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A Dunkl–Type Generalization of Szász–Kantorovich Operators via Post–Quantum Calculus

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Abstract: In this paper, we define the (p, q)-variant of Szász–Kantorovich operators via Dunkl-type generalization generated by an exponential function and study the Korovkin-type results. We also obtain the convergence of our operators in weighted space by the modulus of continuity, Lipschitz class, and Peetre's K-functionals. The extra parameter p provides more flexibility in approximation and plays an important role in symmetrizing these newly-defined operators.

Keywords: (p, q)-integers; Dunkl analogue; generating functions; generalization of exponential function; Szász operator; modulus of continuity

1. Introduction and Preliminaries

Bernstein [1] and q-Bernstein ([2,3]) operators have become very important tools in the study of approximation theory and several branches of applied sciences and engineering. For $[r]_{p,q} = \frac{p^r - q^r}{p-q}$, $r = 0, 1, 2, \cdots,\ 0 < q < p \leqq 1$, the (p, q)-Bernstein operators were introduced by Mursaleen et al. [4]:

$$B_r^{p,q}(g; y) = \frac{1}{p^{\frac{r(r-1)}{2}}} \sum_{m=0}^{r} \left[\begin{array}{c} r \\ m \end{array} \right]_{p,q} p^{\frac{m(m-1)}{2}} y^k \prod_{s=0}^{r-m-1} (p^s - q^s y) g\left(\frac{[m]_{p,q}}{p^{m-r}[r]_{p,q}} \right), \qquad y \in [0, 1], \tag{1}$$

where $[r]_{p,q}$ denotes the (p, q)-integer.

The (p, q)-analogues of exponential functions are defined in two forms as follows:

$$e_r^{p,q}(y) = \sum_{r=0}^{\infty} p^{\frac{r(r-1)}{2}} \frac{y^r}{[r]_{p,q}!}, \quad E_r^{p,q}(y) = \sum_{r=0}^{\infty} q^{\frac{r(r-1)}{2}} \frac{y^r}{[r]_{p,q}!},$$

with the property that $e_r^{p,q}(y) E_r^{p,q}(-y) = 1$. In the case of $p = 1$, $e_r^{p,q}(y)$ and $E_r^{p,q}(y)$ reduce to q-analogues of exponential functions.

The Dunkl-type generalization of Szász operators [5] was introduced by Sucu [6] and the q-analogue by Ben Cheikh et al. [7]. Içöz [8] introduced the q-Dunkl analogue of Szász operators defined by:

$$D_\eta^q(g; y) = \frac{1}{e_\eta^q([r]_q y)} \sum_{m=0}^{\infty} \frac{([r]_q y)^m}{\gamma_\eta^q(m)} g\left(\frac{1 - q^{2\eta\theta_m + m}}{1 - q^r} \right) \tag{2}$$

where $\eta > -\frac{1}{2}$, $y \geqq 0$, $0 < q < 1$, $g \in C[0, \infty)$ and $C[0, \infty)$ is the set of all continuous functions defined on $[0, \infty)$.

The (p,q)- and q-Dunkl analogues have been studied by several authors (see [9–24]). For the most recent work on (p,q)-approximation, we refer to [25–27]. Recently. Alotaibi et al. [28] generalized the q-Dunkl analogue of Szász operators via (pq)-calculus as follows:

$$\mathcal{D}_{\eta}^{p,q}(g;y) = \frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)} p^{\frac{m(m-1)}{2}} g\left(\frac{p^{2\eta\theta_m+m} - q^{2\eta\theta_m+m}}{p^{m-1}(p^r - q^r)}\right) \tag{3}$$

where for $q \in (0,1)$, $p \in (q,1]$, and $\eta > -\frac{1}{2}$, the (p,q)-Dunkl analogue of exponential functions is defined by:

$$e_{\eta}^{p,q} = \sum_{r=0}^{\infty} p^{\frac{r(r-1)}{2}} \frac{y^r}{\gamma_{\eta}^{p,q}(r)}, \qquad y \in [0,\infty) \tag{4}$$

$$\gamma_{\eta}^{p,q}(r) = \frac{\prod_{i=0}^{[\frac{r+1}{2}]-1} p^{2\eta(-1)^{i+1}+1}\left((p^2)^i p^{2\eta+1} - (q^2)^i q^{2\eta+1}\right) \prod_{j=0}^{[\frac{r}{2}]-1} p^{2\eta(-1)^{j}+1}\left((p^2)^j p^2 - (q^2)^j q^2\right)}{(p-q)^r}, \tag{5}$$

