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Methods for Multiple-Attribute Group Decision Making with q-Rung Interval-Valued Orthopair Fuzzy Information and Their Applications to the Selection of Green Suppliers

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Abstract: In the practical world, there commonly exist different types of multiple-attribute group decision making (MAGDM) problems with uncertain information. Symmetry among some attributes' information that is already known and unknown, and symmetry between the pure attribute sets and fuzzy attribute membership sets, can be an effective way to solve this type of MAGDM problem. In this paper, we investigate four forms of information aggregation operators, including the Hamy mean (HM) operator, weighted HM (WHM) operator, dual HM (DHM) operator, and the dual-weighted HM (WDHM) operator with the q-rung interval-valued orthopair fuzzy numbers (q-RIVOFNs). Then, some extended aggregation operators, such as the q-rung interval-valued orthopair fuzzy Hamy mean (q-RIVOFHM) operator; q-rung interval-valued orthopairfuzzy weighted Hamy mean (q-RIVOFWHM) operator; q-rung interval-valued orthopair fuzzy dual Hamy mean (q-RIVOFDHM) operator; and q-rung interval-valued orthopair fuzzy weighted dual Hamy mean (q-RIVOFWDHM) operator are presented, and some of their precious properties are studied in detail. Finally, a real example for green supplier selection in green supply chain management is provided, to demonstrate the proposed approach and to verify its rationality and scientific nature.

Keywords: multiple attribute group decision making (MAGDM); Pythagorean fuzzy set (PFSs); q-rung orthopair fuzzy sets (q-RIVOFSs); q-RIVOFWHM operator; q-RIVOFWDHM operator; green suppliers selection

1. Introduction

For the indeterminacy of decision makers and decision-making issues, we cannot always give accurate evaluation values for alternatives to select the best project in real multiple-attribute decision making (MADM) problems. To overcome this disadvantage, fuzzy set theory, as defined by Zadeh [1] in 1965, originally used the membership function to describe the estimation results, rather than an exact real number. Atanassov [2,3] presents the intuitionistic fuzzy set (IFS) by considering another measurement index which names a non-membership function. Hereafter, the IFS and its extension has aroused the attention of a large number of scholars since its appearance [4–25]. More recently, the Pythagorean fuzzy set (PFS) [26,27] has emerged as a useful tool for describing the indeterminacy of the MADM problems. Zhang and Xu [28] proposed the detailed mathematical expression for PFS and presented the definition of Pythagorean fuzzy numbers (PFNs). Wei and Lu [29] proposed some Maclaurin Symmetric Mean Operators with PFNs. Peng and Yang [30] studied the division and subtraction operations of PFNs. Wei and Lu [31] defined some power aggregation operators with

PFNs based on the traditional power aggregation operators [32–37]. Beliakov and James [38] presented the average aggregation functions of PFNs. Reformat and Yager [39] studied the collaborative-based recommender system under the Pythagorean fuzzy environment. Gou et al. [40] proposed some desirable properties of the continuous Pythagorean fuzzy number. Wei and Wei [41] defined some similar measures of Pythagorean fuzzy sets, based on cosine functions with traditional similarity measures [42–45]. Ren et al. [46] applied the Pythagorean fuzzy TODIM model in MADM. Garg [47] combines the Einstein Operations and Pythagorean fuzzy information to propose a new aggregation operator. Zeng et al. [48] provided a Pythagorean fuzzy hybrid method to study MADM. Garg [49] presents a novel accuracy function based on interval-valued Pythagorean fuzzy information for solving MADM problems. Wei et al. [50] propose the Pythagorean hesitant fuzzy Hamacher operators in MADM. Wei and Lu [51] develop the dual hesitant Pythagorean fuzzy Hamacher operators in MADM. Lu et al. [52] develop the hesitant Pythagorean fuzzy Hamacher aggregation operators in MADM.

In addition to this, based on the fundamental theories of IFS and PFS, Yager [53] further defined the q-rung orthopair fuzzy sets (q-ROFSs), in which the sum of the qth power of the degrees of membership and the qth power of the degrees of non-membership is satisfied the condition $\mu^q + \nu^q \leq 1$. It is clear that the q-ROFSs are more general for IFSs and PFSs, as they are all special cases. Therefore, we can express a wider range of fuzzy information by using q-ROFSs. Liu and Wang [54] develop the q-rung orthopair, fuzzy weighted averaging (q-ROFWA) operator and the q-rung orthopair, fuzzy weighted geometric (q-ROFWG) operator to fuse the evaluation information. Liu and Liu [55] proposes a q-rung orthopair, fuzzy Bonferroni mean (q-ROFBM) aggregation operator, by considering the q-rung orthopair fuzzy information and the Bonferroni mean (BM) operator. Wei et al. [56] combine the q-rung orthopair fuzzy numbers (q-ROFNs) with a generalized Heronian mean (GHM) operator to present some aggregation operators, and applied them into MADM problems. Wei et al. [57] define some q-rung orthopair, fuzzy Maclaurin symmetric mean operators for the potential evaluation of emerging technology commercialization.

Nevertheless, in many practical decision-making problems, for the uncertainty of the decision-making environment and the subjectivity of decision makers (DMs), it is always difficult for DMs to exactly describe their views with a precise number; however, they can be expressed by an interval number within [0, 1]. This denotes that it is necessary to introduce the definition of q-rung interval-valued orthopair fuzzy sets (q-RIVOFSs), of which the degrees of positive membership and negative membership are given by an interval value. This kind of situation is more or less like that encountered in interval-valued, intuitionistic fuzzy environments [58,59]. It should be noted that when the upper and lower limits of the interval values are same, q-RIVOFSs reduce to q-ROFSs, meaning that the latter is a special case of the former.

This research has four main purposes. The first is to develop a comprehensive MAGDM method for selecting the best green supplier with q-RIVOFNs. The second purpose lies in exploring several aggregation operators based on traditional Hamy mean (HM) operators with q-RIVOFNs. The third is to establish an integrated outranking decision-making method by the q-RIVOFWHM (q-RIVOFWDHM) operators. The final purpose is to demonstrate the application, practicality, and effectiveness of the proposed MADM method for selecting the best green supplier.

To further study the q-RIVOFSs, our paper combines the Hamy mean (HM) operator, which considers the relationship between the attribute's estimation values with q-rung interval-valued orthopair fuzzy numbers to investigate MAGDM problems. For the sake of clarity, the rest of this research is organized as follows. Firstly, we briefly introduce the fundamental theories, such as definition, score, and accuracy functions, and operational laws of the q-ROFSs and q-RIVOFSs in Section 2. Then, based on q-RIVOFSs and Hamy mean (HM) operators, we propose four aggregation operators, including the q-rung interval-valued orthopair, fuzzy Hamy mean (q-RIVOFHM) operator; the q-rung interval-valued orthopair, fuzzy weighted Hamy mean (q-RIVOFWHM) operator; the q-rung interval-valued orthopair, fuzzy dual Hamy mean (q-RIVOFDHM) operator; and the q-rung interval-valued orthopair, fuzzy weighted dual Hamy mean (q-RIVOFWDHM) operator in Section 3.

Meanwhile, some important properties of these operators are also studied. Thereafter, the models which apply the proposed aggregation operators to solve MAGDM problems are presented in Section 4, and an illustrative example to select the best green supplier is developed. Some comments are provided to summarize this article in Section 5.

2. Preliminaries

2.1. q-Rung Interval-Valued Orthopair Fuzzy Sets (q-RIVOFSs)

According to the q-rung orthopair fuzzy sets (q-ROFSs) [53] and interval-valued Pythagorean fuzzy sets (IVPFSs) [49], we develop the definition of the q-rung interval-valued orthopair fuzzy sets (q-RIVOFSs).

Definition 1. *Let X be a fixed set. A q-RIVOFS is an object having the form*

$$\widetilde{Q} = \left\{ \left\langle x, \left(\widetilde{\mu}_{\widetilde{Q}}(x), \widetilde{v}_{\widetilde{Q}}(x) \right) \right\rangle | x \in X \right\} \tag{1}$$

where $\widetilde{\mu}_{\widetilde{Q}}(x) \subset [0,1]$ *and* $\widetilde{v}_{\widetilde{Q}}(x) \subset [0,1]$ *are interval numbers, and* $\widetilde{\mu}_{\widetilde{Q}}(x) = \left[\mu^L_{\widetilde{Q}}(x), \mu^R_{\widetilde{Q}}(x) \right]$, $\widetilde{v}_{\widetilde{Q}}(x) = \left[v^L_{\widetilde{Q}}(x), v^R_{\widetilde{Q}}(x) \right]$ *with the condition* $0 \leq \left(\mu^R_{\widetilde{Q}}(x) \right)^q + \left(v^R_{\widetilde{Q}}(x) \right)^q \leq 1$, $\forall x \in X$, $q \geq 1$. *The numbers* $\widetilde{\mu}_{\widetilde{Q}}(x), \widetilde{v}_{\widetilde{Q}}(x)$ *represent, respectively, the function of positive membership degree (PMD) and negative membership degree (NMD) of the element* x *to* $\widetilde{Q}$. *Then, for* $x \in X$, $\widetilde{\pi}_{\widetilde{Q}}(x) = \left[\pi^L_{\widetilde{Q}}(x), \pi^R_{\widetilde{Q}}(x) \right] = \left[\sqrt[q]{1 - \left(\left(\mu^R_{\widetilde{Q}}(x) \right)^q + \left(v^R_{\widetilde{Q}}(x) \right)^q \right)}, \sqrt[q]{1 - \left(\left(\mu^L_{\widetilde{Q}}(x) \right)^q + \left(v^L_{\widetilde{Q}}(x) \right)^q \right)} \right]$ *denotes the function of the refusal membership degree (RMD) of the element* x *to* $\widetilde{Q}$.

As a matter of convenience, we called $\widetilde{q} = \left(\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}} \right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}} \right] \right)$ a q-rung interval-valued orthopair fuzzy number (q-RIVOFN). Let $\widetilde{q} = \left(\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}} \right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}} \right] \right)$ be a q-RIVOFN, then $S(\widetilde{q}) = \frac{1}{4}\left[\left(1 + \left(u^L_{\widetilde{q}} \right)^q - \left(v^L_{\widetilde{q}} \right)^q \right) + \left(1 + \left(u^R_{\widetilde{q}} \right)^q - \left(v^R_{\widetilde{q}} \right)^q \right) \right]$ and $H(\widetilde{q}) = \frac{\left(u^L_{\widetilde{q}} \right)^q + \left(u^R_{\widetilde{q}} \right)^q + \left(v^L_{\widetilde{q}} \right)^q + \left(v^R_{\widetilde{q}} \right)^q}{2}$ are the score and accuracy function of a q-RIVOFN $\widetilde{q}$.

