



Article Starlike Functions Related to the Bell Numbers

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Abstract: The present paper aims to establish the first order differential subordination relations between functions with a positive real part and starlike functions related to the Bell numbers. In addition, several sharp radii estimates for functions in the class of starlike functions associated with the Bell numbers are determined.

Keywords: differential subordination; starlike functions; Bell numbers; radius estimate

MSC: 30C45; 30C55; 30C80

1. Introduction

Let \mathcal{A} be a class of analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions f(0) = 0 and f'(0) = 1. Suppose \mathcal{S} is a subclass of \mathcal{A} consisting of univalent functions. An analytic function f is subordinate to g, written as $f \prec g$, if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $|w(z)| \le |z|$ such that f(z) = g(w(z)) ($z \in \mathbb{D}$). Moreover, if g is univalent in \mathbb{D} , then the equivalent conditions for subordination can be written as f(0) = g(0) and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. By imposing some geometric and analytic conditions over the functions in the class \mathcal{S} , many authors considered several subclasses of \mathcal{S} . Various subclasses of starlike and convex functions were studied in the literature, and they can be unified by considering an analytic univalent function φ with a positive real part in \mathbb{D} , symmetric about the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$. Ma and Minda [1] studied the class

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}.$$

The class $S^*(\varphi)$ for various choice of the domain $\varphi(\mathbb{D})$ was considered in recent years. The class $S^*[A, B] := S^*((1 + Az)/(1 + Bz))(-1 \le B < A \le 1)$ was introduced by Janowski [2]. For $0 \le \alpha \le 1$, the class $S^*(\alpha) := S^*[1 - 2\alpha, -1]$ is the class of starlike functions of order α . Uralegaddi et al. [3] defined the class

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \ (\beta > 1) \right\} = \mathcal{S}^*\left(\frac{1 + (1 - 2\beta)z}{1 - z}\right).$$

Several authors considered various special cases of the class of Janowski starlike functions by considering some specific functions, namely $\varphi_q(z) := z + \sqrt{1+z^2}$, $\varphi_0(z) := 1 + (z/k)((k+z)/(k-z))$ $(k = \sqrt{2}+1)$, $\varphi_s(z) := 1 + \sin z$, and $G_\alpha(z) := 1 + z/(1 - \alpha z^2)$. Some of those classes are: $S_L^* := S^*(\sqrt{1+z})$ [4], $S_q^* := S^*(\varphi_q(z))$ [5], $S_e^* = S^*(e^z)$ [6], $S_R^* = S^*(\varphi_0)$ [7], $S_s^* = S^*(\varphi_s)$ [8]), $\mathcal{BS}^*(\alpha) := S^*(G_\alpha(z)), 0 \le \alpha < 1$ [9,10]. For a brief survey on these classes, readers may refer to [11,12].

It should be noted that the special cases of φ , mentioned above, are univalent in the unit disk. In 2011, Dziok et al. [13,14] considered φ to be a non-univalent function associated with the Fibonacci numbers, defined by

$$\tilde{p}(z) := \varphi(z) = rac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \ \ \tau := \left(1 - \sqrt{5}\right)/2$$

which maps the unit disk \mathbb{D} on to a shell-like domain in the right-half plane. Further, they defined the class $S_F^* := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \tilde{p}(z)\}$. The functions $f \in S_F^*$ are starlike of order $\sqrt{5}/10$.

Motivated by the above defined classes, we consider a function associated with the Bell Numbers. For a fixed non-negative integer *n*, the Bell numbers B_n count the possible disjoint partitions of a set with *n* elements into non-empty subsets or, equivalently, the number of equivalence relations on it. The Bell numbers B_n satisfy a recurrence relation involving binomial coefficients $B_{n+1} = \sum_{k=0}^{n} {n \choose k} B_k$. Clearly $B_0 = B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, and $B_6 = 203$. For more details, see [15–21]. Kumar et al. [22] considered the function

$$Q(z) := e^{e^{z} - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \ (z \in \mathbb{D})$$

which is starlike with respect to 1 and it's coefficients generate the Bell numbers. Kumar et al. [22] defined the class S_B^* by $S_B^* := S^*(Q)$. From [1], note that the function $f \in S_B^*$ if and only if there exists an analytic function q, satisfying $q(z) \prec Q(z)$ ($z \in \mathbb{D}$), such that

$$f(z) = I(q(z)) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} dt\right).$$

The above representation shows that the functions in the class S_B^* can be seen as an integral transform I(q(z)) of the function q with f(0) = 0 and f'(0) = 1. The reader may refer to the paper [23] and the references cited therein for integral transform related works. The authors in [22] determined sharp coefficient bounds on the six initial coefficients, Hankel determinant, and on the first three consecutive higher order Schwarzian derivatives for functions in the class S_B^* .

Let \mathcal{P} be the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with p(0) = 1 and Re p(z) > 0 ($z \in \mathbb{D}$). In 1989, Nunokawa et al. [24] showed that if $1 + zp'(z) \prec 1 + z$, then $p(z) \prec 1 + z$. In 2007, Ali et al. [25] computed the condition on β , in each case, for which

$$1 + \frac{\beta z p'(z)}{p^j(z)} \prec \frac{1 + Dz}{1 + Ez}$$
 $(j = 0, 1, 2)$ implies $p(z) \prec \frac{1 + Az}{1 + Bz}$,

A, *B*, *C*, *D*, *E*, $F \in [-1, 1]$. Further, Ali et al. [26] determined some sufficient conditions for normalized analytic functions to lemniscate starlike functions. Recently, Kumar and Ravichandran [27] obtained sufficient conditions for first order differential subordinations so that the corresponding analytic function belongs to the class \mathcal{P} . In 2016, Tuneski [28] gave a criteria for analytic functions to be Janowski starlike. For more details, see [11,29–33].

Motivated by above works, in Section 2, using the theory of differential subordination developed by Miller and Mocanu, a sharp bound on parameter β is determined in each case so that $p(z) \prec Q(z)$, whenever $1 + \beta z p'(z) / p^j(z)$ (j = 0, 1, 2) is subordinate to the function $\varphi_0(z)$ or $\sqrt{1+z}$ or $G_\alpha(z)$ or (1 + Az) / (1 + Bz) or $\varphi_s(z)$ or $\varphi_q(z)$. Further, various sufficient conditions are obtained for $f \in A$ to be in the class S_B^* as an application of these subordination results. In Section 3, S_B^* -radius for the class of Janowski starlike functions and some other well-known classes of analytic functions are investigated.

