array}\right]\partial_{u_{ij}}. \tag{12}$$

Similar expressions can be obtained for the general vector field in the jet space $\{x^i, u^A\}$.

2.2. *Invariance of Functions*

A differentiable function $F\left(q^i, q^{i(m)}\right)$ on $J^m(M)$ is said to be invariant under the action of X iff

$$X(F) = 0, \tag{13}$$

or, equivalently, iff there exists a function λ such that

$$X(F) = \lambda F\,, modF = 0. \tag{14}$$

In order to determine the invariant functions of a given infinitesimal point generator X one has to solve the associated Lagrange system

$$\frac{dt}{\xi} = \frac{dq^i}{\eta^i} = ... = \frac{dq^{i(m)}}{\eta^{i(m)}}. \tag{15}$$

The characteristic function or zero order invariant w of the vector X is defined as follows:

$$dw = \frac{dt}{\xi} - \frac{dq^i}{\eta^i}. \tag{16}$$

The zero order invariant is indeed invariant under the action of X, that is $X(w) = 0$. Therefore, any function of the form $F = F(w)$ satisfies condition (13) and it is invariant under the action of X. The higher order invariants $v_{(m)}$ are given by the expression

$$v_{(m)} = \frac{dt}{\xi} - \frac{dq^{i(m)}}{\eta^{i(m)}}. \tag{17}$$

3. Symmetries of Differential Equations

Consider an mth order system of differential equations defined on $J^m(M)$ of the form $q^{a[m]} = F(t, q, ..., q^{[m]})$. We say that the point transformation (2) in $J^m(M)$ generated by the vector field $X = \xi\frac{\partial}{\partial t} + \eta^{a[A]}\frac{\partial^{[A]}}{\partial q^{a[A]}}$ is a symmetry of the system of equations if it leaves the *set* of solutions of the system the same. Equivalently, if we consider the function $G = q^{a[m]} - F(t, q, ..., q^{[m]}) = 0$ on $J^m(M)$, then a symmetry of the differential equation is a vector field leaving G invariant.

The main reason for studying the symmetries of a system of differential equations is to find first integrals and/or invariant solutions. Both of these items facilitate the solution and the geometric/physical interpretation of the system of equations.

In the following, we shall be interested in systems of second order differential equations (SODE) of the form $\ddot{q}^a - K^a(t, q, \dot{q}) = 0$, therefore we shall work on the jet space $J^1(M)$ which is essentially the space $R \times TM$. In this case, the infinitesimal point transformation (2) is written

$$
\begin{array}{rcl}
t' & = & t + \varepsilon\xi + O^2\left(\varepsilon^2\right) + \cdots, \\
q^{i'} & = & q^i + \varepsilon\eta^i + O^2\left(\varepsilon^2\right) + \cdots, \\
\dot{q}^{i'} & = & \dot{q}^i + \varepsilon\left(\dot{\eta}^i - \dot{\xi}\dot{q}^i + \phi^i\right) + O\left(\varepsilon^2\right) + \cdots,
\end{array} \tag{18}
$$

and it is generated by the vector field

$$X^W = X^{[1]} + \phi^i \frac{\partial}{\partial \dot{q}^i}, \tag{19}$$

where $\phi^i(t, q, \dot{q})$ are some general (smooth) functions and $X^{[1]}$ is the prolonged vector field:

$$X^{[1]} = \xi(t, q, \dot{q})\frac{\partial}{\partial t} + \eta^a(t, q, \dot{q})\frac{\partial}{\partial q^a} + X^{[1]a}\frac{\partial}{\partial \dot{q}^i}, \tag{20}$$

where

$$X^{[1]a} = \frac{d\eta^a}{dt} - \dot{q}^a \frac{d\xi}{dt}. \tag{21}$$

If $\xi(t, q), \eta^i(t, q)$, then $X = \xi\frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a}$ is defined on the base manifold M and $X^{[1]}$ is called the first prolongation of X in TM.

4. The Conservative Holonomic Dynamical System

Consider the conservative holonomic system (CHS) with Lagrangian $L = \frac{1}{2}\gamma_{ij}\dot{q}^i\dot{q}^j - V(t, q)$, where $V(t, q)$ is the potential of all conservative forces, whose equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{22}$$

These are written

$$E^i(L) = 0, \tag{23}$$

where E^i is the Euler vector field in the jet space $J^1(M)$

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}.$$

Replacing the Lagrangian in Equation (22), we find

$$\ddot{q}^i = \omega^i, \tag{24}$$

where

$$\omega^i(t, q, \dot{q}) = -V^{,i} - \Gamma^i_{jk}\dot{q}^j\dot{q}^k. \tag{25}$$

The CHS defines two important geometric quantities in the jet space $J^1(M)$

a. The kinetic metric (We assume the Lagrangian to be regular, that is $\det \frac{\partial^2 L}{\partial q^i \partial q^j} \neq 0$ so that the kinetic metric is non-degenerate) $\gamma_{ij} = \frac{\partial^2 L}{\partial q^i \partial q^j}$, which is essentially the kinetic energy of the dynamical system. This metric is different from the metric of the space where motion occurs. It is a positive definite metric of dimension equal to the degrees of freedom of the dynamical system.

b. The Hamiltonian vector field Γ

$$\Gamma = \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial q^i}\dot{q}^i + \omega^i \frac{\partial}{\partial \dot{q}^i}. \tag{26}$$

The Hamiltonian vector field is characteristic of the dynamical Equation (24) and can be defined in all cases irrespective of the Lagrangian function.

In the following, we shall restrict our considerations to the symmetries of second order differential equations of the form (24).

5. Types of Symmetries

There are various types of multi parameter point transformations in $J^1(M)$ which generate a symmetry of a system of second order differential equations (SODE). In cosmology—at least at the current status—it appears that two of them are of importance, that is, the Lie symmetries and the Noether symmetries. Both type of symmetries lead to first integrals hence providing ways to solve the considered cosmological equations and shall be discussed in the following. In case the components ξ, η^a of the generators are functions of t, q^a, only the symmetries are called point symmetries.

The requirements for each type of symmetry lead to a set of conditions which when solved give the generators of the corresponding point transformation and consequently the way to determine first integrals. As it has been shown, the generators of these symmetries for autonomous conservative holonomic systems are related to the collineations of the kinetic metric. Specifically, it has been shown [26] that the Lie point symmetries are elements of the special projective algebra and the Noether point symmetries elements of the homothetic subalgebra [27]. Concerning the partial differential equations, the generators are related to the conformal group of the kinetic metric [28,29].

5.1. Lie Symmetries

Definition 1. *The vector field X^w on the jet space $J^1(M)$ is a (dynamical) Lie symmetry of the Equation (24) if it is a symmetry of the Hamiltonian vector field; that is, if the following condition is satisfied,*

$$L_{X^W}\Gamma = \lambda\Gamma, \tag{27}$$

where $\lambda(t, q, \dot{q})$ is a function to be defined.

Geometrically, a Lie symmetry preserves the set of solutions of the Equation (24), in the sense that, under the action of the point transformation generated by X^W, a point of a solution curve is transformed to a point of another solution curve of Equation (24).

The symmetry condition (27) gives:

$$\left[X^W, \Gamma\right] = -\Gamma(\xi)\frac{\partial}{\partial t} + \left(X^W(\dot{q}^a) - \Gamma(\eta^a)\right)\frac{\partial}{\partial q^a} + \left(X^W(F^a) - \Gamma(X^a + \phi^a)\right)\frac{\partial}{\partial \dot{q}^a} = \lambda\Gamma. \tag{28}$$

Condition (28) is equivalent to the following system of equations:

$$\begin{aligned} -\Gamma(\xi) &= \lambda, \\ \Gamma(\eta^a) - \Gamma(\xi)\dot{q}^a - \phi^a &= X^{[1]a}, \end{aligned} \tag{29}$$

$$\xi\frac{\partial\omega^a}{\partial t} + \eta^b\frac{\partial\omega^a}{\partial q^b} + \left(X^{[1]b} + \phi^b\right)\frac{\partial\omega^a}{\partial\dot{q}^b} + \Gamma(\xi)F^a = \Gamma\left(X^{[1]a} + \phi^a\right), \tag{30}$$

where $X^{[1]a}$ is given by (20). The second condition gives $\phi^a = 0$ so that $X^W = X^{[1]}$, where

$$X^{[1]} = \xi\left(t, q^k, \dot{q}^k\right)\partial_t + \eta^a\left(t, q^k, \dot{q}^k\right)\partial_a + \left[\Gamma(\eta^a) - \Gamma(\xi)\dot{q}^a\right]\frac{\partial}{\partial\dot{q}^a}.$$

Then, the third condition (30) becomes

$$\xi\frac{\partial\omega^a}{\partial t} + \eta^b\frac{\partial\omega^a}{\partial q^b} + \left(\Gamma(\eta^a) - \Gamma(\xi)\dot{q}^a\right)\frac{\partial\omega^a}{\partial\dot{q}^b} + \Gamma(\xi)\omega^a = \Gamma\left(\Gamma(\eta^a) - \Gamma(\xi)\dot{q}^a\right)$$

and can be written more compactly as

$$X^{[1]}(\omega^a) + \Gamma(\xi)\,\omega^a = \Gamma\left(\Gamma(\eta^a) - \Gamma(\xi)\,\dot{q}^a\right). \tag{31}$$

We note that the functions ϕ^a do not take part into the conditions of Lie symmetries and can be omitted. The exact form of the Lie symmetry conditions depends on the functional dependence of the functions $\xi(t,q,\dot{q}), \eta^a(t,q,\dot{q})$ and of course on the form of the Hamiltonian field. In the following, we consider two important cases of Lie symmetries.

5.2. Lie Point Symmetries

In case $\xi(t,q), \eta^a(t,q)$, the Lie symmetry is called a Lie point symmetry. For the special class of differential equations of the form

$$\ddot{q} + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + V^{,i}\left(t,q^k\right) = 0, \tag{32}$$

the Lie symmetry condition (30) leads to the following system of covariant conditions [30]

$$L_\eta V^{;a} + \eta^a_{,tt} + 2V^{;a}\xi_{,t} + \xi V^{;a}_{,t} = 0, \tag{33}$$

$$2\eta^a_{;b|t} - \delta^a_b\xi_{,tt} + \left(2\delta^a_c\xi_{;b} + \delta^a_b\xi_{;c}\right)V^{;c} = 0, \tag{34}$$

$$\mathcal{L}_\eta\Gamma^a_{bc} = 2\delta^a_{(b}\xi_{,c)t}, \tag{35}$$

$$\delta^a_{(d}\xi_{;bc)} = 0. \tag{36}$$

The use of an algebraic computing program does not reveal directly the Lie symmetry conditions in this geometric form. Equation (36) implies that $\xi_{,a}$ is a gradient KV of the kinetic metric. Equation (35) means that η^i is a special projective collineation of the metric with projective function $\xi_{,t}$. The remaining two Equations (33) and (34) are constraint conditions, which relate the components ξ, n^i of the Lie point symmetry vector with the potential function $V(t,q)$.

Conditions (33)–(36) can be obtained as special cases from known results. Indeed, in [30], it has been shown that the conditions for a point Lie symmetry of the dynamical equations

$$\ddot{q}^a + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + P^i\left(t,q^k\right) = 0 \tag{37}$$

are the following:

$$L_\eta P^i + 2\xi_{,t}P^i + \xi P^i{}_{,t} + \eta^i{}_{,tt} = 0, \tag{38}$$

$$\left(\xi_{,k}\delta^i_j + 2\xi_{,j}\delta^i_k\right)P^k + 2\eta^i{}_{,t|j} - \xi_{,tt}\delta^i_j = 0, \tag{39}$$

$$L_\eta\Gamma^i_{jk} - 2\xi_{,t(j}\delta^i_{k)} = 0, \tag{40}$$

$$\xi_{(,j|k}\delta^i_{d)} = 0. \tag{41}$$

In order for one to obtain conditions, (33)–(36) simply replaces in (38)–(41) $P^i = V^{,i}\left(t,q^k\right)$.

5.3. Lie Point Symmetries and First Integrals

It is possible that a Lie point symmetry leads to a first integral. These integrals have been called Hojman integrals and belong to the class of non-Noetherian first integrals [31]. As shall be discussed below by means of the Inverse Noether Theorem, one is able to associate a Noether symmetry to a given quadratic first integral. In this sense, as far as the first integrals are concerned, Noether symmetries are the prevailing ones. Concerning the Hojman symmetries, we have the following [31].

Proposition 1. *i. Necessary and sufficient condition that the point transformation* $q^{i'} = q^i + \varepsilon\eta^i\left(t, q^k, \dot{q}^k\right)$ *is a Lie point symmetry of the (SODE)* $\ddot{q}^j - \omega^j\left(t, q^i, \dot{q}^i\right) = 0$ *is that the generator* η^i *satisfies the conditions*

$$\begin{aligned} \Gamma(\Gamma\eta^i) - \eta^{[1]}\left(\omega^j\right) &= 0 \text{ or} \\ \ddot{\eta}^j - \eta^{[1]}(\omega^j) &= 0, \end{aligned} \tag{42}$$

where $\ddot{\eta}^j = \Gamma(\Gamma(\eta^j)$ *and* $\eta^{[1]} = \eta^i \frac{\partial}{\partial q^i} + \dot{\eta}^i \frac{\partial}{\partial \dot{q}^i}$.

ii. The scalar

$$I_2 = \frac{1}{\gamma}\frac{\partial}{\partial q^j}\left(\gamma\eta^j\right) + \frac{\partial\dot{\eta}^j}{\partial\dot{q}^j}$$

is a first integral of the ODE $\ddot{q}^i = \omega^i$ *iff*

a. The vector η^i *is a Lie symmetry of the SODE* $\ddot{q}^i = \omega^i$

b. The function $\gamma(t, q, \dot{q})$ *is defined by the condition*

$$Trace\left(\frac{\partial\omega^i}{\partial\dot{q}^j}\right) = \frac{\partial\omega^j}{\partial\dot{q}^j} = -\frac{d}{dt}\ln\gamma\left(q^k\right).$$

One solution for all η^i *is* $\omega^j = a\,\dot{q}^j + \omega^j(t, q^i)$*, where a =const. For this solution, we have that the SODE has the generic form:*

$$\ddot{q}^j - a\dot{q}^j = \omega^j(t, q^i),$$

which is the equation for forced motion with with linear dumping a.

5.4. Noether Point Symmetries

Noether point symmetries concern Lagrangian dynamical systems and are defined as follows:

Definition 2. *Suppose that* $A\left(q^i\dot{q}^i\right)$ *is the functional (the action integral)*

$$A\left(q^i, \dot{q}^i\right) = \int_{t_1}^{t_2} L\left(t, q^i, \dot{q}^i\right)dt. \tag{43}$$

The vector field $X^W = X^{[1]} + \phi^i \frac{\partial}{\partial \dot{q}^i}$ *in the jet space* $J^1(M)$ *generating the point transformation (18) is said to be a Noether symmetry of the dynamical system with Lagrangian L if*

a. The action integral under the action of the point transformation transforms as follows

$$A'\left(q^{i'}, \dot{q}^{i'}\right) = A\left(q^i, \dot{q}^i\right) + \epsilon\int_{t_1}^{t_2} \frac{df(t, q^i, \dot{q}^i)}{dt}dt, \tag{44}$$

where $f(t, \epsilon)$ *is a smooth function.*

b. The infinitesimal transformation causes a zero end point variation (i.e., the end points of the integral remain fixed).

Noether symmetries use the fact that when we add a perfect differential to a Lagrangian the equations of motion do not change. Hence, the set of solutions remains the same. This is another view of the "invariance" of the dynamical Equation (24).

The condition for a Noether symmetry under the action of the point transformation (18) is

$$X^{[1]}(L) + \phi^i \frac{\partial L}{\partial \dot{q}^i} + L\left(t, q^i, \dot{q}^i\right)\dot{\xi} = \dot{f}. \tag{45}$$

Equation (45) is called the weak Noether condition and it is also known as the First Noether theorem.

5.5. First Integral Defined by a Noether Symmetry

Next, we determine the conditions which a Noether symmetry must satisfy in order to lead to a first integral of Lagrange equations. Expanding the Noether condition (45) and making use of Lagrange Equation (24), we find

$$\phi^i \frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt}\left(f - L\xi - \frac{\partial L}{\partial \dot{q}^i}\left(\eta^i - \xi\dot{q}^i\right)\right). \tag{46}$$

This leads to what is known as the Second Noether Theorem.

Proposition 2. *The quantity*

$$I = f - L\xi - \frac{\partial L}{\partial \dot{q}^i}\left(\eta^i - \xi\dot{q}^i\right) \tag{47}$$

is a first integral for the conservative holonomic system defined by (22) provided the functions ϕ^i (i.e., the variations along the fibers) vanish. In this case, the weak Noether condition (45) becomes

$$X^{[1]}(L) + L\left(t, q^i, \dot{q}^i\right)\dot{\xi} = \dot{f} \tag{48}$$

and the Noether symmetry reduces to a special Lie symmetry (that is a Lie symmetry which in addition satisfies the Noether condition). The function f is called the Noether or the gauge function.

We remark that, contrary to what is generally believed, a Noether point symmetry for a conservative holonomic system does not lead necessarily to a first integral of the equations of motion. Indeed, the weak Noether symmetry condition (45) is more general than the standard Noether condition (48) because it holds for general ϕ^a, whereas the latter holds only for $\phi^a = 0$.