Definition 2. *Let* $\widetilde{q}_1 = \left(\left[u^L_{\widetilde{q}_1}, u^R_{\widetilde{q}_1} \right], \left[v^L_{\widetilde{q}_1}, v^R_{\widetilde{q}_1} \right] \right)$ *and* $\widetilde{q}_2 = \left(\left[u^L_{\widetilde{q}_2}, u^R_{\widetilde{q}_2} \right], \left[v^L_{\widetilde{q}_2}, v^R_{\widetilde{q}_2} \right] \right)$ *be two q-RIVOFNs;* $S(\widetilde{q}_1)$ *and* $S(\widetilde{q}_2)$ *be the scores of* $\widetilde{q}_1$ *and* $\widetilde{q}_2$*, respectively; and let* $H(\widetilde{q}_1)$ *and* $H(\widetilde{q}_2)$ *be the accuracy degrees of* $\widetilde{q}_1$ *and* $\widetilde{q}_2$*, respectively. Then, if* $S(\widetilde{q}_1) < S(\widetilde{q}_2)$*, then* $\widetilde{q}_1 < \widetilde{q}_2$*; if* $S(\widetilde{q}_1) = S(\widetilde{q}_2)$*, then (1) if* $H(\widetilde{q}_1) = H(\widetilde{q}_2)$*, then* $\widetilde{q}_1 = \widetilde{q}_2$*; (2) if* $H(\widetilde{q}_1) < H(\widetilde{q}_2)$*, then* $\widetilde{q}_1 < \widetilde{q}_2$.

Definition 3. *Let* $\widetilde{q}_1 = \left(\left[u^L_{\widetilde{q}_1}, u^R_{\widetilde{q}_1} \right], \left[v^L_{\widetilde{q}_1}, v^R_{\widetilde{q}_1} \right] \right)$, $\widetilde{q}_2 = \left(\left[u^L_{\widetilde{q}_2}, u^R_{\widetilde{q}_2} \right], \left[v^L_{\widetilde{q}_2}, v^R_{\widetilde{q}_2} \right] \right)$, *and* $\widetilde{q} = \left(\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}} \right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}} \right] \right)$ *be three q-RIVOFNs, and some basic operation rules for them are shown as follows:*

$$(1)\ \widetilde{q}_1 \oplus \widetilde{q}_2 = \left(\left[\begin{array}{c} \sqrt[q]{\left(u^L_{\widetilde{q}_1}\right)^q + \left(u^L_{\widetilde{q}_2}\right)^q - \left(u^L_{\widetilde{q}_1}\right)^q \left(u^L_{\widetilde{q}_2}\right)^q}, \\ \sqrt[q]{\left(u^R_{\widetilde{q}_1}\right)^q + \left(u^R_{\widetilde{q}_2}\right)^q - \left(u^R_{\widetilde{q}_1}\right)^q \left(u^R_{\widetilde{q}_2}\right)^q} \end{array} \right], \left[v^L_{\widetilde{q}_1} v^L_{\widetilde{q}_2}, v^R_{\widetilde{q}_1} v^R_{\widetilde{q}_2}\right] \right);$$

$$(2)\widetilde{q}_1 \otimes \widetilde{q}_2 = \left(\left[\mu^L_{\widetilde{q}_1} v^L_{\widetilde{q}_2}, \mu^R_{\widetilde{q}_1} v^R_{\widetilde{q}_2}\right], \left[\begin{array}{c} \sqrt[q]{\left(v^L_{\widetilde{q}_1}\right)^q + \left(v^L_{\widetilde{q}_2}\right)^q - \left(v^L_{\widetilde{q}_1}\right)^q \left(v^L_{\widetilde{q}_2}\right)^q}, \\ \sqrt[q]{\left(v^R_{\widetilde{q}_1}\right)^q + \left(v^R_{\widetilde{q}_2}\right)^q - \left(v^R_{\widetilde{q}_1}\right)^q \left(v^R_{\widetilde{q}_2}\right)^q} \end{array} \right] \right);$$

$$(3)\ \lambda\widetilde{q} = \left(\left[\sqrt[q]{1 - \left(1 - \left(u^L_{\widetilde{q}}\right)^q\right)^\lambda}, \sqrt[q]{1 - \left(1 - \left(u^R_{\widetilde{q}}\right)^q\right)^\lambda} \right], \left[\left(v^L_{\widetilde{q}}\right)^\lambda, \left(v^R_{\widetilde{q}}\right)^\lambda \right] \right), \lambda > 0;$$

$$(4)\ (\widetilde{q})^\lambda = \left(\left[\left(\mu^L_{\widetilde{q}}\right)^\lambda, \left(\mu^R_{\widetilde{q}}\right)^\lambda \right], \left[\sqrt[q]{1 - \left(1 - \left(v^L_{\widetilde{q}}\right)^q\right)^\lambda}, \sqrt[q]{1 - \left(1 - \left(v^R_{\widetilde{q}}\right)^q\right)^\lambda} \right] \right), \lambda > 0;$$

$$(5)\ \widetilde{q}^c = \left(\left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}}\right], \left[\mu^L_{\widetilde{q}}, \mu^R_{\widetilde{q}}\right] \right).$$

2.2. Hamy Mean Operator

Definition 4 [60]. *The HM operator is defined as follows:*

$$\mathrm{HM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \frac{\sum\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\prod\limits_{j=1}^{x} \widetilde{q}_{i_j} \right)^{\frac{1}{x}}}{C_n^x} \tag{2}$$

where x is a parameter and $x = 1, 2, \ldots, n$, $i_1, i_2, \ldots, i_x$ are x integer values taken from the set $\{1, 2, \ldots, n\}$ of k integer values; C_n^x denotes the binomial coefficient and $C_n^x = \frac{n!}{x!(n-x)!}$.

3. Some Hamy Mean Operators with q-RIVOFNs

3.1. q-RIVOFHM Operator

In this chapter, consider both HM operator and q-RIVOFNs, we propose the q-rung interval-valued orthopair fuzzy Hamy mean (q-RIVOFHM) operator.

Definition 5. *Let $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right] \right) (j = 1, 2, \ldots, n)$ be a set of q-RIVOFNs. The q-RIVOFHM operator is*

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \frac{\bigoplus\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\bigotimes\limits_{j=1}^{x} \widetilde{q}_{i_j} \right)^{\frac{1}{x}}}{C_n^x} \tag{3}$$

Theorem 1. *Let $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right] \right) (j = 1, 2, \ldots, n)$ be a set of q-RIVOFNs. The fused value by using q-RIVOFHM operator is also a q-RIVOFN, where*

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n) = \frac{\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j}\right)^{\frac{1}{x}}}{C_n^x}$$
$$= \left\{ \begin{array}{l} \left[\sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^L_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \right], \\ \left[\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}, \left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}} \right] \end{array} \right\} \tag{4}$$

Proof.

$$\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j} = \left\{ \left[\prod_{j=1}^{x} u^L_{\widetilde{q}_j}, \prod_{j=1}^{x} u^R_{\widetilde{q}_j}\right], \left[\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)}, \sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)}\right] \right\} \tag{5}$$

Thus,

$$\left(\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j}\right)^{\frac{1}{x}} = \left\{ \left[\left(\prod_{j=1}^{x} u^L_{\widetilde{q}_j}\right)^{\frac{1}{x}}, \left(\prod_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{1}{x}}\right], \left[\begin{array}{l} \sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}, \\ \sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}} \end{array}\right] \right\} \tag{6}$$

Thereafter,

$$\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j}\right)^{\frac{1}{x}}$$
$$= \left\{ \begin{array}{l} \left[\sqrt[q]{1-\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^L_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)}, \sqrt[q]{1-\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)} \right], \\ \left[\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}, \prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}} \right] \end{array} \right\} \tag{7}$$

Therefore,

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n) = \frac{\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j}\right)^{\frac{1}{x}}}{C_n^x}$$
$$= \left\{ \begin{array}{l} \left[\sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^L_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \right], \\ \left[\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}, \left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}} \right] \end{array} \right\} \tag{8}$$

Hence, Equation (4) is kept.

Then, we need to prove that Equation (4) is a q-RIVOFN. We need to prove $0 \le \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \le 1$.

Let

$$\mu^R_{\widetilde{Q}}(x) = \sqrt[q]{1 - \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\left(1 - \left(\prod_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}$$

$$\nu^R_{\widetilde{Q}}(x) = \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\sqrt[q]{1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}$$

□

Proof.

$$\begin{aligned}
&0 \le \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \\
&= 1 - \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\left(1 - \left(\prod_{j=1}^{x} u^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}} + \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} \\
&\le 1 - \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} + \left(\prod_{1 \le i_1 < \ldots < i_x \le n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} \\
&= 1
\end{aligned}$$

So $0 \le \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \le 1$ is maintained. □

Example 1. *Let* $([0.5, 0.8], [0.4, 0.5]), ([0.3, 0.5], [0.6, 0.7]), ([0.5, 0.7], [0.2, 0.3])$ *and* $([0.4, 0.8], [0.1, 0.2])$ *be four q-RIVOFNs, and suppose* $x = 2$, $q = 3$*—then, according to Equation (4), we have*

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \frac{\underset{1 \le i_1 < \ldots < i_x \le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\widetilde{q}_{i_j}\right)^{\frac{1}{x}}}{C_n^x}$$

$$= \left(\begin{array}{c}
\left[\begin{array}{c}
\sqrt[3]{1 - \left(\begin{array}{c}\left(1 - (0.5 \times 0.3)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.5)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.4)^{\frac{3}{2}}\right) \\ \times\left(1 - (0.3 \times 0.4)^{\frac{3}{2}}\right) \times \left(1 - (0.3 \times 04)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.4)^{\frac{3}{2}}\right)\end{array}\right)^{\frac{1}{C_4^2}}}, \\
\sqrt[3]{1 - \left(\begin{array}{c}\left(1 - (0.8 \times 0.5)^{\frac{3}{2}}\right) \times \left(1 - (0.8 \times 0.7)^{\frac{3}{2}}\right) \times \left(1 - (0.8 \times 0.8)^{\frac{3}{2}}\right) \\ \times\left(1 - (0.5 \times 0.7)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.8)^{\frac{3}{2}}\right) \times \left(1 - (0.7 \times 0.8)^{\frac{3}{2}}\right)\end{array}\right)^{\frac{1}{C_4^2}}}
\end{array}\right], \\
\left[\begin{array}{c}
\left(\sqrt[3]{\left(\begin{array}{c}\left(1 - \left(\left(1 - 0.4^3\right) \times \left(1 - 0.6^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.4^3\right) \times \left(1 - 0.2^3\right)\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.4^3\right) \times \left(1 - 0.1^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.6^3\right) \times \left(1 - 0.2^3\right)\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.6^3\right) \times \left(1 - 0.1^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.2^3\right) \times \left(1 - 0.1^3\right)\right)^{\frac{1}{2}}\right)\end{array}\right)}\right)^{\frac{1}{C_4^2}}, \\
\left(\sqrt[3]{\left(\begin{array}{c}\left(1 - \left(\left(1 - 0.5^3\right) \times \left(1 - 0.7^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.5^3\right) \times \left(1 - 0.3^3\right)\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.5^3\right) \times \left(1 - 0.2^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.7^3\right) \times \left(1 - 0.3^3\right)\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.7^3\right) \times \left(1 - 0.2^3\right)\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.3^3\right) \times \left(1 - 0.2^3\right)\right)^{\frac{1}{2}}\right)\end{array}\right)}\right)^{\frac{1}{C_4^2}}
\end{array}\right]
\end{array}\right)$$

$$= ([\,0.4261, 0.7072], [\,0.3604, 0.4605])$$

The q-RIVOFHM satisfies the following three properties.

