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On Some Fractional Integral Inequalities of Hermite-Hadamard's Type through Convexity

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Abstract: In this paper, we incorporate the notion of convex function and establish new integral inequalities of type Hermite–Hadamard via Riemann—Liouville fractional integrals. It is worth mentioning that the obtained inequalities generalize Hermite–Hadamard type inequalities presented by Özdemir, M.E. et. al. (2013) and Sarikaya, M.Z. et. al. (2011).

Keywords: Hermite–Hadamard's Inequality; Convex Functions; Power-mean Inequality; Jenson Integral Inequality; Riemann—Liouville Fractional Integration

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1. Introduction and Preliminaries

One of the generalizations of classical differentiation and integration is fractional calculus. The contribution of fractional calculus presents in diverse fields, such as pure mathematics, economics, and physical and engineering sciences. The role of inequalities found to be very significant in all fields of mathematics and an attractive and active field of research. Recently, convexity has become the major part in different fields of science. A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is named as convex, if the inequality

$$g(\omega x + (1-\omega) y) \leq \omega g(x) + (1-\omega) g(y)$$

holds for all $x, y \in I$ and $\omega \in [0,1]$. In fact, large number of articles has been written on inequalities using classical convexity, but one of the most important and well known is Hermite– Hadamard's inequality. In [1], this double inequality is stated as: Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $x, y \in I$ with $x < y$. Then,

$$g\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y g(t)\,dt \leq \frac{g(x)+g(y)}{2}.$$

Both inequalities hold in the reversed direction for g to be concave. In the field of mathematical inequalities, Hermite–Hadamard's inequality has been given more attention by many mathematician due to its applicability and usefulness. Many researchers have extended the Hermite–Hadamard's inequality, to different forms using the classical convex function. For further details involving Hermite–Hadamard's type inequality on different concept of convex function and generalizations, the interested reader is referred to [2–12] and references therein.

First, we recall some important definitions and results that will be used in the sequel.

Definition 1. *For $g \in L^1[x,y]$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as $J_{a+}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t) dt, for a < x$, and $J_{b-}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t) dt,\ for\ x < b$, respectively, where $\Gamma(.)$ is Gamma function and is defined as $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$. It is to be noted that $J_{a+}^0 g(x) = J_{b-}^0 g(x) = g(x)$.*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to these operators can be found in [7]. For useful details on Hermite–Hadamard type inequalities connected with fractional integral inequalities, the readers are directed to [8–14].

In [15], Özdemir et. al proved some inequalities related to Hermite–Hadamard's inequalities for functions whose second derivatives in absolute value at certain powers are s-convex functions as follows:

Theorem 1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^0 (where I^0 is the interior of I) such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f''|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| &\leq \frac{(b-a)^2}{8(s+1)(s+2)(s+3)} \times \\ &\quad \left\{ |f''(a)| + (s+1)(s+2)\,|f''(\frac{a+b}{2})| + |f''(b)| \right\} \\ &\leq \frac{\left[1+(s+2)2^{1-s}\right](b-a)^2}{8(s+1)(s+2)(s+3)} \left\{ |f''(a)| + |f''(b)| \right\}.\end{aligned}$$

Corollary 1. *Under the assumptions of Theorem 1, if $s = 1$, then we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)^2\left[|f''(a)| + |f''(b)|\right]}{48}. \tag{1}$$

Theorem 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^0 (where I^0 is the interior of I) such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f''|$ is s-concave on $[a,b]$, for some fixed $s \in (0,1]$ and for $q > 1$ with $\frac{1}{p} + \frac{1}{q}$, then the following inequality holds:*

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{16} \frac{2^{\frac{s}{q}}}{(2p+1)^{1/p}} \left[\left| f''\left(\frac{3a+b}{4}\right) \right| + \left| f''\left(\frac{a+3b}{4}\right) \right| \right]. \tag{2}$$

Corollary 2. *Under the assumptions of Theorem 2, if we choose $s = 1$ and $\frac{1}{3} < \frac{1}{(2p+1)^{1/p}} < 1$, for $p > 1$, we have*