In the following, by a Noether symmetry, we shall mean a Noether symmetry, which leads to a first integral, that is, the quantities $\phi^a = 0$. These Noether symmetries are special Lie symmetries which satisfy condition (48). As has been already mentioned, the Lie point symmetries are the elements of the special projective collineations of the kinetic metric and the Lie point symmetries which are Noether point symmetries are elements of the homothetic subalgebra [27].

6. Generalized Killing Equations

We decompose the Noether condition along the vector $\frac{d}{dt}$ and normal to it. In order to do that, we expand the overdot terms and assume that $\ddot{q}^i$ are independent variables. The right hand side (rhs) is:

$$\begin{aligned}
& L\left(\frac{\partial \xi}{\partial t} + \dot{q}^i\frac{\partial \xi}{\partial q^i} + \ddot{q}^i\frac{\partial \xi}{\partial \dot{q}^i}\right) + \xi\frac{\partial L}{\partial t} + \eta^i\frac{\partial L}{\partial q^i} \\
& + \left(\frac{\partial \eta^i}{\partial t} + \dot{q}^j\frac{\partial \eta^i}{\partial q^j} + \ddot{q}^j\frac{\partial \eta^i}{\partial \dot{q}^j} - \dot{q}^i\frac{\partial \xi}{\partial t} - \dot{q}^i\dot{q}^j\frac{\partial \xi}{\partial q^j} - \dot{q}^i\ddot{q}^j\frac{\partial \xi}{\partial \dot{q}^j}\right)\frac{\partial L}{\partial \dot{q}^i} \\
= \; & L\left(\frac{\partial \xi}{\partial t} + \dot{q}^i\frac{\partial \xi}{\partial q^i}\right) + \xi\frac{\partial L}{\partial t} + \eta^i\frac{\partial L}{\partial q^i} + \left(\frac{\partial \eta^i}{\partial t} + \dot{q}^j\frac{\partial \eta^i}{\partial q^j} - \dot{q}^i\frac{\partial \xi}{\partial t} - \dot{q}^i\dot{q}^j\frac{\partial \xi}{\partial q^j}\right)\frac{\partial L}{\partial \dot{q}^i} \\
& + \ddot{q}^j\left(\frac{\partial \xi}{\partial \dot{q}^j}L + \left(\frac{\partial \eta^i}{\partial \dot{q}^j} - \dot{q}^i\frac{\partial \xi}{\partial \dot{q}^j}\right)\frac{\partial L}{\partial \dot{q}^i}\right),
\end{aligned}$$

while the rhs gives:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q}^i\frac{\partial f}{\partial q^i} + \ddot{q}^i\frac{\partial f}{\partial \dot{q}^i}.$$

Therefore, we obtain the following equivalent system of equations:

$$L\left(\frac{\partial\xi}{\partial t}+\dot{q}^{i}\frac{\partial\xi}{\partial q^{i}}\right)+\xi\frac{\partial L}{\partial t}+\eta^{i}\frac{\partial L}{\partial q^{i}}+\left(\frac{\partial\eta^{i}}{\partial t}+\dot{q}^{j}\frac{\partial\eta^{i}}{\partial q^{j}}-\dot{q}^{i}\frac{\partial\xi}{\partial t}-\dot{q}^{i}\dot{q}^{j}\frac{\partial\xi}{\partial q^{j}}\right)\frac{\partial L}{\partial\dot{q}^{i}} = \frac{\partial f}{\partial t}+\dot{q}^{i}\frac{\partial f}{\partial q^{i}}, \tag{49}$$

$$\frac{\partial\xi}{\partial\dot{q}^{j}}L+\left(\frac{\partial\eta^{i}}{\partial\dot{q}^{j}}-\dot{q}^{i}\frac{\partial\xi}{\partial\dot{q}^{j}}\right)\frac{\partial L}{\partial\dot{q}^{i}} = \frac{\partial f}{\partial\dot{q}^{j}}. \tag{50}$$

These equations have been called the generalized Killing equations (see Equations (17) and (18) of Djukic in [32]).

We note that the generalized Killing equations have $2n+2$ unknowns (the $\xi, \eta^{i}, f, \phi^{i}$) and are only $n+1$ equations. Therefore, there is not a unique solution X^{W} and we are free to fix $n+1$ variables in order to get a solution. However, this is not a problem because all these solutions admit the same first integral I of the dynamical equations (because all satisfy the Noether condition (48)).

6.1. How to Solve the Generalized Killing Equations

Suppose that, by some method, we have determined a quadratic first integral I of the dynamical equations. Our purpose is to determine a gauged Noether symmetry which will admit the given quadratic first integral. Assume the $n+1$ gauge conditions $\xi=0, \phi^{i}=0$. Suppose $L(t,q^{i},\dot{q}^{i})$ is the Lagrangian part of the dynamical equations (Actually, we only need the kinetic energy which will define the kinetic metric). Let $X^{W}=\eta^{i}(t,q^{i},\dot{q}^{i})\frac{\partial}{\partial q^{i}}+\dot{\eta}^{i}\frac{\partial}{\partial\dot{q}^{i}}$ be the vector field generating a Noether symmetry which admits the first integral $I=f-\left(\eta^{i}-\xi\dot{q}^{i}\right)\frac{\partial L}{\partial\dot{q}^{i}}-L\xi$, where $f(t,q^{i},\dot{q}^{i})$ is the Noether gauge function. We compute

$$\frac{\partial I}{\partial\dot{q}^{i}}=\frac{\partial f}{\partial\dot{q}^{i}}-\frac{\partial\eta^{j}}{\partial\dot{q}^{i}}\frac{\partial L}{\partial\dot{q}^{j}}+\gamma_{ij}\eta^{j},$$

where $\gamma_{ij}=\frac{\partial^{2}L}{\partial\dot{q}^{i}\partial\dot{q}^{j}}$ is the kinetic metric determined by the Lagrangian L. Then, the second Equation (50) gives that $\frac{\partial f}{\partial\dot{q}^{i}}-\frac{\partial\eta^{j}}{\partial\dot{q}^{i}}\frac{\partial L}{\partial\dot{q}^{j}}=0$ and we find eventually the expression

$$\eta^{i}=\gamma^{ij}\frac{\partial I}{\partial\dot{q}^{i}}. \tag{51}$$

The vector $X^{W}=\eta^{i}\frac{\partial}{\partial q^{i}}+\dot{\eta}^{i}\frac{\partial}{\partial\dot{q}^{i}}$ we have determined is not the only one possible. For example, one may specify the gauge function f and assume a form for $\xi(t,q)$ (while maintain the gauge $\phi^{i}=0$) and then use Equation (50) to determine the solution X^{W} (see example below). However, in all cases, the first integral I is the same.

The first integral of a Noether point symmetry $X^{W}=X^{[1]}$ is in addition an invariant of the Noether generator, that is,

$$\mathbf{X}^{[1]}(I)=0. \tag{52}$$

The result (52) means that a Noether point symmetry results in a twofold reduction of the order of Lagrange equations. This is done as follows. The first integral $I(t,q^{i},\dot{q}^{i})$ can be used to replace one of the second order equations by the first order ODE $I(t,q^{i},\dot{q}^{i})=I_{0}$, where I_{0} is a constant fixed by the initial (or boundary) conditions. Property (52) says that this new equation admits the Lie symmetry X^{W} (because $X^{W}(I-I_{0})=0$); therefore, it can be used to integrate the equation once more, according to well-known methods.

Noether symmetries are mainly applied to construct first integrals which are important to determine the solution of a given dynamical system. It is possible that there exist different (i.e., not differing by a perfect differential) Lagrangians describing the same dynamical equations. These Lagrangians have different Noether symmetries (see, for example, [33–35]). Therefore, it is clear that, when we refer to a Noether symmetry of a given dynamical system, we should always mention the Lagrangian function assumed.

Let us discuss the above scenario in a practical case.

Consider the Emden–Fauler equation

$$t\ddot{q} + 2\dot{q} + at^{\nu}q^{2\nu+3} = 0,$$

where a, ν are arbitrary constants. This equation defines a conservative holonomic dynamical system with Lagrangian

$$L = \frac{1}{2}\left(t^2\dot{q}^2 - \frac{a}{\nu+2}t^{\nu+1}q^{2\nu+4}\right).$$

Assume now that the function $f(t,q,\dot{q}) = -AtL,$ where A is some constant and assume further that $\xi(t,q)$. Equation (50) gives

$$\frac{\partial\eta}{\partial\dot{q}}\frac{\partial L}{\partial\dot{q}} = -At\frac{\partial L}{\partial\dot{q}} \Rightarrow \eta = -At\dot{q} + F(t,q),$$

where F is an arbitrary function of its arguments. Replacing this in (49), we find

$$L\left(\frac{\partial\xi}{\partial t} + \dot{q}\frac{\partial\xi}{\partial q}\right) + \eta\frac{\partial L}{\partial q} + \xi\frac{\partial L}{\partial t} + \left(\frac{\partial\eta}{\partial t} + \dot{q}\frac{\partial\eta}{\partial q} - \dot{q}\frac{\partial\xi}{\partial t} - \dot{q}^2\frac{\partial\xi}{\partial q}\right)\frac{\partial L}{\partial\dot{q}} = -\frac{\partial f}{\partial t} + \dot{q}\frac{\partial f}{\partial q},$$

which provides the following system

$$\begin{aligned}
\xi &= \xi(t)\,,\\
\left(\frac{\xi}{t} - \frac{1}{2}\xi_{,t}\right) + F_{,q} &= -\frac{1}{2}A,\\
F &= F(q)\,,\\
\frac{1}{2(\nu+2)}\left(\xi_{,t} + \frac{\xi}{t}(\nu+1)\right) + \frac{F}{q} &= -\frac{1}{2}A.
\end{aligned}$$

The solution of the latter system is

$$\xi = -(2c_0 + A)\,t\,,\ \eta = -At\dot{q} + c_0 q. \tag{53}$$

Still, we do not know the parameters A, c_0. In order to compute them, we turn to the first integral $I = f - \left(\eta^i - \xi\dot{q}^i\right)\frac{\partial L}{\partial\dot{q}^i} - L\xi$. Replacing this, we have:

$$\begin{aligned}
I &= -\tfrac{1}{2}A\left(t^2\dot{q}^2 - \tfrac{a}{\nu+2}t^{\nu+1}q^{2\nu+4}\right)t - \left(-At\dot{q} + c_0 q + (2c_0 + A)\,t\dot{q}\right)t^2\dot{q} + \tfrac{1}{2}\left(t^2\dot{q}^2 - \tfrac{a}{\nu+2}t^{\nu+1}q^{2\nu+4}\right)(2c_0 + A)\,t\\
&= -c_0\left(t^2 q\dot{q} + t^3\dot{q}^2 + \tfrac{a}{\nu+2}t^{\nu+2}q^{2(\nu+2)}\right)
\end{aligned}$$

Hence:

$$I = t^3\dot{q}^2 + t^2 q\dot{q} + \frac{a}{2+\nu}t^{\nu+2}q^{2(\nu+2)} = \text{const}.$$

Having computed I, we compute η^i from the relation $\eta^i = \gamma^{ij}\frac{\partial I}{\partial\dot{q}^i}$. $\gamma_{ij} = t^2$ hence $\gamma^{ij} = \frac{1}{t^2}$. Then, $\eta = \frac{1}{t^2}(2t^3\dot{q} + t^2 q) = 2t\dot{q} + q$. Comparing this with what we have already found, we get $A = -2$, $c_0 = 1$. We note that, for these values of A, c_0, the $\xi = 0$ as it is correct because the relation $\eta^i = \gamma^{ij}\frac{\partial I}{\partial\dot{q}^i}$ is valid only under the assumption $\xi = 0$.

7. The Inverse Noether Theorem

One question which arises concerns the extent to which the first integrals provided by different types of symmetry of a system of differential equations are independent. In this section, we show that, to any quadratic first integral, one may associate a Noether symmetry which provides that integral as a Noether integral.

Suppose we have a quadratic first integral I of a Lagrangian system with a non-degenerate kinetic metric. We define a vector $\eta^i(t,q,\dot{q})$ and a function $f(t,q,\dot{q})$ by the requirement

$$I = f - \eta^i \dot{q}^i.$$

Because I is assumed to be quadratic in the velocities η^i must be linear in the velocities and f must be at most quadratic in the velocities. We choose $\eta_i = a_i(t,q) + b_{ij}(t,q)\dot{q}^j$ and $f = \frac{1}{2}c_{ij}(t,q)\dot{q}^i\dot{q}^j + d_i(t,q)\dot{q}^i + e_i(t,q)$. Then, we have

$$\eta_i = -\frac{\partial I}{\partial \dot{q}^i} + \frac{\partial f}{\partial \dot{q}^i} = -\frac{\partial I}{\partial \dot{q}^i} + c_{ij}(t,q)\dot{q}^j + d_i(t,q)$$

and replacing η_i

$$a_i(t,q) + b_{ij}(t,q)\dot{q}^i = -\frac{\partial I}{\partial \dot{q}^i} + c_{ij}(t,q)\dot{q}^j + d_i(t,q).$$

Let us assume that I has the general form

$$I = \frac{1}{2}A_{ij}(t,q)\dot{q}^i\dot{q}^j + B_i(t,q)\dot{q}^i + C(t,q).$$

Then, we have

$$a_i(t,q) + b_{ij}(t,q)\dot{q}^j = A_{ij}(t,q)\dot{q}^j + B_i(t,q) + c_{ij}(t,q)\dot{q}^j + d_i(t,q)$$

from which follows

$$\begin{aligned} b_{ij} &= -A_{ij} + c_{ij}, \\ a_i &= -B_i + d_i. \end{aligned}$$

This system has an infinite number of solutions. To pick up one solution, we have to define c_{ij}, d_i. One choice is $c_{ij} = -A_{ij}$, which gives $b_{ij} = -2A_{ij}$ and $d_i = 0$, which implies $a_i = -B_{i.}$ Therefore, one answer is

$$\begin{aligned} \eta_i &= 2A_{ij}\dot{q}^j + B_i, \\ f &= \frac{1}{2}A_{ij}(t,q)\dot{q}^i\dot{q}^j + e_i(t,q). \end{aligned}$$

Then, the vector $X^W = \eta^i \frac{\partial}{\partial q^i} + \eta^{[1]i}\frac{\partial}{\partial \dot{q}^i}$

$$X^W = \eta^i \frac{\partial}{\partial q^i} + \eta^{[1]i}\frac{\partial}{\partial \dot{q}^i}$$

generates a point transformation which is a gauged Noether symmetry (in the gauge $\xi = 0, \phi^i = 0$) of the conservative holonomic dynamical system admitting I as a Noether integral.

Obviously, all quadratic first integrals of a SODEs correspond to a Noether symmetry which can be computed as indicated above.

8. Symmetries of SODEs in Flat Space

Obviously, an area where symmetries of ODEs play an important role is the cases in which the kinetic metric (not necessarily the spacetime metric) is flat. These cases cover significant part of Newtonian Physics where the kinetic energy is a positive definite metric with constant coefficients;

therefore, there is always a coordinate transformation in the configuration space that brings the metric to the Euclidian metric. Similar remarks apply to Special Relativity and—as we shall see—to Cosmology.

The basic result in these cases is that the maximum number of Lie point symmetries that a SODE can have is $n(n+2)$ and the maximum number of Noether point symmetries $\frac{n(n+1)}{2}+1$. Moreover, the number of point symmetries which a SODE can possess is exactly one of 0, 1, 2, 3, or 8 [36]. Similar results exist for higher-order differential equations [37].

Lie has shown [2] the important result that *"for all the second order ordinary differential equations which are invariant under the elements of the sl* $(3,R)$, *there exists a transformation of variables that brings the equation to the form* $x^{*\prime\prime}=0$ *and vice versa"*.

In current cosmological models of importance, of interest is the case $n=2$, therefore we shall restrict our attention to two cases

i. The case the of systems which admit the maximum number of Lie point symmetries which for $n=2$ is eight and span the algebra $sl(3,R)$.

ii. The case that the Lie point symmetries span the algebra $sl(2,R)$.

8.1. The Case of $sl(3,R)$ *Algebra*

The prototype dynamical system which admits the $sl(3,R)$ algebra of eight Lie point symmetries is the Newtonian free particle moving in one dimension whose dynamical equation is

$$\ddot{x}=0, \tag{54}$$

where $x=x(t)$ and a dot means differentiation with respect to the time parameter t.

Let $X=\xi(t,x)\,\partial_t+\eta(t,x)\,\partial_i$ be the generator of a Lie point symmetry of (54). The Lie point symmetries of (54) are given by the special projective vectors of E^2 (see condition (35))

$$X_1=\partial_t\ ,\ X_2=\partial_x\ ,\ X_3=t\partial_x\ ,\ X_4=t^2\partial_t+tx\partial_x\ ,\ \ X_5=t\partial_t,$$
$$X_6=x\partial_x\ \ ,\ X_7=tx\partial_t+x^2\partial_x\ \ ,\ \ X_8=x\partial_t.$$

To show the validity of the aforementioned Lie's result, we consider the harmonic oscillator

$$\ddot{x}+x=0, \tag{55}$$

which also admits the eight Lie point symmetries [38]

$$\bar{X}_1=\partial_t\ ,\ \bar{X}_2=\cos t\partial_x\ \ ,\ \bar{X}_3=\sin t\partial_x\ \ ,\ \bar{X}_4=x\partial_x\ ,$$

$$\bar{X}_5=\sin 2t\ \partial_t+x\cos 2t\ \partial_x\ ,\ \bar{X}_6=\cos 2t\ \partial_t-x\sin 2t\ \partial_x\ ,$$

$$\bar{X}_7=x\sin t\ \partial_t+x^2\cos t\ \partial_x\ ,\ \bar{X}_8=x\cos t\partial_t-x^2\sin t\ \partial_x,$$

which form another basis of the $Sl\,(3,R)$ Lie algebra. The transformation which relates the two different representations of the $Sl\,(3,R)$ algebra is

$$t\rightarrow\arctan\tau\ ,\ x\rightarrow\frac{y}{\sqrt{1+\tau^2}}. \tag{56}$$

It easy to show that under this transformation Equation (55) becomes $\ddot{y}=0$, which is Equation (54).