Property 1. *Idempotency: if* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1, 2, \ldots, n)$ *are equal, then*

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \widetilde{q} \tag{9}$$

Proof. Since $\widetilde{q}_j = \widetilde{q} = \left(\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}}\right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}}\right]\right)$, then

$$\begin{aligned}
& q\text{-RIVOFHM}^{(x)}(\widetilde{q}, \widetilde{q}, \cdots, \widetilde{q}) = \frac{\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\underset{j=1}{\overset{x}{\otimes}}\widetilde{q}\right)^{\frac{1}{x}}}{C_n^x} \\
&= \left\{\begin{array}{l}
\left[\sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^L_{\widetilde{q}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod\limits_{j=1}^{x} u^R_{\widetilde{q}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right], \\
\left[\left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}, \left(\prod\limits_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}\right]
\end{array}\right\} \\
&= \left\{\begin{array}{l}
\left[\sqrt[q]{1-\left(\left(1-\left(\left(u^L_{\widetilde{q}}\right)^x\right)^{\frac{q}{x}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\left(1-\left(\left(u^R_{\widetilde{q}}\right)^x\right)^{\frac{q}{x}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}}\right], \\
\left[\left(\left(\sqrt[q]{1-\left(\left(1-\left(v^L_{\widetilde{q}}\right)^q\right)^x\right)^{\frac{1}{x}}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}, \left(\left(\sqrt[q]{1-\left(\left(1-\left(v^R_{\widetilde{q}}\right)^q\right)^x\right)^{\frac{1}{x}}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}\right]
\end{array}\right\} \\
&= \left\{\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}}\right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}}\right]\right\} = \widetilde{q}
\end{aligned}$$

□

Property 2. *Monotonicity: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1, 2, \ldots, n)$ *and* $\widetilde{q}'_j = \left(\left[\left(u^L_{\widetilde{q}_j}\right)', \left(u^R_{\widetilde{q}_j}\right)'\right], \left[\left(v^L_{\widetilde{q}_j}\right)', \left(v^R_{\widetilde{q}_j}\right)'\right]\right)(j = 1, 2, \ldots, n)$ *be two sets of q-RIVOFNs. If* $u^L_{\widetilde{q}_j} \le \left(u^L_{\widetilde{q}_j}\right)'$, $u^R_{\widetilde{q}_j} \le \left(u^R_{\widetilde{q}_j}\right)'$, $v^L_{\widetilde{q}_j} \ge \left(v^L_{\widetilde{q}_j}\right)'$ *and* $v^L_{\widetilde{q}_j} \ge \left(v^R_{\widetilde{q}_j}\right)'$ *hold for all j, then*

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \le q\text{-RIVOFHM}^{(x)}(\widetilde{q}'_1, \widetilde{q}'_2, \cdots, \widetilde{q}'_n) \tag{10}$$

Proof. Given that $u^L_{\widetilde{q}_j} \le \left(u^L_{\widetilde{q}_j}\right)'$, we can obtain

$$\left(\prod_{j=1}^{x} u^L_{\widetilde{q}}\right)^{\frac{q}{x}} \le \left(\prod_{j=1}^{x}\left(u^L_{\widetilde{q}}\right)'\right)^{\frac{q}{x}} \tag{11}$$

$$\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x} u^L_{\widetilde{q}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}} \ge \left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u^L_{\widetilde{q}}\right)'\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}} \tag{12}$$

Thereafter,

$$\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x} u^L_{\widetilde{q}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \le \sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u^L_{\widetilde{q}}\right)'\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \tag{13}$$

That means $u^L_{\widetilde{q}} \le \left(u^L_{\widetilde{q}}\right)'$. Similarly, we can obtain $u^R_{\widetilde{q}} \le \left(u^R_{\widetilde{q}}\right)'$, $v^L_{\widetilde{q}} \ge \left(v^L_{\widetilde{q}}\right)'$ and $v^L_{\widetilde{q}} \ge \left(v^R_{\widetilde{q}}\right)'$. Thus, the proof is complete. □

Property 3. *Boundedness: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs. If* $\widetilde{q}^+ = \left(\left[max_i\left(u^L_{\widetilde{q}_j}\right), max_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[min_i\left(v^L_{\widetilde{q}_j}\right), min_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *and* $\widetilde{q}^- = \left(\left[min_i\left(u^L_{\widetilde{q}_j}\right), min_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[max_i\left(v^L_{\widetilde{q}_j}\right), max_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *then*

$$\widetilde{q}^- \leq q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+ \tag{14}$$

From Property 1,

$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1^-, \widetilde{q}_2^-, \cdots, \widetilde{q}_n^-) = \widetilde{q}^-$$
$$q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1^+, \widetilde{q}_2^+, \cdots, \widetilde{q}_n^+) = \widetilde{q}^+$$

From Property 2,

$$\widetilde{q}^- \leq q\text{-RIVOFHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+$$

3.2. The q-RIVOFWHM Operator

In practical MADM problems, it is important to take the attribute weights into account. This section will develop the q-rung interval-valued orthopair, fuzzy weighted Hamy mean (q-RIVOFWHM) operator.

Definition 6. *Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs, with their weight vector as* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0,1]$ *and* $\sum_{i=1}^n w_i = 1$*. Then we can define the q-RIVOFWHM operator as follows:*

$$q\text{-RIVOFWHM}_w^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) = \frac{\underset{1\leq i_1<\ldots<i_x\leq n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}{C_n^x} \tag{15}$$

Theorem 2. *Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs. The fused value obtained by using q-RIVOFWHM operator is also a q-RIVOFN, where*

$$\begin{aligned}
&q\text{-RIVOFWHM}_w^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) = \frac{\underset{1\leq i_1<\ldots<i_x\leq n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}{C_n^x} \\
&= \left\{\begin{array}{l}
\left[\sqrt[q]{1-\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1-\left(\prod\limits_{j=1}^{x}\left(u^L_{\widetilde{q}_j}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1-\left(\prod\limits_{j=1}^{x}\left(u^R_{\widetilde{q}_j}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right], \\
\left[\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}, \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}\right]
\end{array}\right\}
\end{aligned} \tag{16}$$

Proof. From Definition 3, we can obtain

$$\left(\widetilde{q}_{i_j}\right)^{w_{i_j}} = \left\{\left[\left(u^L_{\widetilde{q}_j}\right)^{w_{i_j}}, \left(u^R_{\widetilde{q}_j}\right)^{w_{i_j}}\right], \left[\sqrt[q]{1-\left(1-\left(v^L_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}}, \sqrt[q]{1-\left(1-\left(v^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}}\right]\right\} \tag{17}$$

Thus,

$$\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}=\left\{\begin{array}{c}\left[\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{L}\right)^{w_{i_j}},\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right],\\ \left[\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}},\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}}\right]\end{array}\right\} \tag{18}$$

Therefore,

$$\left(\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}=\left\{\begin{array}{c}\left[\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{1}{x}},\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{1}{x}}\right],\\ \left[\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}},\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right]\end{array}\right\} \tag{19}$$

Thereafter,

$$\begin{array}{l}\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}\\ =\left\{\begin{array}{c}\left[\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)},\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)}\right],\\ \left[\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right),\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right]\end{array}\right\}\end{array} \tag{20}$$

Furthermore,

$$\begin{array}{l}q\text{-RIVOFWHM}_{w}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\ldots,\widetilde{q}_n)=\frac{\underset{1\le i_1<\ldots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}{C_n^x}\\ =\left\{\begin{array}{c}\left[\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right],\\ \left[\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}},\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}\right]\end{array}\right\}\end{array} \tag{21}$$

Hence, Equation (16) is kept.

Then we need to prove that Equation (16) is a q-RIVOFN. We need to prove that $0\le\left(\mu_{\widetilde{Q}}^{R}(x)\right)^{q}+\left(\nu_{\widetilde{Q}}^{R}(x)\right)^{q}\le 1$.

