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Two Variables Shivley's Matrix Polynomials

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Received: 5 January 2019; Accepted: 21 January 2019; Published: 29 January 2019

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Abstract: The principal object of this paper is to introduce two variable Shivley's matrix polynomials and derive their special properties. Generating matrix functions, matrix recurrence relations, summation formula and operational representations for these polynomials are deduced. Finally, Some special cases and consequences of our main results are also considered.

Keywords: Shivley's matrix polynomials; Generating matrix functions; Matrix recurrence relations; summation formula; Operational representations

MSC 2010: 33C05; 33C45; 33E20; 15A60

1. Introduction

Generalized Laguerre polynomials (GLP) are defined explicitly

$$L_n^a(x) = \sum_{r=0}^{n} \frac{(-1)^r \, (1+a)_n \, x^r}{r! \, (n-r)! \, (1+a)_r}, \tag{1}$$

where a is a real -valued parameter, $(a)_r$ is the Pochhammer symbol

$$(a)_r = \begin{cases} a(a+1)\dots(a+(r-1)), & r \geq 1, \\ 1, & r = 0. \end{cases}$$

In confluent hypergeometric notation, we have

$$L_n^a(x) = \frac{(1+a)_n}{n!} \, {}_1F_1\big(-n; a+1; x\big). \tag{2}$$

These polynomials satisfy the second-order linear differential equation (see, for example, [1] p. 298)

$$x \, \mathbf{D}^2 L_n^a(x) + (1+a-x) \, \mathbf{D} L_n^a(x) + n L_n^a(x) = 0, \quad \mathbf{D} = \frac{d}{dx}. \tag{3}$$

The so-called Shively's pseudo-Laguerre polynomials $R_n(a,x)$ are defined by (see, [2])

$$R_n(a,x) = \frac{(a)_{2n}}{(a)_n \, n!} \, {}_1F_1\big(-n; a+n; x\big), \tag{4}$$

which are related to the proper simple Laguerre polynomial (see, [2])

$$L_n(x) = {}_1F_1(-n;1;x), \tag{5}$$

$$R_n(a,x) = \frac{1}{(a-1)_n} \sum_{r=0}^{n} \frac{(a-1)_{n+r}\, L_{n-r}(x)}{r!}. \tag{6}$$

Shivley deduced the generating function for pseudo Laguerre polynomials of one variable as (see, [2])

$$e^{2t}\ {}_0F_1(-;\frac{a}{2}+\frac{1}{2};t^2-xt) = \sum_{n=0}^{\infty} \frac{R_n(a,x)\, t^n}{(\frac{a}{2}+\frac{1}{2})_n}. \tag{7}$$

Now, owing to the significance of the earlier mentioned work related to Laguerre polynomials, we find record that many authors became interested to study the scalar cases of the classical sets of Laguerre polynomials into Laguerre matrix polynomials. Of those authors, we mention [3–7].

Recently, the matrix versions of the classical families orthogonal polynomials such as Jacobi, Hermite, Chebyshev, Legendre, Gegenbauer, Bessel and Humbert polynomials of one variables and some other polynomials were introduced by many authors for matrices in $\mathbb{C}^{N\times N}$ and various properties satisfied by them were given from the scalar case. Rather than giving an exhaustive list of references, we refer the reader to the article [8]. Theory of generalized and multivariable orthogonal matrix polynomials has provided new means of analysis to deal with the majority of problems in mathematical physics which find broad practical applications. In [9,10], Subuhi Khan and others introduced the 2-variable forms of Laguerre and modified Laguerre matrix polynomials and generalized Hermite matrix based polynomials of two variables and Lie algebraic techniques. Furthermore, several papers concerning the orthogonal matrix polynomials for two and multivariables have become more and more relevant, see for example [11–17].

The section-wise treatment is as follows. In Section 2, we deals with some basic facts, notations and results to that are needed in the work. In Section 3, we define Shivleys matrix polynomials of two variables and to study their properties. The generating matrix functions, matrix recurrence relations, summation formula and operational representations these new matrix polynomials are obtained. Some special cases of the established results are also underlined as corollaries. Finally, we give some concluding remarks in Section 4.