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{16} \left[\left| f''\left(\frac{3a+b}{4}\right) \right| + \left| f''\left(\frac{a+3b}{4}\right) \right| \right]. \tag{3}$$

In [6], Sarikaya et al. proved some inequalities related to Hermite–Hadamard's inequalities for functions whose derivatives in absolute value at certain powers are convex as follows:

Theorem 3. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that f'' is integrable and $0 < \lambda \leq 1$ on (a, b) with $a < b$. If $|f''|^q$ is convex on $[a, b]$, for $q \geq 1$, then the following inequality holds*:

$$\left|(\lambda - 1) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{2}\left(\frac{\lambda^3}{3} + \frac{1-3\lambda}{24}\right)^{1-1/q} \\ \times\left\{\left(\left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3\times 2^6}\right]|f''(a)|^q + \left[\frac{(2-\lambda)\lambda^3}{6} + \frac{5-16\lambda}{3\times 2^6}\right]|f''(b)|^q\right)^{1/q}\right. \\ \left. + \left(\left[\frac{(1+\lambda)(1-\lambda)^3}{6} + \frac{48\lambda - 27}{3\times 2^6}\right]|f''(a)|^q + \left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3\times 2^6}\right]|f''(b)|^q\right)^{1/q}\right\}, & \text{for } 0 \leq \lambda \leq 1/2; \\ \\ \frac{(b-a)^2}{2}\left(\frac{3\lambda-1}{24}\right)^{1-1/q} \left\{\left(\frac{8\lambda-3}{3\times 2^6}|f''(a)|^q + \frac{16\lambda-5}{3\times 2^6}|f''(b)|^q\right)^{1/q}\right. \\ \left. + \left(\frac{16\lambda-5}{3\times 2^6}|f''(a)|^q + \frac{8\lambda-3}{3\times 2^6}|f''(b)|^q\right)^{1/q}\right\}, & \text{for } 1/2 \leq \lambda \leq 1. \end{cases}$$

Corollary 3. *Under the assumptions of Theorem 3, if $\lambda = 0$, then we get the following inequality,*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| &\leq \frac{(b-a)^2}{48}\left(\frac{5|f''(a)|^q + 3|f''(b)|^q}{8}\right)^{1/q} \\ &+ \frac{(b-a)^2}{48}\left(\frac{3|f''(a)|^q + 5|f''(a)|^q}{8}\right)^{1/q}. \end{aligned} \tag{4}$$

The aim of this article is to establish Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral using the convexity as well as concavity, for functions whose absolute values of second derivative are convex. We derive a general integral inequality for Riemann–Liouville fractional integral.

2. Main Results

To prove our main results, we need to prove the following lemma, which plays the key role in the next developments:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$ and $n \in N$, then the following equality for fractional integrals holds with $0 < \alpha \leq 1$:*

$$\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)^-} f(a) + J^\alpha_{\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)^+} f(b)\right] - f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)$$
$$= \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left[\int_0^1 (1-\omega)^{\alpha+1}\left(f''(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b + f''(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)\right)d\omega\right.$$
$$\left.\int_0^1 ((1+\omega)^{\alpha+1} - 2^\alpha(1+\omega) + \alpha 2^\alpha(1-\omega))f''(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b) + f''(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega\right].$$

Proof. To compute each integral, we use integration by parts successively and get