Let

$$\mu_{\widetilde{Q}}^{R}(x)=\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(u_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}$$

$$\nu_{\widetilde{Q}}^{R}(x)=\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(v_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}$$

□

Proof.

$$\begin{aligned}
&0 \leq \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \\
&= 1 - \left(\prod_{1 \leq i_1 < \ldots < i_x \leq n}\left(1 - \left(\prod_{j=1}^{x}\left(u^R_{\widetilde{q}_j}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}} + \left(\prod_{1 \leq i_1 < \ldots < i_x \leq n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} \\
&\leq 1 - \left(\prod_{1 \leq i_1 < \ldots < i_x \leq n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} + \left(\prod_{1 \leq i_1 < \ldots < i_x \leq n}\left(1 - \left(\prod_{j=1}^{x}\left(1 - \left(v^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}} \\
&= 1
\end{aligned}$$

Therefore, $0 \leq \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \leq 1$ is maintained. □

Example 2. *Let* $([0.5, 0.8], [0.4, 0.5])$, $([0.3, 0.5], [0.6, 0.7])$, $([0.5, 0.7], [0.2, 0.3])$ *and* $([0.4, 0.8], [0.1, 0.2])$ *be four q-RIVOFNs, and* $w = (0.2, 0.1, 0.3, 0.4)$*; in addition, suppose* $x = 2$, $q = 3$*. Then, according to Equation (16), we have*

$$\begin{aligned}
&q\text{-RIVOFWHM}_w^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) = \frac{\bigoplus_{1 \leq i_1 < \ldots < i_x \leq n}\left(\bigotimes_{j=1}^{x}\left(\widetilde{q}_{i_j}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}{C_n^x} \\
&= \left(\begin{array}{l}
\left[\begin{array}{l}
\sqrt[3]{1 - \left(\begin{array}{l}\left(1 - \left(0.5^{0.2} \times 0.3^{0.1}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.5^{0.2} \times 0.5^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.5^{0.2} \times 0.4^{0.4}\right)^{\frac{3}{2}}\right) \\ \times\left(1 - \left(0.3^{0.1} \times 0.5^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.3^{0.1} \times 0.4^{0.4}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.5^{0.3} \times 0.4^{0.4}\right)^{\frac{3}{2}}\right)\end{array}\right)^{\frac{1}{C_4^2}}}, \\
\sqrt[3]{1 - \left(\begin{array}{l}\left(1 - \left(0.8^{0.2} \times 0.5^{0.1}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.8^{0.2} \times 0.7^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.8^{0.2} \times 0.8^{0.4}\right)^{\frac{3}{2}}\right) \\ \times\left(1 - \left(0.5^{0.1} \times 0.7^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.5^{0.1} \times 0.8^{0.4}\right)^{\frac{3}{2}}\right) \times \left(1 - \left(0.7^{0.3} \times 0.8^{0.4}\right)^{\frac{3}{2}}\right)\end{array}\right)^{\frac{1}{C_4^2}}}
\end{array}\right], \\
\left[\begin{array}{l}
\left(\sqrt[3]{\left(\begin{array}{l}\left(1 - \left(\left(1 - 0.4^3\right)^{0.2} \times \left(1 - 0.6^3\right)^{0.1}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.4^3\right)^{0.2} \times \left(1 - 0.2^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.4^3\right)^{0.2} \times \left(1 - 0.1^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.6^3\right)^{0.1} \times \left(1 - 0.2^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.6^3\right)^{0.1} \times \left(1 - 0.1^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.2^3\right)^{0.3} \times \left(1 - 0.1^3\right)^{0.4}\right)^{\frac{1}{2}}\right)\end{array}\right)}\right)^{\frac{1}{C_4^2}}, \\
\left(\sqrt[3]{\left(\begin{array}{l}\left(1 - \left(\left(1 - 0.5^3\right)^{0.2} \times \left(1 - 0.7^3\right)^{0.1}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.5^3\right)^{0.2} \times \left(1 - 0.3^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 0.5^3\right)^{0.2} \times \left(1 - 0.2^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.7^3\right)^{0.1} \times \left(1 - 0.3^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1 - \left(\left(1 - 07^3\right)^{0.1} \times \left(1 - 0.2^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(\left(1 - 0.3^3\right)^{0.3} \times \left(1 - 0.2^3\right)^{0.4}\right)^{\frac{1}{2}}\right)\end{array}\right)}\right)^{\frac{1}{C_4^2}}
\end{array}\right]
\end{array}\right) \\
&= ([\,0.8204, 0.9266], [\,0.1983, 0.2589])
\end{aligned}$$

The q-RIVOFWHM operator satisfies the following properties.

Property 4. *Monotonicity: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1, 2, \ldots, n)$ *and* $\widetilde{q}'_j = \left(\left[\left(u^L_{\widetilde{q}_j}\right)', \left(u^R_{\widetilde{q}_j}\right)'\right], \left[\left(v^L_{\widetilde{q}_j}\right)', \left(v^R_{\widetilde{q}_j}\right)'\right]\right)(j = 1, 2, \ldots, n)$ *be two sets of q-RIVOFNs. If* $u^L_{\widetilde{q}_j} \leq \left(u^L_{\widetilde{q}_j}\right)'$, $u^R_{\widetilde{q}_j} \leq \left(u^R_{\widetilde{q}_j}\right)'$, $v^L_{\widetilde{q}_j} \geq \left(v^L_{\widetilde{q}_j}\right)'$ *and* $v^L_{\widetilde{q}_j} \geq \left(v^R_{\widetilde{q}_j}\right)'$ *hold for all j, then*

$$q\text{-RIVOFWHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq q\text{-RIVOFWHM}^{(x)}(\widetilde{q}'_1, \widetilde{q}'_2, \cdots, \widetilde{q}'_n) \tag{22}$$

The proof is similar to q-RIVOFHM, so it is omitted here.

Property 5. *Boundedness: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs. If* $\widetilde{q}^+ = \left(\left[max_i\left(u^L_{\widetilde{q}_j}\right), max_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[min_i\left(v^L_{\widetilde{q}_j}\right), min_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *and* $\widetilde{q}^- = \left(\left[min_i\left(u^L_{\widetilde{q}_j}\right), min_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[max_i\left(v^L_{\widetilde{q}_j}\right), max_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *then*

$$\widetilde{q}^- \leq q\text{-RIVOFWHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+ \tag{23}$$

From Theorem 2, we get

$$q\text{-RIVOFWHM}^{(x)}(\widetilde{q}_1^-, \widetilde{q}_2^-, \cdots, \widetilde{q}_n^-) = \left\{ \begin{array}{l} \left[\begin{array}{l} \sqrt[q]{1 - \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1 - \left(\prod\limits_{j=1}^{x}\left(\min\left(u^L_{\widetilde{q}_j}\right)\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \\ \sqrt[q]{1 - \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1 - \left(\prod\limits_{j=1}^{x}\left(\min\left(u^R_{\widetilde{q}_j}\right)\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \end{array} \right], \\ \left[\begin{array}{l} \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x}\left(1 - \left(\max\left(v^L_{\widetilde{q}_j}\right)\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}, \\ \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x}\left(1 - \left(\max\left(v^R_{\widetilde{q}_j}\right)\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}} \end{array} \right] \end{array} \right\} \tag{24}$$

$$q\text{-RIVOFWHM}^{(x)}(\widetilde{q}_1^+, \widetilde{q}_2^+, \cdots, \widetilde{q}_n^+) = \left\{ \begin{array}{l} \left[\begin{array}{l} \sqrt[q]{1 - \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1 - \left(\prod\limits_{j=1}^{x}\left(\max\left(u^L_{\widetilde{q}_j}\right)\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \\ \sqrt[q]{1 - \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1 - \left(\prod\limits_{j=1}^{x}\left(\max\left(u^R_{\widetilde{q}_j}\right)\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}} \end{array} \right], \\ \left[\begin{array}{l} \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x}\left(1 - \left(\min\left(v^L_{\widetilde{q}_j}\right)\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}, \\ \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x}\left(1 - \left(\min\left(v^R_{\widetilde{q}_j}\right)\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}} \end{array} \right] \end{array} \right\} \tag{25}$$

From Property 4, we get

$$\widetilde{q}^- \leq q\text{-RIVOFWHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+ \tag{26}$$

It is obvious that the q-RIVOFWHM operator lacks the property of idempotency.

3.3. The q-RIVOFDHM Operator

Wu et al. [61] define the dual Hamy mean (DHM) operator.

Definition 7 [61]. *The DHM operator can be defined as:*

$$\mathrm{DHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\frac{\sum_{j=1}^{x}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \tag{27}$$

where x is a parameter, and $x=1,2,\ldots,n$, $i_1,i_2,\ldots,i_x$ are x integer values taken from the set $\{1,2,\ldots,n\}$ of k integer values; C_n^x denotes the binomial coefficient and $C_n^x=\frac{n!}{x!(n-x)!}$.

In this section, we will propose the q-rung interval-valued orthopair, fuzzy DHM (q-RIVOFDHM) operator.

Definition 8. *Let $\widetilde{q}_j=\left(\left[u_{\widetilde{q}_j}^L,u_{\widetilde{q}_j}^R\right],\left[v_{\widetilde{q}_j}^L,v_{\widetilde{q}_j}^R\right]\right)(j=1,2,\ldots,n)$ be a set of q-RIVOFNs. The q-RIVOFDHM operator is*

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\bigotimes_{1\le i_1<\ldots<i_x\le n}\left(\frac{\bigoplus_{j=1}^{x}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \tag{28}$$

Theorem 3. *Let $\widetilde{q}_j=\left(\left[u_{\widetilde{q}_j}^L,u_{\widetilde{q}_j}^R\right],\left[v_{\widetilde{q}_j}^L,v_{\widetilde{q}_j}^R\right]\right)(j=1,2,\ldots,n)$ be a set of q-RIVOFNs. The fused value by using q-RIVOFDHM operators is also a q-RIVOFN, where*

$$\begin{gathered}q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\bigotimes_{1\le i_1<\ldots<i_x\le n}\left(\frac{\bigoplus_{j=1}^{x}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}\\
=\left\{\begin{array}{l}\left[\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^L\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}},\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^R\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}\right],\\
\left[\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^L\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^R\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right]\end{array}\right\}\end{gathered} \tag{29}$$

Proof.

$$\bigoplus_{j=1}^{x}\widetilde{q}_{i_j}=\left\{\left[\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^L\right)^q\right)},\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^R\right)^q\right)}\right],\left[\prod_{j=1}^{x}v_{\widetilde{q}_j}^L,\prod_{j=1}^{x}v_{\widetilde{q}_j}^R\right]\right\} \tag{30}$$

Thus,

$$\frac{\bigoplus_{j=1}^{x}\widetilde{q}_{i_j}}{x}=\left\{\left[\begin{array}{c}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^L\right)^q\right)\right)^{\frac{1}{x}}},\\ \sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^R\right)^q\right)\right)^{\frac{1}{x}}}\end{array}\right],\left[\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^L\right)^{\frac{1}{x}},\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^R\right)^{\frac{1}{x}}\right]\right\} \tag{31}$$

Thereafter,

$$\underset{1\le i_1<\ldots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}\widetilde{q}_{i_j}}{x}\right)$$
$$=\left\{\begin{array}{c}\left[\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}},\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right]\\ \left[\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^L_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)},\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)}\right]\end{array}\right\} \tag{32}$$

Therefore,

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\underset{1\le i_1<\ldots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$
$$=\left\{\begin{array}{c}\left[\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u^L_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}},\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}\right],\\ \left[\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^L_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right]\end{array}\right\} \tag{33}$$

Hence, Equation (29) is kept.

Then, we need to prove that Equation (29) is a q-RIVOFN. We need to prove that $0\le\left(\mu^R_{\widetilde{Q}}(x)\right)^q+\left(\nu^R_{\widetilde{Q}}(x)\right)^q\le 1$.