$$\gamma_{\eta}^{p,q}(r+1) = \frac{p^{2\eta(-1)^{r+1}+1}\left(p^{2\eta\theta_{r+1}+r+1} - q^{2\eta\theta_{r+1}+r+1}\right)}{(p-q)}\gamma_{\eta}^{p,q}(r), \tag{6}$$

$$\theta_r = \begin{cases} 0 & \text{for } r = 2\ell,\ \ell = 1,2,\cdots,n \\ 1 & \text{for } r = 2\ell+1,\ \ell = 1,2,\cdots,n. \end{cases} \tag{7}$$

and $[\frac{r}{2}]$ denotes the greatest integer function; also, we have:

$$(\alpha-\beta)_{p,q}^r = \begin{cases} \prod_{j=0}^{r-1}(p^j\alpha - q^j\beta) & \text{if } r = 1,2,\cdots,n \\ 1 & \text{if } r = 0. \end{cases}$$

Lemma 1. *For $g(t) = 1,\ t,\ t^2$*
1^*. $\mathcal{D}_{\eta}^{p,q}(1;y) = 1$;
2^*. $\mathcal{D}_{\eta}^{p,q}(t;y) = y$;
3^*. $y^2 + \frac{q^{2\eta}}{[r]_{p,q}}[1-2\eta]_{p,q}\frac{e_{\eta}^{p,q}(\frac{q}{p}[r]_{p,q}y)}{e_{\eta}^{p,q}([r]_{p,q}y)}y \leqq \mathcal{D}_{\eta}^{p,q}(t^2;y) \leqq y^2 + \frac{1}{[r]_{p,q}}[1+2\eta]_{p,q}y.$

2. New Operators and Estimations of Moments

In this section, we construct the (p,q)-variant of Szász–Kantorovich operators via Dunkl-type generalization as follows.

Definition 1. *For any $y \in [0,\infty)$, $g \in C[0,\infty)$ $r \in \mathbb{N}$ and $0 < q < p \leqq 1$, we define:*

$$\mathcal{K}_{\eta}^{p,q}(g;y) = \frac{[r]_{p,q}}{e_{\eta}^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)} p^{-(m+2\eta\theta_m)} p^{\frac{m(m-1)}{2}} \int_{q\mathcal{A}}^{q\mathcal{A}+\mathcal{B}} g\left(\frac{t}{qp^{m-1}}\right) d_{p,q}t. \tag{8}$$

We use the following relation:

$$[m+1+2\eta\theta_m]_{p,q} = q[m+2\eta\theta_m]_{p,q} + p^{m+2\eta\theta_m}, \tag{9}$$

$$\mathcal{A} = \frac{[m+2\eta\theta_m]_{p,q}}{[r]_{p,q}}, \quad \mathcal{B} = \frac{p^{m+2\eta\theta_m}}{[r]_{p,q}} \tag{10}$$

where the parameter $\eta \geqq 0$.

To show the uniform convergence of operators $\mathcal{K}_\eta^{p,q}(\cdot\,;\cdot)$, we take $q = q_r,\ \ p = p_r$ with $0 < q_r < 1$ and $q_r < p_r \leqq 1$ such that:

$$\lim_{r\to\infty} p_r \to 1, \quad \lim_{r\to\infty} q_r \to 1, \quad \lim_{r\to\infty} p_r^r \to u, \quad \lim_{r\to\infty} q_r^r \to v, \quad (0 < u, v \leqq 1). \tag{11}$$

For $p = 1$, these operators reduce to the operators defined in [29]. For $\eta = 0$, these are reduced to the (p,q)-variant of Kantorovich-type operators defined by [30].

Lemma 2. *Let $g(t) = g_i$ such that $g_i = t^{i-1}$ for $i = 1,2,3$. Then, we have:*
(1) $\mathcal{K}_\eta^{p,q}(g_1;y) = 1$
(2) $\mathcal{K}_\eta^{p,q}(g_2;y) \leqq \frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[r]_{p,q}}$
(3) $\mathcal{K}_\eta^{p,q}(g_3;y) \leqq \frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[r]_{p,q}}\left([1+2\eta]_{p,q} + \frac{1}{q[r]_{p,q}}\right)y + \frac{1}{[3]_{p,q}q^2[r]_{p,q}^2}.$