In order to calculate the Noether point symmetries of (54), we have to define a Lagrangian. We recall that the Noether symmetries depend on the particular Lagrangian we consider. Let us assume the classical Lagrangian $L_1(t,x,\dot{x}) = \frac{1}{2}\dot{x}^2$. Replacing in the Noether condition, we find the associated Noether conditions

$$\xi_{,x} = 0\ ,\ \eta_{,xx} - \xi_{,t} = 0, \tag{57}$$

$$\eta_{,t} - f_{,x} = 0\ ,\ f_{,t} = 0, \tag{58}$$

whose solution shows that the Noether point symmetries of (54) for the Lagrangian L_1 are the vector fields $X_1,\ X_2,\ X_3,\ X_4$ and $X_N = 2X_5 + X_6$, with corresponding non-constant Noether functions the $f_3 = x$ and $f_4 = \frac{1}{2}x^2$.

Furthermore, from the second theorem of Noether, the corresponding first integrals are calculated easily. The vector X_1 provides the conservation law of energy, X_2 the conservation law of momentum, while X_3 gives the Galilean invariance [39]. Finally, the vector fields X_4 and X_N are also important because they can be used to construct higher-order conservation laws.

A more intriguing example is the slowly lengthening pendulum whose equation of motion in the linear approximation is (The time dependence in the 'spring constant' is due to the length of the pendulum's string increasing slowly [40].),

$$\ddot{x} + \omega^2(t)\,x = 0, \tag{59}$$

which also admits eight Lie point symmetries. According to Lie's result, there is a transformation which brings (59) to the form (54). In order to find this transformation, one considers the Noether point symmetries and shows that (59) admits the quadratic first integral [41]

$$I = \frac{1}{2}\left\{(\rho\dot{x} - \dot{\rho}x)^2 + \left(\frac{x}{\rho}\right)^2\right\}, \tag{60}$$

where $\rho = \rho(t)$, is a solution of the second-order differential equation

$$\ddot{\rho} + \omega^2(t)\,\rho = \frac{1}{\rho^3}. \tag{61}$$

The first integral (60) is known as the Lewis invariant.

On the other hand, Equation (61) is the well-known Ermakov–Pinney Equation [42] whose solution has been given by Pinney [43] and it is

$$\rho(t) = \sqrt{Av_1^2 + 2Bv_1v_2 + Cv_2^2}, \tag{62}$$

where $A,\ B,\ C$ are constants of which only two are independent, and the functions $v_1(t)\,,\ v_2(t)\,,$ are two linearly independent solutions of (59).

Finally, the transformation which connects the time dependent linear Equation (59) with (54) is the following

$$y = \frac{x}{\dot{x}}\ ,\ P = \rho\dot{x} - \dot{\rho}x\ ,\ \tau = \int^t \rho^{-2}(\eta)\,d\eta, \tag{63}$$

where $\rho(t)$ is given by (62).

8.2. The Case of the $sl(2,R)$ Algebra

The prototype system in this case is the Ermakov system defined by Equations (59) and (60) whose Lie point symmetries span the $sl(2,R)$ algebra for arbitrary function $\omega(t)$, while any solution for a specific $\omega(t)$ can be transformed to a solution for another $\omega(t)$ by a coordinate transformation. The Ermakov system has numerous applications in diverting areas of Physics (see, for instance [44–46]).

Let us restrict our considerations to the autonomous case, with $\omega(t) = \mu^2$ a constant. Equation (61) becomes

$$\ddot{\rho} + \mu^2 \rho = \frac{1}{\rho^3}, \tag{64}$$

while the elements of the admitted $sl(2,R)$ Lie algebra are

$$Z_1 = \partial_t\ ,\ Z_2 = 2t\partial_t + \rho\partial_\rho \text{ and } Z_3 = t^2\partial_t + t\rho\partial_\rho, \text{ when } \mu = 0$$

and

$$Z_1 = \partial_t\ ,\ Z_2 = \frac{1}{\mu}e^{+2i\mu t}\,\partial_t + e^{+2i\mu t}\rho t\,\partial_\rho\ \ ,\ Z_3 = \frac{1}{\mu}e^{-2i\mu t}\partial_t - e^{+2i\mu t}\rho\partial_\rho\ \ , \text{ when } \mu \neq 0.$$

Concerning the Noether point symmetries, we consider the Lagrangian

$$L(t,\rho,\dot{\rho}) = \frac{1}{2}\dot{\rho}^2 - \frac{\mu^2}{2}\rho^2 - \frac{1}{2}\frac{1}{\rho^2} \tag{65}$$

and find that the Lie symmetries Z_1, Z_2, Z_3 satisfy the Noether condition, hence they are also Noether point symmetries, and lead to the quadratic first integral of energy

$$E = \frac{1}{2}\dot{\rho}^2 + \frac{\mu^2}{2}\rho^2 + \frac{1}{2}\frac{1}{\rho^2} \tag{66}$$

and the time dependent first integrals

$$I_1 = 2tE - \rho\dot{\rho}, \tag{67}$$

$$I_2 = t^2 E - t\rho\dot{\rho} + \frac{1}{2}\rho^2, \text{ when } \mu = 0, \tag{68}$$

or

$$I_+ = \frac{1}{2\mu}e^{+2i\mu t}E - e^{+2i\mu t}\,\rho\dot{\rho} + \mu e^{+2i\mu t}\,\rho^2, \tag{69}$$

$$I_- = \frac{1}{2\mu}e^{-2i\mu t}E + e^{-2i\mu t}\,\rho\dot{\rho} + \mu e^{-2i\mu t}\,\rho^2\ , \text{ when } \mu \neq 0. \tag{70}$$

While the first integrals $I_1,\ I_2$ and $I_+,\ I_-$ are time-dependent, we can easily construct the time-independent Lewis invariant [47]. For instance, $I_+ I_-$ is a time-independent first integral.

The two-dimensional system with Lagrangian

$$L\left(t,\rho,\theta,\dot{\rho},\dot{\theta}\right) = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho^2\dot{\theta} - \frac{\mu^2}{2}\rho^2 - \frac{V(\theta)}{\rho^2} \tag{71}$$

describes the simplest generalization of the Ermakov–Pinney system in two-dimensions. It can be shown that the Lie point symmetries Z_1, Z_2, Z_3 are Noether point symmetries of (71) with the same first integrals. Again with the use of the time dependent Noether integrals $I_1,\ I_2$ and $I_+,\ I_-$, we are able to construct the autonomous conservation laws [48]

$$\Phi = 4I_2 E - I_1^2 = \rho^4\dot{\phi}^2 + 2V(\phi) \tag{72}$$

and

$$\bar{\Phi} = E^2 -\ I_+ I_- = \rho^4\dot{\phi}^2 + 2V(\phi)\,. \tag{73}$$

As we shall see below, the Ermakov–Pinney system and its generalizations are used in the dark energy models [47].

9. Symmetries of SODEs and the Geometry of the Underline Space

As it has been mentioned, the symmetries of a SODE concern the kinetic metric, which is independent of the metric of the space where motion occurs. On the other hand, one of the basic principles of General Relativity (and Newtonian Physics) is the Principle of Equivalence according to which the trajectories of free fall are the geodesics of the space where motion occurs. This means that the Principle of Relativity locks the Lie point symmetries of the geodesic equations (= free fall) with the collineations of the geometry (=metric) of space-time where the particle moves. The relation between the Lie and the Noether point symmetries and the collineations of the space where motion occurs has been given not only for the case of geodesics, but also for a general autonomous conservative dynamical system (see [26–29]).

Below, we briefly discuss the collineations of Riemannian manifolds and also the results of [27] because they are used in the construction of cosmological models.

9.1. Collineations

Consider a Riemannian manifold M of dimension n and metric g_{ij}. Let $\mathbf{A}$ be a geometric object (not necessarily a tensor) defined in terms of the metric (This is not necessary, but it is enough for the cases we consider in this work), X a vector field in $F_0^1(M)$ and $\mathbf{B}$ a tensor field on M which has the same number of indices as $\mathbf{A}$ and with the same symmetries of the indices. We say that X is a collineation of $\mathbf{A}$ if the following condition holds:

$$\mathcal{L}_X \mathbf{A} = \mathbf{B}, \tag{74}$$

where $\mathcal{L}_X$ denotes the Lie derivative. The collineations of a geometric object form a Lie algebra. The classification of the possible collineations in a Riemannian space can be found in [49]. The most important are the collineations given in Table 1.

Table 1. Collineations of the metric and of the connection in a Riemannian space.

Collineation	A	B
Killing vector (KV)	g_{ij}	0
Homothetic vector (HV)	g_{ij}	$\psi g_{ij}, \psi_{,i} = 0$
Conformal Killing vector (CKV)	g_{ij}	$\psi g_{ij}, \psi_{,i} \neq 0$
Affine collineation	Γ^i_{jk}	0
Projective collineation (PC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)}, \phi_{,i} \neq 0$
Special Projective collineation (SPC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)}, \phi_{,i} \neq 0$ and $\phi_{,jk} = 0$

Some general results concerning collineations of a Riemannian manifold M of dimension n are the following:

a. M can have at most $n(n+1)/2$ KVs and when this is the case M is called a maximally symmetric space. The curvature tensor of a a maximally symmetric space is given by the expression

$$R_{abcd} = R\left(g_{ac}g_{db} - g_{ad}g_{cb}\right),$$

where R is the curvature scalar, which is a constant. Flat space is a maximally symmetric space for which $R = 0$.

b. M can have at most one proper HV.

c. M can have at most $\frac{(n+1)(n+2)}{2}$ proper CKVs, $n\left(n+1\right)$ proper ACs, and $n(n+2)$ proper PCs. A two-dimensional space has infinite CKVs.

d. If the metric admits a SCKV, then it also admits a SPC, a gradient HV and a gradient KV [50]

Other properties of the collineations can be found in [1].

What shall be important in our discussions are the collineations of the n-dimensional flat space. These collineations we summarize in Table 2. It is important to note which collineations are gradient.

Table 2. The elements for the projective algebra of the Euclidian space.

Collineation	Gradient	Non-Gradient
Killing vectors (KV)	$\mathbf{S}_I = \delta_I^i \partial_i$	$\mathbf{X}_{IJ} = \delta_{[I}^j \delta_{j]}^i x_j \partial_i$
Homothetic vector (HV)	$\mathbf{H} = x^i \partial_i$	
Affine Collineation (AC)	$\mathbf{A}_{II} = x_I \delta_I^i \partial_i$	$\mathbf{A}_{IJ} = x_J \delta_I^i \partial_i$
Special Projective collineation (SPC)		$\mathbf{P}_I = S_I \mathbf{H}$.

10. Motion and Symmetries in a Riemannian Space

The equation of motion of a particle moving in a Riemannian space is given by the SODE

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = F^i, \tag{75}$$

where $\Gamma^i_{jk}(x^r)$ are the connection coefficients and the field F^i stands for the forces acting on the particle. The Lie symmetries of (75) are defined by the system of equations [27]

$$L_\eta F^i + 2\xi_{,t} F^i + \xi F^i_{,t} + \eta^i{}_{,tt} = 0, \tag{76}$$

$$\left(\xi_{,k} \delta^i_j + 2\xi_{,j} \delta^i_k\right) F^k + 2\eta^i{}_{,t|j} - \xi_{,tt} \delta^i_j = 0, \tag{77}$$

$$L_\eta \Gamma^i_{(jk)} = 2\xi_{,t(j} \delta^i_{k)}, \tag{78}$$

$$\xi_{(,i|j} \delta^k_{r)} = 0. \tag{79}$$

From (78), it follows that the Lie point symmetries of the SODE (75) are generated from the special projective algebra of the space.

10.1. Lie Point Symmetries of (75)

In the case where the force is autonomous and conservative, that is, $F^i = g^{ij} V_{,j}\left(x^k\right)$ and $V_{,j}$ is not a gradient KV of the metric, the general solution of the system of Equations (76)–(79) has been given in [27] and has the following:

- *Case I* Lie point symmetries due to the affine algebra. The resulting Lie point symmetries are

$$\mathbf{X} = \left(\frac{1}{2} d_1 a_1 t + d_2\right) \partial_t + a_1 Y^i \partial_i, \tag{80}$$

where a_1 and d_1 are constants, provided the potential satisfies the condition

$$L_Y V^{,i} + d_1 V^{,i} = 0. \tag{81}$$

- *Case IIa* The Lie point symmetries are generated by the gradient homothetic algebra and $Y^i \neq V^{,i}$. The Lie point symmetries are

$$\mathbf{X} = 2\psi \int T(t)\, dt \partial_t + T(t)\, Y^i \partial_i, \tag{82}$$

where the function $T(t)$ is the solution of the equation $T_{,tt} = a_1 T$ provided the potential $V(x^i)$ satisfies the condition

$$\mathcal{L}_{\mathbf{Y}} V^{,i} + 4\psi V^{,i} + a_1 Y^i = 0. \tag{83}$$

- *Case IIb* The Lie point symmetries are generated by the gradient HV $Y^i = \kappa V^{,i}$, where κ is a constant. In this case, the potential is the function generating the gradient HV and the Lie symmetry vectors are

$$\mathbf{X} = \left(-c_1\sqrt{\psi k}\cos\left(2\sqrt{\tfrac{\psi}{k}}t\right) + c_2\sqrt{\psi k}\sin\left(2\sqrt{\tfrac{\psi}{k}}t\right)\right)\partial_t + \left(c_1\sin\left(2\sqrt{\tfrac{\psi}{k}}t\right) + c_2\cos\left(2\sqrt{\tfrac{\psi}{k}}t\right)\right)H^{,i}\partial_i. \quad (84)$$

- *Case IIIa* The Lie point symmetries are due to the proper special projective algebra. In this case, the Lie symmetry vectors are (the index J counts the gradient KVs)

$$\mathbf{X}_J = \left(C\left(t\right)S_J + D\left(t\right)\right)\partial_t + T\left(t\right)Y^i\partial_i, \quad (85)$$

where the functions $C(t), T(t), D(t)$ are solutions of the system of the system of equations

$$D_{,t} = \frac{1}{2}d_1 T \ , \quad T_{,tt} = a_1 T \ , \quad T_{,t} = c_2 C \ , \quad D_{,tt} = d_c C \ , \quad C_{,t} = a_0 T \quad (86)$$

and, in addition, the potential satisfies the conditions

$$\mathcal{L}_Y V^{,i} + 2a_0 S V^{,i} + d_1 V^{,i} + a_1 Y^i = 0, \quad (87)$$

$$\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right)V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right)c_2 - d_c\delta^i_j = 0. \quad (88)$$

- *Case IIIb* Lie point symmetries are due to the proper special projective algebra and $Y^i_J = \lambda S_J V^{,i}$, in which $V^{,i}$ is a gradient HV, and S^i_J is a gradient KV. The Lie point symmetry vectors are

$$X_J = \left(C\left(t\right)S_J + d_1\right)\partial_t + T(t)\lambda S_J V^{,i}\partial_i, \quad (89)$$

where the functions $C\left(t\right)$ and $T(t)$ are computed from the relations

$$T_{,tt} + 2C_{,t} = \lambda_1 T \ , \quad T_{,t} = \lambda_2 C \ , \quad C_{,t} = a_0 T \quad (90)$$

and the potential satisfies the conditions

$$\mathcal{L}_{Y_J} V^{,i} + \lambda_1 S_J V^{,i} = 0, \quad (91)$$

$$C\left(\lambda_1 S_J\delta^i_j + 2S_{J,j}\,V^{,i}\right) + \lambda_2\left(2\lambda S_{J,j}V^{,i} + \left(2\lambda S_J - a_0 S_J\right)\delta^i_j\right) = 0. \quad (92)$$

The general case for time dependent conservative forces has been given in [51]. In the following, only the autonomous case will be considered.

10.2. Noether Point Symmetries of (75)

In case the force is autonomous and conservative, the SODE (75) follows from the regular Lagrangian

$$L\left(t, x^k, \dot{x}^k\right) = \frac{1}{2}g_{ij}\left(x^k\right)\dot{x}^i\dot{x}^j - V\left(x^k\right). \quad (93)$$

The Noether point symmetries of Lagrangian (93) as well as the corresponding first integrals have been given in [27] and are generated by the homothetic algebra of the metric g_{ij} as follows:

- *Case I.* The HV satisfies the condition:

$$V_{,k}Y^k + 2\psi_Y V + c_1 = 0. \quad (94)$$

The Noether point symmetry vector is

$$\mathbf{X} = 2\psi_Y t\partial_t + Y^i\partial_i,\;\; f = c_1 t, \tag{95}$$

where $T(t) = a_0 \neq 0$ and the corresponding first integral is

$$\Phi = 2\psi_Y tE - g_{ij}Y^i\dot{x}^j + c_1 t. \tag{96}$$

- *Case II.* The metric admits the gradient KVs S_J, the gradient HV $H^{,i}$ and the potential satisfies the condition

$$V_{,k}Y^{,k} + 2\psi_Y V = c_2 Y + d. \tag{97}$$

In this case, the Noether point symmetry vector and the Noether function are

$$\mathbf{X} = 2\psi_Y \int T(t)\, dt\partial_t + T(t)\, S_J^{,i}\partial_i \;\;,\;\; f\left(t,x^k\right) = T_{,t}S_J\left(x^k\right) \;+d\int Tdt, \tag{98}$$

where the functions $T(t)$ and $K(t)$ ($T_{,t} \neq 0$) are computed from the relations

$$T_{,tt} = c_2 T\;,\; K_{,t} = d\int Tdt + \text{constant}, \tag{99}$$

where c_2 is a constant. The Noether integral in this case is

$$\bar{\Phi} = \psi_Y E\int Tdt \;- Tg_{ij}H^i\dot{x}^j + T_{,t}H + d\int Tdt. \tag{100}$$

In addition to the above cases, there is also the Noether point symmetry ∂_t whose first integral is the energy (i.e., the Hamiltonian) E.