Throughout this paper, for $\mathbb{C}^N$ denote the N-dimensional complex vector space and $\mathbb{C}^{N\times N}$ denote all square matrices with N rows and N columns with entries are complex numbers, $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$ denote the real and imaginary parts of a complex number z, respectively. For any matrix A in $\mathbb{C}^{N\times N}$, $\sigma(A)$ is the spectrum of A, the set of all eigenvalues of A, which will be denoted by $\|A\|$, is defined by

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2},$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^H y)^{\frac{1}{2}}$ is Euclidean norm of y. I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively.

2. Preliminaries

We shall adopt in this work a somewhat different notation and facts from that used throughout this work.

For $A \in \mathbb{C}^{N\times N}$, the matrix version of the symbol is

$$(A)^{(n)} = (A)(A-I)\cdots(A-(n-1)I), \quad n \geq 1,$$

and the Pochhammer symbol (the shifted factorial) is

$$(A)_n = A(A+I)\cdots(A+(n-1)I), \qquad n \geq 1; \;\; (A)_0 \equiv I.$$

Note that if $A = -jI$, where j is a positive integer, then $(A)_n = \mathbf{0}$ whenever $n > j$ (cf. [18]).

The reciprocal scalar Gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Thus, for any $A \in \mathbb{C}^{N\times N}$, Riesz-Dunford functional calculus [18–20] shows that $\Gamma^{-1}(A)$ is well defined and is, indeed, the inverse of $\Gamma(A)$. Furthermore, if

$$A + nI \quad \text{is invertible for all integer } n \geq 0, \tag{8}$$

then

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A). \tag{9}$$

Form (9), it is easily to find that

$$(A)_{2n} = 2^{2n}(\frac{1}{2}(A+I))_n(\frac{1}{2}(A))_n,$$

and

$$(A)_{n+k} = (A)_n(A+nI)_k.$$

In 1731, Euler defined the derivative formula

$$D_x^{\nu}\, x^{\alpha} = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha-\nu+1)}\, x^{\alpha-\nu}, \quad D_x \equiv \frac{d}{dx},$$

where α and ν are arbitrary complex numbers. By application of the matrix functional calculus to this definition, for any matrix $A \in \mathbb{C}^{N\times N}$, one gets (see [5,18])

$$\mathbf{D}_t^n[t^{A+mI}] = (A+I)_m[(A+I)_{m-n}]^{-1}\, t^{A+(m-n)I}, \;\; \mathbf{D}_t = \frac{d}{dt}, \;\; n = 0,1,2,3,...$$

On other hand, if $\mathbf{D}_z = \frac{\partial}{\partial z}$, $\mathbf{D}_z = \frac{\partial}{\partial w}$ and $\mathbf{D}_v = \frac{\partial}{\partial v}$ the trinomial expansion for $(\mathbf{D}_z + \mathbf{D}_w + \mathbf{D}_v)^n$ is given by (see [21,22])

$$(\mathbf{D}_z + \mathbf{D}_w + \mathbf{D}_v)^n = \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-1)^{r+s}\,(-n)_{r+s}}{r!\; s!}\, \mathbf{D}_z^{n-r-s}\, \mathbf{D}_w^r\, \mathbf{D}_v^s, \tag{10}$$

operating (10) on $F(z,w,v)$, we get

$$\begin{aligned}&(\mathbf{D}_z + \mathbf{D}_w + \mathbf{D}_v)^n F(z,w,v) = \\ &\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-1)^{r+s}\,(-n)_{r+s}}{r!\; s!}\, \mathbf{D}_z^{n-r-s}\, \mathbf{D}_w^r\, \mathbf{D}_v^s F(z,w,v),\end{aligned} \tag{11}$$

in particular, if $F(z,w,v) = f(z)g(w)h(v)$, then (11) gives

$$\begin{aligned}&(\mathbf{D}_z + \mathbf{D}_w + \mathbf{D}_v)^n \Big\{ f(z)g(w)h(v) \Big\} = \\ &\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-1)^{r+s}\,(-n)_{r+s}}{r!\; s!}\, \mathbf{D}_z^{n-r-s} f(z)\; \mathbf{D}_w^r g(w)\; \mathbf{D}_v^s h(v).\end{aligned} \tag{12}$$