$$
\begin{aligned}
P_1 &= \int_0^1 (1-\omega)^{\alpha+1} f''(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega \\
&= \frac{(n+1)(1-\omega)^{\alpha+1} f'(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega}{a-b}\Bigg|_0^1 \\
&\qquad + \frac{(n+1)(a+1)}{a-b}\int_0^1 (1-\omega)^{\alpha} f'(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega \\
&= \frac{(n+1)}{b-a} f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) + \frac{(n+1)(\alpha+1)}{a-b}\left[\frac{2(1-\omega)^{\alpha} f'(\frac{1+w}{n+1}a + \frac{1-\omega}{n+1}b)d\omega}{a-b}\Bigg|_0^1\right. \\
&\qquad \left. + \frac{(n+1)\alpha}{a-b}\int_0^1 (1-\omega)^{\alpha} f(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega\right] \\
&= \frac{(n+1)}{b-a} f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) + \frac{(n+1)(\alpha+1)}{a-b}\left[\frac{-2}{a-b} f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\
&\qquad \left. - \frac{(n+1)\alpha}{b-a}\int_0^1 (1-\omega)^{\alpha-1} f(\frac{n+\omega}{n+1}a + \frac{1-\omega}{n+1}b)d\omega\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n+1)}{b-a} f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) - \frac{(n+1)^2(\alpha+1)}{(b-a)^2} f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) \\
&+ \frac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}}\Gamma(\alpha+2) J^{\alpha}_{\frac{n}{n+1}a + \frac{1}{n+1}b^-} f(a). \\
P_2 &= \int_0^1 ((1+\omega)^{\alpha+1} - 2^{\alpha}(1+\omega) + \alpha 2^{\alpha}(1-\omega)) f''(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)d\omega \\
&= \frac{2\left[(1+\omega)^{\alpha+1} - 2^{\alpha}(1+\omega) + \alpha 2^{\alpha}(1-\omega)\right] f'(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)d\omega)}{b-a}\Bigg|_0^1 \\
&\qquad + \frac{2(\alpha+1)}{b-a}\int_0^1 \left[(\alpha+1)(1+\omega)^{\alpha} - 2^{\alpha} - \alpha 2^{\alpha}\right] f'(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)d\omega) \\
&= -\frac{n+1}{b-a}\left[1 + 2^{\alpha}(\alpha-1)\right] f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) \\
&\qquad + \frac{(n+1)(\alpha+1)}{b-a}\int_0^1 \left[(\alpha+1)(1+\omega)^{\alpha} - 2^{\alpha} - \alpha 2^{\alpha}\right] f'(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)d\omega) \\
&= -\frac{n+1}{b-a}\left[1 + 2^{\alpha}(\alpha-1)\right] f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) \\
&\qquad + \frac{(n+1)(\alpha+1)}{b-a}\left[\frac{(n+1)\left[(\alpha+1)(1+\omega)^{\alpha} - 2^{\alpha} - \alpha 2^{\alpha}\right] f'(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b)d\omega)}{b-a}\Bigg|_0^1\right. \\
&\qquad \left. - \frac{\alpha(n+1)(\alpha+1)}{b-a}\int_0^1 (1+\omega)^{\alpha-1} f\left(\frac{1-\omega}{n+1}a + \frac{n+\omega}{n+1}b\right)d\omega\right] \\
&= -\frac{n+1}{b-a}\left[1 + 2^{\alpha}(\alpha-1)\right] f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) + \frac{(n+1)^2(\alpha+1)}{(b-a)^2}(2^{\alpha}-1) \\
&\times f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) - \frac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}}\Gamma(\alpha+2) J^{\alpha}_{\frac{1}{n+1}a + \frac{n}{n+1}b^-} f(b).
\end{aligned}
$$

Analogously: $P_3 = -\frac{n+1}{b-a} f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) - \frac{(n+1)^2(\alpha+1)}{(b-a)^2} f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) + \frac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}}\Gamma(\alpha + 2) J^{\alpha}_{\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)^-} f(a)$.

$P_4 = -\frac{n+1}{b-a}\left[2^{\alpha}(1-\alpha) - 1\right] f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) - \frac{(n+1)^2(\alpha+1)}{(b-a)^2}$

$$\times\left(2^{\alpha}-1\right) f\left(\tfrac{1}{n+1} a+\tfrac{n}{n+1} b\right)+\tfrac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}} \Gamma(\alpha+2) J_{\left(\frac{1}{n+1} a+\frac{n}{n+1} b\right)^{+}}^{\alpha} f(b).$$