2. Differential Subordinations

Theorem 1 provides estimate on β so that $p(z) \prec Q(z)$ holds, whenever $1 + \beta z p'(z) \prec \varphi_0(z)$ or $\varphi_s(z)$ or $\sqrt{1+z}$ or $G_{\alpha}(z)$ or (1+Az)/(1+Bz) or $\varphi_s(z)$ or $\varphi_q(z)$ or e^z .

To prove our main results, we need the following lemma due to Miller and Mocanu:

Lemma 1. ([32] Theorem 3.4h, p. 132) Let q be analytic in \mathbb{D} and let ψ and v be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set

$$Q(z) := zq'(z)\psi(q(z))$$
 and $h(z) := \nu(q(z)) + Q(z)$

Suppose that

- (i) either h is convex, or Q is starlike univalent in \mathbb{D} and (ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbb{D}$.

If p is analytic in \mathbb{D} *, with* p(0) = q(0)*,* $p(\mathbb{D}) \subseteq U$ *and*

$$\nu(p(z)) + zp'(z)\psi(p(z)) \prec \nu(q(z)) + zq'(z)\psi(q(z)),$$

then $p \prec q$ *, and* q *is most dominant.*

Theorem 1. Let $l(e) = (1 - e^{(1-e)/e})^{-1}$, $0 < \alpha < 1$, 0 < B < A < 1, and *p* be an analytic function defined in \mathbb{D} with p(0) = 1.

Set

$$Y_{\beta}(z, p(z)) = 1 + \beta z p'(z).$$

Then, the following are sufficient for $p(z) \prec Q(z)$.

- $Y_{\beta}(z, p(z)) \prec \varphi_0(z)$ for $\beta \ge l(e)(1 \sqrt{2} + \log 2) \approx 0.59533.$ *(a)*
- (b) $Y_{\beta}(z, p(z)) \prec \sqrt{1+z}$ for $\beta \ge l(e)(2(1-log2)) \approx 1.30984$.

- $\begin{array}{ll} \text{(c)} & Y_{\beta}(z, \, p(z)) \prec G_{\alpha}(z) \text{ for } \beta \geq l(e) \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.\\ \text{(d)} & Y_{\beta}(z, \, p(z)) \prec \frac{1+Az}{1+Bz} \text{ for } \beta \geq l(e) \frac{A-B}{B} \log (1-B)^{-1}.\\ \text{(e)} & Y_{\beta}(z, \, p(z)) \prec \varphi_{s}(z) \text{ for } \beta \geq l(e) \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!(2n+1)} \approx 2.01905. \end{array}$
- (f) $Y_{\beta}(z, p(z)) \prec \varphi_{q}(z)$ for $\beta \ge l(e)(2 \sqrt{2} \log 2 + \log (1 + \sqrt{2})) \approx 1.65198.$ (g) $Y_{\beta}(z, p(z)) \prec e^{z}$ for $\beta \ge l(e) \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!n} \approx 0.785166.$

The lower bound on β *in each case is sharp.*

Proof. Let the functions ν and ψ be defined by $\nu(w) = 1$ and $\psi(w) = \beta$. (a) Define the function $q_{\beta} : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_{\beta}(z) = 1 - rac{1}{eta k} \left(z + 2k \log \left(1 - rac{z}{k}
ight)
ight)$$

is a solution of the differential equation $\beta zq'(z) = \varphi_0(z) - 1$ and is analytic in \mathbb{D} . Now consider the function

$$Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \varphi_0(z) - 1 = \frac{k + z - 2k^2}{k - z}$$

It can be easily seen that Q is starlike in \mathbb{D} and the function *h* is defined by

$$h(z) := \nu(q(z)) + \mathcal{Q}(z) = 1 + \mathcal{Q}(z)$$

satisfies the following inequality

$$\operatorname{Re}\left(rac{zh'(z)}{\mathcal{Q}(z)}
ight) = \operatorname{Re}\left(rac{z\mathcal{Q}'(z)}{\mathcal{Q}(z)}
ight) > 0 \ (z \in \mathbb{D}).$$

Therefore, from Lemma 1, we conclude that

$$1 + \beta z p'(z) \prec 1 + \beta z q'_{\beta}(z) \text{ implies } p \prec q_{\beta}.$$
(1)

Now the subordination $p \prec Q$ holds if subordination $q_{\beta} \prec Q$. Thus, the subordination $q_{\beta} \prec Q$ holds if the inequalities

$$Q(-1) \le q_{\beta}(-1) \le q_{\beta}(1) \le Q(1)$$

hold and these yield a necessary condition for subordination $p \prec Q$ to hold. In view of the graph of the respective function, the necessary condition is also sufficient condition. The inequalities $q_{\beta}(-1) \ge Q(-1)$ and $q_{\beta}(1) \le Q(1)$ yield $\beta \ge \beta_1$ and $\beta \ge \beta_2$, where

$$\beta_1 = \frac{1 - \sqrt{2} + \log 2}{1 - e^{(1-e)/e}}$$
 and $\beta_2 = \frac{1 - \sqrt{2} - 2\log(2 - \sqrt{2})}{e^{(e-1)/e} - 1}$.

Now the subordination $q_{\beta} \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$. (b) The function

$$q_{\beta}(z) = \frac{\beta + 2(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1)}{\beta}$$

is an analytic solution of the first order differential equation $\beta zq'(z) = \sqrt{1+z} - 1$ in \mathbb{D} . The function \mathcal{Q} defined by $\mathcal{Q}(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \sqrt{1+z} - 1$ is starlike in \mathbb{D} and the function $h(z) := \nu(q(z)) + \mathcal{Q}(z)$ satisfies Re $(zh'(z)/\mathcal{Q}(z)) = \text{Re}(z\mathcal{Q}'(z)/\mathcal{Q}(z)) > 0, z \in \mathbb{D}$. Therefore, in view of the subordination relation 1, the required subordination $p \prec Q$ holds if subordination $q_{\beta} \prec Q$ holds. Thus, the subordination $q_{\beta} \prec Q$ holds if the inequalities

$$Q(-1) \le q_{\beta}(-1) \le q_{\beta}(1) \le Q(1)$$

hold which in-turn yield a necessary condition for subordination $p \prec Q$. The inequalities $q_{\beta}(-1) \ge Q(-1)$ and $q_{\beta}(1) \le Q(1)$ yield $\beta \ge \beta_1 = 2(1 - \log 2)/1 - e^{(1-e)/e}$ and $\beta \ge \beta_2 = 2(\sqrt{2} - 1 + \log 2 - \log(1 + \sqrt{2}))/(e^{(1-e)/e} - 1)$, respectively. Therefore, the subordination $q_{\beta} \prec Q$ holds if $\beta \ge \max \{\beta_1, \beta_2\} = \beta_1$.