Similarly,

$$
\begin{aligned}
&(\mathbf{D}_z\mathbf{D}_w + \mathbf{D}_z\mathbf{D}_v + \mathbf{D}_w\mathbf{D}_v)^n\Big\{f(z)g(w)h(v)\Big\} = \\
&\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-1)^{r+s}\,(-n)_{r+s}}{r!\; s!}\;\mathbf{D}_z^{n-s}f(z)\;\mathbf{D}_w^{n-r}g(w)\;\mathbf{D}_v^{r+s}h(v).
\end{aligned}
\tag{13}
$$

Moreover, if $A \in \mathbb{C}^{N\times N}$, and z is any complex number, then the matrix exponential e^{Az} is defined to be

$$
e^{Az} = I + Az + ... + \frac{A^n}{n!}\,z^n + ...,
$$

$$
\frac{d^n}{dz^n}[e^{Az}] = A^n\, e^{Az} = e^{Az}\, A^n, \quad n = 0, 1, 2, 3, ...
$$

Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ and C satisfy condition (8), then the hypergeometric matrix function of 2-numerator and 1-denominator for $|z| < 1$ is defined by the matrix power series (see [20,23])

$$
{}_2F_1(A, B; C; z) = \sum_{n\geq 0} \frac{(A)_n(B)_n[(C)_n]^{-1}}{n!} z^n, \tag{14}
$$

For an arbitrary matrix $A \in \mathbb{C}^{N\times N}$, satisfy condition (8) then the n-th Laguerre matrix polynomials $L_n^A(z)$ is defined by (see [8])

$$
L_n^A(z) = \frac{(A+I)_n}{n!}\;{}_1F_1(-nI; A+I; z). \tag{15}
$$

Therefore, the Shively's pseudo Laguerre matrix polynomials are reduced in the form

$$
R_n^A(z) = \frac{(A)_{2n}\;[(A)_n]^{-1}}{n!}\;{}_1F_1(-nI; A+nI; z). \tag{16}
$$

For matrices $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geq 0, k \geq 0$, the following relations are satisfied (see [24])

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k, n-2k) \tag{17}
$$

and

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k). \tag{18}
$$

Similarly, we can write

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n+2k), \tag{19}
$$

and

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+k). \tag{20}
$$

3. Two Variables Shivley's Matrix Polynomials

In this section we define two variables Shively's matrix polynomials and several properties for these polynomials as given below:

Definition 1. *For an arbitrary matrix $A \in \mathbb{C}^{N\times N}$, with $A + mI$ invertible for every integer $m \geq 1$, then the m-th Shively's matrix polynomials of two variables $R_m^A(z,w)$ is defined by*

$$R_m^A(z,w) = \frac{(A+mI)_m}{m!} \sum_{n=0}^{m} \sum_{k=0}^{m-n} [(A+mI)_k]^{-1} (-mI)_{n+k} \frac{z^k w^n}{n!\, k!}. \tag{21}$$

Remark 1. *For simplicity, we consider only two complex variables Shively's matrix polynomials, though the results can easily be extended to several complex variables.*

3.1. Generating Functions and Recurrence Relations

Two more basic properties of two variables Shively's matrix polynomials are developed in this subsection. The generating matrix functions which is obtained from Theorem 1 and with the help of Definition 1. Also, some matrix recurrence relations for two variables Shively's matrix polynomials are given.