From the above results, it is clear that, in order a given dynamical system to admit Lie/Noether point symmetries, the underlying space must admit collineations. This means that, by studying the collineations of the underlying geometry, we can infer important information for the existence or not and also compute the Lie/Noether point symmetries and the associated first integrals. In that respect, we may say that geometry determines the evolution of the dynamical systems.

10.3. Point Symmetries of Constrained Lagrangians

The previous analysis holds for regular dynamical systems while when the dynamical system is constrained extra conditions are introduced.

Consider the constrained Lagrangian

$$\bar{L}\left(t,x^k,\dot{x}^k,N\right) = \frac{1}{2N}g_{ij}\dot{x}^i\dot{x}^j - NV\left(x^k\right), \tag{101}$$

where $N = N(t)$ is a singular degree of freedom with Euler–Lagrange equation $\frac{\partial L}{\partial N} = 0$. The latter equation is a constraint of the system. Lagrangian functions of the form of (101) are provided in cosmological studies.

10.3.1. Lie Point Symmetries of (101)

The generic Lie point symmetry vector for the Euler–Lagrange equations of (101) is [52]

$$X_L = X_N - 2a_3\partial_N, \tag{102}$$

where

$$X_N = \alpha_2\chi(t)\,\partial_t + \left(2\alpha_1\tau\left(x^k\right) + \alpha_2\chi_{,t}(t)\,N\right)N\partial_N - \alpha_1\eta^i\partial_i, \tag{103}$$

and where η^i is a CKV of the metric g_{ij} with conformal factor $\tau\left(x^k\right)$ which is related to the potential by the following condition/constraint:

$$\mathcal{L}_\eta V\left(x^k\right) + 2\left(\tau\left(x^k\right) + a_3\right) V\left(x^k\right) = 0. \tag{104}$$

We note that the Lie point symmetries of the constrained Lagrangian are generated by the elements of the conformal algebra, whereas for regular systems these symmetries are generated by the elements of the special projective algebra. There are also differences in the constraint condition for the potential.

10.3.2. Noether Point Symmetries of (101)

In order to compute the Noether point symmetries of (101), we consider the Noether condition (48). We find one Noether point symmetry that is again the vector X_N [52]; the potential satisfies the following condition/constraint:

$$\mathcal{L}_\eta V\left(x^k\right) + 2\tau\left(x^k\right) V\left(x^k\right) = 0. \tag{105}$$

The first integral defined by the Noether point symmetry X_N is given by the formula [52]

$$\Phi^* = \frac{1}{N} g_{ij}\eta^i \dot{x}^j - \chi(t)\left(\frac{\partial L}{\partial N}\right) \simeq \frac{1}{N} g_{ij}\eta^i \dot{x}. \tag{106}$$

The function Φ^* is a "weak" conservation law in the sense that someone has to impose the constraint condition $\left(\frac{\partial L}{\partial N}\right) = 0$, in order $\frac{d\Phi}{dt} = 0$; this is because $\frac{d\Phi}{dt} = 2\frac{\partial L}{\partial N}$. Furthermore, we note that there is a difference in the Noether point symmetries between the regular and the constrained Lagrangian. For instance, while the regular Lagrangian $L(t,x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$ possesses five Noether point symmetries [53], for the singular Lagrangian $\bar{L}(t,x,\dot{x},N) = \frac{1}{2N}\dot{x}^2 - \frac{1}{2}Nx^2$, we found only the X_N Noether symmetry.

We note that if $a_3 = 0$ then there is only one Lie point symmetry which is also a Noether point symmetry.

11. Symmetries in Cosmology

The nature of the source which drives the late-acceleration phase of the universe is an important problem of modern cosmology. Currently, the late-acceleration phase of the universe is attributed to a perfect fluid with a negative equation of state parameter, which has been named the dark energy. The simplest dark energy candidate is the cosmological constant model leading to the ΛCDM cosmology. In this model, the gravitational field equations can be linearized and one is able to write the analytic solution in closed-form. However, in spite of its simplicity, the ΛCDM cosmological model suffers from two major problems: the fine-tuning problem and the coincidence problem [54–56]. In order to overcome these problems, cosmologists introduced dynamical evolving dark energy models. In these models, the dark energy fluid can be an exotic matter source like the Chaplygin gas, quintessence, $k-$essence, tachyons or it can be of a geometric origin provided by a modification of Einstein's General Relativity [57–70]. The dark energy components introduce new terms in the gravitational field equations which are nonlinear or increase the degrees of freedom. Thereafter, the linearization process applied in the case of the ΛCDM model fails and other mathematical methods must be applied in the study of integrability of the field equations and the construction of analytic solutions.

Two different groups, de Ritis et al. [71] and Rosquist et al. [72], applied independently the symmetries of differential equations in order to construct first integrals in scalar field cosmology. In particular, they determined the forms of the scalar field potential, which drives the dynamics of the dark energy, in order for the field equations to admit Noether point symmetries. The classification scheme is based on an idea proposed by Ovsiannikov [5]. Since then, the classification scheme has been applied to various dark energy models and modified theories of gravity. Some of these classifications are complete while some others lack mathematical completeness leading to incorrect results. The purpose of the current review is to present the application of symmetries of differential equations in modern cosmology.

A cosmological model is a relativistic model and therefore requires two assumptions:

a. A specification of the metric, which is achieved mainly by the collineations for the comoving observers we discussed above and

b. Equations of state which specify the matter of the model universe and are mathematically compatible with the assumed collineations defining the metric. This is done by the introduction of a potential function in the action integral from which the field equations follow.

One important class of cosmological models are the ones in which spacetime brakes in 1 + 3 parts, that is, the cosmic time and the spatial universe, respectively. The latter is realized geometrically by three-dimensional spacelike hypersurfaces which are generated by the orbits of the KVs of a three-dimensional Lie algebra. In 1898, Luigi Bianchi [73] classified all possible real three-dimensional real Lie algebras in nine types. Each Lie algebra leads to a (hypersurface orthogonal) cosmological model called a Bianchi Spatially homogeneous cosmological model. These nine models have been studied extensively in the literature over the years and have resulted in many important cosmological solutions.

The principal advantage of Bianchi cosmological models is that, due to the geometric structure of spacetime, the physical variables depend only on the time, thus reducing the Einstein and the other governing equations to ordinary differential equations [74]. However, the gravitational field equations in General Relativity for the Bianchi cosmologies are ordinary second-order differential equations, due to the existence of nonlinear terms, exact solutions have been determined only for a few of them [75–80], while there was a debate a few years ago on the integrability or not of the Mixmaster universe (Bianchi type IX model) [80–84].

In order to get detailed information on these alternative models, one has to find an analytical solution of the field equations. This can be a formidable task depending on the form of the potential function and the free parameters that it has. The standard method to find an analytical solution is to use Noether point symmetries and compute first integrals of the field equations. Indeed, the application of symmetries of differential equations in the dark energy models started with the use of the first Noether theorem in [71] and with the consideration of the second Noether theorem in [72]. Both approaches are equivalent. Since then, Noether point symmetries have been applied to a plethora of models for the determination of first integrals, and consequently analytical solutions. We refer the reader to some of them [85–116]. It is important to note that some of the published results are mathematically incorrect. For instance, in [117], the authors used the Noether conditions in order to solve the dynamical equations of the model, and a posteriori, they determined the symmetries of the field equations. This is not correct because Noether symmetries are imposed by the requirement that they transform the Action Integral in a certain way and not as extra conditions on the Euler–Lagrange equations.

The difficulty with the above approach is that one has to work in spacetime where the geometry is not simple and the field equations are rather complex. To bypass that difficulty, a new scenario has been developed in which one transforms the problem to a minisuperspace defined by the dynamical variables through a Lagrangian which produces the field equations in that space [118]. Then, one considers the Lagrangian in two parts: the kinematic part which defines the kinetic metric and the remaining part which defines the effective potential. If one knows the homothetic algebra of the kinetic metric [27], then the application of the results of Section 10 provide the Noether point symmetries and the corresponding Noether first integrals of the field equations in mini superspace. Therefore, the solution of the field equations is made possible, and, by the inverse transformation, one finds the solution of the original field equations in the original dynamic variables in spacetime.

This approach brought new results in various dark energy models and modified theories of gravity [119–122]. Some of these results are discussed in the following.

Before we enter into detailed discussion, it is useful to state the scenario of this method of work in order to provide a working tool to new cosmologists not experienced in this field.

11.0.1. Method of Work—Scenario

1. Consider the Action Integral of the model in spacetime and produce the field equations.
2. Change variables and give a new set of field equations in the minisuperspace of dynamical variables in a convenient form.
3. Define a Lagrangian for the field equations in the mini superspace.
4. Read from the Lagrangian the kinetic metric and the effective potential. The new variables must be such that the kinetic metric will be flat or at least one for which one knows already the homothetic algebra. This defines the phrase "convenient form " stated in step 2 above.
5. Apply the results of Section 3 to get a classification of Noether point symmetries of the field equations and compute the corresponding first integrals in the mini superspace.
6. Using the first integrals, solve the field equations for the various cases of the effective potential and other possible parameters.
7. Apply the inverse transformation and get the solution of the original field equations in terms of the original dynamical variables in spacetime.

In the sections which follow, we apply this scenario to the major cosmological models proposed so far and give the detailed results in each case.

11.1. FRW Spacetime and the ΛCDM Cosmological Model

The FRW spacetime is a decomposable 1 + 3 spacetime in which the three-dimensional hypersurfaces are maximally symmetric spacelike hypersurfaces of constant curvature, which are normal to the time coordinate. The metric of a FRW spacetime is specified modulo a function of time, the scale factor $a(t)$. In comoving coordinates $\{t, x, y, z\}$, it has the form

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{107}$$

In the Bianchi classification, it is a Type IX spacetime. This spacetime in classical General Relativity for comoving observers (non-comoving observers can support all types of matter) $u^a = \frac{\partial}{\partial t}$ $(u^a u_a = -1)$ can support matter that is a perfect fluid, that is, the energy momentum tensor, is

$$T_{ab} = \rho u_a u_b + p h_{ab},$$

where ρ, p are the energy density and the isotropic pressure of matter as measured by the comoving observers. $h_{ab} = g_{ab} + u_a u_b$ is the tensor projecting normal to the vector u^a. This spacetime has been used in the early steps of relativistic cosmology.

The first cosmological model using this spacetime was the ΛCDM cosmology, which was a vacuum spatially flat FRW spacetime with matter generated by a non-vanishing cosmological constant Λ. For this model, Einstein field equations are

$$-3a\dot{a}^2 + 2a^3\Lambda = 0, \tag{108}$$

and

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 - a\Lambda = 0, \tag{109}$$

whose solution is the well-known de Sitter solution $a(t) = a_0 e^{\sqrt{\frac{2\Lambda}{3}}t}$ that is a maximally symmetric spacetime (not only the maximally symmetric 3d hypersurfaces). According to earlier comments, there exists a coordinate transformation that brings the system to the linear Equation (110) [123]. Indeed, if we introduce the new variable $r(t) = a(t)^{3/2}$, the field Equation (109) becomes [124]

$$-\frac{1}{2}\dot{r}^2 + \frac{3}{2}\Lambda \dot{r}^2 = 0 \ , \ \ddot{r} - \frac{3}{2}\Lambda r = 0, \tag{110}$$

which is the one-dimensional hyperbolic linear oscillator, which admits eight Lie point symmetries.

Let us demonstrate the geometric scenario mentioned above in this simple case. For this, we need to find the maximum number of Noether point symmetries admitted by the field Equation (109). We choose the variables $\{t, a\}$ and have a two-dimensional mini superspace. A Lagrangian for Equation (109) in the minisuperspace $\{t, a\}$ is

$$L(t, a, \dot{a}) = 3a\dot{a}^2 + 2\Lambda a^3 \tag{111}$$

from where we read the kinetic metric $s^2(t) = 3a\dot{a}^2$ and the effective potential $V(a) = -2\Lambda a^3$. The Noether condition (48) for the Lagrangian (111) gives that it admits five Noether point symmetries, which is the maximum number of Noether symmetries for a two-dimensional space. These Noether symmetries are

$$X_{\Lambda(1)} = \partial_a \ , \ X_{\Lambda(2)} = \frac{e^{\frac{\sqrt{6\Lambda}}{2}t}}{\sqrt{a}}\partial_a \ , \ X_{\Lambda(3)} = \frac{e^{-\frac{\sqrt{6\Lambda}}{2}t}}{\sqrt{a}}\partial_a,$$

$$X_{\Lambda(4)} = e^{\frac{\sqrt{6\Lambda}}{2}t}\left(\frac{3}{\sqrt{\Lambda}}\partial_t + \sqrt{6}a\partial_a\right) \ , \ X_{\Lambda(5)} = e^{-\frac{\sqrt{6\Lambda}}{2}t}\left(\frac{3}{\sqrt{\Lambda}}\partial_t - \sqrt{6}a\partial_a\right).$$

Having the Noether symmetries, we continue with the first integrals and finally get the solution we have already found. Note that not all first integrals are independent. This simple application shows how the geometry of the kinetic metric can be used to recognize the equivalence of well-known systems of classical mechanics with dark energy models.

11.2. Scalar-Field Cosmology

In the case of classical General Relativity with a minimally coupled scalar field (quintessence or phantom), the Action Integral in spacetime is considered to be

$$S_M = \int dx^4 \sqrt{-g}\left[R + \frac{1}{2}g_{ij}\phi^{;i}\phi^{;j} - V(\phi)\right]. \tag{112}$$

Assuming a spatially flat FLRW background and comoving observers, the field equations are

$$-3a\dot{a}^2 + \frac{\varepsilon}{2}a^2\dot{\phi}^2 + a^3 V(\phi) = 0, \tag{113}$$

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 + \frac{\varepsilon}{4}\dot{\phi}^2 - \frac{1}{2}aV = 0, \tag{114}$$

$$\ddot{\phi} + \frac{3}{a}\dot{a}\dot{\phi} + \varepsilon V_{,\phi} = 0. \tag{115}$$

We consider the mini superspace defined by the dynamic variables $\{a, \phi\}$. A point-like Lagrangian in the mini superspace of the for field Equations (114) and (115) is

$$L(t, a, \dot{a}, \phi, \dot{\phi}) = -3a\dot{a}^2 + \frac{\varepsilon}{2}a^2\dot{\phi}^2 - a^3 V(\phi). \tag{116}$$

To bring the Lagrangian in the "convenient form", we consider the coordinate transformation $\{a, \phi\}$ to $\{r, \theta\}$

$$r = \sqrt{\frac{8}{3}}a^{3/2} \ , \ \theta = \sqrt{\frac{3\varepsilon}{8}}\phi, \tag{117}$$

and again $\{r,\theta\}$ to $\{x,y\}$, where

$$x = r\cosh\theta \ , \ y = r\sinh\theta, \tag{118}$$

and the new variables have to satisfy the following inequality $x \geq |y|$. In the coordinates $\{x,y\}$, the scale factor becomes

$$a = \left[\frac{3(x^2-y^2)}{8}\right]^{1/3}. \tag{119}$$

Under the coordinate transformation $\{a,\phi\} \rightarrow \{x,y\}$, the point-like Lagrangian takes the simpler form

$$L(t,x,\dot{x},y,\dot{y}) = \frac{1}{2}\left(\dot{y}^2 - \dot{x}^2\right) - V_{eff}(x,y) \tag{120}$$

in which the metric in the coordinates $\{x,y\}$ is the Lorentzian 2d metric $diag(-1,1)$, which is the metric of a flat space while the effective potential is

$$V_{eff}(x,y) = \frac{3}{8}\left(x^2-y^2\right)\tilde{V}(\theta). \tag{121}$$

Application of the previous analysis gives the following classification of Noether point symmetries of the model for various forms of the effective potential [118]:

- For arbitrary potential $V_{eff}(x,y)$, Lagrangian (120) admits the Noether point symmetry ∂_t with first integral the constraint Equation (113).
- For constant potential $V(\theta) = V_0$, the system admits the extra Noether point symmetry $x\partial_y - y\partial_x$ with first integral the angular momentum $r^2\dot{\theta} = const$.
- For the exponential potential $V_{eff}(x,y) = r^2 e^{-d\theta}$, Lagrangian (120) admits an extra Noether point symmetry provided by the proper HV of the two-dimensional flat space,

$$X_{(\phi)1} = 2t\partial_t + \left(x + \frac{4}{d}y\right)\partial_x + \left(y + \frac{4}{d}x\right)\partial_y,$$

with first integral

$$\Phi_{(\phi)1} = \left(x + \frac{4}{d}y\right)\dot{x} - \left(y + \frac{4}{d}x\right)\dot{y} \tag{122}$$

while when $d = 2$, the Lagrangian admits the additional Noether point symmetry $\partial_x + \partial_y$, with corresponding first integral

$$\Phi_{(\phi)2} = \dot{x} - \dot{y}. \tag{123}$$

- Finally, when $V_{eff}(x,y) = \frac{1}{2}\left(\omega_1 x^2 - \omega_2 y^2\right)$, that is, $\tilde{V}(\theta) = \frac{1}{2}\left(\omega_1\cosh^2(\theta) - \omega_2\sinh^2(\theta)\right)$, the dynamical system admits four extra Noether point symmetries

$$X_{(\phi)_2} = \sinh\left(\sqrt{\omega_1}t\right)\partial_x \ , \ X_{(\phi)_3} = \cosh\left(\sqrt{\omega_1}t\right)\partial_x \ ,$$
$$X_{(\phi)_4} = \sinh\left(\sqrt{\omega_2}t\right)\partial_y \ , \ X_{(\phi)_5} = \cosh\left(\sqrt{\omega_2}t\right)\partial_y \ ,$$

with corresponding first integrals

$$I_{n_2} = \sinh\left(\sqrt{\omega_1}t\right)\dot{x} - \sqrt{\omega_1}\cosh\left(\sqrt{\omega_1}t\right)x,$$
$$I_{n_3} = \cosh\left(\sqrt{\omega_1}t\right)\dot{x} - \sqrt{\omega_1}\sinh\left(\sqrt{\omega_1}t\right)x,$$
$$I_{n_4} = \sinh\left(\sqrt{\omega_2}t\right)\dot{y} - \sqrt{\omega_2}\cosh\left(\sqrt{\omega_2}t\right)y,$$
$$I_{n_5} = \cosh\left(\sqrt{\omega_2}t\right)\dot{y} - \sqrt{\omega_2}\sinh\left(\sqrt{\omega_2}t\right)y.$$

The latter dynamical hyperbolic dynamical system reduces to that of the unified dark matter potential (UDM) when $\omega_1 = 2\omega_2$ [125]. The amount of information one receives by the direct application of the geometric symmetries of the kinetic metric is noticeable.