(c) The analytic function

$$q_{\beta}(z) = \frac{2\sqrt{\alpha}\beta + \log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}}{2\sqrt{\alpha}\beta}$$

is a solution of the differential equation $\beta z q'_{\beta}(z) = G_{\alpha}(z) - 1$ in \mathbb{D} . Now computation shows that

$$\mathcal{Q}(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \frac{z}{1 - \alpha z^2}$$

is starlike in \mathbb{D} . Note that the function $h(z) := \nu(q(z)) + \mathcal{Q}(z) = 1 + \mathcal{Q}(z)$ satisfies Re $(zh'(z)/\mathcal{Q}(z)) =$ Re $(z\mathcal{Q}'(z)/\mathcal{Q}(z)) > 0$ in \mathbb{D} . Therefore, in view of the subordination relation 1, the required subordination $p \prec Q$ holds if subordination $q_{\beta} \prec Q$. Similar to as in part (a), the desired subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$, where $\beta_1 = l(e)g(\alpha)$ and $\beta_2 = -l(e)g(\alpha)$ such that

$$g(\alpha) = \frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

(d) Consider the analytic function

$$q_{\beta}(z) = \frac{B\beta + (A - B)\log(1 + Bz)}{B\beta}$$

which is a solution of differential equation

$$\beta zq'(z) = \frac{(A-B)z}{1+Bz}.$$

Since the function (A - B)z/(1 + Bz) is starlike in \mathbb{D} , it follows that $\mathcal{Q}(z) = zq'_{\beta}(z)\psi(q_{\beta}(z))$ is starlike in \mathbb{D} . The function $h : \mathbb{D} \to \mathbb{C}$ defined by $h(z) := \nu(q_{\beta}(z)) + Q(z) = 1 + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ ($z \in \mathbb{D}$). Thus, as in previous case, the subordination $p \prec \mathcal{Q}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$, where

$$\beta_1 = \frac{(A-B)\log(1-B)^{-1}}{B(1-e^{(1-e)/e})}$$
 and $\beta_2 = \frac{(A-B)\log(1+B)}{B(e^{(1-e)/e}-1)}$.

(e) The differential equation

$$\frac{dq}{dz} = \frac{\sin z}{\beta z}$$

has an analytic solution

$$q_{\beta}(z) = 1 + \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}$$

in \mathbb{D} . Now the function $Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \sin z$ is starlike in \mathbb{D} and the function $h(z) := \nu(q(z)) + Q(z) = 1 + Q(z)$, satisfies Re (zh'(z)/Q(z)) = Re(zQ'(z)/Q(z)) > 0 holds. As in part (a), the desired subordination $p(z) \prec Q(z)$ holds if $\beta \ge \max\{\beta_1, \beta_2\} = \beta_1$, where

$$\beta_1 = \frac{1}{(1 - e^{(1 - e)/e})} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 2.01905$$

and

$$\beta_2 = \frac{1}{(e^{(e-1)} - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 0.206779.$$

(f) The differential equation

$$\frac{dq}{dz} = \frac{z + \sqrt{1 + z^2} - 1}{\beta z}$$

has an analytic solution

$$q_{\beta}(z) = \frac{\beta + (z + \sqrt{1 + z^2} - \log(1 + \sqrt{1 + z^2}) - 1 + \log 2)}{\beta}.$$

Computation shows that the function

$$\mathcal{Q}(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = z + \sqrt{1+z^2} - 1$$

is starlike in \mathbb{D} . As before, the function $h(z) := \nu(q(z)) + \mathcal{Q}(z)$ satisfies Re $(zh'(z)/\mathcal{Q}(z)) > 0$, $z \in \mathbb{D}$. Therefore, the desired subordination $p \prec Q$ holds if $\beta \ge \max\{\beta_1, \beta_2\} = \beta_1$, where

$$\beta_1 = \frac{2 - \sqrt{2} - \log 2 + \log(1 + \sqrt{2})}{1 - e^{(1 - e)/e}} \approx 1.65198$$

and

$$\beta_2 = \frac{\sqrt{2} + \log 2 - \log(1 + \sqrt{2})}{e^{(1-e)/e} - 1} \approx 0.267979.$$

(g) The differential equation

$$\frac{dq}{dz} = \frac{e^z - 1}{\beta z}$$

has an analytic solution

$$q_{\beta}(z) = 1 + \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{z^n}{n!n}.$$

Note that the function $Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = e^{z}$ is starlike in the unit disk \mathbb{D} and the function $h(z) := \nu(q(z)) + \mathcal{Q}(z) = 1 + \mathcal{Q}(z)$ satisfies $\operatorname{Re}(zh'(z)/\mathcal{Q}(z)) = \operatorname{Re}(z\mathcal{Q}'(z)/\mathcal{Q}(z)) > 0$. Now the subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$, where

$$\beta_1 = \frac{1}{(1 - e^{(1 - e)/e})} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!n} \approx 0.785166 \quad \text{and} \quad \beta_2 = \frac{1}{(e^{(e-1)} - 1)} \sum_{n=0}^{\infty} \frac{1}{n!n} \approx 0.288069.$$

This ends the proof. \Box

Theorem 1 also provides the following various sufficient conditions for the normalized analytic functions *f* to be in the class S_B^* .

Let function $f \in A$ and set

$$Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) = 1 + \beta \frac{zf'(z)}{f(z)} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right).$$

If either of the following subordination holds

(a)
$$Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{0}(z) \ (\beta \ge 0.59533),$$

(b) $Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \ (\beta \ge 1.30984),$
(c) $Y_{\beta}\left(z, \frac{f'(z)}{f(z)}\right) \prec G_{\alpha}(z) \ (\beta \ge \frac{1}{(1-e^{(1-e)/e})} \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}),$
(d) $Y_{\beta}\left(z, \frac{f'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \ (\beta \ge \frac{1}{(1-e^{(1-e)/e})} \frac{A-B}{B} \log (1-B)^{-1}),$
(e) $Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{s}(z) \ (\beta \ge 2.01905),$
(f) $Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{q}(z) \ (\beta \ge 1.65198),$
(g) $Y_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec e^{z} \ (\beta \ge 0.785166),$

(g)
$$I_{\beta}\left(z, \frac{f(z)}{f(z)}\right) \prec e^{z} \ (p \geq 0.785)$$

then $f \in \mathcal{S}_B^*$.

The next result gives sharp lower bound on β such that subordination $p \prec Q$ holds, whenever $1 + \beta z p'(z) / p(z) \prec \varphi_0(z)$ or $\varphi_s(z)$ or $\sqrt{1+z}$ or $G_\alpha(z)$ or (1+Az)/(1+Bz) or $\varphi_s(z)$ or $\varphi_q(z)$ or e^z .