Theorem 1. *The generating matrix function of $R_m^A(z,w)$ is given by*

$$\sum_{m=0}^{\infty} R_m^A(z,w)\, [(\frac{A+I}{2})_m]^{-1}\, t^m = e^{2t}\, {}_0F_1\Big(-; \frac{A+I}{2}; t^2(1-w) - tz\Big). \tag{22}$$

Proof. From Definition 1 in the left hand side of (22), we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} R_m^A(z,w)\, [(\frac{A+I}{2})_m]^{-1}\, t^m = \\
&\sum_{m=0}^{\infty} [(\frac{A+I}{2})_m]^{-1}\, t^m\, \frac{(A+mI)_m}{m!} \\
&\times \sum_{n=0}^{m} \sum_{k=0}^{m-n} [(A+mI)_k]^{-1}\, (-mI)_{n+k}\, \frac{z^k w^n}{n!\, k!} \\
&= \sum_{m=0}^{\infty} 2^{2m}\, (\frac{A}{2})_m\, (\frac{A+I}{2})_m\, [(A)_m]^{-1}\, [(\frac{A+I}{2})_m]^{-1}\, t^m \\
&\times \sum_{n=0}^{m} \sum_{k=0}^{m-n} [(A+mI)_k]^{-1}\, (-mI)_{n+k}\, \frac{z^k w^n}{n!\, k!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\frac{A}{2})_n\, (\frac{A}{2} + nI)_{m+k}\, [(A)_{2n}]^{-1}\, [(A+2n)_{m+k}]^{-1} \\
&\times \frac{(-1)^{n+k} z^n w^k (4t)^{m+n+k}}{m!n!k!} \\
&= e^{2t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [(\frac{A+I}{2})_{m+n}]^{-1}\, \frac{(t^2(1-w))^m\, (-zt)^n}{m!\, n!}.
\end{aligned} \tag{23}$$

Further simplification yields

$$\begin{aligned}
\sum_{m=0}^{\infty} R_m^A(z,w)\, [(\frac{A+I}{2})_m]^{-1}\, t^m = \quad & e^{2t} \sum_{m=0}^{\infty} [(\frac{A+I}{2})_m]^{-1}\, \frac{\big(t^2(1-w) - zt\big)^m}{m!} \\
= e^{2t}\, {}_0F_1\Big(-; \frac{A+I}{2}; t^2(1-w) - tz\Big).&
\end{aligned} \tag{24}$$

This completes the proof of Theorem 1. □

Theorem 1 leads to the following corollaries:

Corollary 1. *The generating matrix function for the Shively's pseudo Laguerre matrix polynomials $R_m^A(z)$ is given by*

$$\sum_{m=0}^{\infty} R_m^A(z)\ [(\frac{A+I}{2})_m]^{-1}\ t^m = \quad e^{2t}\ {}_0F_1\Big(-;\frac{A+I}{2};t^2-tz\Big). \tag{25}$$

Proof. Follows by successive application of Theorem 1. □

Corollary 2. *From the generating matrix function (22), we can deduce that*

$$R_m^A(0,w) = \frac{1}{m}\ {}_2F_1\Big(-\frac{m}{2}I, \frac{-m+1}{2}I; I-(\frac{A}{2}+mI); w\Big). \tag{26}$$

Proof. By putting $z=0$ in (22), we find

$$\begin{aligned}
&\sum_{m=0}^{\infty} R_m^A(0,w)\ [(\frac{A+I}{2})_m]^{-1}\ t^m \\
=&e^{2t}\ {}_0F_1\Big(-;\frac{A+I}{2};t^2(1-w)\Big) \\
=&e^{2t}\ \sum_{m=0}^{\infty} [(\frac{A+I}{2})_m]^{-1}\ t^{2m}\ \frac{(1-w)^m}{m!} \\
=&e^{2t}\ \sum_{m=0}^{\infty} [(\frac{A+I}{2})_m]^{-1}\ \frac{t^{2m}}{m!}\ \sum_{k=0}^{m} \frac{(-mI)_k\ w^k}{k!}.
\end{aligned}$$