Proof. Using (9) and (10), we get:

$$\int_{q\mathcal{A}}^{q\mathcal{A}+\mathcal{B}} f\left(\frac{t}{qp^{k-1}}\right) d_{p,q}t = \begin{cases} \mathcal{B} & \text{for} \quad g(t) = g_1 \\ \frac{\mathcal{B}}{[2]_{p,q}p^{m-1}q}\,(2q\mathcal{A}+\mathcal{B}) & \text{for} \quad g(t) = g_2 \\ \frac{\mathcal{B}}{[3]_{p,q}p^{2(m-1)}q^2}\,(3q^2\mathcal{A}^2 + 3q\mathcal{A}\mathcal{B} + \mathcal{B}^2) & \text{for} \quad g(t) = g_3 \end{cases} \tag{12}$$

If we take $g(t) = g_1$, then from (12), we have:

$$\begin{aligned} \mathcal{K}_\eta^{p,q}(g_1;y) &= \frac{[r]_{p,q}}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{-(m+2\eta\theta_m)} p^{\frac{m(m-1)}{2}} \int_{q\mathcal{A}}^{q\mathcal{A}+\mathcal{B}} d_{p,q}t \\ &= 1. \end{aligned}$$

For $g(t) = g_2$, (12) implies:

$$\begin{aligned} \mathcal{K}_\eta^{p,q}(g_2;y) &= \frac{1}{[2]_{p,q}q[r]_{p,q}} \frac{1}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{-(m+2\eta\theta_m)} p^{\frac{m(m-1)}{2}} \\ &\times\ p^{1+2\eta\theta_m}\left(2q[m+2\eta\theta_m]_{p,q} + p^{m+2\eta\theta_m}\right) \\ &= \frac{2}{[2]_{p,q}[r]_{p,q}} \frac{1}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{\frac{(m-1)(m-2)}{2}} [m+2\eta\theta_m]_{p,q} \\ &+ \frac{1}{[2]_{p,q}q[r]_{p,q}} \frac{1}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{1+2\eta\theta_m} \\ &= \frac{2}{[2]_{p,q}} \frac{1}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{\frac{m(m-1)}{2}} \left(\frac{p^{m+2\eta\theta_m} - q^{m+2\eta\theta_m}}{p^{m-1}(p^r - q^r)}\right) \\ &+ \frac{1}{[2]_{p,q}q[r]_{p,q}} \frac{1}{e_\eta^{p,q}([r]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([r]_{p,q}y)^m}{\gamma_\eta^{p,q}(m)} p^{1+2\eta\theta_m}. \end{aligned}$$

Separating into even and odd terms, we get:

$$\begin{aligned} \mathcal{K}_\eta^{p,q}(g_2;y) &= \frac{2}{[2]_{p,q}}y + \frac{p}{[2]_{p,q}q[r]_{p,q}} && \text{for} \quad r = 0,2,4,\cdots \\ \mathcal{K}_\eta^{p,q}(g_2;y) &= \frac{2}{[2]_{p,q}}y + \frac{p^{1+2\eta}}{[2]_{p,q}q[r]_{p,q}} && \text{for} \quad r = 1,3,5,\cdots. \end{aligned}$$

Since $0 < q < p \leqq 1$, $\eta \geqq 0$, and $p^{1+2\eta} \leqq 1$, we have:

$$\mathcal{K}_{\eta}^{p,q}(g_2;y) \leqq \frac{2}{[2]_{p,q}}y + \frac{1}{q[2]_{p,q}[r]_{p,q}}.$$

Similarly for $g(t) = g_3$, we have:

$$\begin{aligned}
\mathcal{K}_{\eta}^{p,q}(g_3;y) &= \frac{3}{[3]_{p,q}[r]_{p,q}^3}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{\frac{m(m-1)}{2}-2(m-1)}[m+2\eta\theta_m]_{p,q}^2 \\
&+ \frac{3}{[3]_{p,q}q[r]_{p,q}^3}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{m+2\eta\theta_m}p^{\frac{m(m-1)}{2}-2(m-1)}[m+2\eta\theta_m]_{p,q} \\
&+ \frac{1}{[3]_{p,q}q^2[r]_{p,q}^2}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{2(m+2\eta\theta_m)}p^{\frac{m(m-1)}{2}-2(m-1)} \\
&= \frac{3}{[3]_{p,q}[r]_{p,q}^2}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{\frac{m(m-1)}{2}}\left(\frac{p^{m+2\eta\theta_m}-q^{m+2\eta\theta_m}}{p^{m-1}(p^r-q^r)}\right)^2 \\
&+ \frac{3}{[3]_{p,q}q[r]_{p,q}^2}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{\frac{m(m-1)}{2}}\left(\frac{p^{m+2\eta\theta_m}-q^{m+2\eta\theta_m}}{p^{m-1}(p^r-q^r)}\right) \\
&+ \frac{1}{[3]_{p,q}q^2[r]_{p,q}^2}\frac{1}{e_{\eta}^{p,q}([r]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([r]_{p,q}y)^m}{\gamma_{\eta}^{p,q}(m)}p^{\frac{m(m-1)}{2}}p^{2(1+2\eta\theta_m)}.
\end{aligned}$$

