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Some Real-Life Applications of a Newly Constructed Derivative Free Iterative Scheme

Ramandeep Behl ^{1,*}, M. Salimi ², M. Ferrara ^{3,4} , S. Sharifi ⁵ and Samaher Khalaf Alharbi ¹

¹ Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; samaher271@gmail.com

² Center for Dynamics and Institute for Analysis, Department of Mathematics, Technische Universität Dresden, 01062 Dresden, Germany; msalimi1@yahoo.com or mehdi.salimi@tu-dresden.de

³ Department of Law, Economics and Human Sciences, University Mediterranea of Reggio Calabria, 89125 Reggio Calabria, Italy; massimiliano.ferrara@unirc.it

⁴ ICRIOS—The Invernizzi Centre for Research on Innovation, Organization, Strategy and Entrepreneurship, Department of Management and Technology, Bocconi University, Via Sarfatti, 25, 20136 Milano, Italy

⁵ MEDAlics, Research Center at Università per Stranieri Dante Alighieri, 89125 Reggio Calabria, Italy; somayeh.sharifi69@yahoo.com

* Correspondence: ramanbehl87@yahoo.in

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Abstract: In this study, we present a new higher-order scheme without memory for simple zeros which has two major advantages. The first one is that each member of our scheme is derivative free and the second one is that the present scheme is capable of producing many new optimal family of eighth-order methods from every 4-order optimal derivative free scheme (available in the literature) whose first substep employs a Steffensen or a Steffensen-like method. In addition, the theoretical and computational properties of the present scheme are fully investigated along with the main theorem, which demonstrates the convergence order and asymptotic error constant. Moreover, the effectiveness of our scheme is tested on several real-life problems like Van der Waal's, fractional transformation in a chemical reactor, chemical engineering, adiabatic flame temperature, etc. In comparison with the existing robust techniques, the iterative methods in the new family perform better in the considered test examples. The study of dynamics on the proposed iterative methods also confirms this fact via basins of attraction applied to a number of test functions.

Keywords: scalar equations; computational convergence order; Steffensen's method; basins of attraction

1. Introduction

In the last few years, several scholars introduced the concept of how to remove derivatives from the iteration functions. The main practical difficulty associated with iterative methods involving derivatives is to calculate first and/or high-order derivatives at each step, which is quite difficult and time-consuming. Computing derivatives of standard nonlinear equations (which are generally considered for academic purposes) is an easy task. On the other hand, in regard to practical problems of calculating the derivatives of functions, it is either very expensive or requires a huge amount of time. Therefore, we need derivative free methods, software or tools which are capable of generating derivatives automatically (for a detailed explanation, please see [1]).

There is no doubt that optimal 8-order multi-point derivative free methods are one of the important classes of iterative methods. They have faster convergence towards the required root and a better efficiency index as compared to Newton/Steffensen's method. In addition, one can easily attain the desired accuracy of any specific number of digits within a small number of iterations with the help of these iterative methods.

In recent years, many scholars have proposed a big number of 8-order derivative free schemes in their research articles [2–18]. However, most of these eighth-order methods are the extensions or modifications of particularly well-known or unknown existing optimal fourth-order derivative free methods; for detailed explanations, please see [5,6,14,16,17]. However, there is no optimal derivative free scheme in a general way that is capable of producing optimal eighth-order convergence from every optimal fourth-order derivative free scheme to date, according to our knowledge.

In this paper, we present a new optimal scheme that doesn't require any derivative. In addition, the proposed scheme is capable of generating new optimal 8-order methods from the earlier optimal fourth-order schemes whose first substep employs Steffensen's or a Steffensen-type method. In this way, our scheme is giving the flexibility in the choice of a second-step to the scholars who can pick any existing optimal derivative free fourth-order method (available in the literature) unlike the earlier studies. The construction of the presented scheme is based on a technique similar to Sharma et al. [19] along with some modifications that can be seen in the next section. We tested the applicability of a newly proposed scheme on a good variety of numerical examples. The obtained results confirm that our methods are more efficient and faster as compared to existing methods in terms of minimum residual error, least asymptotic error constants, minimum error between two consecutive iterations, etc. Moreover, we investigate their dynamic behavior in the complex plane adopting basins of attraction. Dynamic behavior provides knowledge about convergence, and stability of the mentioned methods also supports the theoretical aspects.