Adding above equalities, we get

$$\begin{aligned}
& \frac{n+1}{b-a} f\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)-\frac{\Gamma(\alpha+2)}{(n+1)^{2}(b-a)^{\alpha}}\left[J_{\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)^{-}}^{\alpha} f(a)+J_{\left(\frac{1}{n+1} a+\frac{n}{n+1} b\right)^{+}}^{\alpha} f(b)\right] \\
= & P_{1}+P_{2}+P_{3}+P_{4}.
\end{aligned}$$

This completes the proof. □

Theorem 4. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a twice differentiable function on (a,b) with $a<b$ and $n\in N^{*}$. If $f''\in L[a,b]$ and $|f''|$ is convex on $[a,b]$, then the following inequality for Riemann–Liouville fractional integrals holds:*

$$\begin{aligned}
& \left|\frac{\Gamma(\alpha+2)}{(n+1)(b-a)^{\alpha}}\left[J_{\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)^{-}}^{\alpha} f(a)+J_{\left(\frac{1}{n+1} a+\frac{n}{n+1} b\right)^{+}}^{\alpha} f(b)\right]-f\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)\right| \\
& \qquad \leq(b-a)^{2} \frac{2^{\alpha-1}(2+(\alpha-1) \alpha)}{(n+1)^{\alpha+3}(\alpha+1)(\alpha+2)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right). \qquad (5)
\end{aligned}$$

Proof. Using Lemma 1 and properties of modulus, we have

$$\begin{aligned}
& \left|\frac{\Gamma(\alpha+2)}{(n+1)(b-a)^{\alpha}}\left[J_{\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)^{-}}^{\alpha} f(a)+J_{\left(\frac{1}{n+1} a+\frac{n}{n+1} b\right)^{+}}^{\alpha} f(b)\right]-f\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)\right| \\
\leq & \frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \sum_{i=1}^{4}\left|P_{i}\right|.
\end{aligned}$$

Now, using convexity of $|f''|$, we have

$$\begin{aligned}
& \left|\frac{\Gamma(\alpha+2)}{(n+1)(b-a)^{\alpha}}\left[J_{\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)^{-}}^{\alpha} f(a)+J_{\left(\frac{1}{n+1} a+\frac{n}{n+1} b\right)^{+}}^{\alpha} f(b)\right]-f\left(\frac{n}{n+1} a+\frac{1}{n+1} b\right)\right| \\
& \leq\left|\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}(1-\omega)^{\alpha+1} f''\left(\frac{n+\omega}{n+1} a+\frac{1-\omega}{n+1} b\right) d \omega\right| \\
& +\left|\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}(1-\omega)^{\alpha+1} f''\left(\frac{1-\omega}{n+1} a+\frac{n+\omega}{n+1} b\right) d \omega\right| \\
& +\left|\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}\left((1+\omega)^{\alpha+1}-2^{\alpha}(1+\omega)+\alpha 2^{\alpha}(1-\omega)\right) f''\left(\frac{n+\omega}{n+1} a+\frac{1-\omega}{n+1} b\right) d \omega\right| \\
& +\left|\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}\left((1+\omega)^{\alpha+1}-2^{\alpha}(1+\omega)+\alpha 2^{\alpha}(1-\omega)\right) f''\left(\frac{1-\omega}{n+1} a+\frac{n+\omega}{n+1} b\right) d \omega\right| \\
& =\frac{(bv-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}(1-\omega)^{\alpha+1}\left|f''\left(\frac{n+\omega}{n+1} a+\frac{1-\omega}{n+1} b\right)\right| d \omega \\
& +\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}(1-\omega)^{\alpha+1}\left|f''\left(\frac{1-\omega}{n+1} a+\frac{n+\omega}{n+1} b\right)\right| d \omega \\
& +\frac{(b-a)^{2}}{(n+1)^{\alpha+3}(\alpha+1)} \int_{0}^{1}\left((1+\omega)^{\alpha+1}-2^{\alpha}(1+\omega)+\alpha 2^{\alpha}(1-\omega)\right)\left|f''\left(\frac{n+\omega}{n+1} a+\frac{1-\omega}{n+1} b\right)\right| d \omega
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))|f''(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)|d\omega \\
&\le \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\int_0^1(1-\omega)^{\alpha+1}\left[\left(\frac{n+\omega}{n+1}\right)|f''(a)|+\left(\frac{1-\omega}{n+1}\right)|f''(b)|\right]d\omega \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\int_0^1(1-\omega)^{\alpha+1}\left[\left(\frac{1-\omega}{n+1}\right)|f''(a)|+\left(\frac{n+\omega}{n+1}\right)|f''(b)|\right]d\omega \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega)) \\
&\times\left[\left(\frac{n+\omega}{n+1}\right)|f''(a)|+\left(\frac{1-\omega}{n+1}\right)|f''(b)|\right]d\omega \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega)) \\
&\times\left[\left(\frac{1-\omega}{n+1}\right)|f''(a)|+\left(\frac{n+\omega}{n+1}\right)|f''(b)|\right]d\omega.
\end{aligned}
$$