Theorem 2. Let $0 < \alpha < 1$, 0 < B < A < 1, and p be an analytic function defined in \mathbb{D} with p(0) = 1. Set

$$\Omega_{\beta}(z, p(z)) = 1 + \beta \frac{zp'(z)}{p(z)}.$$

Then, the following conditions are sufficient for subordination $p \prec Q$ *.*

(a)
$$\Omega_{\beta}(z, p(z)) \prec \varphi_0(z)$$
 for $\beta \ge \frac{e(2(1+\sqrt{2})\log\sqrt{2}-1)}{2(e-1)(1+\sqrt{2})} \approx 0.441266.$

(b)
$$\Omega_{\beta}(z, p(z)) \prec \sqrt{1+z} \text{ for } \beta \geq \frac{2e(1-\log 2)}{e-1} \approx 0.970868.$$

- (c) $\Omega_{\beta}(z, p(z)) \prec G_{\alpha}(z)$ for $\beta \geq \frac{e}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}$. (d) $\Omega_{\beta}(z, p(z)) \prec \frac{1+Az}{1+Bz}$ for $\beta \geq \frac{e}{B(e-1)}(A-B)\log(1-B)^{-1}$.

(e) $\Omega_{\beta}(z, p(z)) \prec \varphi_{s}(z)$ for $\beta \geq \frac{e}{e-1} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!(2n+1)} \approx 1.49655.$

(f)
$$\Omega_{\beta}(z, p(z)) \prec \varphi_{\alpha}(z)$$
 for $\beta \geq \frac{e}{e^{-1}}(2 - \sqrt{2} + \log(1 + \sqrt{2}) - \log 2) \approx 1.22447$.

(i) $\Omega_{\beta}(z, p(z)) \prec \varphi_{q}(z) \text{ for } \beta \geq \frac{1}{e-1} \sum_{n=0}^{\infty} \frac{1}{n!n} \approx 0.766987.$

The lower bound on β *in each case is sharp.*

Proof. Let us define $\nu(w) = 1$ and $\psi(w) = \beta/w$ for all $w \in \mathbb{C}$.

(a) The function

$$q_{\beta}(z) = \exp\left(-\frac{1}{\beta k}\left(z + 2k\log\left(1 - \frac{z}{k}\right)\right)\right)$$

satisfies the differential equation $\beta z q'(z)/q(z) = \varphi_0(z) - 1$. Clearly, the function $Q : \overline{\mathbb{D}} \to$ defined by $Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = (z - 2k^2 + k)/(k - z)$ is starlike in \mathbb{D} . Further, the function $h(z) := \nu(q_{\beta}(z)) + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ ($z \in \mathbb{D}$). Thus, using Lemma 1, it follows that

$$1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \beta \frac{zq'_{\beta}(z)}{q_{\beta}(z)} \quad \text{implies} \quad p \prec q_{\beta}.$$
⁽²⁾

Now using Theorem 1 (a), the subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$, where

$$\beta_1 = \frac{(-1+2(1+\sqrt{2})\log\sqrt{2})e}{(e-1)(1+\sqrt{2})}$$

and

$$\beta_2 = -\frac{(1+2(1+\sqrt{2})\log{(2-\sqrt{2})})}{(e-1)(1+\sqrt{2})}.$$

(b) The function

$$q_{\beta}(z) = \exp\left(\frac{2}{\beta}\left(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1\right)\right)$$

is a solution of the differential equation

$$\beta \frac{zq'(z)}{q(z)} = \sqrt{1+z} - 1.$$

Moreover, the function $Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \sqrt{1+z}-1$ is starlike in \mathbb{D} and a computation shows that the function $h(z) := \nu(q(z)) + Q(z)$ satisfies Re (zh'(z)/Q(z)) > 0 $(z \in \mathbb{D})$. Now the desired subordination $p \prec Q$ holds if $\beta \ge \max \{\beta_1, \beta_2\} = \beta_1$, where $\beta_1 = 2e(1 - \log 2)/(e - 1)$ and $\beta_2 = 2(-1 + \sqrt{2} + \log 2 - \log(1 + \sqrt{2}))/(e - 1)$.

(c) Consider the function q_{β} defined by

$$q_{\beta}(z) = \exp\left(\frac{1}{2\sqrt{\alpha}\beta}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right).$$

It can be verified that the function q_β is a solution of the differential equation

$$\beta \frac{zq'(z)}{q(z)} = \frac{1}{1 - \alpha z^2}.$$

Now the function $Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = 1/(1 - \alpha z^2)$ is starlike in \mathbb{D} and the function $h(z) := \nu(q(z)) + Q(z)$ satisfies Re (zh'(z)/Q(z)) > 0 $(z \in \mathbb{D})$. Now, as in previous cases, $p \prec Q$ holds only if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$, where

$$\beta_1 = \frac{e}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \text{ and } \beta_2 = \frac{1}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}$$

(d) Let the function $q_{\beta}(z) = \exp(((A - B)\log(1 + Bz)/\beta B))$ be an analytic solution of the differential equation

$$1 + \beta \frac{zq'(z)}{q(z)} = \frac{1 + Az}{1 + Bz}.$$

Now the desired subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$, where $\beta_1 = e(A - B)\log(1 - B)^{-1}/B(e - 1)$ and $\beta_2 = e(A - B)\log(1 + B)/B(e - 1)$.

(e) The differential equation $\beta z q'(z) / q(z) = \sin z$ has an analytic solution given by

$$q_{\beta}(z) = \exp\left(\frac{1}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}\right).$$

As in part Theorem 2 (a), the subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_1$ where

$$\beta_1 = rac{e}{e-1} \sum_{n=0}^{\infty} rac{(-1)^n}{(2n+1)!(2n+1)} \approx 1.49655$$

and

$$\beta_2 = \frac{1}{e-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 0.55055.$$

(f) The solution of the differential equation

$$\frac{dq}{dz} = \frac{z + \sqrt{1 + z^2} - 1}{\beta z}$$

is given by

$$q_{\beta}(z) = \exp\left(\frac{z + \sqrt{1 + z^2} - \log(1 + \sqrt{1 + z^2}) - 1 + \log 2}{\beta}\right).$$