To find the solution in the original dynamical variables $\{a, \phi\}$, we apply the inverse transformation. The result is

$$a^3(t) = \frac{3}{8}[\sinh^2(\sqrt{\omega_1}t + \theta_1) - \frac{\omega_1}{\omega_2}\sinh^2(\sqrt{\omega_2}t + \theta_2)], \tag{124}$$

$$\phi(t) = \sqrt{\frac{8}{3\varepsilon}} \arctan h\left[\sqrt{\frac{\omega_1}{\omega_2}} \frac{\sinh(\sqrt{\omega_2}t + \theta_2)}{\sinh(\sqrt{\omega_1}t + \theta_1)}\right]. \tag{125}$$

11.3. Brans–Dicke Cosmology

The Brans–Dicke action is [126]

$$S_{NM} = \int dt dx^3 \sqrt{-g}\left[F_0\psi^2 R - \frac{1}{2}\bar{g}_{ij}\psi^{;i}\psi^{;j} + \bar{V}(\psi)\right], \tag{126}$$

where F_0 is related to the Brans–Dicke parameter.

In the case of the spatially flat FRW background and comoving observers, the point-like Lagrangian in the mini superspace defined by the variables $\{a, \psi\}$, which describes the gravitational field equations is

$$L(t, a, \dot{a}, \psi, \dot{\psi}) = 6F_0\psi^2 a\dot{a}^2 - 12F_0\psi a^2\dot{a}\dot{\psi} - \frac{1}{2}a^3\dot{\psi}^2 + a^3 V(\psi). \tag{127}$$

If one performs the coordinate transformation $\{a, \psi\}$ to $\{r, \theta\}$ by the equations

$$a \simeq r^{\frac{2}{3}}\ ,\ \ \theta \simeq \ln\psi, \tag{128}$$

Lagrangian (127) becomes

$$L\left(t, r, \dot{r}, \theta, \dot{\theta}\right) = e^{k\theta}\left(-dr^2 + r^2 d\theta^2\right) - r^2 V(\theta) \tag{129}$$

from which we have that the kinetic metric of the minisuperspace is the conformally flat Lorentzian 2d metric $e^{k\theta}\left(-dr^2 + r^2 d\theta^2\right)$ whose symmetry algebra depends on the values $|k| \neq 1$ $|k| = 1$ while the effective potential is $V_{effec.} = -r^2 V(\theta)$.

We consider cases.

11.3.1. Case $|k| \neq 1$.

For $|k| \neq 1$, the homothetic algebra of the minisuperspace consists of the gradient KVs

$$K^1 = \frac{e^{(1-k)\theta} r^k}{N_0^2}\left(-\partial_r + \frac{1}{r}\partial_\theta\right), \tag{130}$$

$$K^2 = \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2}\left(\partial_r + \frac{1}{r}\partial_\theta\right), \tag{131}$$

the non-gradient KV

$$K^3 = r\partial_r - \frac{1}{k}\partial_\theta \tag{132}$$

and the gradient HV

$$H^i = \frac{1}{N_0^2\left(k^2-1\right)}\left(-r\partial_r + k\partial_\theta\right)\ ,\ H\left(r,\theta\right) = \frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}. \tag{133}$$

The symmetry classification provides the following results [121]:

- For arbitrary potential $V\left(\theta\right)$, the dynamical system admits the Noether point symmetry ∂_t.
- For $V\left(\theta\right) = V_0e^{2\theta}$ there are two additional Noether point symmetries $K^1,\ tK^1$ with first integrals

$$I_1 = \frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right)\ ,\ I_2 = t\frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right) - \left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right). \tag{134}$$

- For $V\left(\theta\right) = V_0e^{2\theta} - \frac{mN_0^2}{2(k^2-1)}e^{2k\theta}$, there are two additional Noether point symmetries $e^{\pm\sqrt{m}t}K^1$, with corresponding first integrals

$$I'_\pm = e^{\pm\sqrt{m}t}\left[\frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right) \mp \sqrt{m}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right)\right]. \tag{135}$$

- For $V\left(\theta\right) = V_0e^{-2\theta}$, we have the extra Noether point symmetries $K^2,\ tK^2$ with Noether first integrals

$$J_1 = \frac{d}{dt}\left(\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right)\ ,\ J_2 = t\frac{d}{dt}\left(\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right) - \frac{r^{1-k}e^{-(1-k)\theta}}{k-1}. \tag{136}$$

- For $V\left(\theta\right) = V_0e^{-2\theta} - \frac{mN_0^2}{2(k^2-1)}e^{2k\theta}$, we have the extra Noether symmetries $e^{\pm\sqrt{m}t}K^2$ m =constant, with first integrals

$$J'_\pm = e^{\pm\sqrt{m}t}\left(\frac{d}{dt}\left(\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right) \mp \sqrt{m}\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right). \tag{137}$$

- For the potential $V\left(\theta\right) = V_0e^{2k\theta}$, the additional symmetry is the vector field K^3 with first integrals

$$I_3 = \frac{re^{2k\theta}}{k}\left(k\dot{r} + r\dot{\theta}\right). \tag{138}$$

- For $V\left(\theta\right) = V_0e^{-2\frac{\left(k^2-2\right)}{k}\theta}, k^2-2 \neq 0$, the extra Noether point symmetries are $2t\partial_t + H^i\ ,\ t^2\partial_t + tH^i$ with first integrals

$$I_{H_1} = -\frac{d}{dt}\left(\frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}\right)\ ,\ I_{H_2} = -t\frac{d}{dt}\left(\frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}\right) + \frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}. \tag{139}$$

We note that in this case the system is the Ermakov–Pinney dynamical system (because it admits the Noether point symmetry algebra the $sl(2,R)$, hence the Lie symmetry algebra is at least $sl(2,R)$) .

- For $V\left(\theta\right) = V_0e^{-2\frac{\left(k^2-2\right)}{k}\theta} - \frac{N_0^2m}{k^2-1}e^{2k\theta}$, $k^2-2 \neq 0$, we have the Noether point symmetries $\frac{2}{\sqrt{m}}e^{\pm\sqrt{m}t}\partial_t \pm e^{\pm\sqrt{m}t}H^i$, $m =$ constant with corresponding first integral

$$I_{+,-} = e^{\pm2\sqrt{m}t}\left(\mp\frac{d}{dt}\left(\frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}\right) + 2\sqrt{m}\left(\frac{1}{2}\frac{r^2e^{2k\theta}}{k^2-1}\right)\right). \tag{140}$$

For this potential, the Noether point symmetries form the $sl\left(2,R\right)$ Lie algebra, i.e., the dynamical system is the two-dimensional Kepler–Ermakov system.

- The case $V\left(\theta\right)=0$ corresponds the two-dimensional free particle in flat space and the dynamical system admits seven additional Noether point symmetries.

11.3.2. Case $|k|=1$

We have to consider two cases i.e., $k=1$ and $k=-1$. It is enough to consider the case $k=1$, because the results for $k=-1$ are obtained directly from those for $k=1$ if we make the substitution $\theta_{(k=-1)}=-\bar{\theta}$.

For $k=1$, the homothetic algebra of the minisuperspace is given by the vector fields $K_{k=1}^{1,2}$ of (130,131) and the vector field

$$K_{k=1}^{3}=-r\left(\ln\left(re^{-\theta}\right)-1\right)\partial_{r}+\ln\left(re^{-\theta}\right)\partial_{\theta}. \tag{141}$$

Hence, the symmetry classification provides the following cases:

- For arbitrary potential $V\left(\theta\right)$, the dynamical system admits the Noether point symmetry ∂_{t}.

  All the rest of cases admit additional symmetries.
- If $V\left(\theta\right)=V_{0}e^{2\theta}$, we have the extra Noether point symmetries K^{1} , tK^{1} with first integrals (134) with $k=1$.
- If $V\left(\theta\right)=V_{0}e^{2\theta}-\frac{m}{2}\theta e^{2\theta}$, we have the Noether point symmetries $e^{\pm\sqrt{m}t}K^{1}$ with first integrals the (135) with $k=1$.
- Noether point symmetries generated by the KV K^{2}.
- If $V\left(\theta\right)=V_{0}e^{-2\theta}$, then we have the Noether point symmetries K^{2} , tK^{2} with first integrals

$$I_{2}^{\prime}=\frac{d}{dt}\left(\theta-\ln r\right)\ ,\ I_{2}^{\prime}=t\left[\frac{d}{dt}\left(\theta-\ln r\right)\right]-\left(\theta-\ln r\right).$$

- If $V\left(\theta\right)=V_{0}e^{-2\theta}-\frac{1}{4}pe^{2\theta}$ then we have the Noether point symmetries K^{2} , tK^{2} with first integrals

$$I_{1}^{\prime}=\frac{d}{dt}\left(\theta-\ln r\right)-pt\ ,\ I_{2}^{\prime}=t\left[\frac{d}{dt}\left(\theta-\ln r\right)\right]-\left(\theta-\ln r\right)-\frac{1}{2}pt^{2}.$$

- If $V\left(\theta\right)=0$, then the system becomes the free particle and admits seven extra Noether point symmetries.

The exact solutions of the models and their physical properties can be found in [121]. The results from the classification analysis are presented in Tables 3 and 4. For the notation of the admitted Lie algebra, we follow the Mubarakzyanov Classification Scheme [127–129].

Table 3. Noether symmetry classification for the Brans–Dicke action in a spatially flat Friedmann–Lemaître–Robertson–Walker metric FLRW spacetime (I).

$\lvert\mathbf{k}\rvert$	Potential	# Symmetries	Lie Algebra	Symmetries
	$V\left(\theta\right)$	1	A_{1}	∂_{t} ,
$\neq 1$	$V_{0}e^{2\theta}$	3	$A_{1}\otimes_{s}\left(2A_{1}\right)$	$\partial_{t},\ K^{1},\ tK^{1}$
$\neq 1$	$V_{0}e^{2\theta}-\frac{mN_{0}^{2}}{2(k^{2}-1)}e^{2k\theta}$	3	$A_{1}\otimes_{s}\left(2A_{1}\right)$	$\partial_{t},\ e^{\pm\sqrt{m}t}K^{1}$
$\neq 1$	$V_{0}e^{-2\theta}$	3	$A_{1}\otimes_{s}\left(2A_{1}\right)$	$\partial_{t},\ K^{2},\ tK^{2}$
$\neq 1$	$V_{0}e^{-2\theta}-\frac{mN_{0}^{2}}{2(k^{2}-1)}e^{2k\theta}$	3	$A_{1}\otimes_{s}\left(2A_{1}\right)$	$\partial_{t},\ e^{\pm\sqrt{m}t}K^{21}$
$\neq 1$	$V_{0}e^{2k\theta}$	2	$2A_{1}$	$\partial_{t},\ K^{3}$
$\neq 1$	$V_{0}e^{-2\frac{\left(k^{2}-2\right)}{k}\theta},k^{2}-2$	3	$Sl\left(2,R\right)$	$\partial_{t},\ 2t\partial_{t}+H^{i}\ ,\ t^{2}\partial_{t}+tH^{i}$
$\neq 1$	$V_{0}e^{-2\frac{\left(k^{2}-2\right)}{k}\theta}-\frac{N_{0}^{2}m}{k^{2}-1}e^{2k\theta}\ ,k^{2}-2\neq 0$	3	$Sl\left(2,R\right)$	$\partial_{t},\ \frac{2}{\sqrt{m}}e^{\pm\sqrt{m}t}\partial_{t}\pm e^{\pm\sqrt{m}t}H^{i}$

Table 4. Noether symmetry classification for the Brans–Dicke action in a spatially flat FLRW spacetime (II).

$\lvert\mathbf{k}\rvert$	Potential	# Symmetries	Lie Algebra	Symmetries
=1	$V_0 e^{2\theta}$	3	$A_1 \otimes_s (2A_1)$	$\partial_t,\ K^1,\ tK^1$
=1	$V_0 e^{2\theta} - \frac{m}{2}\theta e^{2\theta}$	3	$A_1 \otimes_s (2A_1)$	$\partial_t,\ e^{\pm\sqrt{m}t}K^1$
=1	$V_0 e^{-2\theta}$	3	$A_1 \otimes_s (2A_1)$	$\partial_t,\ K^2,\ tK^2$
=1	$V_0 e^{-2\theta} - \frac{1}{4} p e^{2\theta}$	3	$A_1 \otimes_s (2A_1)$	$\partial_t,\ K^2,\ tK^2$

11.4. $f(R)$-Gravity

$f(R)$-gravity (in the metric formalism) is a fourth-order theory where the Action Integral in spacetime is assumed to be

$$S = \int d^4x \sqrt{-g} f(R) . \tag{142}$$

In the case of FRW background and comoving observers, the resulting field equations follow from the Lagrangian [130]

$$L\left(t,a,\dot{a},R,\dot{R}\right) = 6af^{'}\dot{a}^2 + 6a^2 f^{''}\dot{a}\dot{R} + a^3\left(f^{'}R - f\right) - 6Kaf', \tag{143}$$

where $K = 0, \pm 1$ is the spatial curvature of the FRW spacetime, and a prime denotes derivative with respect to the dynamical parameter R, that is $f'(R) = \frac{df(R)}{dR}$. Since $f(R)$ theory can be written as a special case of Brans–Dicke theory, the so-called O'Hanlon gravity [130], the results will be similar to that of the previous analysis. However, for completeness, we present them below.

The classification scheme provides the following cases [119]:

- For arbitrary function $f(R)$, there exists the autonomous symmetry ∂_t, which derives the constraint equation.
- For $f(R) = R^{\frac{3}{2}}$, the theory admits the additional Noether point symmetries

$$K_1 = 2t\partial_t + \frac{4}{3}\partial_a - \frac{9}{2}\frac{f'}{f''}\partial_R,$$

$$K_2 = \frac{1}{a}\partial_a - \frac{1}{a^2}\frac{f'}{f''}\partial_R \ , \ K_2^* = t\left(\frac{1}{a}\partial_a - \frac{1}{a^2}\frac{f'}{f''}\partial_R\right),$$

with first integrals

$$\Phi_1 = 6a^2\dot{a}\sqrt{R} + 6\frac{a^3}{\sqrt{R}}\dot{R}, \tag{144}$$

$$\Phi_2 = \frac{d}{dt}\left(a\sqrt{R}\right) \ , \ \Phi_2^* = t\frac{d}{dt}\left(a\sqrt{R}\right) - a\sqrt{R}. \tag{145}$$

- For $f(R) = R^{\frac{7}{8}}$ and $K = 0$, the theory admits the additional Noether point symmetries

$$K_3 = 2t\partial_t + \frac{a}{2}\partial_a + \frac{1}{2}\frac{f'}{f''}\partial_R \ , \ K_3^* = t^2\partial_t + t\left(\frac{a}{2}\partial_a + \frac{1}{2}\frac{f'}{f''}\partial_R\right), \tag{146}$$

with corresponding first integrals

$$\Phi_3 = \frac{d}{dt}\left(a^3 R^{-\frac{1}{8}}\right) \ , \ \Phi_3^* = t\frac{d}{dt}\left(a^3 R^{-\frac{1}{8}}\right) - a^3 R^{-\frac{1}{8}}. \tag{147}$$

- The power-law theory $f(R) = R^n$ (with $n \neq 0, 1, \frac{3}{2}, \frac{7}{8}$) and for $K = 0$, or with K arbitrary and $n = 2$, the system admits the extra Noether point symmetry

$$K_1^* = 2t\partial_t + \left(\frac{4n}{3} - \frac{2}{3}\right)a\partial_t - 3\frac{f'}{f''}\partial_R,$$

with first integral

$$\Phi_1^* = a^2 R^{n-1} \dot{a} \left(2-n\right) + \frac{1}{2} a^3 R^{n-2} \dot{R} \left(2n-1\right) \left(n-1\right). \tag{148}$$

- For $f\left(R\right) = \left(R - 2\Lambda\right)^{3/2}$ the extra Noether point symmetries are

$$K_{(\pm)2} = e^{\pm\sqrt{m}t} \left(\frac{1}{a} \partial_a - \frac{1}{a^2} \frac{f'}{f''} \partial_R \right), \tag{149}$$

with corresponding first integral

$$\Phi_{(\pm)2} = e^{\pm\sqrt{m}t} \left(\frac{d}{dt} \left(a\sqrt{R-2L} \right) \mp 9\sqrt{m} a \sqrt{R-2\Lambda} \right). \tag{150}$$

- Finally, when $f\left(R\right) = \left(R - 2\Lambda\right)^{7/8}$, the field equations admit the Noether point symmetries

$$K_{(\pm)4} = \pm \frac{1}{\sqrt{m}} e^{\pm 2\sqrt{m}t} \partial_t + e^{\pm 2\sqrt{m}t} \left(\frac{a}{2} \partial_a + \frac{1}{2} \frac{f'}{f''} \partial_R \right), \tag{151}$$

with corresponding first integrals

$$\Phi_{(\pm)4} = \frac{d}{dt} \left(a^3 \left(R - 2\Lambda\right)^{-\frac{1}{8}} \right) \mp \frac{1}{2} \sqrt{m} a^3 \left(R - 2\Lambda\right)^{-\frac{1}{8}}. \tag{152}$$

In Table 5, we collect the results of the classification scheme for $f\left(R\right)$-gravity.