This completes the proof. □

Remark 1. *If we take $\alpha = n = 1$ in Theorem 4, then the inequality (5) reduces to the inequality (1). The inequality (1) was obtained by Ozdemir [15].*

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) with $a < b$ and $n \in N$. If $f'' \in L[a,b]$ and $|f''|^q$ is convex on $[a,b]$, then the following inequality for Riemann–Liouville fractional integrals holds with $0 < \alpha \le 1$:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a)+J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b)\right]-f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right| \\
&\le \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left[(U_5)^{1-1/q}\left\{\left(\frac{U_1|f''(a)|^q+U_2|f''(b)|^q}{(n+1)}\right)^{1/q}+\left(\frac{U_2|f''(a)|^q+U_1|f''(b)|^q}{(n+1)}\right)^{1/q}\right\}+\right. \\
&\left.(U_6)^{1-1/q}\left\{\left(\frac{U_3|f''(a)|^q+U_4|f''(b)|^q}{(n+1)}\right)^{1/q}+\left(\frac{U_4|f''(a)|^q+U_3|f''(b)|^q}{(n+1)}\right)^{1/q}\right\}\right].
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
U_1 &= \frac{n\alpha+3n+1}{(\alpha+2)(\alpha+3)},\quad U_2=\frac{1}{\alpha+3}, \\
U_3 &= \frac{6+2^\alpha\left(\alpha^3+5\alpha-6\right)+3n(\alpha+3)\left(-2+2^\alpha(2+(\alpha-1)\alpha)\right)}{6(\alpha+2)(\alpha+3)} \\
U_4 &= \frac{-3(\alpha+4)+2^\alpha(12+(\alpha-1)\alpha(\alpha+4))}{3(\alpha+2)(\alpha+3)},\qquad U_5=\frac{1}{\alpha+2}, \\
\text{and}\quad U_6 &= \frac{2^{\alpha+2}-1}{\alpha+2}-2^{\alpha+1}+\alpha 2^{\alpha-1}+2^{\alpha-1}-3\alpha 2^{\alpha-1}.
\end{aligned}
$$

Proof. Using Lemma 1, well-known power-mean integral inequality and the fact that $|f''|^q$ is convex , we have

$$\left|\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a)+J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b)\right]-f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right|$$