2. Construction of the Proposed Scheme

This section is devoted to 8-order derivative free schemes for nonlinear equations. In order to obtain this scheme, we consider a general fourth-order method $\eta(v_j, x_j, y_j)$ in the following way:

$$\begin{cases} y_j = x_j - \frac{f(x_j)}{f[v_j, x_j]}, \\ z_j = \eta(v_j, x_j, y_j), \end{cases} \quad (1)$$

where $v_j = x_j + \lambda f(x_j)$, $\lambda \in \mathbb{R}$ and $f[v_j, x_j] = \frac{f(v_j) - f(x_j)}{v_j - x_j}$ are the first-order finite difference. We can simply obtain eighth-order convergence by applying the classical Newton's technique, which is given by

$$x_{j+1} = z_j - \frac{f(z_j)}{f'(z_j)}. \quad (2)$$

The above scheme is non optimal because it does not satisfy the Kung–Traub conjecture [7]. Thus, we have to reduce the number of evaluations of functions or their derivatives. In this regard, we some approximation of the first-order derivative For this purpose, we need a suitable approximation approach of functions that can approximate the derivatives. Therefore, we choose the following rational functional approach

$$\Omega(x) = \Omega(x_j) - \frac{(x - x_j) + \theta_1}{\theta_2(x - x_j)^2 + \theta_3(x - x_j) + \theta_4}, \quad (3)$$

where $\theta_i, i = 1, 2, 3, 4$ are free parameters. This approach is similar to Sharma et al. [19] along with some modifications. We can determine these disposable parameters θ_i by adopting the following tangency constraints

$$\Omega(x_j) = f(x_j), \quad \Omega(v_j) = f(v_j), \quad \Omega(y_j) = f(y_j), \quad \Omega(z_j) = f(z_j). \quad (4)$$

The number of tangency conditions depends on the number of undetermined parameters. If we increase the number of undetermined parameters in the above rational function, then we can also attain high-order convergence (for the detailed explanation, please see Jarratt and Nudds [20]).

By imposing the first tangency condition, we have

$$\theta_1 = 0. \quad (5)$$

The last three tangency conditions provide us with the following three linear equations:

$$\begin{aligned} \theta_2(v_j - x_j)^2 + \theta_3(v_j - x_j) + \theta_4 &= \frac{1}{f[v_j, x_j]}, \\ \theta_2(y_j - x_j)^2 + \theta_3(y_j - x_j) + \theta_4 &= \frac{1}{f[y_j, x_j]}, \\ \theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4 &= \frac{1}{f[z_j, x_j]}, \end{aligned} \quad (6)$$

with three unknowns θ_2 , θ_3 and θ_4 .

After some simplification, we further yield

$$\begin{aligned} \theta_2 &= \frac{f(v_j) + \theta_4 f[x_j, v_j] f[x_j, y_j] (y_j - v_j) - f(y_j)}{(f(v_j) - f(x_j))(f(x_j) - f(y_j))(v_j - y_j)}, \\ \theta_3 &= -\frac{\theta_2(f(v_j) - f(x_j))(v_j - x_j) + \theta_4 f[x_j, v_j] - 1}{f(v_j) - f(x_j)}, \\ \theta_4 &= \frac{(f(x_j) - f(y_j))(f(x_j) - f(z_j))(y_j - z_j) - a}{f[x_j, v_j] f[x_j, y_j] f[x_j, z_j] (v_j - y_j)(v_j - z_j)(y_j - z_j)}, \end{aligned} \quad (7)$$

where $a = (f(v_j) - f(x_j))[(f(x_j) - f(y_j))(v_j - y_j) + (f(z_j) - f(x_j))(v_j - z_j)]$ and $f[\cdot, \cdot]$ are the finite difference of first order. Now, we differentiate the expression (3) with respect to x at the point $x = z_j$, which further provides

$$f'(z_j) \approx \Omega'(z_j) = \frac{\theta_4 - \theta_2(z_j - x_j)^2}{[\theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4]^2}. \quad (8)$$

Finally, by using the expressions (1), (2) and (8), we have

$$\begin{cases} y_j = x_j - \frac{f(x_j)}{f[v_j, x_j]}, \\ z_j = \eta(v_j, x_j, y_j), \\ x_{j+1} = z_j - \frac{f(z_j) [\theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4]^2}{\theta_4 - \theta_2(z_j - x_j)^2}, \end{cases} \quad (9)$$

where v_j and θ_i , $i = 2, 3, 4$ was already explained earlier in the same section. Now, we demonstrate in the next Theorem 1 how a rational function of the form (2) plays an important role in the development of a new derivative free technique. In addition, we confirm the eighth-order of convergence of (9) without considering any extra functional evaluation/s.