Table 5. Noether symmetry classification for $f(R)$ in FLRW spacetime.

Sp. Curv. K	$\mathbf{f}\left(\mathbf{R}\right)$	# Symmetries	Lie Algebra	Symmetries
$=0,\pm1$	Arbitrary	1	A_1	∂_t ,
$=0,\pm1$	$R^{\frac{3}{2}}$	4	$(2A_1) \otimes_s (2A_1)$	$\partial_t,\ K_1,\ K_2,\ K_2^*$
$=0$	$R^{\frac{7}{8}}$	3	$Sl\left(2,R\right)$	$\partial_t,\ K_3,\ K_3^*$
$=0$	R^n (with $n \neq 0,1,\frac{3}{2},\frac{7}{8}$)	2	$2A_1$	$\partial_t,\ K_1^*$
$=1$	R^2	2	$A_1 \otimes_s (2A_1)$	$\partial_t,\ K_{1(n=2)}^*$
$=0,\pm1$	$(R-2\Lambda)^{3/2}$	3	$A_1 \otimes_s (2A_1)$	$\partial_t,\ K_{(\pm)2}$
$=0$	$(R-2\Lambda)^{7/8}$	3	$Sl\left(2,R\right)$	$\partial_t,\ K_{(\pm)4}$

11.5. Two-Scalar Field Cosmology

We consider now a two-scalar field cosmological model in General Relativity with Action Integral

$$S = \int dx^4 \sqrt{-g} \left(R - \frac{1}{2} g_{ij} \left(\Phi^C\right) \Phi^{A,i} \Phi^{B,i} + V\left(\Phi^C\right) \right), \tag{153}$$

where H_{AB} describes the coupling between the two scalar fields $\Phi^A = (\phi, \psi)$ in the kinematic part. Moreover, we assume the metric tensor H_{AB} to be a maximally symmetric metric of constant curvature [122]. In such a scenario, it is not possible to define new fields in order to remove the coupling in the kinematic part.

Assuming again a spatially flat FRW spacetime and comoving observers the field equations are

$$-3a\dot{a}^2 + \frac{1}{2} a^3 H_{AB} \dot{\Phi}^A \dot{\Phi}^B + a^3 V\left(\Phi^C\right) = 0, \tag{154}$$

$$\ddot{a} + \frac{1}{2a} \dot{a}^2 + \frac{a}{4} H_{AB} \dot{\Phi}^A \dot{\Phi}^B - \frac{1}{2} a V = 0, \tag{155}$$

$$\ddot{\Phi}^A + \frac{3}{2a} \dot{a} \dot{\Phi}^A + \tilde{\Gamma}^A_{BC} \dot{\Phi}^B \dot{\Phi}^C + H^{AB} V_{,B} = 0, \tag{156}$$

where $\tilde{\Gamma}^{A}_{BC}$ are the connection coefficients for the metric $H_{AB}\left(\Phi^{C}\right)$.

In the mini superspace defined by the variables $\{a,\Phi\}$, we introduce the new variables $\{u,\phi\}$ by the requirements $a=\left(\frac{3}{8}\right)^{\frac{1}{3}}u^{\frac{2}{3}}$, $diag\left(1,e^{2\phi}\right)=h_{AB}\dot{\Phi}^{A}\dot{\Phi}^{B}$ and the field equations become

$$-\frac{1}{2}\dot{u}^{2}+\frac{1}{2}u^{2}\left(\dot{\phi}^{2}+e^{2\phi}\,\dot{\psi}^{2}\right)+u^{2}V\left(\phi,\psi\right)=0, \tag{157}$$

$$\ddot{u}+u\dot{\phi}^{2}+ue^{2\phi}\,\dot{\psi}^{2}-2uV=0, \tag{158}$$

$$\ddot{\phi}+\frac{2}{u}\dot{u}\dot{\phi}-e^{2\phi}\dot{\psi}^{2}+V_{,\phi}=0, \tag{159}$$

$$\ddot{\psi}+\frac{2}{u}\dot{u}\dot{\psi}+2\dot{\phi}\dot{\psi}+e^{-2\phi}V_{,\psi}=0. \tag{160}$$

These follow from the point-like Lagrangian

$$L\left(t,u,\dot{u},\phi,\dot{\phi},\psi,\dot{\psi}\right)=-\frac{1}{2}\dot{u}^{2}+\frac{1}{2}u^{2}\left(\dot{\phi}^{2}+e^{2\phi}\,\dot{\psi}^{2}\right)-u^{2}V\left(\phi,\psi\right), \tag{161}$$

which defines the 3d flat Lorentzian kinetic metric $-\frac{1}{2}\dot{u}+\frac{1}{2}u^{2}\left(\dot{\phi}^{2}+e^{2\phi}\,\dot{\psi}^{2}\right)$ and the effective potential $V_{eff.}=u^{2}V\left(\phi,\psi\right)$.

The kinetic metric admits a seven-dimensional homothetic algebra consisting of the three gradient KVs (translations)

$$K^{1}=-\frac{1}{2}\left(e^{\phi}\left(1+\psi^{2}\right)+e^{-\phi}\right)\partial_{u}+\frac{1}{2u}\left(e^{\phi}\left(1+\psi^{2}\right)-e^{-\phi}\right)\partial_{\phi}+\frac{1}{u}\psi e^{-\phi}\partial_{\psi},$$

$$K^{2}=-\frac{1}{2}\left(e^{\phi}\left(1-\psi^{2}\right)-e^{-\phi}\right)\partial_{u}+\frac{1}{2u}\left(e^{\phi}\left(1-\psi^{2}\right)+e^{-\phi}\right)\partial_{\phi}-\frac{1}{u}\psi e^{-\phi}\partial_{\psi},$$
$$K^{3}=-\psi e^{\theta}\partial_{u}+\frac{1}{u}\psi e^{\phi}\partial_{\phi}+\frac{1}{u}e^{-\phi}\partial_{\psi},$$

with corresponding gradient functions $S_{(1-3)}$ given by

$$S_{(1)}=\frac{1}{2}u\left(e^{\phi}\left(1+\psi^{2}\right)+e^{-\phi}\right),$$
$$S_{(2)}=\frac{1}{2}u\left(e^{\phi}\left(1-\psi^{2}\right)-e^{-\phi}\right),$$
$$S_{(3)}=u\psi e^{\theta},$$

the three non-gradient KVs (rotations) which span the $SO\left(3\right)$ algebra

$$X_{12}=\partial_{\psi}\ ,\ \ X_{23}=\partial_{\phi}+\psi\partial_{\psi},X_{13}=\psi\partial_{\phi}+\frac{1}{2}\left(\psi^{2}-e^{2\phi}\right)\partial_{\phi}$$

and the gradient proper HV

$$H_{V}=u\partial_{u}\ ,\ \ \psi_{H_{V}}=1.$$

The classification of the Noether symmetries for the various potentials $V\left(\phi,\psi\right)$ is as follows [122]:

- For arbitrary potential $V\left(\phi,\psi\right)$, the field equations admit the Noether point symmetry ∂_{t} which provides the constraint equation of General Relativity.
- For $V\left(\phi,\psi\right)=0$, the dynamical system is maximally symmetric and admits in total twelve Noether point symmetries.
- For $V\left(\phi,\psi\right)=V\left(\phi\right)$, there exists the additional Noether point symmetry, the vector field X_{12}, with conservation law the angular momentum on the two-dimensional sphere, that is,

$$\Phi_{12}=e^{2\phi}\dot{\psi}.$$

- For $V_A(\phi,\psi) = \frac{\omega_0^2}{2}u^2 + \frac{\mu^2}{2\left(1-a_0^2\right)}\left(S_{(\mu)} + a_0 S_{(\nu)}\right)^2 - \frac{\omega_3^2}{2}S_{(\sigma)}^2$, $a_0 \neq 1$, the system admits six additional Noether point symmetries given by the vector fields

$$T_1(t)K^1 , \; T_2(t)K^2 , T_3(t)K^3,$$

where

$$T^A_{,tt} = \omega^\gamma_{\ \delta} T^\delta \; , \; \omega^\gamma_{\ \delta} = diag\left((\omega_1)^2, (\omega_2)^2, (\omega_3)^3\right)$$

and $\mu, \nu, \sigma = 1,2,3$. The corresponding Noether integrals are expressed as follows:

$$I_C^\gamma = T_\gamma \frac{d}{dt} S_{(\gamma)} - T_{\gamma,t} S_{(\gamma)}. \tag{162}$$

However, when two constants ω_A are equal, for instance, $\omega_\mu = \omega_\nu$, then the dynamical system admits an extra Noether symmetry. That is, it admits the rotation normal to the plane defined by the axes x^μ, x^ν given by the vector

$$X = x^\nu \partial_\mu - \varepsilon x^\mu \partial_\nu,$$

where $\varepsilon = -1$ if $x^\nu / x^\mu = x$ and $\varepsilon = 1$ if $x^\nu / x^\mu \neq 1$.

- For the potential being $V_B(\phi,\psi) = \frac{\omega_0^2}{2}u^2 + \frac{\mu^2}{2\left(1-a_0^2\right)}\left(S_{(\mu)} + a_0 S_{(\nu)}\right)^2 - \frac{\omega_3^2}{2}S_{(\sigma)}^2$, $a_0 \neq 1$, the dynamical system admits the extra Noether symmetries

$$\bar{T}(t)\left(K^\mu + a_0 K^\nu\right) \; , \; T'(t)K^\sigma \; , \; T^*(t)\left(a_0 K^\mu + K^\nu\right),$$

where the functions T, T' and $\bar{T}$ are given by the linear second-order differential equations

$$\bar{T}_{,tt} = \left(\mu^2 + \omega_0^2\right)\bar{T} \; , \; T_{\sigma,tt} = \left(\omega_3^2 + \omega_0^2\right)T_\sigma \; , \; T^*_{,tt} = \omega_0^2 \bar{T}, \tag{163}$$

and $\mu, \nu, \sigma = 1,2,3$. Finally, the corresponding Noether integrals are as follows:

$$\Phi_{1a2} = \bar{T}\frac{d}{dt}\left(S_{(\mu)} + a_0 S_{(\nu)}\right) - \bar{T}_{,t}\left(S_{(\mu)} + a_0 S_{(\nu)}\right), \tag{164}$$

$$\Phi_3 = T_\sigma \frac{d}{dt} S_{(\sigma)} - T_{\sigma,t} S_{(\sigma)}, \tag{165}$$

$$\Phi_{a12} = T^* \frac{d}{dt}\left(a_0 S_{(\mu)} + S_{(\nu)}\right) - T^*_{,t}\left(a_0 S_{(\mu)} + S_{(\nu)}\right). \tag{166}$$

In both of the last cases, from the admitted algebras of Lie symmetries, it is easy to recognize that the gravitational field equations can be linearized. Indeed, for the potential $V_A(\phi,\psi)$ under the coordinate transformation

$$x = \frac{1}{2}u\left(e^\phi\left(1+\psi^2\right) + e^{-\phi}\right), \tag{167}$$

$$y = \frac{1}{2}u\left(e^\phi\left(1-\psi^2\right) - e^{-\phi}\right), \tag{168}$$

$$z = u\psi e^\phi, \tag{169}$$

the field equations become

$$\ddot{x} - (\omega_1)^2 x = 0, \tag{170}$$

$$\ddot{y} - (\omega_2)^2 y = 0, \tag{171}$$

$$\ddot{z} - (\omega_3)^2 z = 0, \tag{172}$$

$$-\frac{1}{2}\dot{x}^2+\frac{1}{2}\dot{y}+\frac{1}{2}\dot{z}^2+\frac{\omega_1^2}{2}x^2-\frac{\omega_2^2}{2}y^2-\frac{\omega_3^2}{2}z^2=0, \tag{173}$$

which is the three-dimensional "unharmonic-oscillator".

On the other hand, for the potential $V_B(\phi,\psi)$, we perform the additional transformation

$$x=(w+v)\ ,\ y=\frac{1}{a_0}(w-v)\ ,\ z=z, \tag{174}$$

and the field equations are linearized as follows:

$$\ddot{w}-\left(\mu^2+\omega_0^2\right)w=0, \tag{175}$$

$$\ddot{v}+\frac{a_0^2+1}{a_0^2-1}\mu^2 w-\omega_0^2 \nu=0, \tag{176}$$

$$\ddot{z}-\left(\omega_3^2+\omega_0^2\right)z=0, \tag{177}$$

$$0=\frac{1}{2}\left[\left(\frac{1}{a_0^2}-1\right)\dot{w}^2-\left(\frac{1}{a_0^2}+1\right)dwdv+\left(\frac{1}{a_0^2}-1\right)dv^2+\frac{1}{2}z^2\right]+ \\ -\frac{2\mu^2}{\left(a_0^2-1\right)}w^2-\frac{1}{2}\left(\omega_3^2+\omega_0^2\right)z^2+\frac{\omega_0^2}{2}\left((w+v)^2-\frac{1}{a_0^2}(w-v)^2\right). \tag{178}$$

11.6. Galilean Cosmology

The cubic Galilean cosmological model in a spatially flat FRW spacetime with comoving observers is defined by the Lagrangian [131]

$$L\left(a,\dot{a},\phi,\dot{\phi}\right)=3\,a\,\dot{a}^2-\frac{1}{2}a^3\dot{\phi}^2+a^3V(\phi)+g(\phi)a^2\dot{a}\,\dot{\phi}^3-\frac{g'(\phi)}{6}a^3\,\dot{\phi}^4. \tag{179}$$

From the symmetry condition, we should determine two functions, $V(\phi)$ and $g(\phi)$. Indeed, we find that, when [104]

$$V(\phi)=V_0e^{-\lambda\phi}\quad\text{and}\quad g(\phi)=g_0e^{\lambda\phi}, \tag{180}$$

Lagrangian (179) admits the Noether point symmetries

$$X_1=\partial_t\ ,\ X_2=t\partial_t+\frac{a}{3}\partial_a+\frac{2}{\lambda}\partial_\phi, \tag{181}$$

which form the $2A_1$ Lie algebra.

The Noether point symmetry X_1 provides as first integral the constraint equation, while X_2 gives the first integral

$$\Phi_2=-\left(2\,a^2\dot{a}-\frac{2}{\lambda}a^3\dot{\phi}\right)+g_0e^{\lambda\phi}a^3\dot{\phi}^3-\frac{6}{\lambda}g_0a^2e^{\lambda\phi}\dot{a}\dot{\phi}^2. \tag{182}$$

Furthermore, the same first integral exists in the limit in which $V_0=0$. In addition, we remark that when the universe is dominated by the potential of the scalar field, then $g(\phi)\to 0$, and the model reduces to that of a minimally coupled scalar field.

12. Higher-Order Symmetries in Cosmology

In the previous section, we presented classification of cosmological models based on point symmetries. However, these are not the only cases where first integrals are used. Indeed, it is possible for one to extend the classification scheme by applying non-point symmetries, such as the contact symmetries.

In particular, for Lagrangians of the form (93), it has been found that the vector field $X = K^i_j\left(t, q^k\right)\dot{x}^i\partial_i$ is a contact symmetry for the Action Integral iff the following conditions are satisfied [10]

$$K_{(ij;k)} = 0, \tag{183}$$

$$K_{ij,t} = 0 \ , \ f_{,t} = 0, \tag{184}$$

$$K^{ij}V_j + f_{,i} = 0. \tag{185}$$

Conditions (183)–(185) follow directly from the application of the weak Noether condition. From the symmetry condition (184), it follows that $K^i_j = K^i_j\left(q^k\right)$ and $f = f\left(q^k\right)$. Furthermore, the second-theorem of Noether provides the first integral

$$I = K_{ij}\dot{x}^i\dot{x}^j - f\left(x^i\right). \tag{186}$$

Condition (183) means that the second rank tensor $K^i_j\left(q^k\right)$ is a Killing tensor of order 2 of the metric g_{ij}. Condition (185) is a constraint relating the potential with the Killing tensor K^{ij} and the Noether function f. Application of contact symmetries in cosmological studies can be found in [72,132–135]. In the following, we present the results for the classification of contact symmetries in scalar-field cosmology and $f(R)$-gravity.