$$\begin{aligned}
\leq\ & \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1(1-\omega)^{\alpha+1}\left|f''(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1(1-\omega)^{\alpha+1}\left|f''(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))\left|f''(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))\left|f''(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
=\ & \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)^{1-1/q}\left(\int_0^1(1-\omega)^{\alpha+1}\left|f''(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)^{1-1/q}\left(\int_0^1(1-\omega)^{\alpha+1}\left|f''(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))\left|f''(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b)\right|^q d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))d\omega\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha 2^\alpha(1-\omega))\left|f''(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)\right|^q d\omega\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
\leq \quad & \frac{(b-a)^2}{(n+1)^{\frac{\alpha+3}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-1/q}\left(\int_0^1 (1-\omega)^{\alpha+1}\left((n+\omega)\left|f''(a)\right|^q+(1-\omega)\left|f''(b)\right|^q\right)d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\frac{\alpha+3}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-1/q}\left(\int_0^1 (1-\omega)^{\alpha+1}\left((1-\omega)\left|f''(a)\right|^q+(n+\omega)\left|f''(b)\right|^q\right)d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\frac{\alpha+3}{q}}(\alpha+1)}\left(\frac{2^{\alpha+2}-1}{\alpha+2}-2^{\alpha+1}+\alpha2^{\alpha-1}+2^{\alpha-1}-3\alpha2^{\alpha-1}\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))\left((n+\omega)\left|f''(a)\right|^q+(1-\omega)\left|f''(b)\right|^q\right)d\omega\right)^{1/q} \\
&+\frac{(b-a)^2}{(n+1)^{\frac{\alpha+3}{q}}(\alpha+1)}\left(\frac{2^{\alpha+2}-1}{\alpha+2}-2^{\alpha+1}+\alpha2^{\alpha-1}+2^{\alpha-1}-3\alpha2^{\alpha-1}\right)^{1-1/q} \\
&\times\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))\left((n+\omega)\left|f''(a)\right|^q+(1-\omega)\left|f''(b)\right|^q\right)d\omega\right)^{1/q}.
\end{aligned}
$$

Simple computations give

$$
\begin{aligned}
\int_0^1 (1-\omega)^{\alpha+1}(n+\omega)\,d\omega &= \frac{n\alpha+3n+1}{(\alpha+2)(\alpha+3)} = U_1, \\
\int_0^1 (1-\omega)^{\alpha+1}(1-\omega)\,d\omega &= \frac{1}{\alpha+3} = U_2,
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))(n+\omega)\,d\omega \\
= \quad & \frac{6+2^\alpha\left(\alpha^3+5\alpha-6\right)+3n(\alpha+3)(-2+2^\alpha(2+(\alpha-1)\alpha))}{6(\alpha+2)(\alpha+3)} = U_3,
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))(1-\omega)\,d\omega \\
&= \frac{-3(\alpha+4)+2^\alpha(12+(\alpha-1)\alpha(\alpha+4))}{3(\alpha+2)(\alpha+3)} = U_4, \\
\int_0^1 (1-\omega)^{\alpha+1}\,d\omega &= \frac{1}{\alpha+2} = U_5, \\
& \int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))d\omega \\
&= \frac{2^{\alpha+2}-1}{\alpha+2}-2^{\alpha+1}+\alpha2^{\alpha-1}+2^{\alpha-1}-3\alpha2^{\alpha-1} = U_6.
\end{aligned}
$$

This completes the proof. □

Remark 2. *If we take $\alpha = n = 1$ in Theorem 5, then the inequality (6) reduces to the inequality (4). The inequality in (1) was obtained by Sarikaya [6].*

In the following theorem, we obtain estimate of Hermite–Hadamard inequality for concave function.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) with $a < b$ and $n \in N$. If $f'' \in L[a,b]$ and $|f''|^q$ is concave on $[a,b]$, then the following inequality for Riemann–Liouville fractional integrals holds with $0 < \alpha \leq 1$:*

$$\left|\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a)+J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b)\right]-f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right|$$
$$\leq \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\times\left[U_5\left\{\left|f''\left(\left\{\frac{U_1(a)+U_2(b)}{U_5(n+1)}\right\}\right)\right|+\left|f''\left(\left\{\frac{U_1(b)+U_2(a)}{U_5(n+1)}\right\}\right)\right|\right\}\right. \tag{7}$$
$$\left.+U_6\left|f''\left(\left\{\frac{(U_3(a)+U_4(b)}{U_6(n+1)}\right\}\right)\right|+\left|f''\left(\left\{\frac{(U_3(b)+U_4(a)}{U_6(n+1)}\right\}\right)\right|\right].$$

Proof. Using the concavity of $|f''|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}|f''(\lambda a+(1-\lambda)b)|^q &> \lambda|f''(a)|^q+(1-\lambda)|f''(b)|^q\\ &\geq \left(\lambda|f''(a)|+(1-\lambda)|f''(b)|\right)^q.\end{aligned}$$