3. Convergence Analysis

Theorem 1. We assume that the function $f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic in the neighborhood of simple zero ξ . In addition, we consider that $\eta(v_j, x_j, y_j)$ is any 4-order optimal derivative free iteration function and initial guess $x = x_0$ is close enough to the required zero ξ for the ensured convergence. The scheme (9) reaches an eighth-order convergence.

Proof. We assume that $e_j = x_j - \zeta$ is the error at j th point. We expand the function $f(x_j)$ around the point $x = \zeta$ by adopting Taylor's series expansion. Then, we have

$$f(x_j) = c_1 e_j + c_2 e_j^2 + c_3 e_j^3 + c_4 e_j^4 + c_5 e_j^5 + c_6 e_j^6 + c_7 e_j^7 + c_8 e_j^8 + O(e_j^9), \quad (10)$$

where $c_n = \frac{f^{(n)}(\zeta)}{n!}$ for $n = 1, 2, \dots, 8$.

By using the above expression (10), we further obtain

$$v_j - \zeta = (1 + \lambda c_1) e_j + \lambda (c_2 e_j^2 + c_3 e_j^3 + c_4 e_j^4 + c_5 e_j^5 + c_6 e_j^6 + c_7 e_j^7 + c_8 e_j^8) + O(e_j^9). \quad (11)$$

Again, we have the following expansion of $f(v_j)$ by adopting the Taylor's series expansion

$$f(v_j) = c_1 (1 + \lambda c_1) e_j + c_2 \left\{ (1 + \lambda c_1)^2 + \lambda c_1 \right\} e_j^2 + \sum_{m=1}^6 G_m e_j^{m+2} + O(e_j^9), \quad (12)$$

where $G_m = G_m(\lambda, c_1, c_2, \dots, c_8)$.

By using the expressions (10) and (12), we have

$$\begin{aligned} y_j - \zeta &= \left(\frac{1}{c_1} + \lambda \right) c_2 e_j^2 + \frac{c_1 c_3 (\lambda^2 c_1^2 + 3\lambda c_1 + 2) - c_2^2 (\lambda^2 c_1^2 + 2\lambda c_1 + 2)}{c_1^2} e_j^3 \\ &+ \sum_{m=1}^5 \bar{G}_m e_j^{m+3} + O(e_j^9). \end{aligned} \quad (13)$$

Once again, the Taylor's series expansion of $f(y_j)$ about $x = \zeta$ provide

$$\begin{aligned} f(y_j) &= c_2 (1 + \lambda c_1) e_j^2 + \frac{c_1 c_3 (\lambda^2 c_1^2 + 3\lambda c_1 + 2) - c_2^2 (\lambda^2 c_1^2 + 2\lambda c_1 + 2)}{c_1} e_j^3 \\ &+ \sum_{m=1}^5 \bar{\bar{G}}_m e_j^{m+3} + O(e_j^9). \end{aligned} \quad (14)$$

With the help of of expressions (10)–(14), we further obtain

$$\begin{aligned} \frac{f(x_j) - f(v_j)}{x_j - v_j} &= c_1 + c_2 (2 + \lambda c_1) e_j + \left\{ c_3 (\lambda^2 c_1^2 + 3\lambda c_1 + 3) + \lambda c_2^2 \right\} e_j^2 \\ &+ \sum_{i=1}^6 H_i e_j^{i+2} + O(e_j^9) \end{aligned} \quad (15)$$

and

$$\frac{f(x_j) - f(y_j)}{x_j - y_j} = c_1 + c_2 e_j + \left(c_2^2 \left(\frac{1}{c_1} + \lambda \right) + c_3 \right) e_j^2 + \sum_{i=1}^6 \bar{H}_i e_j^{i+2} + O(e_j^9), \quad (16)$$

where H_i and \bar{H}_i are the constant functions of some constants λ and c_i , $1 \leq i \leq 8$.