Further simplification yields

$$\begin{aligned}
&\sum_{m=0}^{\infty} R_m^A(0,w)\ [(\frac{A+I}{2})_m]^{-1}\ t^m \\
=&\sum_{m=0}^{\infty} [(\frac{A+I}{2})_m]^{-1}\ t^m\ \frac{1}{m}\ {}_2F_1\Big(-\frac{m}{2}I, \frac{-m+1}{2}I; I-(\frac{A}{2}+mI); w\Big),
\end{aligned} \tag{27}$$

and the relation (27) evidently leads us to the required result. □

Among the infinitely many recurrence relations for two variables Shively's matrix polynomials, we list the following two as being the most useful or interesting ones.

$$AR_m^A(z,w) + z\mathbf{D}_z\ R_m^A(z,w) = (A+mI)\ R_m^{A-I}(z,w), \tag{28}$$

$$(A+mI)\ R_{m-1}^A(z,w) + (z\mathbf{D}_z + w\mathbf{D}_w)\ R_m^A(z,w) = m\ R_m^{A-I}(z,w),\ \ \mathbf{D}_z \equiv \frac{\partial}{\partial z},\ \ \mathbf{D}_w \equiv \frac{\partial}{\partial w}. \tag{29}$$

It can be easily verifying these relations from the Definition 1.

3.2. Summation Formulas and Operational Representation

We are now in a position to obtain some series expansion formulae involving partial derivatives for the $R_m^A(z,w)$, these series expansion formulae are given by the following theorem:

Theorem 2. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (8) and $u \in \mathbb{C}$. Then the Shively's matrix polynomials of two variables has the following summation formulas*

$$\sum_{k=0}^{m} \frac{u^k}{k!}\, \mathbf{D}_z^k\, R_m^A(z,w) = R_m^A(z+u,w), \tag{30}$$

$$\sum_{k=0}^{m} \frac{u^k}{k!}\, \mathbf{D}_w^k\, R_m^A(z,w) = R_m^A(z,w+u), \tag{31}$$

$$\begin{aligned}&\sum_{k=0}^{m}[(A+I)_{m-k}]^{-1}\, \frac{(-u)^k}{k!}\, \mathbf{D}_w^k\, R_m^{A-kI}(z,w)\\ &= (1+u)^m\, [(A+I)_m]^{-1}\, R_m^A\Big(\frac{z}{1+u}, \frac{w}{1+u}\Big),\end{aligned} \tag{32}$$

$$\begin{aligned}&\sum_{k=0}^{m} \frac{(-u)^k((m+k)!)^2(-mI)_{-k}}{k!}\, \mathbf{D}_z^k\, \mathbf{D}_w^k\, R_{m+k}^{A-kI}(z,w)\\ &= m!\, (1+u)^m\, R_m^A\Big(\frac{z}{1+u}, \frac{w}{1+u}\Big),\end{aligned} \tag{33}$$

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k}\, [(I+A-nI)_k]^{-1}\, z^k\, \mathbf{D}_z^k\, R_m^A(z,w)\\ &=(A+I)_m\, [(I+A-nI)_m]^{-1}\, R_m^{A-nI}(z,w); \quad n \le m.\end{aligned} \tag{34}$$

Proof. Taking the left hand side of (30) and substituting the value of $R_m^A(z,w)$ from (21), we get

$$\begin{aligned}&\sum_{k=0}^{m} \frac{u^k(A+I)_m}{m!k!}\, \mathbf{D}_z^k \sum_{n=0}^{m}\sum_{r=0}^{m-n} [(A+mI)_r]^{-1}\, (-mI)_{n+r}\, \frac{z^r\, w^n}{n!\, r!}\\ =&\frac{(A+I)_m}{m!} \sum_{n=0}^{m}\sum_{r=0}^{m-n} [(A+mI)_r]^{-1}\, (-mI)_{n+r}\, \frac{z^r\, w^n}{n!\, r!} \sum_{k=0}^{m} \frac{(-r)_k(\frac{-u}{z})^k}{k!}\\ =&\frac{(A+I)_m}{m!} \sum_{n=0}^{m}\sum_{r=0}^{m-n} [(A+mI)_r]^{-1}\, (-mI)_{n+r}\, \frac{(z+u)^r\, w^n}{n!\, r!} = R_m^A(z+u,w).\end{aligned} \tag{35}$$

This completes the proof of (30). Similarly, we can prove (31).