Hence,

$$|f''(\lambda a+(1-\lambda)b)| \geq \lambda|f''(a)|+(1-\lambda)|f''(b)|,$$

thus $|f''|$ is also concave. Using Jensen integral inequality, we have

$$\left|\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a)+J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b)\right]-f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right|$$
$$\leq \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)\left|f''\left(\frac{\int_0^1(1-\omega)^{\alpha+1}\left|\left(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b\right)\right|^q d\omega}{\int_0^1(1-\omega)^{\alpha+1}d\omega}\right)\right|^q$$
$$+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1(1-\omega)^{\alpha+1}d\omega\right)\left|f''\left(\frac{\int_0^1(1-\omega)^{\alpha+1}\left|\left(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b\right)\right|^q d\omega}{\int_0^1(1-\omega)^{\alpha+1}d\omega}\right)\right|^q$$
$$+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))d\omega\right)$$
$$\times\left|f''\left(\frac{\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))(\frac{n+\omega}{n+1}a+\frac{1-\omega}{n+1}b)d\omega}{\int_0^1((1+\omega)^{\alpha+1}-2^\alpha1+\omega+\alpha2^\alpha(1-\omega))d\omega}\right)\right|^q$$
$$+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\left(\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))d\omega\right)$$
$$\times\left|f''\left(\frac{\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1+\omega)+\alpha2^\alpha(1-\omega))(\frac{1-\omega}{n+1}a+\frac{n+\omega}{n+1}b)d\omega}{\int_0^1((1+\omega)^{\alpha+1}-2^\alpha(1-\omega)+\alpha2^\alpha(1-\omega))d\omega}\right)\right|^q$$
$$\leq \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}(U_5)\left|f''\left(\frac{U_1(a)+U_2(b)}{U_5(n+1)}\right)\right|^q+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}(U_5)\left|f''\left(\frac{U_1(b)+U_2(a)}{U_5(n+1)}\right)\right|^q$$
$$+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}(U_6)\left|f''\left(\frac{(U_3(a)+U_4(b)}{U_6(n+1)}\right)\right|^q+\frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}(U_6)\left|f''\left(\frac{(U_3(b)+U_4(a)}{U_6(n+1)}\right)\right|^q$$

$$\left|\frac{\Gamma(\alpha+2)}{(n+1)^2(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a)+J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b)\right]-f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right|$$
$$\leq \frac{(b-a)^2}{(n+1)^{\alpha+3}(\alpha+1)}\times\left[U_5\left\{\left|f''\left(\left\{\frac{U_1(a)+U_2(b)}{U_5(n+1)}\right\}\right)\right|+\left|f''\left(\left\{\frac{U_1(b)+U_2(a)}{U_5(n+1)}\right\}\right)\right|\right\}\right.$$
$$\left.+U_6\left|f''\left(\left\{\frac{(U_3(a)+U_4(b)}{U_6(n+1)}\right\}\right)\right|+\left|f''\left(\left\{\frac{(U_3(b)+U_4(a)}{U_6(n+1)}\right\}\right)\right|\right].$$

The proof is completed. □

Corollary 4. *On letting $\alpha = n = 1$ in Theorem 6, the inequality in Equation (8) becomes:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{48} \left| f''\left(\frac{5a+3b}{8}\right) \right| + \left| f''\left(\frac{3a+5b}{8}\right) \right|. \tag{8}$$

Remark 3. *The inequality in (8) is an improvement of the obtained inequality in Corollary 4 of [15]. This gives us a comparatively better estimate.*

3. Conclusions

We have derived some inequalities of Hermite–Hadamard type by establishing more general inequalities for functions that possesses second derivative on interior of an interval of real numbers, by using the Holder inequality and the assumptions that the mappings $|(f'')|^q$, for $q \geq 1$ are convex, as well as concave. The results presented here, certainly, provided refinements of those results proved in [6,15], since, by putting $\alpha = n = 1$ in our obtained inequalities, we achieve the already-presented inequalities in [6,15].

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