Hence, for $m = 0, 2, 4, \cdots$, we have:

$$\mathcal{K}_{\eta}^{p,q}(g_3;y) \leqq \frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[r]_{p,q}}\left([1+2\eta]_{p,q} + \frac{p}{q[r]_{p,q}}\right)y + \frac{p^2}{q^2[3]_{p,q}[r]_{p,q}^2},$$

and for $m = 1, 3, 5, \cdots$,

$$\mathcal{K}_{\eta}^{p,q}(g_3;y) \leqq \frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[r]_{p,q}}\left([1+2\eta]_{p,q} + \frac{p}{q[r]_{p,q}}\right)y + \frac{p^2}{[3]_{p,q}q^2[r]_{p,q}^2}.$$

Therefore,

$$\mathcal{K}_{\eta}^{p,q}(g_3;y) \leqq \frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[r]_{p,q}}\left([1+2\eta]_{p,q} + \frac{1}{q[r]_{p,q}}\right)y + \frac{1}{[3]_{p,q}q^2[r]_{p,q}^2}.$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $\chi_i = (t-y)^i$ for $i = 1,\ 2$. Then, we have:*

$$\mathcal{K}_{\eta}^{p,q}(\chi_i;y) \leqq \begin{cases} \left(\dfrac{2}{[2]_{p,q}} - 1\right)y + \dfrac{1}{[2]_{p,q}[r]_{p,q}} & \text{for} \quad i = 1 \\[2ex] \left(\dfrac{3}{[3]_{p,q}} + 1 - \dfrac{4}{[2]_{p,q}}\right)y^2 & \\ +\dfrac{1}{q[r]_{p,q}}\left(\dfrac{3}{[3]_{p,q}}\left(\dfrac{1}{[r]_{p,q}} + q[1+2\eta]_{p,q}\right) - \dfrac{2}{[2]_{p,q}}\right)y & \\ +\dfrac{1}{[3]_{p,q}q^2[r]_{p,q}^2} & \text{for} \quad i = 2. \end{cases} \tag{13}$$

3. Main Results

In this section, we study the Korovkin-type approximation theorems for positive linear operators $\mathcal{K}_{\eta}^{p,q}(\,\cdot\,;\,\cdot\,)$ defined by (8). We denote the set of all bounded and continuous functions by $C_B[0,\infty)$ equipped with norm $\parallel g \parallel_{C_B} = \sup_{y\in[0,\infty)} \mid g(y) \mid$. We write:

$$\mathfrak{E} := \{g(y) : y \in [0,\infty), \frac{g(y)}{1+y^2} \text{ is convergent as } y \to \infty\}.$$

Let:

$$B_{\sigma}[0,\infty) = \left\{g : |g(y)| \leqq \mathcal{M}_g \sigma(y)\right\},$$
$$C_{\sigma}[0,\infty) = \{g : g \in B_{\sigma}[0,\infty) \cap C[0,\infty)\},$$
$$C_{\sigma}^{k}[0,\infty) = \left\{g : g \in C_{\sigma}[0,\infty) \quad \text{and} \quad \lim_{y\to\infty} \frac{g(y)}{\sigma(y)} = k\right\},$$

where $\sigma(y)$ is the weight function given by $\sigma(y) = 1 + y^2$, k is a constant, and $\mathcal{M}_g$ depends on g. $C_{\sigma}[0,\infty)$ is equipped with the norm $||g||_{\sigma} = \sup_{y\in[0,\infty)} \frac{|g(y)|}{\sigma(y)}$.