Let

$$\mu^R_{\widetilde{Q}}(x)=\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}$$
$$\nu^R_{\widetilde{Q}}(x)=\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}$$

□

Proof.

$$\begin{aligned}&0\le\left(\mu^R_{\widetilde{Q}}(x)\right)^q+\left(\nu^R_{\widetilde{Q}}(x)\right)^q\\ &=1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v^R_{\widetilde{q}_j}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}+\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}}\\ &\le 1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}}+\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)\right)^{\frac{1}{x}}\right)\right)^{\frac{1}{C_n^x}}\\ &=1\end{aligned}$$

Therefore, $0\le\left(\mu^R_{\widetilde{Q}}(x)\right)^q+\left(\nu^R_{\widetilde{Q}}(x)\right)^q\le 1$ is maintained. □

Example 3. *Let* $([0.5, 0.8], [0.4, 0.5])$, $([0.3, 0.5], [0.6, 0.7])$, $([0.5, 0.7], [0.2, 0.3])$ *and* $([0.4, 0.8], [0.1, 0.2])$ *be four q-RIVOFNs, and suppose* $x = 2$, $q = 3$*; then according to Equation (29), we have*

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \left(\underset{1 \le i_1 < \ldots < i_x \le n}{\otimes} \left(\frac{\overset{x}{\underset{j=1}{\oplus}} \widetilde{q}_{i_j}}{x} \right) \right)^{\frac{1}{C_n^x}}$$

$$= \left(\begin{array}{l} \left[\begin{array}{l} \left(\sqrt[3]{ \left(\begin{array}{l} \left(1 - \left((1-0.5^3) \times (1-0.3^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.5^3) \times (1-0.5^3)\right)^{\frac{1}{2}}\right) \\ \times \left(1 - \left((1-0.5^3) \times (1-0.4^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.3^3) \times (1-0.5^3)\right)^{\frac{1}{2}}\right) \\ \times \left(1 - \left((1-0.3^3) \times (1-0.4^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.5^3) \times (1-0.4^3)\right)^{\frac{1}{2}}\right) \end{array} \right) } \right)^{\frac{1}{C_4^2}}, \\ \left(\sqrt[3]{ \left(\begin{array}{l} \left(1 - \left((1-0.8^3) \times (1-0.5^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.8^3) \times (1-0.7^3)\right)^{\frac{1}{2}}\right) \\ \times \left(1 - \left((1-0.8^3) \times (1-0.8^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.5^3) \times (1-0.7^3)\right)^{\frac{1}{2}}\right) \\ \times \left(1 - \left((1-0.5^3) \times (1-0.8^3)\right)^{\frac{1}{2}}\right) \times \left(1 - \left((1-0.7^3) \times (1-0.8^3)\right)^{\frac{1}{2}}\right) \end{array} \right) } \right)^{\frac{1}{C_4^2}} \end{array} \right], \\ \left[\begin{array}{l} \sqrt[3]{1 - \left(\begin{array}{l} \left(1 - (0.4 \times 0.6)^{\frac{3}{2}}\right) \times \left(1 - (0.4 \times 0.2)^{\frac{3}{2}}\right) \times \left(1 - (0.4 \times 0.1)^{\frac{3}{2}}\right) \\ \times \left(1 - (0.6 \times 0.2)^{\frac{3}{2}}\right) \times \left(1 - (0.6 \times 0.1)^{\frac{3}{2}}\right) \times \left(1 - (0.2 \times 0.1)^{\frac{3}{2}}\right) \end{array} \right)^{\frac{1}{C_4^2}}}, \\ \sqrt[3]{1 - \left(\begin{array}{l} \left(1 - (0.5 \times 0.7)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.3)^{\frac{3}{2}}\right) \times \left(1 - (0.5 \times 0.2)^{\frac{3}{2}}\right) \\ \times \left(1 - (0.7 \times 0.3)^{\frac{3}{2}}\right) \times \left(1 - (0.7 \times 0.2)^{\frac{3}{2}}\right) \times \left(1 - (0.3 \times 0.2)^{\frac{3}{2}}\right) \end{array} \right)^{\frac{1}{C_4^2}}} \end{array} \right] \end{array} \right)$$

$$= ([0.4348, 0.7214], [0.3283, 0.4291])$$

The q-RIVOFDHM has the following three operators.

Property 6. *Idempotency: if* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j} \right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j} \right] \right) (j = 1, 2, \ldots, n)$ *are equal, then*

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \widetilde{q} \tag{34}$$

Proof. Since $\widetilde{q}_j = \widetilde{q} = \left(\left[u^L_{\widetilde{q}}, u^R_{\widetilde{q}} \right], \left[v^L_{\widetilde{q}}, v^R_{\widetilde{q}} \right] \right)$, then

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\underset{1\le i_1<\ldots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$

$$=\left\{\begin{array}{l}\left[\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}},\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}\right],\\ \left[\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^{L}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}v_{\widetilde{q}_j}^{R}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right]\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left[\left(\left(\sqrt[q]{1-\left(\left(1-\left(u_{\widetilde{q}}^{L}\right)^q\right)^x\right)^{\frac{1}{x}}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}},\left(\left(\sqrt[q]{1-\left(\left(1-\left(u_{\widetilde{q}}^{R}\right)^q\right)^x\right)^{\frac{1}{x}}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}\right],\\ \left[\sqrt[q]{1-\left(\left(1-\left(\left(v_{\widetilde{q}}^{L}\right)^x\right)^{\frac{q}{x}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\left(1-\left(\left(v_{\widetilde{q}}^{R}\right)^x\right)^{\frac{q}{x}}\right)^{C_n^x}\right)^{\frac{1}{C_n^x}}}\right],\end{array}\right\}$$

$$=\left\{\left[u_{\widetilde{q}}^{L},u_{\widetilde{q}}^{R}\right],\left[v_{\widetilde{q}}^{L},v_{\widetilde{q}}^{R}\right]\right\}=\widetilde{q}$$

□

Property 7. *Monotonicity: let* $\widetilde{q}_j=\left(\left[u_{\widetilde{q}_j}^{L},u_{\widetilde{q}_j}^{R}\right],\left[v_{\widetilde{q}_j}^{L},v_{\widetilde{q}_j}^{R}\right]\right)(j=1,2,\ldots,n)$ *and* $\widetilde{q}_j'=\left(\left[\left(u_{\widetilde{q}_j}^{L}\right)',\left(u_{\widetilde{q}_j}^{R}\right)'\right],\left[\left(v_{\widetilde{q}_j}^{L}\right)',\left(v_{\widetilde{q}_j}^{R}\right)'\right]\right)(j=1,2,\ldots,n)$ *be two sets of q-RIVOFNs. If* $u_{\widetilde{q}_j}^{L}\le\left(u_{\widetilde{q}_j}^{L}\right)',u_{\widetilde{q}_j}^{R}\le\left(u_{\widetilde{q}_j}^{R}\right)',v_{\widetilde{q}_j}^{L}\ge\left(v_{\widetilde{q}_j}^{L}\right)'$ *and* $v_{\widetilde{q}_j}^{L}\ge\left(v_{\widetilde{q}_j}^{R}\right)'$ *hold for all j, then*

$$q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)\le q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1',\widetilde{q}_2',\cdots,\widetilde{q}_n') \quad (35)$$

Proof. Given that $u_{\widetilde{q}_j}^{L}\le\left(u_{\widetilde{q}_j}^{L}\right)'$, we can obtain

$$\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^q\right)\ge\prod_{j=1}^{x}\left(1-\left(\left(u_{\widetilde{q}_j}^{L}\right)'\right)^q\right) \quad (36)$$

$$1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^q\right)\right)^{\frac{1}{x}}\le 1-\left(\prod_{j=1}^{x}\left(1-\left(\left(u_{\widetilde{q}_j}^{L}\right)'\right)^q\right)\right)^{\frac{1}{x}} \quad (37)$$

Thereafter,

$$\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}}\le\left(\prod_{1\le i_1<\ldots<i_x\le n}\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(\left(u_{\widetilde{q}_j}^{L}\right)'\right)^q\right)\right)^{\frac{1}{x}}}\right)^{\frac{1}{C_n^x}} \quad (38)$$

That means that $u_{\widetilde{q}}^{L}\le\left(u_{\widetilde{q}}^{L}\right)'$. Similarly, we can obtain $u_{\widetilde{q}}^{R}\le\left(u_{\widetilde{q}}^{R}\right)',v_{\widetilde{q}}^{L}\ge\left(v_{\widetilde{q}}^{L}\right)'$ and $v_{\widetilde{q}}^{L}\ge\left(v_{\widetilde{q}}^{R}\right)'$. Thus, the proof is complete. □

Property 8. *Boundedness: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs. If* $\widetilde{q}^+ = \left(\left[max_i\left(u^L_{\widetilde{q}_j}\right), max_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[min_i\left(v^L_{\widetilde{q}_j}\right), min_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *and* $\widetilde{q}^- = \left(\left[min_i\left(u^L_{\widetilde{q}_j}\right), min_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[max_i\left(v^L_{\widetilde{q}_j}\right), max_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *then*

$$\widetilde{q}^- \leq q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+ \tag{39}$$

From Property 6,

$$q\text{-RIVOFDHM}^{(x)}\left(\widetilde{q}_1^-, \widetilde{q}_2^-, \cdots, \widetilde{q}_n^-\right) = \widetilde{q}^-$$
$$q\text{-RIVOFDHM}^{(x)}\left(\widetilde{q}_1^+, \widetilde{q}_2^+, \cdots, \widetilde{q}_n^+\right) = \widetilde{q}^+$$

From Property 7,

$$\widetilde{q}^- \leq q\text{-RIVOFDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \leq \widetilde{q}^+$$

3.4. The q-RIVOFWDHM Operator

In real MADM problems, it's of necessity to take attribute weights into account; we will propose the q-rung interval-valued orthopair fuzzy weighted DHM (q-RIVOFWDHM) operator in this chapter.