As in proof of Theorem 2 (a), the desired result holds if $\beta \ge \max\{\beta_1, \beta_2\} = \beta_1$, where $\beta_1 = e(2 - \sqrt{2} + \log(1 + \sqrt{2}) - \log 2)/(e - 1)$ and $\beta_2 = (\sqrt{2} - \log(1 + \sqrt{2}) + \log 2)/(e - 1)$. (g) The differential equation $\beta zq'(z)/q(z) = e^z - 1$ has a solution

$$q_{\beta}(z) = \exp\left(\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{z^n}{n!n}\right)$$

analytic in \mathbb{D} . Thus, as previous, the subordination $p \prec Q$ holds if $\beta \ge \max{\{\beta_1, \beta_2\}} = \beta_2$, where

$$\beta_1 = \frac{e}{e-1} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!n} \approx 0.581976 \quad \text{and} \quad \beta_2 = \frac{1}{e-1} \sum_{n=0}^{\infty} \frac{1}{n!n} \approx 0.766987$$

This ends the proof. \Box

Next, Theorem 2 also provides the following various sufficient conditions for the normalized analytic functions f to be in the class S_B^* . Let the function $f \in A$ and set

$$\Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) = 1 + \beta\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right).$$

If either of the following subordination conditions are fulfilled:

$$\begin{array}{ll} (a) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{0}(z) \ (\beta \geq 0.441266), \\ (b) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \ (\beta \geq 0.970868), \\ (c) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec G_{\alpha}(z) \ (\beta \geq \frac{e}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}), \\ (d) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \ (\beta \geq \frac{e}{B(e-1)}(A-B) \log(1-B)^{-1}), \\ (e) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{s}(z) \ (\beta \geq 1.49655), \\ (f) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{q}(z) \ (\beta \geq 1.22447), \\ (g) & \Omega_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec e^{z} \ (\beta \geq 0.766987), \end{array}$$

then $f \in \mathcal{S}_B^*$.

In the following theorem, the sharp lower bound on β is obtained so that the subordination $p \prec Q$ holds, whenever $1 + \beta z p'(z) / p^2(z) \prec \varphi_0(z)$ or $\varphi_s(z)$ or $\sqrt{1+z}$ or $G_\alpha(z)$ or (1+Az)/(1+Bz) or $\varphi_s(z)$ or $\varphi_a(z)$ or e^z . These results can be proved by defining the functions $\nu, \psi : \mathbb{D} \to$ defined by $\nu(w) = 1$ and $\psi(w) = \beta/w^2$ and proceeding in a similar fashion as in the proofs of Theorems 1 and 2.

Theorem 3. Let $0 < \alpha < 1$, 0 < B < A < 1, and p be an analytic function defined in \mathbb{D} with p(0) = 1. Set

$$\Xi_{\beta}(z, p(z)) = 1 + \beta \frac{zp'(z)}{p^2(z)}.$$

Then, the following conditions are sufficient for $p \prec Q$.

- $\begin{array}{ll} (a) & \Xi_{\beta}(z,\,p(z)) \prec \varphi_{0}(z) \, for \, \beta \geq \frac{1+2(\sqrt{2}+1)\log(2-\sqrt{2})}{(1+\sqrt{2})(e^{(1-e)}-1)} \approx 0.798642. \\ (b) & \Xi_{\beta}(z,\,p(z)) \prec \sqrt{1+z} \, for \, \beta \geq \frac{2(-1+\sqrt{2}+\log 2-\log(1+\sqrt{2}))}{1-e^{1-e}} \approx 0.550768. \\ (c) & \Xi_{\beta}(z,\,p(z)) \prec G_{\alpha}(z) \, for \, \beta \geq \frac{e^{e-1}}{e^{e-1}-1} \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}. \\ (d) & \Xi_{\beta}(z,\,p(z)) \prec \frac{1+Az}{1+Bz} \, for \, \beta \geq \frac{e^{(1-e)/e}}{1-e^{(1-e)/e}} \frac{(A-B)\log(1-B)^{-1}}{B}. \\ (e) & \Xi_{\beta}(z,\,p(z)) \prec \varphi_{s}(z) \, for \, \beta \geq \frac{e^{e-1}}{e^{e-1}-1} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!(2n+1)} \approx 1.15278. \\ (f) & \Xi_{\beta}(z,\,p(z)) \prec \varphi_{q}(z) \, for \, \beta \geq \frac{e^{e-1}}{e^{e-1}-1} (\sqrt{2}-\log(1+\sqrt{2})+\log 2) \approx 1.49397. \\ (g) & \Xi_{\beta}(z,\,p(z)) \prec e^{z} \, for \, \beta \geq \frac{e^{e-1}}{e^{e-1}-1} \sum_{n=0}^{\infty} \frac{1}{n!n} \approx 1.60597. \end{array}$

The lower bound on β *in each case is sharp.*

Let $f \in \mathcal{A}$ and set

$$\Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) = 1 + \beta \left(\frac{zf'(z)}{f(z)}\right)^{-1} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right).$$

If either of the following subordination holds

$$\begin{array}{ll} \text{(a)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{0}(z) \ (\beta \geq 0.798642), \\ \text{(b)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \ (\beta \geq 0.550768), \\ \text{(c)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec G_{\alpha}(z) \ (\beta \geq \frac{e^{e^{-1}}}{e^{e^{-1}-1}}\frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}), \\ \text{(d)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \ (\beta \geq \frac{e^{(1-e)/e}}{1-e^{(1-e)/e}}\frac{(A-B)\log(1-B)^{-1}}{B}), \\ \text{(e)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{s}(z) \ (\beta \geq 1.15278), \\ \text{(f)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec \varphi_{q}(z) \ (\beta \geq 1.49397), \\ \text{(g)} & \Xi_{\beta}\left(z, \frac{zf'(z)}{f(z)}\right) \prec e^{z} \ (\beta \geq 1.60597), \\ \text{then } f \in \mathcal{S}_{\mathcal{B}}^{*}. \end{array}$$

3. Radius Estimates

Let θ_1 and θ_2 be two sub-families of \mathcal{A} . The θ_1 radius of θ_2 is the largest number $\rho \in (0, 1)$ such that $r^{-1}f(rz) \in \theta_1$, $0 < r \le \rho$ for all $f \in \theta_2$. Grunsky [34] obtained the radius of starlikeness for functions in the class \mathcal{S} . Sokół [35] computed the radius of α -convexity and α -starlikeness for a class \mathcal{S}_L^* . In 2016, authors [7] determined the \mathcal{S}_R^* -radius for various subclasses of starlike functions. For more results on radius problems, see [36–41].