Taking the left hand side of (32), substituting the value of Shively's matrix polynomials of two variables from (21) and differentiating, we get

$$\begin{aligned}&\sum_{k=0}^{m}[(A+I)_{m-k}]^{-1}\, \frac{(-u)^k}{k!}\, \mathbf{D}_w^k\, R_m^{A-kI}(z,w)\\ &= \frac{1}{m!} \sum_{k=0}^{m} \frac{(-u)^k[(A+I)_{-k}]^{-1}}{k!} \sum_{n=0}^{m}\sum_{r=0}^{m-n} [(I+A-kI)_r]^{-1}\, (-mI)_{n+r}\, \frac{z^{r-k}\, w^n}{n!\, (r-k)!}.\end{aligned} \tag{36}$$

Putting $r = \mu + k$ where μ is new parameter of summation and changing the order of summation so that the first summation becomes last,

$$
\begin{aligned}
&\sum_{k=0}^{m}[(A+I)_{m-k}]^{-1}\ \frac{(-u)^k}{k!}\ \mathbf{D}_w^k\ R_m^{A-kI}(z,w)\\
&=\frac{1}{m!}\ \sum_{n=0}^{m}\ \sum_{\mu=0}^{m-n}\ [(I+A-kI)_\mu]^{-1}\ (-mI)_{n+\mu}\ \frac{z^\nu\ w^n}{n!\ \mu!}\\
&\times\sum_{k=0}^{m-n-\mu}\ \frac{(-m+n+\mu)(-u)^k}{k!}\\
&=\frac{1}{m!}\ \sum_{n=0}^{m}\ \sum_{\mu=0}^{m-n}\ [(I+A-kI)_\mu]^{-1}\ (-mI)_{n+\mu}\ \frac{z^\nu\ w^n}{n!\ \mu!}\ (1+u)^{(m-n-\mu)},
\end{aligned}
\tag{37}
$$

which in view of (21), gives us the right hand-side of assertion (32).

Also, from the left hand side of (33) and substituting the value of Shively's matrix polynomials of two variables from (21), we obtain

$$
\begin{aligned}
&\sum_{k=0}^{m}\ \frac{(-u)^k(-mI)_{-k}(I+A-kI)_{m+k}}{k!}\ \mathbf{D}_z^k\ \mathbf{D}_w^k\\
&\times\sum_{n=0}^{m+k}\ \sum_{r=0}^{m+k-n}\ [(I+A-kI)_r]^{-1}\ (-mI-kI)_{n+r}\ \frac{z^r\ w^n}{n!\ r!}.
\end{aligned}
\tag{38}
$$

Now, differentiating and substituting $p=r-k$, $q=n-k$, we have

$$
\begin{aligned}
&(A+I)_m\sum_{k=0}^{m}\sum_{q=0}^{m-k}\ \sum_{p=0}^{m+k-q}\ [(A+I)_p]^{-1}\ \frac{(-u)^k(-mI)_{p+q+k}}{k!}\ \frac{z^p\ w^q}{p!\ q!}\\
&=(A+I)_m\ \sum_{q=0}^{m}\ \sum_{p=0}^{m-q}\ [(A+I)_p]^{-1}\ \frac{(-mI)_{p+q}\ z^p\ w^q}{p!\ q!}\ (1+u)^{m-p-q}.
\end{aligned}
\tag{39}
$$

Again, using the expression (21), we arrive at the right-hand side of (33).