Theorem 1. *Let q_r, p_r be the real numbers, with $q_r \in (0,1)$ and $p_r \in (q_r,1]$ for every integer r, satisfying $(q_r) \to 1$ and $(p_r) \to 1$ as $r \to \infty$. Then, for every $g \in C[0,\infty) \cap \mathfrak{E}$,*

$$\lim_{r\to\infty} \mathcal{K}_{\eta}^{p_r,q_r}(g;y) = g(y)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. For the proof of the uniform convergence of the operators $\mathcal{K}_{\eta}^{p_r,q_r}$ on each compact subset of $[0,\infty)$, we apply the well-known Korovkin theorem [31]. It is sufficient to show that $\lim_{r\to\infty} \mathcal{K}_{\eta}^{p_r,q_r}(g_i;y) = y^{i-1}$, where $g_i = t^{i-1}$ for $i = 1,2,3$.

Clearly, if $q_r \to 1$, $p_r \to 1$ as $r \to \infty$, then $\frac{1}{[r]_{p_r,q_r}} \to 0$, $\frac{r}{[r]_{p_r,q_r}} \to 1$. This yields that:

$$\lim_{r\to\infty} \mathcal{K}_{\eta}^{p_r,q_r}(g_1;y) = 1, \quad \lim_{r\to\infty} \mathcal{K}_{\eta}^{p_r,q_r}(g_2;y) = y, \quad \lim_{r\to\infty} \mathcal{K}_{\eta}^{p_r,q_r}(g_3;y) = y^2.$$

□

Theorem 2. *Let q_r, p_r be the real numbers, with $q_r \in (0,1)$ and $p_r \in (q_r,1]$ for every integer r, satisfying $(q_r) \to 1$ and $(p_r) \to 1$ as $r \to \infty$. Then, for every $g \in C_{\sigma}^{k}[0,\infty)$, we have:*

$$\lim_{r\to\infty} \left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y) - g\right|\right|_{\sigma} = 0. \tag{14}$$

Proof. Suppose $g(t) \in C_{\sigma}^{k}[0,\infty)$ and $g(t) = g_{\tau}$, where $g_{\tau} = t^{\tau-1}$ for $\tau = 1,2,3$. Then, from the well-known Korovkin theorem, we have $\mathcal{K}_{\eta}^{p_r,q_r}(g_{\tau};y) \to y^{\tau-1}$ $(r \to \infty)$ uniformly for each $\tau = 1,2,3$. Hence, from Lemma 2, we have:

$$\lim_{r\to\infty} \left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_1;y) - 1\right|\right|_{\sigma} = 0. \tag{15}$$

For $\tau = 2$,

$$\begin{aligned}
&\left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_2;y) - y\right|\right|_{\sigma} \\
&= \sup_{y \geqq 0} \frac{\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_2;y) - y\right|}{1+y^2} \\
&\leqq \left(\frac{2}{[2]_{p_r,q_r}} - 1\right) \sup_{y \geqq 0} \frac{y}{1+y} + \frac{1}{q_r[2]_{p_r,q_r}[r]_{p_r,q_r}} \sup_{y \geqq 0} \frac{1}{1+y}.
\end{aligned}$$

Then:

$$\lim_{r\to\infty} \left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_2;y) - y\right|\right|_{\sigma} = 0. \tag{16}$$

Similarly, if we take $\tau = 3$,

$$\begin{aligned}
&\left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_3;y) - y^2\right|\right|_{\sigma} \\
&= \sup_{y \geqq 0} \frac{\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_3;y) - y^2\right|}{1+y^2} \\
&\leqq \left(\frac{3}{[3]_{p_r,q_r}} - 1\right) \sup_{y \geqq 0} \frac{y^2}{1+y^2} \\
&+ \frac{3}{[3]_{p_r,q_r}[r]_{p_r,q_r}} \left([1+2\eta]_{p_r,q_r} + \frac{1}{q_r[r]_{p_r,q_r}}\right) \sup_{y \geqq 0} \frac{y}{1+y^2} \\
&+ \frac{1}{[3]_{p_r,q_r} q_r^2 [r]_{p_r,q_r}^2} \sup_{y \geqq 0} \frac{1}{1+y^2},
\end{aligned}$$

$$\lim_{r\to\infty} \left|\left|\mathcal{K}_{\eta}^{p_r,q_r}(g_3;y) - y^2\right|\right|_{\sigma} = 0. \tag{17}$$