The main technique involved in tackling the S_B^* -radius estimates for classes of functions f is the determination of the disk that contains the values of zf'(z)/f(z). The associated technical lemma is achieved as:

Lemma 2. Let $Q(z) := e^{e^z - 1}$, $z \in \mathbb{D}$. Define the function $r : [e^{1/e - 1}, e^{e^{-1}}] \to \mathbb{R}^+$ by $\int e^{a - e^{1/e}} e^{1/e^{-1}} dx = e^{1/e + e^e}$

$$r(a) := \begin{cases} \frac{ea - e^{r}}{e}, & e^{\frac{1}{e} - 1} \le a \le \frac{e^{r/2} + e^{e}}{2e}; \\ \frac{e^{e} - ea}{e}, & \frac{e^{1/e} + e^{e}}{2e} \le a \le e^{e-1}. \end{cases}$$

Then, the following holds:

$$\{w \in \mathbb{C} : |w-a| < r(a)\} \subset \Omega_B \subset \left\{w \in \mathbb{C} : |w-1| < \frac{e^e - e}{e}\right\}.$$

Proof. To prove the assertion, we let $z = e^{it}$, $t \in (-\pi, \pi]$. Therefore,

$$Q(e^{it}) = e^{e^{e^{it}} - 1} = u(t) + iv(t)$$

with

$$u(t) := \cos\left(\sin(\sin t)e^{\cos t}\right)\exp\left(e^{\cos t}\cos(\sin t) - 1\right)$$

and

$$v(t) := \sin\left(\sin(\sin t)e^{\cos t}\right)\exp\left(e^{\cos(t)}\cos(\sin t) - 1\right)$$

Now, consider the square of the distance of an arbitrary point (u(t), v(t)) on the boundary of $\partial Q(\mathbb{D})$ from (a, 0) and is given by

$$h(t) = d^{2}(t) = a^{2} - 2ae^{e^{\cos t}\cos(\sin t) - 1}\cos\left(\sin(\sin t)e^{\cos t}\right) + e^{2e^{\cos t}\cos(\sin t) - 2}.$$

Now we need to prove |w - a| < r(a) is the largest disk contained in $Q(\mathbb{D})$. For this, we need to show that $\min_{-\pi \le t \le \pi} d(t) = r(a)$. Since *h* is an even function, i.e., h(t) = h(-t), we need to only consider the case when $t \in [0, \pi]$. Now h'(t) = 0 has three roots viz. 0, π and $t_0(a) \in (0, \pi)$. Among these roots, the root $t_0(a)$ depends on *a* and graphics reveals that *h* is increasing in the interval $[0, t_0(a)]$ and decreasing in $[t_0(a), \pi]$, and therefore, *h* attains its minimum either at 0 or π . Further computations give $h(\pi) = (ea - e^{1/e})^2 / e^2$ and $h(0) = (e^e - ea)^2 / e^2$. Hence, we have

$$\min_{-\pi \le t \le \pi} h(t) = \min \left\{ h(0), h(\pi) \right\} = \begin{cases} h(\pi), & e^{\frac{1}{e} - 1} \le a \le \frac{e^{1/e} + e^e}{2e}; \\ h(0), & \frac{e^{1/e} + e^e}{2e} \le a \le e^{e-1}. \end{cases}$$

Therefore, we can write

$$\min_{-\pi \le t \le \pi} d(t) = \begin{cases} \frac{ea - e^{1/e}}{e}, & e^{\frac{1}{e} - 1} \le a \le \frac{e^{1/e} + e^e}{2e};\\ \frac{e^e - ea}{e}, & \frac{e^{1/e} + e^e}{2e} \le a \le e^{e - 1}. \end{cases}$$

To find the circle of minimum radius with center at (1,0) containing the domain $Q(\mathbb{D})$, we need to find the maximum distance from (1,0) to an arbitrary point on the boundary of the domain $Q(\mathbb{D})$. The square of this distance function is given by

$$\phi(t) = -2e^{e^{\cos t}\cos(\sin t) - 1}\cos\left(\sin(\sin t)e^{\cos t}\right) + e^{2e^{\cos t}\cos(\sin t) - 2} + 1$$

The equation $\phi'(t) = 0$ has two roots in $[0, \pi]$, namely 0 and π . It is easy to see that $\phi(0) = (e - e^e)^2 / e^2$ and $\phi(\pi/2) = (e - e^{1/e})^2 / e^2$. Therefore,

$$\max \left\{ \phi(0), \phi(\pi) \right\} = \phi(0) = \frac{(e - e^e)^2}{e^2}.$$

Hence, the radius of the smallest disk containing $Q(\mathbb{D})$ is $(e - e^e) / e$. This ends the proof. \Box

We now recall some classes and results related to them which are to be used for further development of this section. For $-1 \le B < A \le 1$, let

$$\mathcal{P}_n[A,B] := \left\{ p(z) = 1 + \sum_{k=n}^{\infty} c_n z^n : \ p(z) \prec \frac{1+Az}{1+Bz} \right\}$$

Let us denote $\mathcal{P}_n(\alpha) := \mathcal{P}_n[1 - 2\alpha, -1]$ and $\mathcal{P}_1(0) =: \mathcal{P}$. For $f \in \mathcal{A}$, if we set p(z) = zf'(z)/f(z)and p(z) = 1 + zf''(z)/f'(z), then the class $\mathcal{P}[A, B]$ is denoted by $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$, respectively. These classes were introduced and studied by [2]. Further, let $\mathcal{S}^*(\alpha) := \mathcal{S}^*[1 - 2\alpha, -1]$.

The following results will be needed:

Lemma 3. [42] *If* $p \in P_n[A, B]$ *, then, for* |z| = r*,*

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \le \frac{(A - B)r^n}{1 - B^2 r^{2n}}$$

In particular, if $p \in \mathcal{P}_n(\alpha)$, then, for |z| = r,

$$\left| p(z) - \frac{(1 + (1 - 2\alpha))r^{2n}}{1 - r^{2n}} \right| \le \frac{2(1 - \alpha)r^n}{1 - r^{2n}}$$

Lemma 4. [43] If $p \in \mathcal{P}_n(\alpha)$, then, for |z| = r,

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)nr^n}{(1-r^n)(1+(1-2\alpha)r^n)}.$$

The main objective of this section is to determine the S_B^* -radii constants for functions belonging to certain well-known subclasses of A. Let \mathcal{G} denote the class of functions $f \in S$ for which $f(z)/z \in \mathcal{P}$. The following theorem gives the sharp S_B^* -radius for the class \mathcal{G} .