Consider the series

$$
\begin{aligned}
&\sum_{k=0}^{n}\ \binom{n}{k}\ [(I+A-nI)_k]^{-1}\ z^k\ \mathbf{D}_z^k\ R_m^A(z,w)\\
&=\frac{(A+I)_m}{m!}\ \sum_{k=0}^{n}\ \binom{n}{k}\ [(I+A-nI)_k]^{-1}\\
&\times\sum_{n=0}^{m}\ \sum_{r=0}^{m-n}\ [(A+mI)_r]^{-1}\ (-mI)_{n+r}\ \frac{z^r\ w^n}{n!\ (r-k)!}\\
&=\frac{(A+I)_m}{m!}\ \sum_{n=0}^{m}\ \sum_{r=0}^{m-n}\ [(A+mI)_r]^{-1}\ (-mI)_{n+r}\ \frac{z^r\ w^n}{n!\ (r!)^2}\ {}_2F_1\Big(-rI,-nI;I+A-nI;1\Big).
\end{aligned}
$$

In light of the relationship (see, [25])

$$
F(-nI,B:C;1)=\Gamma(C)\Gamma(C-B+nI)\Gamma^{-1}(C+nI)\Gamma^{-1}(C-B),
\tag{40}
$$

where $B,C\in\mathbb{C}^{N\times N}$, we obtain the required result in (34). □

Remark 2. *Setting $n=1$ in (34) we have the recurrence relations for the $R_m^A(z,w)$ in (28).*

Next, according to (13), we have the following operational representation for the $R_m^A(z,w)$:

$$
\begin{aligned}
&\Big(\mathbf{D}_z\ \mathbf{D}_w + \mathbf{D}_z\ \mathbf{D}_v + \mathbf{D}_w\ \mathbf{D}_v\Big)^m \Big\{ z^{A+(2m-1)I}\ w^m\ e^{-v} \Big\} \\
&= \sum_{n=0}^{m} \sum_{k=0}^{m-n} \frac{(-1)^{n+k}\ (-mI)_{n+k}}{n!\ k!} \\
&\times \mathbf{D}_z^{m-k}\big(z^{A+(2m-1)I}\big)\ \mathbf{D}_w^{m-n}\big(w^m\big)\ \mathbf{D}_v^{n+k}\big(e^{-v}\big) \\
&= \frac{(m!)^2}{n!}\ z^{A+(m-1)I}\ e^{-v} \Big\{ \frac{(A+mI)_m}{m!} \sum_{n=0}^{m} \sum_{k=0}^{m-n} \frac{[(A+kI)_k]^{-1} (-mI)_{n+k}\ z^k w^n}{n!k!} \Big\} \\
&= \frac{(m!)^2}{n!}\ z^{A+(m-1)I}\ e^{-v}\ R_m^A(z,w),
\end{aligned}
\tag{41}
$$

thus, we get

$$
\begin{aligned}
&\Big(\mathbf{D}_z\ \mathbf{D}_w + \mathbf{D}_w\ \mathbf{D}_v + \mathbf{D}_v\ \mathbf{D}_z\Big)^m \Big\{ z^{A+(2m-1)I}\ w^m\ e^{-v} \Big\} \\
&= \frac{(m!)^2}{n!}\ z^{A+(m-1)I}\ e^{-v}\ R_m^A(z,w).
\end{aligned}
\tag{42}
$$

Summarizing, the following result has been obtained:

Theorem 3. *Let $R_m^A(z,w)$ be given in (21). The operational representation in (42) holds true.*

4. Concluding Remarks

This paper is to define a new matrix polynomial, say, Shivley's matrix polynomials of two complex variables and to study their properties. Some formulas related to an explicit representation, generating matrix functions, matrix recurrence relations, series expansion and operational representations are deduced. Also, some interested particular cases and consequences of our results have been discussed. Within such a context, new matrix polynomial structures emerge with wide possibilities of applications in physics and engineering. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

Author Contributions: All authors contributed equally.

Funding: This Work is Supported by National Natural Science Foundation of China (11601525), Natural Science Foundation of Hunan Province (2017JJ3406).

Acknowledgments: The third and fourth authors wish to acknowledge the approval and the support of this research study from the Deanship of Scientific Research in King Khalid University, Abha, Saudi Arabia under grant (R.G.P-215-39). The authors would like to thank the referee for the valuable comments, which helped to improve this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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