This completes the proof. □

The modulus of continuity $\omega_b(g;\delta)$ of the function $g \in \tilde{C}[0,\infty)$ is defined by:

$$\omega_b(g;\delta) = \sup_{|t-y|\leq\delta;} \sup_{y,t\in[0,b]} \mid g(t) - g(y) \mid \tag{18}$$

where $\tilde{C}[0,\infty)$ denotes the space of uniformly-continuous functions on $[0,\infty)$. It is obvious that $\lim_{\delta\to 0+} \omega_b(g;\delta) = 0$ and for $g \in C[0,\infty)$:

$$\mid g(t) - g(y) \mid \leq \left(\frac{\mid t-y \mid}{\delta} + 1\right) \omega_b(g;\delta). \tag{19}$$

Theorem 3. *Let q_r, p_r be the real numbers, with $q_r \in (0,1)$ and $p_r \in (q_r,1]$ for every integer r, satisfying $(q_r) \to 1$ and $(p_r) \to 1$ as $r \to \infty$. Then, for every $g \in C_\sigma[0,\infty)$:*

$$\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y) - g(y)\right| \leqq 2\left(\omega_{b+1}(g;\delta_\eta(y)) + \mathcal{M}_g(1+b^2)\left(\delta_\eta(y)\right)^2\right),$$

where $\delta_\eta(y) = \sqrt{\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2;y)}$, $\mathcal{M}_g$ is a constant depending only on g and $\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2;y)$ is defined by Lemma 3; and $[0,b+1] \subset [0,\infty)$, $b > 0$.

Proof. Let $y \in [0,b]$ and $t > b+1$, with $t > 0$. Then, for $\delta > 0$, we have:

$$| g(t) - g(y) | \leqq \omega_{b+1}(g; | t - y |) \leqq \left(1 + \frac{| t - y |}{\delta}\right) \omega_{b+1}(g; \delta). \tag{20}$$

By applying the Cauchy–Schwarz inequality and the linearity of $\mathcal{K}_{\eta}^{p_r,q_r}$:

$$\mathcal{K}_{r,\eta}^{p_r,q_r} |g(t) - g(y); y| \leqq \left(\left(1 + \frac{1}{\delta}\mathcal{K}_{\eta}^{p_r,q_r}\left((t-y)^2; y\right)\right)^{\frac{1}{2}}\right)\omega_{b+1}(g; \delta). \tag{21}$$

For $t - y > 1$, we have:

$$\begin{aligned}
&|g(t) - g(y)| \\
&\quad \leqq \mathcal{M}_g\left(2 + y^2 + t^2\right) \\
&\quad \leqq \mathcal{M}_g\left(2 + 3y^2 + 2(t-y)^2\right) \leqq 2\mathcal{M}_g(1 + b^2)(t-y)^2
\end{aligned}$$

$$\mathcal{K}_{\eta}^{p_r,q_r}\left(|g(t) - g(y)|;\right) \leqq 2\mathcal{M}_g(1 + b^2)\mathcal{K}_{\eta}^{p_r,q_r}\left((t-y)^2; y\right). \tag{22}$$

From (21) and (22), we easily see that:

$$\begin{aligned}
&\left|\mathcal{K}_{\eta}^{p_r,q_r}(g; y) - g(y)\right| \\
&\qquad \leqq \mathcal{K}_{\eta}^{p_r,q_r} |g(t) - g(y); y| \\
&\qquad \leqq \left(\left(1 + \frac{1}{\delta}\mathcal{K}_{\eta}^{p_r,q_r}\left((t-y)^2; y\right)\right)^{\frac{1}{2}}\right)\omega_{b+1}(g; \delta) \\
&\qquad + 2\mathcal{M}_g(1 + b^2)\mathcal{K}_{\eta}^{p_r,q_r}\left((t-y)^2; y\right) \\
&\qquad = \left(1 + \frac{1}{\delta}\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2; y)\right)^{\frac{1}{2}}\omega_{b+1}(g; \delta) \\
&\qquad + 2\mathcal{M}_g(1 + b^2)\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2; y)
\end{aligned}$$

If we choose $\delta = \delta_\eta(y) = \sqrt{\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2; y)}$, then we get our result. $\square$

For any $g \in C[0, \infty]$, $\mathcal{L} > 0$, $0 < \nu \leq 1$ and $\gamma_1, \gamma_2 \in [0, \infty)$, we recall that:

$$Lip_{\mathcal{L}}(\nu) = \{g :| g(\gamma_1) - g(\gamma_2) | \leq \mathcal{L} | \gamma_1 - \gamma_2 |^{\nu}\}. \tag{23}$$

Theorem 4. *Let q_r, p_r be the real numbers, with $q_r \in (0,1)$ and $p_r \in (q_r, 1]$ for every integer r, satisfying $(q_r) \to 1$ and $(p_r) \to 1$ as $r \to \infty$. Then, for each $g \in Lip_{\mathcal{L}}(\nu)$, we have:*

$$\left|\mathcal{K}_{\eta}^{p_r,q_r}(g; y) - g(y)\right| \leqq \mathcal{L}\left(\delta_\eta(y)\right)^{\nu},$$

where $\delta_\eta(y)$ is defied by Theorem 3.