12.1. Scalar-Field Cosmology from Contact Symmetries

In the polar coordinates (117), the Lagrangian of the field equations in scalar field cosmology become

$$L\left(r, \theta, \dot{r}, \dot{\theta}\right) = -\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - r^2V\left(\theta\right) \tag{187}$$

while the Killing tensors of rank two for a two-dimensional flat space in Cartesian coordinates $\{x, y\}$ are

$$K_{ij} = \begin{pmatrix} c_1y^2 + 2c_2y + c_3 & c_6 - c_1yx - c_2x - c_4y \\ c_6 - c_1yx - c_2x - c_4y & c_1x^2 + 2c_4x + c_5 \end{pmatrix}. \tag{188}$$

Thus, condition (185) provides the following cases [132]:

Case A: For the hyperbolic Potential $V\left(\theta\right) = c_1 + c_2\cosh^2\theta$, the field equations admit the contact symmetry

$$X_1 = -\left(\cosh^2\theta\dot{r} + \frac{1}{2}r\sinh\left(2\theta\right)\dot{\theta}\right)\partial_r + \frac{1}{r}\left(\frac{1}{2}\sinh\left(2\theta\right)\dot{r} + r\sinh\theta\,\dot{\theta}\right)\partial_{\dot{r}} \tag{189}$$

with corresponding Noether Integral

$$I_1 = \left(\cosh\theta\dot{r} + r\sinh\theta\,\dot{\theta}\right)^2 - 2r^2\left(c_1 + c_2\right)\cosh^2\theta. \tag{190}$$

In the special case where $c_2 = 3c_1$, the field equations admit the second contact symmetry

$$\bar{X} = -r^2\sinh\theta\,\dot{\theta}\partial_r + \left(\sinh\theta\dot{r} + 2r\cosh\theta\,\dot{\theta}\right)\partial_\theta \tag{191}$$

with corresponding Noether Integral

$$\bar{I}_1 = \left(r^2\sinh\theta\,\dot{r}\dot{\theta} + r^3\cosh\theta\,\dot{\theta}^2\right) + 2c_1r^3\cosh\theta\sinh^2\theta. \tag{192}$$

Case B: For the potential $V(\theta) = c_1\left(1 + 3\cosh^2\theta\right) + c_2\left(3\cosh\theta + \cosh^3\theta\right)$, there exists the contact symmetry

$$I_2 = \left(r^2 \sinh\theta\, \dot{r}\dot{\theta} + r^3 \cosh\theta\, \dot{\theta}^2\right) + r^3 \sinh^2\theta\left(2c_1\cosh\theta + c_2\left(1 + \cosh^2\theta\right)\right) \tag{193}$$

with Noether Integral

$$I_2 = \left(r^2 \sinh\theta\, \dot{r}\dot{\theta} + r^3 \cosh\theta\, \dot{\theta}^2\right) + r^3 \sinh^2\theta\left(2c_1\cosh\theta + c_2\left(1 + \cosh^2\theta\right)\right). \tag{194}$$

Case C: For potential $V(\theta) = c_1\left(1 - 3\sinh^2\theta\right) + c_2\left(3\sinh\theta - \sinh^3\theta\right)$, the admitted contact symmetry is

$$\bar{X}_2 = -r^2\cosh\theta\, \dot{\theta}\partial_r + \left(\cosh\theta\, \dot{r} + 2r\sinh\theta\, \dot{\theta}\right)\partial_\theta \tag{195}$$

with corresponding Noether Integral

$$\bar{I}_2 = \left[\cosh\theta\, \dot{r} + r\sinh\theta\, \dot{\theta}\right) r^2\dot{\theta} - r^3\cosh^2\theta\left(2c_1\sinh\theta - c_2\left(1 - \sinh^2\theta\right)\right]. \tag{196}$$

This potential is equivalent to case B under the transformation $\theta = \bar{\theta} + i\frac{\pi}{2}$.
Case D: For $V(\theta) = c_1 + c_2 e^{2\theta}$ the admitted contact symmetry is

$$X_3 = -e^{2\theta}\left(\dot{r} + r\dot{\theta}\right)\partial_r + \frac{e^{2\theta}}{r}\left(\dot{r} + r\dot{\theta}\right)\partial_\theta \tag{197}$$

with corresponding Noether Integral

$$I_3 = e^{2\theta}\left(\left(\dot{r} + r\dot{\theta}\right)^2 - 2r^2 c_1\right). \tag{198}$$

Moreover, when $c_1 = 0$, the dynamical system admits the additional contact symmetry

$$X_3 = -r^2 e^{\theta}\dot{\theta}\partial_r + e^{\theta}\left(\dot{r} + 2r\dot{\theta}\right)\partial_\theta \tag{199}$$

with corresponding Noether Integral

$$\bar{I}_3 = r^2 e^{\theta}\left(\dot{r}\dot{\theta} + r\dot{\theta}^2\right) + \frac{2}{3}c_2 r^3 e^{3\theta}. \tag{200}$$

Case E: Finally, for the potential

$$V(\theta) = c_1 e^{2\theta} + c_2 e^{3\theta}, \tag{201}$$

the field equations admit the first integral

$$I_3 = r^2 e^{\theta}\left(\dot{r}\dot{\theta} + r\dot{\theta}^2\right) + r^3 e^{3\theta}\left(\frac{2}{3}c_1 + c_2 e^{\theta}\right) \tag{202}$$

generated by the contact symmetry (199). It is important to note that in all cases the results remain the same under the transformation $\theta \to -\theta$.

12.2. $f(R)$-Gravity from Contact Symmetries

Without loss of generality, we define $\phi = f'(R)$, where now $f(R)$-gravity can be written in its equivalent form as a Brans–Dicke scalar field cosmological model. Specifically, the Lagrangian of the field equations is written equivalently as

$$L(a, \dot{a}, \phi, \dot{\phi}) = 6a\phi\dot{a}^2 + 6a^2\dot{a}\dot{\phi} + a^3 V(\phi), \tag{203}$$

where

$$V(\phi) = \left(f'R - f\right) \text{ or } V\left(f'(R)\right) = \left(f'R - f\right). \quad (204)$$

The classification in terms of the contact symmetries provides the following five cases for the potential $V(\phi)$ and the corresponding first integrals.

Case A: For $V_I(\phi) = V_1\phi + V_2\phi^3$, the field equations admit the quadratic first integral

$$I_I = 3(\phi\dot{a} + a\dot{\phi})^2 - V_1\, a^2\phi^2. \quad (205)$$

Case B: For $V_{II}(\phi) = V_1\phi - V_2\phi^{-7}$, the field equations admit the quadratic first integral

$$I_{II} = 3a^4(\phi\dot{a} - a\dot{\phi})^2 + 4V_2 a^6\phi^{-6}, \quad (206)$$

Case C: For $V_{III}(\phi) = V_1 - V_2\phi^{-\frac{1}{2}}$, the field equations admit the quadratic first integral

$$I_{III} = 6a^3\dot{a}(a\dot{\phi} - \phi\dot{a}) - a^5\left(\frac{3}{5}V_1 - V_2\phi^{-\frac{1}{2}}\right). \quad (207)$$

Case D: For $V_{IV}(\phi) = V_1\phi^3 + V_2\phi^4$, the field equations admit the quadratic first integral

$$I_{IV} = 12a^2\left(a^2\dot{\phi}^2 - \phi^2\dot{a}^2\right) + (a\phi)^4(3V_1 + 4V_2\phi). \quad (208)$$

Case E: For $V_V(\phi) = V_1\left(\phi^3 + \beta\phi\right) + V_2\left(\phi^4 + 6\beta\phi^2 + \beta^2\right)$, the field equations admit the quadratic first integral

$$\begin{aligned} I_V = {} & 12a^2\left[\left(\beta - \phi^2\right)\dot{a}^2 + a^2\dot{\phi}^2\right] + \\ & - a^4\left(\beta - \phi^2\right)\left[V_1\left(\beta + 3\phi^2\right) + 4V_2\left(3\beta\phi + \phi^3\right)\right]. \end{aligned} \quad (209)$$

However, in order to derive the function $f(R)$, one has to solve the Clairaut Equation (204). For the above cases, Clairaut equation has a closed-form solution only for some particular forms of $V(\phi)$. The analytic forms of $f(R)$ functions that admit contact symmetries are presented in Table 6.

Table 6. Analytic forms of $f(R)$ theory where the field equations admit contact symmetries.

Potential $V(\phi)$	Function $f(R)$
$V_I(\phi)$	$(R - V_1)^{\frac{3}{2}}$
$V_{II}(\phi)$	$(R - V_1)^{\frac{7}{8}}$,
$V_{III}(\phi)$	$R^{\frac{1}{3}} - V_1$
$V_{IV}(\phi)$, $V_1 = 0$	R^4
$V_{IV}(\phi)$, $V_2 = 0$	$R^{\frac{3}{2}}$
$V_V(\phi)$, $V_1 = \pm 4V_2\sqrt{\beta}$	$\mp\sqrt{\beta}R + R^{\frac{4}{3}}$

13. Conclusions

In this work, we discussed the dark energy problem using the classification of the cosmological models which are based on the FRW background for comoving observers using the first integrals provided by the Noether symmetries of the kinetic metric. We discussed the definitions for the Lie and Noether symmetries (point and generalized) for conservative holonomic dynamical systems. Moreover, we established the relation of Lie and Noether symmetries with the properties of the underlying geometry for singular and regular dynamical systems. In particular, for regular dynamical systems, we found that the generators of Noether symmetries are the elements of the homothetic algebra of the mini superspace metric, whereas, for the singular systems, the generators of Noether symmetries are constructed by the CKVs of the mini superspace metric.

These geometric results have been used to develop a geometric scenario based on the admitted Noether point symmetries of the mini superspace metric which leads to the classification scheme of the dark energy models. We demonstrated the application of this scenario to the most well known cosmological models including the modified theories of gravity and derived in many of them analytical cosmological solutions. This scenario is not limited to the case of cosmological models and can be applied to other areas of study of dynamical equations, especially in the case of general holonomic systems.

Author Contributions: The authors contributed equally in all the sections of the paper.

Funding: This research was funded by Fondo Nacional de Desarrollo Científico y Tecnológico: 3160121.

Acknowledgments: A.P. acknowledges the financial support of FONDECYT Grant No. 3160121.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hall, G.S. *Symmetries and Curvature Structure in General Relativity*; World Scientific Lecture Notes in Physics 46; World Scientific: Singapore, 2004.
2. Lie, S. *Theorie der Transformationsgruppen I*; B. G. Teubner: Leipzig, Germany, 1888.
3. Lie, S. *Theorie der Transformationsgruppen II*; B. G. Teubner: Leipzig, Germany, 1888.
4. Lie, S. *Theorie der Transformationsgruppen III*; B. G. Teubner: Leipzig, Germany, 1888.
5. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
6. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
7. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws*; CRS Press LLC: Boca Raton, FL, USA, 2000.
8. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
9. Crampin, M. Hidden symmetries and Killing tensors. *Rep. Math. Phys.* **1984**, *20*, 31–40. [CrossRef]
10. Kalotas, T.M.; Wybourne, B.G. Dynamical noether symmetries. *J. Phys. A Math. Gen.* **1982**, *15*, 2077. [CrossRef]
11. Prince, G.E.; Eliezer, C.J. On the Lie symmetries of the classical Kepler problem. *J. Phys. A Math. Gen.* **1981**, *14*, 587. [CrossRef]
12. Lutzky, M. Symmetry groups and conserved quantities for the harmonic oscillator. *J. Phys. A Math. Gen.* **1978**, *11*, 249. [CrossRef]
13. Govinder, K.S.; Leach, P.G.L. On the determination of non-local symmetries. *J. Phys. A Math. Gen.* **1995**, *28*, 5349. [CrossRef]
14. Sarlet, W.; Cantrijin, F. Generalizations of Noether's theorem in classical mechanics. *SIAM Rev.* **1981**, *23*, 467–494. [CrossRef]
15. Mahomed, F.M. Symmetry group classification of ordinary differential equations: Survey of some results. *Math. Meth. Appl. Sci.* **2007**, *30*, 1995–2012. [CrossRef]
16. Nucci, M.C. The role of symmetries in solving differential equations. *Math. Comp. Mod.* **1997**, *25*, 181–193. [CrossRef]
17. Harrison, B.K. The differential form method for finding symmetries. *Sigma* **2005**, *1*, 001. [CrossRef]
18. Abraham-Shrauner, B.; Guo, A. Hidden symmetries associated with the projective group of nonlinear first-order ordinary differential equations. *J. Phys. A Math. Gen.* **1992**, *25*, 5597. [CrossRef]
19. Popovych, R.O.J. On Lie reduction of the Navier-Stokes equations. *Nonlinear Math. Phys.* **1995**, *2*, 301–311. [CrossRef]
20. Katzin, G.H.; Levine, J. Geodesic first integrals with explicit path-parameter dependence in Riemannian space-times. *J. Math. Phys.* **1981**, *22*, 1878–1891. [CrossRef]
21. Noether, E. Invariante variationsprobleme, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen. *Math.-Phys. Klasse* **1918**, 235–257.
22. Singer, S.F. *Symmetry in Mechanics*; Birkhauser Boston: Boston, MA, USA, 2004.

23. Costa, G.; Fogli, G. *Symmetry and Group Theory in Particle Physics*; Lecture Notes in Physics; Springer: Berlin, Germany, 2012.
24. Sundermeyer, K. *Symmetries in Fundamental Physics*; Springer: Heidelberg, Germany, 2014.
25. Trautman, A. *Conservation Laws in General Relativity, in Gravitation, and Introduction to Current Research*; Witten, L., Ed.; Willey: New York, NY, USA, 1962.
26. Tsamparlis, M.; Paliathanasis, A. Lie symmetries of geodesic equations and projective collineations. *Nonlinear Dyn.* **2010**, *62*, 203–214. [CrossRef]
27. Tsamparlis, M.; Paliathanasis, A. Lie and Noether symmetries of geodesic equations and collineations. *Gen. Relativ. Gravit.* **2010**, *42*, 2957–2980. [CrossRef]
28. Paliathanasis, A.; Tsamparlis, M. Lie point symmetries of a general class of PDEs: The heat equation. *J. Geom. Phys.* **2012**, *62*, 2443–2456. [CrossRef]
29. Paliathanasis, A.; Tsamparlis, M. Lie and Noether point symmetries of a class of quasilinear systems of second-order differential equations. *J. Geom. Phys.* **2016**, *107*, 45–59. [CrossRef]
30. Tsamparlis, M.; Paliathanasis, A. Two-dimensional dynamical systems which admit Lie and Noether symmetries. *J. Phys. A Math. Theor.* **2011**, *44*, 175202. [CrossRef]
31. Hojman, S.A. A new conservation law constructed without using either Lagrangians or Hamiltonians. *J. Phys. A Math. Gen.* **1992**, *25*, L291. [CrossRef]
32. Djukic, D.C. A procedure for finding first integrals of mechanical systems with gauge-variant Lagrangians. *Int. J. Non-Linear Mech.* **1973**, *8*, 479–488. [CrossRef]
33. Nucci, M.C.; Leach, P.G.L. Jacobi's last multiplier and the complete symmetry group of the Euler–Poinsot system. *J. Non. Math. Phys.* **2002**, *9*, 110–121. [CrossRef]
34. Nucci, M.C.; Leach, P.G.L. The Jacobi Last Multiplier and its applications in mechanics. *Phys. Scr.* **2008**, *78*, 065011. [CrossRef]
35. Nucci, M.C.; Tamazhmani, K.M. Lagrangians for dissipative nonlinear oscillators: The method of Jacobi last multiplier. *J. Non. Math. Phys.* **2010**, *17*, 167–178. [CrossRef]
36. Mahomed, F.M.; Leach, P.G.L. Lie algebras associated with scalar second-order ordinary differential equations. *J. Math. Phys.* **1989**, *30*, 2770–2777. [CrossRef]
37. Mahomed, F.M.; Leach, P.G.L. Symmetry Lie algebras of nth order ordinary differential equations. *J. Math. Anal. Appl.* **1990**, *151*, 80–107. [CrossRef]
38. Wulfman, C.E.; Wybourne, B.G. The Lie group of Newton's and Lagrange's equations for the harmonic oscillator. *J. Phys. A Math. Gen.* **1976**, *9*, 507. [CrossRef]
39. Levy-Leblond, J.M. Conservation laws for gauge-variant Lagrangians in classical mechanics. *Am. J. Phys.* **1971**, *39*, 502–506. [CrossRef]
40. Werner, A.C.J.; Eliezer, J. The lengthening pendulum. *Aust. Math. Soc.* **1969**, *9*, 331–336. [CrossRef]
41. Lewis, H.; Ralph, J. Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians. *Phys. Rev. Lett.* **1967**, *18*, 510. [CrossRef]
42. Ermakov, V. Second order differential equations. Conditions of complete integrability. *Universita Izvestia Kiev* **1880**, *9*, 1–25. [CrossRef]
43. Pinney, E. The nonlinear differential equation $y'' + p(x)\,y + cy^{-3} = 0$. *Proc. Am. Math. Soc.* **1950**, *1*, 681. [CrossRef]
44. Rogers, C.; Hoenselaers, C.; Ray, J.R. On (2+ 1)-dimensional Ermakov systems. *J. Phys. A Math. Gen.* **1993**, *26*, 2625. [CrossRef]
45. Schief, W.K.; Rogerts, C.; Bassom, A.P. Ermakov systems of arbitrary order and dimension: Structure and linearization. *J. Phys. A Math. Gen.* **1996**, *29*, 903. [CrossRef]
46. Leach, P.G.L.; Andriopoulos, K. The Ermakov equation: A commentary. *Appl. Anal. Discrete Math.* **2008**, *2*, 146–157. [CrossRef]
47. Tsamparlis, M.; Paliathanasis, A. Generalizing the autonomous Kepler–Ermakov system in a Riemannian space. A note on the construction of the Ermakov–Lewis invariant. *J. Phys. A Math. Theor.* **2012**, *45*, 275202. [CrossRef]
48. Moyo, S.; Leach, P.G.L. A note on the construction of the Ermakov–Lewis invariant. *J. Phys. A Math. Gen.* **2002**, *35*, 5333. [CrossRef]