Definition 9. *Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs, with their weight vector as* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0,1]$ *and* $\sum_{i=1}^n w_i = 1$*. If*

$$q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \left(\underset{1\leq i_1<\ldots<i_x\leq n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}} w_{i_j}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \tag{40}$$

Theorem 4. *Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1,2,\ldots,n)$ *be a set of q-RIVOFNs. The fused value by using q-RIVOFWDHM operators is also a q-RIVOFN, where*

$$\begin{aligned} & q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \left(\underset{1\leq i_1<\ldots<i_x\leq n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}} w_{i_j}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \\ & = \left\{\begin{array}{l} \left[\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(u^L_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}, \left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(\sqrt[q]{1-\left(\prod\limits_{j=1}^{x}\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}\right] \\ \left[\sqrt[q]{1-\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1-\left(\prod\limits_{j=1}^{x}\left(v^L_{\widetilde{q}_j}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}, \sqrt[q]{1-\left(\prod\limits_{1\leq i_1<\ldots<i_x\leq n}\left(1-\left(\prod\limits_{j=1}^{x}\left(v^R_{\widetilde{q}_j}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right] \end{array}\right\} \end{aligned} \tag{41}$$

Proof.

$$w_{i_j}\widetilde{q}_{i_j} = \left\{\left[\sqrt[q]{1-\left(1-\left(u^L_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}}, \sqrt[q]{1-\left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}}}\right], \left[\left(v^L_{\widetilde{q}_j}\right)^{w_{i_j}}, \left(v^R_{\widetilde{q}_j}\right)^{w_{i_j}}\right]\right\} \tag{42}$$

Thus,

$$\overset{x}{\underset{j=1}{\oplus}}\left(w_{i_j}\widetilde{q}_{i_j}\right)=\left\{\begin{array}{l}\left[\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}},\sqrt[q]{1-\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}}\right],\\ \left[\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{L}\right)^{w_{i_j}},\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right]\end{array}\right\} \tag{43}$$

Therefore,

$$\frac{\overset{x}{\underset{j=1}{\oplus}}\left(w_{i_j}\widetilde{q}_{i_j}\right)}{x}=\left\{\begin{array}{l}\left[\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}},\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right]\\ \left[\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{1}{x}},\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{1}{x}}\right]\end{array}\right\} \tag{44}$$

Thereafter,

$$\begin{array}{l}\underset{1\le i_1<\ldots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}w_{i_j}\widetilde{q}_{i_j}}{x}\right)\\ =\left\{\begin{array}{l}\left[\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right),\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right]\\ \left[\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)},\sqrt[q]{1-\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)}\right]\end{array}\right\}\end{array} \tag{45}$$

Furthermore,

$$\begin{array}{l}q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1,\widetilde{q}_2,\cdots,\widetilde{q}_n)=\left(\underset{1\le i_1<\ldots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}w_{i_j}\widetilde{q}_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}\\ =\left\{\begin{array}{l}\left[\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{L}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}},\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}\right]\\ \left[\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{L}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}},\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}\right]\end{array}\right\}\end{array} \tag{46}$$

Hence, Equation (41) is kept.

Then, we need to prove that Equation (41) is a q-RIVOFN. We need to prove that $0\le\left(\mu_{\widetilde{Q}}^{R}(x)\right)^{q}+\left(\nu_{\widetilde{Q}}^{R}(x)\right)^{q}\le 1$.

Let

$$\mu_{\widetilde{Q}}^{R}(x)=\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\sqrt[q]{1-\left(\prod_{j=1}^{x}\left(1-\left(u_{\widetilde{q}_j}^{R}\right)^{q}\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right)\right)^{\frac{1}{C_n^x}}$$

$$\nu_{\widetilde{Q}}^{R}(x)=\sqrt[q]{1-\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(1-\left(\prod_{j=1}^{x}\left(v_{\widetilde{q}_j}^{R}\right)^{w_{i_j}}\right)^{\frac{q}{x}}\right)\right)^{\frac{1}{C_n^x}}}$$

□

Proof.

$$
\begin{aligned}
&0 \le \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \\
&= \left(\prod_{1\le i_1<\ldots<i_x\le n} \left(1 - \left(\prod_{j=1}^{x} \left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}} \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_n^x}} + 1 - \left(\prod_{1\le i_1<\ldots<i_x\le n} \left(1 - \left(\prod_{j=1}^{x} \left(v^R_{\widetilde{q}_j}\right)^{w_{i_j}} \right)^{\frac{q}{x}} \right) \right)^{\frac{1}{C_n^x}} \\
&\le \left(\prod_{1\le i_1<\ldots<i_x\le n} \left(1 - \left(\prod_{j=1}^{x} \left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}} \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_n^x}} + 1 - \left(\prod_{1\le i_1<\ldots<i_x\le n} \left(1 - \left(\prod_{j=1}^{x} \left(1-\left(u^R_{\widetilde{q}_j}\right)^q\right)^{w_{i_j}} \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_n^x}} \\
&= 1
\end{aligned}
$$

Therefore, $0 \le \left(\mu^R_{\widetilde{Q}}(x)\right)^q + \left(\nu^R_{\widetilde{Q}}(x)\right)^q \le 1$ is maintained. □

Example 4. *Let* $([0.5, 0.8], [0.4, 0.5])$, $([0.3, 0.5], [0.6, 0.7])$, $([0.5, 0.7], [0.2, 0.3])$ *and* $([0.4, 0.8], [0.1, 0.2])$ *be four q-RIVOFNs; suppose* $x = 2$, $q = 3$, *and* $\omega = (0.2, 0.1, 0.3, 0.4)$. *Then, based on Equation (41), we can get*

$$
q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) = \left(\bigotimes_{1\le i_1<\ldots<i_x\le n} \left(\frac{\overset{x}{\underset{j=1}{\oplus}} w_{i_j}\widetilde{q}_{i_j}}{x} \right) \right)^{\frac{1}{C_n^x}}
$$

$$
= \left(\begin{array}{l}
\left[\begin{array}{l}
\left(\sqrt[3]{ \left(\begin{array}{l} \left(1-\left(\left(1-0.5^3\right)^{0.2}\times\left(1-0.3^3\right)^{0.1}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.5^3\right)^{0.2}\times\left(1-0.5^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1-\left(\left(1-0.5^3\right)^{0.2}\times\left(1-0.4^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.3^3\right)^{0.1}\times\left(1-0.5^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1-\left(\left(1-0.3^3\right)^{0.1}\times\left(1-0.4^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.5^3\right)^{0.3}\times\left(1-0.4^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \end{array} \right) } \right)^{\frac{1}{C_4^2}}, \\
\left(\sqrt[3]{ \left(\begin{array}{l} \left(1-\left(\left(1-0.8^3\right)^{0.2}\times\left(1-0.5^3\right)^{0.1}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.8^3\right)^{0.2}\times\left(1-0.7^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1-\left(\left(1-0.8^3\right)^{0.2}\times\left(1-0.8^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.5^3\right)^{0.1}\times\left(1-0.7^3\right)^{0.3}\right)^{\frac{1}{2}}\right) \\ \times\left(1-\left(\left(1-0.5^3\right)^{0.1}\times\left(1-0.8^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \times \left(1-\left(\left(1-0.7^3\right)^{0.3}\times\left(1-0.8^3\right)^{0.4}\right)^{\frac{1}{2}}\right) \end{array} \right) } \right)^{\frac{1}{C_4^2}}
\end{array} \right], \\
\left[\begin{array}{l}
\sqrt[3]{ 1 - \left(\begin{array}{l} \left(1-\left(0.4^{0.2}\times 0.6^{0.1}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.4^{0.2}\times 0.2^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.4^{0.2}\times 0.1^{0.4}\right)^{\frac{3}{2}}\right) \\ \times\left(1-\left(0.6^{0.1}\times 0.2^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.6^{0.1}\times 0.1^{0.4}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.2^{0.3}\times 0.1^{0.4}\right)^{\frac{3}{2}}\right) \end{array} \right)^{\frac{1}{C_4^2}} }, \\
\sqrt[3]{ 1 - \left(\begin{array}{l} \left(1-\left(0.5^{0.2}\times 0.7^{0.1}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.5^{0.2}\times 0.3^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.5^{0.2}\times 0.2^{0.4}\right)^{\frac{3}{2}}\right) \\ \times\left(1-\left(0.7^{0.1}\times 0.3^{0.3}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.7^{0.1}\times 0.2^{0.4}\right)^{\frac{3}{2}}\right) \times \left(1-\left(0.3^{0.3}\times 0.2^{0.4}\right)^{\frac{3}{2}}\right) \end{array} \right)^{\frac{1}{C_4^2}} }
\end{array} \right]
\end{array} \right)
$$

$$
= ([0.2819, 0.4954], [0.7249, 0.7855])
$$

We will then study some precious properties of q-RIVOFWDHM operator.

Property 9. *Monotonicity: let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right) (j = 1, 2, \ldots, n)$ *and* $\widetilde{q}'_j = \left(\left[\left(u^L_{\widetilde{q}_j}\right)', \left(u^R_{\widetilde{q}_j}\right)'\right], \left[\left(v^L_{\widetilde{q}_j}\right)', \left(v^R_{\widetilde{q}_j}\right)'\right]\right) (j = 1, 2, \ldots, n)$ *be two sets of q-RIVOFNs. If* $u^L_{\widetilde{q}_j} \le \left(u^L_{\widetilde{q}_j}\right)'$, $u^R_{\widetilde{q}_j} \le \left(u^R_{\widetilde{q}_j}\right)'$, $v^L_{\widetilde{q}_j} \ge \left(v^L_{\widetilde{q}_j}\right)'$ *and* $v^L_{\widetilde{q}_j} \ge \left(v^R_{\widetilde{q}_j}\right)'$ *hold for all j, then*

$$
q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \le q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}'_1, \widetilde{q}'_2, \cdots, \widetilde{q}'_n) \tag{47}
$$

This proof is similar to q-RIVOFDHM, so it is omitted here.