Theorem 4. Let $f \in \mathcal{G}$. Then, the sharp \mathcal{S}_{B}^{*} -radius is

$$R_{\mathcal{S}^*_B}(\mathcal{G}) := rac{e - e^{1/e}}{\sqrt{2e^2 - 2e^{1 + rac{1}{e}} + e^{2/e}} + e} pprox 0.222654.$$

Proof. Since $f \in \mathcal{G}$, therefore, $f(z)/z \in \mathcal{P}$. Then, from Lemma 2, we must have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{2r}{1 - r^2}$$

Therefore, $f \in \mathcal{S}^*_B$ if $2r/(1-r^2) \le (e-e^{1/e})/e$, or equivalently if

$$(e - e^{1/e})r^2 + 2er + e^{1/e} - e \le 0$$

which holds for all

$$r \leq rac{e - e^{1/e}}{\sqrt{2e^2 - 2e^{1 + rac{1}{e}} + e^{2/e}} + e} =: R_{\mathcal{S}^*_B}(\mathcal{G}) pprox 0.222654.$$

For verification of sharpness, consider the function f(z) = z(1+z)/(1-z). Then, $f(z)/z \in \mathcal{P}$ and at $z = R_{\mathcal{S}^*_B}(\mathcal{G})$, we have

$$\frac{R_{\mathcal{S}_B^*}(\mathcal{G})f'(R_{\mathcal{S}_B^*}(\mathcal{G}))}{f(R_{\mathcal{S}_B^*}(\mathcal{G}))} - 1 = \frac{R_{\mathcal{S}_B^*}(\mathcal{G})}{1 - R_{\mathcal{S}_B^*}(\mathcal{G})} = 1 - e^{\frac{1}{e} - 1}.$$

Hence the result is sharp. \Box

In the following theorem, we shall investigate sharp S_B^* -radius for the class $S^*[A, B]$.

Theorem 5. Let $f \in S^*[A, B]$. Then,

1. for $0 \le B < A \le 1$, the sharp S_B^* -radius for the class $S^*[A, B]$ is

$$R_{\mathcal{S}_{B}^{*}}(\mathcal{S}^{*}[A,B]) = \min\left\{1; \frac{\sqrt{e-e^{1/e}}}{\sqrt{eAB - e^{1/e}B^{2}}}; \frac{e^{1/e} - e}{e^{1/e}B - eA}\right\}$$

2. for $-1 \le B < 0 \le A \le 1$, the sharp S_B^* -radius for the class $S^*[A, B]$ is

$$R_{\mathcal{S}_B^*}(\mathcal{S}^*[A,B]) = \min\left\{1; \sqrt{\frac{-2e + e^{1/e} + e^e}{-2eAB + e^{1/e}B^2 + e^eB^2}}; \frac{e^{1/e} - e}{e^{1/e}B - eA}\right\}$$

Proof. Let $f \in S^*[A, B]$. Then using Lemma 4, we see that f maps the disk $|z| \le r$ onto the disk

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 - ABr^2}{1 - B^2r^2}\right| \le \frac{(A - B)r}{1 - B^2r^2}.$$

The center of the above disk is at (c, 0) and the radius is *R*, where

$$c := \frac{1 - ABr^2}{1 - B^2r^2}$$
 and $R := \frac{(A - B)r}{1 - B^2r^2}$.

(1) We see that $c \leq (e^{1/e} + e^e)/(2e)$ holds for all $0 \leq B < A \leq 1$ and 0 < r < 1. Further, the condition $1 - e^{1/e} \leq c$ is equivalent to

$$-eABr^2 + e^{1/e}B^2r^2 - e^{1/e} + e \ge 0$$

which holds for all

$$r \le \sqrt{\frac{e - e^{1/e}}{eAB - e^{1/e}B^2}} =: r_1.$$

Further computation shows that the condition $R \le (e^e a - e^{1/e})/e$ is equivalent to $eAr - e^{1/e}Br + e^{1/e} - e \le 0$ which holds for all

$$r \le \frac{e^{1/e} - e}{e^{1/e}B - eA} =: r_2.$$

Now from Lemma 2, $f \in \mathcal{S}^*_B$ for all $|z| \le R_{\mathcal{S}^*_B}(\mathcal{S}^*[A, B]) = \min\{1; r_1; r_2\}$.

(2) Let $-1 \le B < 0 \le A \le 1$. Then we see that $e^{1/e-1} \le c$ holds for all 0 < r < 1. Further, $c \le (e^e + e^{1/e})/2e$ is equivalent to

$$-2eABr^2 + e^{1/e}B^2r^2 + e^eB^2r^2 - e^{1/e} - e^e + 2e \le 0$$

which holds for

$$r \leq \sqrt{\frac{-2e + e^{1/e} + e^e}{-2eAB + e^{1/e}B^2 + e^eB^2}} =: r_3.$$

Now, as in the previous case $R < (ec - e^{1/e})/e$ holds if $r \le r_2$. Therefore, S_B^* -radius for the class $S^*[A, B]$ is $R_{S_R^*}(S^*[A, B]) = \min\{1; r_2; r_3\}$.

The equality holds in case of the function f_0 defined by

$$f_0(z) = \begin{cases} z(1+Bz)^{\frac{A}{B}-1}, & B \neq 0; \\ ze^{Az}, & B = 0. \end{cases}$$

This ends the proof. \Box

Remark 1. Let $f \in S^*$. Then, since $S^* = S^*[0, -1]$, it follows from the above theorem, that the S^*_B -radius for starlike functions is $r_4 := (e - e^{1/e})/(e + e^{1/e}) \approx 0.30594$. To see the sharpness, consider the Koebe function $k(z) = z/(1-z)^2$. Then, at $z = r_4$, we have

$$\frac{r_4 f'(r_4)}{f(r_4)} = \frac{1+r_4}{1-r_4} = e^{1-\frac{1}{e}}.$$

Because the function k is univalent too, it follows that the S_B^* -radius for the class S and S^* is r_4 . Therefore, the radius r_4 can not be increased. Thus, we have the following:

Corollary 1. The sharp S_B^* -radius for the classes S and S^* is $(e - e^{1/e})/(e + e^{1/e}) \approx 0.30594$.

Let the class \mathcal{F}_1 be defined by

$$\mathcal{F}_1 := \left\{ f \in \mathcal{A} : \operatorname{Re} rac{f(z)}{g(z)} > 0 ext{ and } \operatorname{Re} rac{g(z)}{z} > 0, \ g \in \mathcal{A}
ight\}.$$

The following theorem gives the sharp S_B^* -radius for the class \mathcal{F}_1 .