Since we assumed earlier that $\eta(v_j, x_j, y_j)$ is any 4-order optimal derivative free scheme, it is therefore undeniable that it will satisfy the error equation of the following form

$$z_j - \zeta = \tau_1 e_j^4 + \tau_2 e_j^5 + \tau_3 e_j^6 + \tau_4 e_j^7 + \tau_5 e_j^8 + O(e_j^9), \quad (17)$$

where $\tau_1 \neq 0$ and τ_i ($1 \leq i \leq 4$) are asymptotic error constants which may depend on some constants λ and c_i , $1 \leq i \leq 8$.

Now, we obtain the following expansion of $f(z_j)$ about $z = \zeta$

$$f(z_j) = c_1 \tau_1 e_j^4 + c_1 \tau_2 e_j^5 + c_1 \tau_3 e_j^6 + c_1 \tau_4 e_j^7 + (c_2 \tau_1^2 + c_1 \tau_4) e_j^8 + O(e_j^9). \quad (18)$$

By using (10), (17) and (18), we obtain

$$\begin{aligned} \frac{f(x_j) - f(z_j)}{x_j - z_j} &= c_1 + c_2 e_j + c_3 e_j^2 + c_4 e_j^3 + (c_2 \tau_1 + c_5) e_j^4 + (c_3 \tau_1 + c_2 \tau_2 \\ &+ c_6) e_j^5 + (c_4 \tau_1 + (c_2 + c_3) \tau_2 + c_7) e_j^6 + (c_5 \tau_1 + c_3 \tau_2 \\ &+ c_4 \tau_2 + c_2 \tau_3 + c_8) e_j^7 + O(e_j^8). \end{aligned} \quad (19)$$

By using the expressions (10)–(19), we have

$$\begin{aligned} \frac{f(z_j) [(z_j - x_j)^2 \theta_2 + (z_j - x_j) \theta_3 + \theta_4]^2}{\theta_4 - (z_j - x_j)^2 \theta_2} &= \tau_1 e_j^4 + \tau_2 e_j^5 + \tau_3 e_j^6 + \tau_4 e_j^7 \\ &- \frac{c_2 \tau_1 [c_1^3 \tau_1 + (1 + \lambda c_1)^2 (c_1^2 c_4 + c_2^3 - 2c_1 c_2 c_3)]}{c_1^4} e_j^8 + O(e_j^9). \end{aligned} \quad (20)$$

Finally, by inserting the expressions (17) and (20) in the last sub step of scheme (9), we have

$$e_{j+1} = \frac{c_2 \tau_1 [c_1^3 \tau_1 + (1 + \lambda c_1)^2 (c_1^2 c_4 + c_2^3 - 2c_1 c_2 c_3)]}{c_1^4} e_j^8 + O(e_j^9). \quad (21)$$

It is straightforward to say from the expression (21) that the scheme (9) has 8-order convergence. Since the scheme (9) uses only four values of function (viz. $f(x_j)$, $f(v_j)$, $f(y_j)$ and $f(z_j)$) per step, this is therefore an optimal scheme according to the Kung–Traub conjecture. A single coefficient τ_1 from $\eta(x_j, v_j, y_j)$ occurs in the above error equation and also plays an important role in the development of our scheme. Hence, this completes the proof. \square

Remark 1. In general, it is quite obvious that one thinks that the asymptotic error constant in the error equation of scheme (9) may rely on some other constants λ , c_i , $1 \leq i \leq 8$ and τ_j , $1 \leq j \leq 5$. There is no doubt that the expression (21) confirms that the asymptotic error constant is dependent only on λ , c_1, c_2, c_3, c_4 and τ_1 . This clearly demonstrates that our current rational function approach with the tangency constraints contributes a significant role in the construction of a new scheme with 8-order convergence.