Proof. Using Theorem 4, (23), and the well-known Hölder's inequality, we get:

$$\begin{aligned}
\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y)-g(y)\right| &\leqq \left|\mathcal{K}_{r,\eta}^{p_r,q_r}(g(t)-g(y);y)\right| \\
&\leqq \mathcal{K}_{\eta}^{p_r,q_r}\left(|g(t)-g(y)|\,;y\right) \\
&\leqq |\,\mathcal{L}\mathcal{K}_{\eta}^{p_r,q_r}\left(|t-y|^{\nu}\,;y\right) \\
&\leqq \mathcal{L}\left(\mathcal{K}_{\eta}^{p_r,q_r}(g_1;y)\right)^{\frac{2-\nu}{2}}\left(\mathcal{K}_{\eta}^{p_r,q_r}(|t-y|^2\,;y)\right)^{\frac{\nu}{2}} \\
&= \mathcal{L}\left(\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2;y)\right)^{\frac{\nu}{2}}.
\end{aligned}$$

This completes the proof of the theorem. □

We denote:

$$C_B^2[0,\infty)=\left\{\psi:\psi\in C_B[0,\infty)\quad\text{and}\quad\psi',\psi''\in C_B[0,\infty)\right\}, \tag{24}$$

$$||\psi||_{C_B^2(\mathbb{R}^+)}=||\psi||_{C_B[0,\infty)}+||\psi'||_{C_B[0,\infty)}+||\psi''||_{C_B[0,\infty)}, \tag{25}$$

$$||\psi||_{C_B[0,\infty)}=\sup_{y\in[0,\infty)}|\psi(y)|. \tag{26}$$

Theorem 5. *Let q_r, p_r be the real numbers, with $q_r\in(0,1)$ and $p_r\in(q_r,1]$ for every integer r, satisfying $(q_r)\to 1$ and $(p_r)\to 1$ as $r\to\infty$. Then:*

$$\left|\mathcal{K}_{\eta}^{p_r,q_r}(\psi;y)-\psi(y)\right|\leqq \Upsilon_{\eta}(y)||\psi||_{C_B^2[0,\infty)}, \tag{27}$$

where $\Upsilon_{\eta}(y)=\delta_n(y)\left(1+\frac{\delta_{\eta}(y)}{2}\right)$ and $\delta_{\eta}(y)$ is defined by Theorem 3.

Proof. From the Taylor series expansion for any $\psi\in C_B^2[0,\infty)$, we have:

$$\psi(t)=\psi(y)+\psi'(y)(t-y)+\psi''(\varphi)\,\frac{(t-y)^2}{2}\quad\text{for}\quad\varphi\in(y,t),$$

$$|\psi(t)-\psi(y)|\leqq\mathfrak{P}\mid t-y\mid+\frac{1}{2}\mathfrak{Q}(t-y)^2,$$

where:

$$\mathfrak{P}=\sup_{y[0,\infty)}|\psi'(y)|=||\psi'||_{C_B[0,\infty)}\leqq||\psi||_{C_B^2[0,\infty)},$$

$$\mathfrak{Q}=\sup_{y[0,\infty)}|\psi''(y)|=||\psi''||_{C_B[0,\infty)}\leqq||\psi||_{C_B^2[0,\infty)}.$$

Therefore,

$$|\psi(t)-\psi(y)|\leqq\left(\mid t-y\mid+\frac{1}{2}(t-y)^2\right)||\psi||_{C_B^2[0,\infty)}.$$