Theorem 6. Let $f \in \mathcal{F}_1$. Then, the sharp \mathcal{S}_B^* -radius is

$$R_{\mathcal{S}_B^*}(\mathcal{F}_1) = \frac{e - e^{1/e}}{\sqrt{5e^2 - 2e^{1 + \frac{1}{e}} + e^{2/e}}} \approx 0.11557.$$

Proof. Since $f \in \mathcal{F}_1$, there is $g \in \mathcal{A}$ such that $\operatorname{Re}(g(z)/z) > 0$. Define the functions $p, h : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{g(z)}{z}$$
 and $h(z) = \frac{f(z)}{g(z)}$

Then, through some assumptions, we have $p, h \in \mathcal{P}$. Now using Lemma 4, we get

$$\begin{array}{ll} \left| \frac{zf'(z)}{f(z)} - 1 \right| & \leq & \left| \frac{zh'(z)}{h(z)} \right| + \left| \frac{zp'(z)}{p(z)} \right| \\ \\ & \leq & \frac{4r}{1 - r^2} \leq \frac{e - e^{1/e}}{e}, \end{array}$$

this holds if and only if $(e - e^{1/e})r^2 + 4er + e^{1/e} - e \le 0$, that is if

$$r \leq rac{e - e^{1/e}}{\sqrt{5e^2 - 2e^{1 + rac{1}{e}} + e^{2/e}}} =: R_{\mathcal{S}^*_B}(\mathcal{F}_1) pprox 0.11557.$$

Consider the functions f_2 and g_2 defined by

$$f_2(z) = z \left(\frac{1+z}{1-z}\right)^2$$
 and $g_2(z) = z \left(\frac{1+z}{1-z}\right)$

Further, we have $\operatorname{Re}(f_2(z)/g_2(z)) > 0$ and $\operatorname{Re}(g_2(z)/z) > 0$, and therefore $f \in \mathcal{F}_1$. Now a computation shows that, for $z = R_{\mathcal{S}_R^*}(\mathcal{F}_1)$,

$$\frac{R_{\mathcal{S}_B^*}(\mathcal{F}_1)f_2'(R_{\mathcal{S}_B^*}(\mathcal{F}_1))}{f_2(R_{\mathcal{S}_B^*}(\mathcal{F}_1))} - 1 = \frac{4R_{\mathcal{S}_B^*}(\mathcal{F}_1)}{1 - R_{\mathcal{S}_B^*}(\mathcal{F}_1)^2} = 1 - e^{\frac{1}{e} - 1}$$

Hence the result is sharp. \Box

Let us define the class \mathcal{F}_2 by

$$\mathcal{F}_2 := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{g(z)} > 0 \text{ and } \operatorname{Re} \frac{g(z)}{z} > 1/2, \ g \in \mathcal{A} \right\}.$$

The following theorem gives the sharp S_B^* -radius for the class \mathcal{F}_2 .

Theorem 7. Let $f \in \mathcal{F}_2$. Then, the sharp \mathcal{S}_B^* -radius is

$$\mathcal{S}^*_B(\mathcal{F}_2) = rac{2\left(e - e^{1/e}
ight)}{\sqrt{17e^2 - 12e^{1 + rac{1}{e}} + 4e^{2/e}} + 3e} pprox 0.145776.$$

Proof. Since $f \in \mathcal{F}_2$ and $g \in \mathcal{A}$ satisfies $\operatorname{Re}(g(z)/z) > 1/2$. Now define the functions $p, h : \mathbb{D} \to \mathbb{C}$ by p(z) = g(z)/z and h(z) = f(z)/g(z). Then, it is clear that $p \in \mathcal{P}(1/2)$ and $h \in \mathcal{P}$. Further, since f(z) = zp(z)h(z), it follows from Lemma 4, get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{3r + r^2}{1 - r^2} \le \frac{e - e^{1/e}}{e}$$

provided $-e^{1/e}r^2 + 2er^2 + 3er + e^{1/e} - e \le 0$. This holds for

$$r \leq \frac{2\left(e - e^{1/e}\right)}{\sqrt{17e^2 - 12e^{1 + \frac{1}{e}} + 4e^{2/e}} + 3e} =: \mathcal{S}_B^*(\mathcal{F}_2) \approx 0.145776.$$

Thus, $f \in \mathcal{S}^*_B$ for $r \leq \mathcal{S}^*_B(\mathcal{F}_2)$.

For the sharpness of the result, consider the functions

$$f_3(z) = \frac{z(1+z)}{(1-z)^2}$$
 and $g_3(z) = \frac{z}{1-z}$

Then, we see that $\operatorname{Re}(f_3(z)/g_3(z)) > 0$ and $\operatorname{Re}(g_3(z)/z) > 1/2$, and therefore, $f \in \mathcal{F}_2$. Now from the definition of f_0 , we see that at $z = S_B^*(\mathcal{F}_2)$,

$$\frac{\mathcal{S}_B^*(\mathcal{F}_2)f_3'(\mathcal{S}_B^*(\mathcal{F}_2))}{f_3(\mathcal{S}_B^*(\mathcal{F}_2))} - 1 = \frac{3\mathcal{S}_B^*(\mathcal{F}_2) + \mathcal{S}_B^*(\mathcal{F}_2)^2}{1 - \mathcal{S}_B^*(\mathcal{F}_2)^2} = 1 - e^{\frac{1}{e} - 1}.$$

This confirms the sharpness of the result. \Box

Define the class \mathcal{F}_3 by

$$\mathcal{F}_3 := \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ and } \operatorname{Re} \frac{g(z)}{z} > 0, \ g \in \mathcal{A} \right\}.$$

The next result gives the sharp S_B^* -radius for the class \mathcal{F}_3 .

Theorem 8. Let $f \in \mathcal{F}_3$. Then, the sharp \mathcal{S}_B^* -radius is

$$\mathcal{S}_B^*(\mathcal{F}_3) = rac{2\left(e - e^{1/e}\right)}{\sqrt{17e^2 - 12e^{1 + rac{1}{e}} + 4e^{2/e}} + 3e} \approx 0.145776.$$

Proof. Since $f \in \mathcal{F}_3$, it follows that $p \in \mathcal{P}$ and $h \in \mathcal{P}(1/2)$, where the functions $p, h : \mathbb{D} \to \mathbb{C}$ are defined by p(z) = g(z)/z and h(z) = g(z)/f(z). Now since f(z) = zp(z)/h(z) from Lemma 4, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{3r + r^2}{1 - r^2} \le \frac{e - e^{1/e}}{e}$$

which holds for all $r \leq S_B^*(\mathcal{F}_3)$.

Consider the functions f_4 and g_4 defined by

$$f_4(z) = rac{z(1+z)^2}{(1-z)}$$
 and $g_4(z) = rac{z(1+z)}{1-z}$.

The results are sharp, since at $z = S_B^*(\mathcal{F}_3)$, we have

$$\frac{\mathcal{S}_{B}^{*}(\mathcal{F}_{3})f_{4}'(\mathcal{S}_{B}^{*}(\mathcal{F}_{3}))}{f_{4}(\mathcal{S}_{B}^{*}(\mathcal{F}_{3}))} = 2 - e^{\frac{1}{e} - 1}.$$

This completes the proof. \Box

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