Property 10. *(Boundedness) Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right)(j = 1, 2, \ldots, n)$ *be a set of q-RIVOFNs. If* $\widetilde{q}^+ = \left(\left[max_i\left(u^L_{\widetilde{q}_j}\right), max_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[min_i\left(v^L_{\widetilde{q}_j}\right), min_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *and* $\widetilde{q}^- = \left(\left[min_i\left(u^L_{\widetilde{q}_j}\right), min_i\left(u^R_{\widetilde{q}_j}\right)\right], \left[max_i\left(v^L_{\widetilde{q}_j}\right), max_i\left(v^R_{\widetilde{q}_j}\right)\right]\right)$ *then*

$$\widetilde{q}^- \le q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \le \widetilde{q}^+ \tag{48}$$

From Theorem 4, we get

$$q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1^-, \widetilde{q}_2^-, \cdots, \widetilde{q}_n^-)$$
$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x} \left(1 - \left(\min\left(u^L_{\widetilde{q}_j}\right)\right)^q \right)^{w_{i_j}} \right)^{\frac{1}{x}}} \right) \right)^{\frac{1}{C_n^x}}, \\ \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x} \left(1 - \left(\min\left(u^R_{\widetilde{q}_j}\right)\right)^q \right)^{w_{i_j}} \right)^{\frac{1}{x}}} \right) \right)^{\frac{1}{C_n^x}} \end{array} \right] \\ \left[\begin{array}{l} \sqrt[q]{1 - \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(1 - \left(\prod\limits_{j=1}^{x} \left(\max\left(v^L_{\widetilde{q}_j}\right)\right)^{w_{i_j}} \right)^{\frac{q}{x}} \right) \right)^{\frac{1}{C_n^x}}}, \\ \sqrt[q]{1 - \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(1 - \left(\prod\limits_{j=1}^{x} \left(\max\left(v^R_{\widetilde{q}_j}\right)\right)^{w_{i_j}} \right)^{\frac{q}{x}} \right) \right)^{\frac{1}{C_n^x}}} \end{array} \right] \end{array} \right\} \tag{49}$$

$$q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1^+, \widetilde{q}_2^+, \cdots, \widetilde{q}_n^+)$$
$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x} \left(1 - \left(\max\left(u^L_{\widetilde{q}_j}\right)\right)^q \right)^{w_{i_j}} \right)^{\frac{1}{x}}} \right) \right)^{\frac{1}{C_n^x}}, \\ \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(\sqrt[q]{1 - \left(\prod\limits_{j=1}^{x} \left(1 - \left(\max\left(u^R_{\widetilde{q}_j}\right)\right)^q \right)^{w_{i_j}} \right)^{\frac{1}{x}}} \right) \right)^{\frac{1}{C_n^x}} \end{array} \right] \\ \left[\begin{array}{l} \sqrt[q]{1 - \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(1 - \left(\prod\limits_{j=1}^{x} \left(\min\left(v^L_{\widetilde{q}_j}\right)\right)^{w_{i_j}} \right)^{\frac{q}{x}} \right) \right)^{\frac{1}{C_n^x}}}, \\ \sqrt[q]{1 - \left(\prod\limits_{1 \le i_1 < \ldots < i_x \le n} \left(1 - \left(\prod\limits_{j=1}^{x} \left(\min\left(v^R_{\widetilde{q}_j}\right)\right)^{w_{i_j}} \right)^{\frac{q}{x}} \right) \right)^{\frac{1}{C_n^x}}} \end{array} \right] \end{array} \right\} \tag{50}$$

From Property 9, we get

$$\widetilde{q}^- \le q\text{-RIVOFWDHM}^{(x)}(\widetilde{q}_1, \widetilde{q}_2, \cdots, \widetilde{q}_n) \le \widetilde{q}^+ \tag{51}$$

It is obvious that the q-RIVOFWDHM operator is short of the property of idempotency.

4. Application of Green Supplier Selection

4.1. Numerical Example

With the rapid development of economic globalization, and the growing enterprise competition environment, the competition between modern enterprises has become the competition between supply chains. The diversity of the people consuming is increasing, and the new product life cycles are getting shorter. The volatility of the demand market and from external factors drives enterprises for effective supply chain integration and management, as well as strategic alliances with other enterprises to enhance core competitiveness and resist external risk. The key measure to achieving this goal is supplier selection. Therefore, the supplier selection problem has gained a lot of attention, whether in regard to supply chain management theory or in actual production management problems [62–70]. In order to illustrate our proposed method in this article, we provide a numerical example for selecting green suppliers in green supply chain management using q-RIVOFNs. There is a panel with five possible green suppliers in green supply chain management to select: $\widetilde{Q}_i (i = 1, 2, 3, 4, 5)$. The experts select four attributes to evaluate the five possible green suppliers: (1) C_1 is the product quality factor; (2) C_2 is the environmental factors; (3) C_3 is the delivery factor; and (4) C_4 is the price factor. The five possible green suppliers $\widetilde{Q}_i (i = 1, 2, 3, 4, 5)$ are to be evaluated by the decision maker using the q-RIVOFNs, under the above four attributes (whose weighting vector $\omega = (0.3, 0.2, 0.3, 0.2)$, and expert weighting vector $\omega = (0.2, 0.2, 0.6)$) which are listed in Tables 1–3.

Table 1. The q-RIVOFN decision matrix 1 (R_1) by expert one.

Alternatives	C_1	C_2	C_3	C_4
$\widetilde{Q}_1$	([0.4,0.5],[0.5,0.7])	([0.6,0.7],[0.2,0.3])	([0.3,0.5],[0.4,0.6])	([0.7,0.8],[0.2,0.4])
$\widetilde{Q}_2$	([0.2,0.3],[0.4,0.5])	([0.1,0.2],[0.6,0.7])	([0.6,0.8],[0.2,0.3])	([0.5,0.6],[0.5,0.7])
$\widetilde{Q}_3$	([0.7,0.9],[0.1,0.2])	([0.4,0.5],[0.2,0.3])	([0.5,0.7],[0.3,0.4])	([0.6,0.7],[0.1,0.2])
$\widetilde{Q}_4$	([0.3,0.5],[0.4,0.6])	([0.2,0.3],[0.1,0.2])	([0.5,0.6],[0.1,0.5])	([0.3,0.4],[0.2,0.3])
$\widetilde{Q}_5$	([0.3,0.6],[0.2,0.4])	([0.4,0.6],[0.2,0.3])	([0.1,0.2],[0.4,0.5])	([0.2,0.4],[0.1,0.3])

Table 2. The q-RIVOFN decision matrix 1 (R_2) by expert two.

Alternatives	C_1	C_2	C_3	C_4
$\widetilde{Q}_1$	([0.3,0.4],[0.4,0.6])	([0.7,0.8],[0.3,0.4])	([0.2,0.4],[0.3,0.5])	([0.8,0.9],[0.3,0.5])
$\widetilde{Q}_2$	([0.1,0.2],[0.3,0.4])	([0.2,0.3],[0.7,0.8])	([0.5,0.7],[0.1,0.2])	([0.6,0.7],[0.6,0.8])
$\widetilde{Q}_3$	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.4,0.6],[0.2,0.3])	([0.7,0.8],[0.2,0.3])
$\widetilde{Q}_4$	([0.2,0.4],[0.3,0.5])	([0.3,0.4],[0.2,0.3])	([0.4,0.5],[0.1,0.4])	([0.4,0.5],[0.3,0.4])
$\widetilde{Q}_5$	([0.2,0.5],[0.1,0.3])	([0.5,0.7],[0.3,0.4])	([0.1,0.2],[0.3,0.4])	([0.3,0.5],[0.2,0.4])

Table 3. The q-RIVOFN decision matrix 1 (R_3) by expert three.

Alternatives	C_1	C_2	C_3	C_4
$\widetilde{Q}_1$	([0.5,0.6],[0.6,0.8])	([0.5,0.6],[0.1,0.2])	([0.4,0.6],[0.5,0.7])	([0.6,0.7],[0.1,0.3])
$\widetilde{Q}_2$	([0.3,0.4],[0.5,0.6])	([0.1,0.2],[0.5,0.6])	([0.7,0.9],[0.3,0.4])	([0.4,0.5],[0.4,0.6])
$\widetilde{Q}_3$	([0.8,0.9],[0.2,0.3])	([0.3,0.4],[0.1,0.2])	([0.6,0.8],[0.4,0.5])	([0.5,0.6],[0.1,0.2])
$\widetilde{Q}_4$	([0.4,0.6],[0.5,0.7])	([0.1,0.2],[0.1,0.2])	([0.6,0.7],[0.2,0.6])	([0.2,0.3],[0.1,0.2])
$\widetilde{Q}_5$	([0.4,0.7],[0.3,0.5])	([0.3,0.5],[0.1,0.2])	([0.2,0.3],[0.5,0.6])	([0.1,0.3],[0.1,0.2])

In the following, we utilize the approach developed to select green suppliers in green supply chain management.

Step 1. According to q-RIVOFNs $\widetilde{q}_{ij} (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, we can aggregate all q-RIVOFNs $\widetilde{q}_{ij}$ by using the q-RIVOFWA (q-RIVOFWG) operator, to get the overall q-RIVOFNs $\widetilde{Q}_i (i = 1, 2, 3, 4, 5)$ of the green suppliers $\widetilde{Q}_i$. Then, the fused values are given in Table 4. (Let $q = 3$).

Definition 10. *Let* $\widetilde{q}_j = \left(\left[u^L_{\widetilde{q}_j}, u^R_{\widetilde{q}_j}\right], \left[v^L_{\widetilde{q}_j}, v^R_{\widetilde{q}_j}\right]\right) (j = 1, 2, \ldots, n)$ *be a set of q-RIVOFNs, with their weight vector as* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^n w_i = 1$*. Then we can obtain*

$$
\begin{aligned}
&q\text{-RIVOFWA}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) = \sum_{j=1}^n w_j \widetilde{q}_j \\
&= \left\langle \left[\sqrt[q]{1 - \prod_{j=1}^n \left(1 - u^L_{\widetilde{q}_j}\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^n \left(1 - u^R_{\widetilde{q}_j}\right)^{w_j}} \right], \left[\prod_{j=1}^n \left(v^L_{\widetilde{q}_j}\right)^{w_j}, \prod_{j=1}^n \left(v^R_{\widetilde{q}_j}\right)^{w_j} \right] \right\rangle
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&q\text{-RIVOFWG}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) = \prod_{j=1}^n \left(\widetilde{q}_j\right)^{w_j} \\
&= \left\langle \left[\prod_{j=1}^n \left(u^L_{\widetilde{q}_j}\right)^{w_j}, \prod_{j=1}^n \left(u^R_{\widetilde{q}_j}\right)^{w_j} \right], \left[\sqrt[q]{1 - \prod_{j=1}^n \left(1 - v^L_{\widetilde{q}_j}\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^n \left(1 - v^R_{\widetilde{q}_j}\right)^{w_j}} \right] \right\rangle
\end{aligned}
\tag{53}
$$

Table 4. The fused results from the q-RIVOFWA operator.