49. Katzin, G.H.; Levine, J.; Davis, R.W. Curvature Collineations: A Fundamental Symmetry Property of the Space-Times of General Relativity Defined by the Vanishing Lie Derivative of the Riemann Curvature Tensor. *J. Math. Phys.* **1969**, *10*, 617–629. [CrossRef]
50. Hall, G.S.; Roy, I.M. Some remarks on special conformal and special projective symmetries in general relativity. *Gen. Relativ. Gravit.* **1997**, *29*, 827–835. [CrossRef]
51. Karpathopoulos, L.; Paliathanasis, A.; Tsamparlis, M. Lie and Noether point symmetries for a class of nonautonomous dynamical systems. *Math. Phys.* **2017**, *58*, 082901. [CrossRef]
52. Christodoulakis, T.; Dimakis, N.; Terzis, P.A. Lie point and variational symmetries in minisuperspace Einstein gravity. *J. Phys. A Math. Theor.* **2014**, *47*, 095202. [CrossRef]
53. Gordon, T.J. On the symmetries and invariants of the harmonic oscillator. *J. Phys. A Math. Theor.* **1986**, *19*, 183. [CrossRef]
54. Weinberg, S. The cosmological constant problem. *Rev. Mod. Phys.* **1989**, *61*, 1. [CrossRef]
55. Peebles, P.J.; Ratra, B. The cosmological constant and dark energy. *Rev. Mod. Phys.* **2003**, *75*, 559. [CrossRef]
56. Padmanabhan, T. Cosmological constant—The weight of the vacuum. *Phys. Rept.* **2003**, *380*, 235–320. [CrossRef]
57. Ratra, B.; Peebles, P.J.E. Cosmological consequences of a rolling homogeneous scalar field. *Phys. Rev D* **1988**, *37*, 3406. [CrossRef]
58. Overduin, J.M.; Cooperstock, F.I. Evolution of the scale factor with a variable cosmological term. *Phys. Rev. D* **1998**, *58*, 043506. [CrossRef]
59. Basilakos, S.; Plionis, M.; Solà, J. Hubble expansion and structure formation in time varying vacuum models. *Phys. Rev. D* **2009**, *80*, 3511. [CrossRef]
60. Starobinsky, A.A. A new type of isotropic cosmological models without singularity. *Phys. Lett. B* **1980**, *91*, 99–102. [CrossRef]
61. Gorini, V.; Kamenshchik, A.; Moshella, U. Can the Chaplygin gas be a plausible model for dark energy? *Phys. Rev. D* **2003**, *67*, 063509. [CrossRef]
62. Barrow, J.D.; Saich, P. Scalar-field cosmologies. *Class. Quantum Grav.* **1993**, *10*, 279. [CrossRef]
63. Rubano, C.; Barrow, J.D. Scaling solutions and reconstruction of scalar field potentials. *Phys. Rev. D* **2001**, *64*, 127301. [CrossRef]
64. Gariel, J.; le Denmat, G. Matter creation and bulk viscosity in early cosmology. *Phys. Lett. A* **1995**, *200*, 11–16. [CrossRef]
65. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified gravity and cosmology. *Phys. Rep.* **2012**, *513*, 1–189. [CrossRef]
66. Joyce, A.; Lombriser, L.; Schmidt, F. Dark energy versus modified gravity. *Annu. Rev. Nucl. Part. Sci.* **2016**, *66*, 95–122. [CrossRef]
67. Amendola, L.; Polarski, D.; Tsujikawa, S. Are f(R)Dark Energy Models Cosmologically Viable? *Phys. Rev. Lett.* **2007**, *98*, 131302. [CrossRef] [PubMed]
68. Battye, R.A.; Bolliet, B.; Pearson, J.A. f(R) gravity as a dark energy fluid. *Phys. Rev. D* **2016**, *93*, 044026. [CrossRef]
69. Basilakos, S.; Kouretsis, A.P.; Saridakis, E.N.; Stavrinos, P. Resembling dark energy and modified gravity with Finsler-Randers cosmology. *Phys. Rev. D* **2013**, *88*, 123510. [CrossRef]
70. Geng, C.-Q.; Lee, C.-C.; Saridakis, E.N.; Wu, Y.P. "Teleparallel" dark energy. *Phys. Lett. B* **2011**, *704*, 384–387. [CrossRef]
71. De Ritis, R.; Marmo, G.; Platania, G.; Rubano, C.; Scudellaro, P.; Stornaiolo, C. New approach to find exact solutions for cosmological models with a scalar field. *Phys. Rev. D.* **1990**, *42*, 1091. [CrossRef]
72. Rosquist, K.; Uggla, C. Killing tensors in two-dimensional space-times with applications to cosmology. *J. Math. Phys.* **1991**, *32*, 3412–3422. [CrossRef]
73. Bianchi, L. On the spaces of three dimensions that admit a continuous group of movements. *Soc. Ita. Mem. di Mat.* **1898**, *11*, 268.
74. Rayan, M.P., Jr.; Shepley, L.C. *Homogeneous Relativistic Cosmologies*; Princeton University Press: Princeton, NJ, USA, 1975.
75. Kasner, E. Geometrical theorems on Einstein's cosmological equations. *Am. J. Math.* **1921**, *43*, 217–221. [CrossRef]

76. Moussiaux, A.; Tombal, P.; Demaret, J. Exact solution for vacuum Bianchi type III model with a cosmological constant. *J. Phys. A Math. Gen.* **1981**, *14*, L277. [CrossRef]
77. Christodoulakis, T.; Terzis, P.A. The general solution of Bianchi type III vacuum cosmology. *Class. Quantum Grav.* **2007**, *24*, 875. [CrossRef]
78. Terzis, P.A.; Christodoulakis, T. The general solution of Bianchi type VII h vacuum cosmology. *Gen. Relat. Gravit.* **2009**, *41*, 469–495. [CrossRef]
79. Harvey, A.; Tsoubelis, D. Exact Bianchi IV cosmological model. *Phys. Rev. D* **1977**, *15*, 2734. [CrossRef]
80. Cotsakis, S.; Leach, P.G.L. Painlevé analysis of the Mixmaster universe. *J. Phys. A Math. Gen.* **1994**, *27*, 1625. [CrossRef]
81. Maciejewski, A.; Szydlowski, M. On the integrability of Bianchi cosmological models. *J. Phys. A Math. Gen.* **1998**, *31*, 2031. [CrossRef]
82. Libre, J.; Valls, C.J. Integrability of the Bianchi IX system. *Math. Phys.* **2005**, *46*, 0742901. [CrossRef]
83. Libre, J.; Valls, C.J. Formal and analytical integrability of the Bianchi IX system. *Math. Phys.* **2006**, *47*, 022704. [CrossRef]
84. Christiansen, F.; Rugh, H.H.; Rugh, S.E. Non-integrability of the mixmaster universe. *J. Phys. A Math. Gen.* **1995**, *28*, 657. [CrossRef]
85. Capozziello, S.; de Ritis, R. Nöther's symmetries in fourth-order cosmologies. *Nuovo Cimento B* **1994**, *109*, 795–802. [CrossRef]
86. Capozziello, S.; de Ritis, R.; Marino, A.A. Conformal equivalence and Noether symmetries in cosmology. *Class. Quantum Grav.* **1997**, *14*, 3259. [CrossRef]
87. Modak, B.; Kamilya, S.; Biswas, S. Evolution of dynamical coupling in scalar tensor theory from Noether symmetry. *Gen. Relativ. Gravit.* **2000**, *32*, 1615–1626. [CrossRef]
88. Sanyal, A.K. Noether and some other dynamical symmetries in Kantowski-Sachs model. *Phys. Lett. B* **2002**, *524*, 177–184. [CrossRef]
89. Motavali, H.; Golshani, M. Exact solutions for cosmological models with a scalar field. *IJMPA* **2002**, *17*, 375–381. [CrossRef]
90. Kamilya, S.; Modak, B. Beyond Einstein gravity: A Survey of gravitational theories for cosmology and astrophysics. *Gen. Relativ. Gravit.* **2004**, *36*, 676.
91. Bonanno, A.; Esposito, G.; Rubano, C. Noether symmetry approach in matter-dominated cosmology with variable G and Λ. *Gen. Relativ. Gravit.* **2007**, *39*, 189–209. [CrossRef]
92. Camci, U.; Kucukakca, Y. Noether symmetries of Bianchi I, Bianchi III, and Kantowski-Sachs spacetimes in scalar-coupled gravity theories. *Phys. Rev. D* **2007**, *76*, 084023. [CrossRef]
93. Capozziello, S.; Nesseris, S.; Perivolaropoulos, L. Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry. *JCAP* **2007**, *12*, 009. [CrossRef]
94. Vakili, B. Noether symmetry in f (R) cosmology. *Phys. Lett. B* **2008**, *664*, 16–20. [CrossRef]
95. Capozziello, S.; Felice, A. f (R) cosmology from Noether's symmetry. *JCAP* **2008**, *8*, 16. [CrossRef]
96. Capozziello, S.; Martin-Moruno, P.; Rubano, C. Dark energy and dust matter phases from an exact f (R)-cosmology model. *Phys. Lett. B* **2008**, *664*, 12–15. [CrossRef]
97. Capozziello, S.; Piedipalumbo, E.; Rubano, C.; Scudellaro, P. Noether symmetry approach in phantom quintessence cosmology. *Phys. Rev. D* **2009**, *80*, 104030. [CrossRef]
98. Mubasher, J.; Mahomed, F.M.; Momeni, D. Noether symmetry approach in f (R)-tachyon model. *Phys. Lett. B* **2011**, *702*, 315–319.
99. Cotsakis, S.; Leach, P.G.L.; Pantazi, C. Symmetries of homogeneous cosmologies. *Gravit. Cosmol.* **1998**, *4*, 314.
100. Dimakis, N.; Christodoulakis, T.; Terzis, P.A. FLRW metric f (R) cosmology with a perfect fluid by generating integrals of motion. *J. Geom. Phys.* **2012**, *77*, 97. [CrossRef]
101. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Barrow, J.D. Classical and quantum solutions in Brans-Dicke cosmology with a perfect fluid. *Phys. Rev. D* **2016**, *93*, 043528. [CrossRef]
102. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Barrow, J.D. Dynamical analysis in scalar field cosmology. *Phys. Rev. D* **2015**, *91*, 123535. [CrossRef]
103. Paliathanasis, A.; Vakili, B. Closed-form solutions of the Wheeler–DeWitt equation in a scalar-vector field cosmological model by Lie symmetries. *Gen. Relativ. Gravit.* **2016**, *48*, 13. [CrossRef]
104. Dimakis, N.; Giacomini, A.; Jamal, S.; Leon, G.; Paliathanasis, A. Noether symmetries and stability of ideal gas solutions in Galileon cosmology. *Phys. Rev. D* **2017**, *95*, 064031. [CrossRef]

105. Paliathanasis, A.; Tsamparlis, M. Exact solution of the Einstein-Skyrme model in a Kantowski-Sachs spacetime. *J. Geom. Phys.* **2017**, *114*, 1–11. [CrossRef]
106. Paliathanasis, A. Symmetries of Differential Equations and Applications in Relativistic Physics. Ph.D. Thesis, University of Athens, Athens, Greece, 2014, arXiv:1501.05129.
107. Belinchon, J.A.; Harko, T.; Mak, M.K. Exact Scalar-Tensor Cosmological Solutions via Noether Symmetry. *Astrophys. Space Sci.* **2016**, *361*, 52. [CrossRef]
108. Christodoulakis, T.; Dimakis, N.; Terzis, P.A.; Vakili, B.; Melas, E.; Grammenos, T. Minisuperspace canonical quantization of the Reissner-Nordström black hole via conditional symmetries. *Phys. Rev. D* **2014**, *89*, 044031. [CrossRef]
109. Zampeli, A.; Pailas, T.; Terzis, P.A.; Christodoulakis, T. Conditional symmetries in axisymmetric quantum cosmologies with scalar fields and the fate of the classical singularities. *JCAP* **2016**, *16*, 066. [CrossRef]
110. Capozziello, S.; de Rittis, R. Minisuperspace and Wheeler-DeWitt equation for string dilaton cosmology. *IJMPD* **1993**, *2*, 373. [CrossRef]
111. Capozziello, S.; Lambiase, G. Selection rules in minisuperspace quantum cosmology. *Gen. Relativ. Gravit.* **2000**, *32*, 673–696. [CrossRef]
112. Capozziello, S.; Lambiase, G. Higher-order corrections to the effective gravitational action from Noether symmetry approach. *Gen. Relativ. Gravit.* **2000**, *32*, 295–311. [CrossRef]
113. Paliathanasis, A.; Capozziello, S. Noether symmetries and duality transformations in cosmology. *MPLA* **2016**, *31*, 1650183. [CrossRef]
114. Gionti, G.; Paliathanasis, A. Duality transformation and conformal equivalent scalar-tensor theories. *MPLA* **2018**, *33*, 1850093. [CrossRef]
115. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Barrow, J.D. Dynamical analysis in scalar field cosmology. *Phys. Rev. D* **2015**, *91*, 123535. [CrossRef]
116. Paliathanasis, A. Dust fluid component from Lie symmetries in scalar field cosmology. *MPLA* **2017**, *32*, 1750206. [CrossRef]
117. Motavali, H.; Capozziello, S.; Rorwshan, M.; Jog, A. Scalar-tensor cosmology with R- 1 curvature correction by Noether symmetry. *Phys. Lett. B* **2008**, *666*, 10–15. [CrossRef]
118. Basilakos, S.; Tsamparlis, M.; Paliathanasis, A. Using the Noether symmetry approach to probe the nature of dark energy. *Phys. Rev. D* **2011**, *83*, 103512. [CrossRef]
119. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S. Constraints and analytical solutions of theories of gravity using Noether symmetries. *Phys. Rev. D* **2011**, *84*, 123514. [CrossRef]
120. Tsamparlis, M.; Paliathanasis, A. Three-fluid cosmological model using Lie and Noether symmetries. *Class. Quantum Grav.* **2012**, *29*, 015006. [CrossRef]
121. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Capozziello, S. New Schwarzschild-like solutions in f(T) gravity through Noether symmetries. *Phys. Rev. D* **2014**, *89*, 104042. [CrossRef]
122. Paliathanasis, A.; Tsamparlis, M. Two scalar field cosmology: Conservation laws and exact solutions. *Phys. Rev. D* **2014**, *90*, 043529. [CrossRef]
123. Mahomed, F.M.; Kara, A.H.; Leach, P.G.L. Lie and Noether counting theorems for one-dimensional systems. *J. Math. An. Appl.* **1993**, *178*, 116–129. [CrossRef]
124. Paliathanasis, A.; Leach, P.G.L.; Capozziello, S. On the Hojman conservation quantities in Cosmology. *Phys. Lett. B* **2016**, *755*, 8–12. [CrossRef]
125. Bertacca, D.; Matarrese, S.; Pietroni, M. Unified dark matter in scalar field cosmologies. *Mod. Phys. Lett. A* **2007**, *22*, 2893–2907. [CrossRef]
126. Brans, C.; Dicke, R.H. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.* **1961**, *124*, 195. [CrossRef]
127. Mubarakzyanov, G.M. On solvable Lie algebras. *Izv. Vyss. Uchebn Zavendeniĭ Mat.* **1963**, *32*, 114.
128. Mubarakzyanov, G.M. Classification of real structures of Lie algebras of fifth order. *Izv. Vyss. Uchebn Zavendeniĭ Mat.* **1963**, *34*, 99.
129. Mubarakzyanov, G.M. Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izv. Vyss. Uchebn Zavendeniĭ Mat.* **1963**, *35*, 104.
130. Sotiriou, T.P.; Faraoni, V. f(R) theories of gravity. *Rev. Mod. Phys.* **2010**, *82*, 451. [CrossRef]
131. Deffayet, C.; Esposito-Farese, G.; Vikman, A. Covariant galileon. *Phys. Rev. D* **2009**, *79*, 084003. [CrossRef]

132. Paliathanasis, A.; Tsamparlis, M.; Basilakos, S. Dynamical symmetries and observational constraints in scalar field cosmology. *Phys. Rev. D* **2014**, *10*, 103524. [CrossRef]
133. Paliathanasis, A. f (R)-gravity from Killing tensors. *Class. Quantum Gravit.* **2016**, *33*, 075012. [CrossRef]
134. Papagiannopoulos, G.; Barrow, J.D.; Basilakos, S.; Giacomini, A.; Paliathanasis, A. Dynamical symmetries in Brans-Dicke cosmology. *Phys. Rev. D* **2017**, *95*, 024021. [CrossRef]
135. Sadjadi, H.M. Generalized Noether symmetry in f (T) gravity. *Phys. Lett. B* **2012**, *718*, 270–275. [CrossRef]