By applying the linearity of $\mathcal{K}_{\eta}^{p_r,q_r}$, we get:

$$\begin{aligned}
&\left|\mathcal{K}_{\eta}^{p_r,q_r}(\psi;y)-\psi(y)\right| \\
&\leqq \left(\mathcal{K}_{\eta}^{p_r,q_r}(\mid t-y\mid;y)+\frac{1}{2}\mathcal{K}_{\eta}^{p_r,q_r}((t-y)^2;y)\right)||\psi||_{C_B^2[0,\infty)} \\
&\leqq \left(\left(\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2;y)\right)^{\frac{1}{2}}+\frac{1}{2}\mathcal{K}_{\eta}^{p_r,q_r}(\chi_2;y)\right)||\psi||_{C_B^2[0,\infty)} \\
&= \left(\delta_\eta(y)+\frac{(\delta_\eta(y))^2}{2}\right)||\psi||_{C_B^2[0,\infty)}.
\end{aligned}$$

This completes the proof of the theorem. □

Peetre's K-functional $K_2(g;\delta)$ for $\delta>0$ (see [32]) is defined by:

$$K_2(g;\delta)=\inf_{y\in[0,\infty)}\left\{\left(\delta\left|\left|\psi''+||g-\psi||_{C_B[0,\infty)}\right|\right|_{C_B[0,\infty)}\right)\right\} \tag{28}$$

for all $\psi\in C_B^2[0,\infty)$.

For a given positive constant $\mathfrak{L}>0$:

$$K_2(g;\delta)\leqq \mathfrak{L}\omega_2(g;\delta^{\frac{1}{2}}),$$

where the second-order modulus of continuity denoted by $\omega_2(g;\delta)$ is defined as:

$$\omega_2(g;\delta)=\sup_{0<h<\delta},\sup_{y\in[0,\infty)}|g(y)+g(y+2h)-2g(y+h)|. \tag{29}$$

Theorem 6. *Let q_r, p_r be the real numbers, with $q_r\in(0,1)$ and $p_r\in(q_r,1]$ for every integer r, satisfying $(q_r)\to 1$ and $(p_r)\to 1$ as $r\to\infty$. Then, for all $g\in C_B[0,\infty)$, we have:*

$$\begin{aligned}
&\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y)-g(y)\right| \\
&\leqq 2\mathcal{A}\left\{\omega_2\left(g;\sqrt{\frac{\Upsilon_\eta(y)}{2}}\right)+\min\left(1;\frac{\Upsilon_\eta(y)}{2}\right)||g||_{C_B[0,\infty)}\right\},
\end{aligned}$$

where $\mathcal{A}$ is a positive constant and $\Upsilon_\eta(y)$ is given in Theorem 5.

Proof. We take $\psi\in C_B^2[0,\infty)$ and apply Theorem (5). Thus:

$$\begin{aligned}
\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y)-g(y)\right| &\leqq \left|\mathcal{K}_{\eta}^{p_r,q_r}(g-\psi;y)\right|+\left|\mathcal{K}_{\eta}^{p_r,q_r}(\psi;y)-\psi(y)\right|+|g(y)-\psi(y)| \\
&\leqq 2||g-\psi||_{C_B[0,\infty)}+\Upsilon_\eta(y)||\psi||_{C_B^2[0,\infty)} \\
&= 2\left(||g-\psi||_{C_B[0,\infty)}+\frac{\Upsilon_\eta(y)}{2}||\psi||_{C_B^2[0,\infty)}\right).
\end{aligned}$$

By taking the infimum over all $\psi\in C_B^2[0,\infty)$ and using (28), we get:

$$\left|\mathcal{K}_{\eta}^{p_r,q_r}(g;y)-g(y)\right|\leqq 2K_2\left(g;\frac{\Upsilon_\eta(y)}{2}\right).$$

Now, from [33] for all $g \in C_B[0,\infty)$, we have the relation:

$$K_2(g;\delta) \leqq \mathcal{A}\{\min(1;\delta) + \omega_2(g;\sqrt{\delta})||g||_{C_B[0,\infty)}\},$$

where $\mathcal{A} > 0$ is an absolute constant. If we choose $\delta = \frac{\Upsilon_\eta(y)}{2}$, then we get the desired result. $\square$

4. Conclusions

In this paper, we have studied the approximation results via Dunkl generalization of the Szász–Kantorovich operators in (p,q)-calculus. These types of modifications enable us to generalize error estimation rather than the classical and q-calculus on the interval $[0,\infty)$ obtained in [29]. We have also proven the Korovkin-type results and obtained the convergence of our operators in weighted space by the modulus of continuity, Lipschitz class, and Peetre's K-functionals. We have a more generalized version of the operators [29,30], and if we take $\eta = 0$ in (8), then the operators $\mathcal{K}_\eta^{p,q}$ reduce to the operators defined by [30].

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