Alternatives	C_1	C_2
$\widetilde{Q}_1$	([0.7637,0.8175],[0.5335,0.7354])	([0.8283,0.8756],[0.1431,0.2491])
$\widetilde{Q}_2$	([0.6249,0.7011],[0.4317,0.5335])	([0.4945,0.6047],[0.5547,0.6554])
$\widetilde{Q}_3$	([0.9089,0.9601],[0.1516,0.2551])	([0.7149,0.7756],[0.1431,0.2491])
$\widetilde{Q}_4$	([0.7011,0.8175],[0.4317,0.6346])	([0.5474,0.6420],[0.1149,0.2169])
$\widetilde{Q}_5$	([0.7011,0.8654],[0.2221,0.4317])	([0.7149,0.8283],[0.1431,0.2491])
Alternatives	C_3	C_4
$\widetilde{Q}_1$	([0.7011,0.8175],[0.4317,0.6346])	([0.8756,0.9197],[0.1431,0.3519])
$\widetilde{Q}_2$	([0.8654,0.9498],[0.2221,0.3288])	([0.7756,0.8283],[0.4536,0.6554])
$\widetilde{Q}_3$	([0.8175,0.9089],[0.3288,0.4317])	([0.8283,0.8756],[0.1149,0.2169])
$\widetilde{Q}_4$	([0.8175,0.8654],[0.1516,0.5335])	([0.6420,0.7149],[0.1431,0.2491])
$\widetilde{Q}_5$	([0.5445,0.6396],[0.4317,0.5335])	([0.5474,0.7149],[0.1149,0.2491])

Step 2. Based on Table 4, we can fuse all q-RIVOFNs $\widetilde{q}_{ij}$ by the q-RIVOFWHM (q-RIVOFWDHM) operator to get the results of q-RIVOFNs. Let $x = 2$, then the fused values are given in Table 5.

Table 5. The fused values of the q-rung interval-valued orthopair, fuzzy weighted Hamy mean (q-RIVOFWHM) and the q-rung interval-valued orthopair, fuzzy weighted dual Hamy mean (q-RIVOFWDHM)) operators.

Alternatives	***q*-RIVOFWHM**	***q*-RIVOFWDHM**
$\widetilde{Q}_1$	([0.9422,0.9616],[0.2248,0.3558])	([0.5409,0.6039],[0.7423,0.8415])
$\widetilde{Q}_2$	([0.9148,0.9418],[0.2710,0.3562])	([0.4842,0.5611],[0.7959,0.8530])
$\widetilde{Q}_3$	([0.9536,0.9720],[0.1237,0.1901])	([0.5790,0.6546],[0.6536,0.7346])
$\widetilde{Q}_4$	([0.9112,0.9379],[0.1415,0.2910])	([0.4637,0.5318],[0.6697,0.8006])
$\widetilde{Q}_5$	([0.8903,0.9356],[0.1575,0.2505])	([0.4140,0.5250],[0.6861,0.7805])

Step 3. Based on the fused values given in Table 5, and the score functions of q-RIVOFNs, the green suppliers' scores are shown in Table 6.

Table 6. The score values $s\left(\widetilde{Q}_i\right)$ of the green suppliers.

Alternatives	q-RIVOFWHM	q-RIVOFWDHM
$\widetilde{Q}_1$	0.9172	0.3434
$\widetilde{Q}_2$	0.8840	0.2914
$\widetilde{Q}_3$	0.9442	0.4497
$\widetilde{Q}_4$	0.8885	0.3591
$\widetilde{Q}_5$	0.8762	0.3543

Step 4. Rank all the alternatives by the values of Table 6, and the ordering results are shown in Table 7. Obviously, the best selection is $\widetilde{Q}_3$.

Table 7. Ordering of the green suppliers.

Methods	Ordering
q-RIVOFWHM	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_2 > \widetilde{Q}_5$
q-RIVOFWDHM	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$

4.2. Influence of the Parameter x

In order to show the effects on the ranking results, by changing parameters of x in the q-RIVOFWHM (q-RIVOFWDHM) operators, all of the results are shown in Tables 8 and 9. (Let $q = 3$).

Table 8. Ordering results for different x values by the q-RIVOFWHM operator.

Parameters	$S(\widetilde{Q}_1)$	$S(\widetilde{Q}_2)$	$S(\widetilde{Q}_3)$	$S(\widetilde{Q}_4)$	$S(\widetilde{Q}_5)$	Ordering
$x = 1$	0.9306	0.8993	0.9476	0.8941	0.8844	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$x = 2$	0.9172	0.8840	0.9442	0.8885	0.8762	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_2 > \widetilde{Q}_5$
$x = 3$	0.9290	0.8959	0.9454	0.8947	0.8786	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$x = 4$	0.9080	0.8772	0.9419	0.8839	0.8703	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_2 > \widetilde{Q}_5$

Table 9. Ordering results for different x values by the q-RIVOFWDHM operator.

Parameters	$S(\widetilde{Q}_1)$	$S(\widetilde{Q}_2)$	$S(\widetilde{Q}_3)$	$S(\widetilde{Q}_4)$	$S(\widetilde{Q}_5)$	Ordering
$x = 1$	0.3330	0.2579	0.4340	0.3424	0.3464	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$x = 2$	0.3434	0.2914	0.4497	0.3591	0.3543	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$
$x = 3$	0.2557	0.2292	0.3406	0.3005	0.3024	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$x = 4$	0.3486	0.3150	0.4585	0.3679	0.3586	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$

4.3. Influence of the Parameter q

In order to show the effects on the ranking results by changing the parameters of q in the q-RIVOFWHM (q-RIVOFWDHM) operators, all of the results are shown in Tables 10 and 11. (Let $x = 2$).

Table 10. Ordering results for different q by the q-RIVOFWHM operator.

Parameters	$S(\widetilde{Q}_1)$	$S(\widetilde{Q}_2)$	$S(\widetilde{Q}_3)$	$S(\widetilde{Q}_4)$	$S(\widetilde{Q}_5)$	Ordering
$q = 1$	0.9090	0.8899	0.9481	0.9147	0.9121	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 2$	0.9244	0.8982	0.9555	0.9109	0.9031	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_2$
$q = 3$	0.9172	0.8840	0.9442	0.8885	0.8762	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_2 > \widetilde{Q}_5$
$q = 4$	0.9033	0.8634	0.9293	0.8627	0.8469	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 5$	0.8872	0.8412	0.9139	0.8371	0.8187	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 6$	0.8705	0.8193	0.8989	0.8127	0.7926	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 7$	0.8540	0.7983	0.8844	0.7899	0.7687	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 8$	0.8380	0.7785	0.8704	0.7687	0.7468	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 9$	0.8225	0.7600	0.8570	0.7490	0.7270	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$
$q = 10$	0.8078	0.7427	0.8441	0.7308	0.7089	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_4 > \widetilde{Q}_5$

Table 11. Ordering results for different q by the q-RIVOFWDHM operator.

Parameters	$S(\widetilde{Q}_1)$	$S(\widetilde{Q}_2)$	$S(\widetilde{Q}_3)$	$S(\widetilde{Q}_4)$	$S(\widetilde{Q}_5)$	Ordering
$q = 1$	0.2814	0.2415	0.3520	0.2756	0.2655	$\widetilde{Q}_3 > \widetilde{Q}_1 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_2$
$q = 2$	0.3107	0.2617	0.4074	0.3188	0.3110	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 3$	0.3434	0.2914	0.4497	0.3591	0.3543	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 4$	0.3722	0.3204	0.4788	0.3913	0.3893	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 5$	0.3962	0.3464	0.4978	0.4161	0.4163	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 6$	0.4157	0.3689	0.5098	0.4350	0.4369	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 7$	0.4314	0.3881	0.5171	0.4494	0.4524	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 8$	0.4441	0.4044	0.5211	0.4604	0.4641	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 9$	0.4543	0.4181	0.5230	0.4688	0.4729	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$
$q = 10$	0.4625	0.4297	0.5235	0.4754	0.4795	$\widetilde{Q}_3 > \widetilde{Q}_5 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2$

4.4. Comparative Analysis

In this chapter, we compare the q-RIVOFWHM and q-RIVOFWDHM operators with the q-RIVOFWA and q-RIVOFWG operators. The comparative results are shown in Table 12.

Table 12. Comparative results.

Methods	Ordering
q-RIVOFWA	$\widetilde{Q}_3 > \widetilde{Q}_4 > \widetilde{Q}_1 > \widetilde{Q}_2 > \widetilde{Q}_5$
q-RIVOFWG	$\widetilde{Q}_3 > \widetilde{Q}_2 > \widetilde{Q}_5 > \widetilde{Q}_1 > \widetilde{Q}_4$

From above, we can see that we get the same optimal green suppliers, which shows the practicality and effectiveness of the proposed approaches. However, the q-RIVOFWA operator and q-RIVOFWG operator do not consider the information about the relationship between arguments being aggregated, and thus cannot eliminate the influence of unfair arguments on decision results. Our proposed q-RIVOFWHM and q-RIVOFWDHM operators consider the information about the relationship among arguments being aggregated.

At the same time, Liu and Wang [54] develop the q-rung orthopair, fuzzy weighted averaging (q-ROFWA) operator, as well as the q-rung orthopair, fuzzy weighted geometric (q-ROFWG) operator. Liu and Liu [55] propose some q-rung orthopair, fuzzy Bonferroni mean (q-ROFBM) aggregation operators. Wei et al. [56] define the generalized Heronian mean (GHM) operator to present some aggregation operators, and apply them into MADM problems. Wei et al. [57] define some q-rung orthopair, fuzzy Maclaurin symmetric mean operators. However, all of these operators can only deal with q-rung orthopair fuzzy sets (q-ROFSs), and cannot deal with q-rung interval-valued orthopair fuzzy sets (q-RIVOFSs). The main contribution of this paper is to study the MAGDM problems based on the q-rung interval-valued orthopair fuzzy sets (q-RIVOFSs), and to utilize the Hamy mean

(HM) operator, weighted Hamy mean (WHM) operator, dual Hamy mean (DHM) operator, and weighted dual Hamy mean (WDHM) operator, to develop some Hamy mean aggregation operators with q-RIVOFNs.

5. Conclusions

In this paper, we study the MAGDM problems with q-RIVOFNs. Then, we utilize the Hamy mean (HM) operator, weighted Hamy mean (WHM) operator, dual Hamy mean (DHM) operator, and weighted dual Hamy mean (WDHM) operator, in order to develop some Hamy mean aggregation operators with q-RIVOFNs. The prominent characteristic of each of these proposed operators is studied. Then, we have utilized these operators to develop some approaches to solve the MAGDM problems with q-RIVOFNs. Finally, a practical example for green supplier selection is given to show the developed approach. Using the illustrated example, we have roughly shown the effects on the ranking results by changing parameters in the q-RIVOFWHM (q-RIVOFWDHM) operators. In the future, the application of the proposed fused operators of q-RIVOFNs needs to be explored in decision making [71–74], risk analysis [75,76], and many other fields under uncertain environments [77–81].

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