4. Numerical Examples

Here, we checked the effectiveness, convergence behavior and efficiency of our schemes with the other existing optimal eighth-order schemes without derivatives. Therefore, we assume that, out of five problems, four of them are from real-life problems, e.g., a fractional conversion problem of the chemical reactor, Van der Waal's problem, the chemical engineering problem and the adiabatic flame temperature problem. The fifth one is a standard nonlinear problem of a piecewise continuous function, which is displayed in the following Examples (1)–(5). The desired solutions are available up to many significant digits (minimum thousand), but, due to the page restriction, only 30 significant places are also listed in the corresponding example.

For comparison purposes, we require the second sub-step in the presented technique. We can choose any optimal derivative free method from the available literature whose first sub-step employs Steffensen's or a Steffensen-type method. Now, we assume some special cases of our scheme that are given as below:

1. We choose an optimal derivative free fourth-order method (6) suggested by Cordero and Torregrosa [3]. Then, we have

$$\begin{cases} y_j = x_j - \frac{f(x_j)^2}{f(x_j + f(x_j)) - f(x_j)}, \\ z_j = y_j - \frac{f(y_j)}{\frac{af(y_j)-bf(v_j)}{y_j-v_j} + \frac{cf(y_j)-df(x_j)}{y_j-x_j}}, \\ x_{j+1} = z_j - \frac{f(z_j) [\theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4]^2}{\theta_4 - \theta_2(z_j - x_j)^2}, \end{cases} \tag{22}$$

where $a, b, c, d \in \mathbb{R}$ such that $a = c = 1$ and $b + d = 1$. We consider $a = b = c = 1$ and $d = 0$ in expression (26) for checking the computational behavior, denoted by (PM1₈).

2. We consider another 4-order optimal method (11) presented by Liu et al. in [8]. Then, we obtain the following new optimal 8-order derivative free scheme

$$\begin{cases} y_j = x_j - \frac{f(x_j)^2}{f(x_j + f(x_j)) - f(x_j)}, \\ z_j = y_j - \frac{f[y_j, x_j] - f[v_j, y_j] + f[v_j, x_j]}{(f[y_j, x_j])^2} f(y_j), \\ x_{j+1} = z_j - \frac{f(z_j) [\theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4]^2}{\theta_4 - \theta_2(z_j - x_j)^2}, \end{cases} \tag{23}$$

Let us call the above expression (PM2₈) for computational experimentation.

3. Once again, we pick expression (12) from a scheme given by Ren et al. in [10]. Then, we obtain another interesting family

$$\begin{cases} y_j = x_j - \frac{f(x_j)^2}{f(x_j + f(x_j)) - f(x_j)}, \\ z_j = y_j - \frac{f(y_j)}{f[y_j, x_j] + f[v_j, y_j] - f[v_j, x_j] + a(y_j - x_j)(y_j - v_j)}, \\ x_{j+1} = z_j - \frac{f(z_j) [(z_j - x_j)^2\theta_2 + (z_j - x_j)\theta_3 + \theta_4]^2}{\theta_4 - (z_j - x_j)^2\theta_2}, \end{cases} \tag{24}$$

where $a \in \mathbb{R}$. We choose $a = 1$ in (30), known as (PM3₈).

4. Now, we assume another 4-order optimal method (12), given by Zheng et al. in [18], which further produces

$$\begin{cases} y_j = x_j - \frac{f(x_j)^2}{f(x_j + f(x_j)) - f(x_j)}, \\ z_j = y_j - \left[\frac{f[y_j, x_j] + (p-1)f[v_j, y_j] - (p-1)f[v_j, x_j] - b(y_j - x_j)(y_j - v_j)}{f[y_j, x_j] + pf[v_j, y_j] - pf[v_j, x_j] + a(y_j - x_j)(y_j - v_j)} \right] \times \frac{f(y_j)}{f[y_j, x_j]}, \\ x_{j+1} = z_j - \frac{f(z_j) [\theta_2(z_j - x_j)^2 + \theta_3(z_j - x_j) + \theta_4]^2}{\theta_4 - \theta_2(z_j - x_j)^2}, \end{cases} \tag{25}$$

where $a, b, p \in \mathbb{R}$. We choose $p = 2$ and $a = b = 0$ in (31), called (PM4₈).

Now, we compare them with iterative methods presented by Kung–Traub [7]. Out of these, we considered an optimal eighth-order method, called KT₈. We also compare them with a derivative free optimal family of 8-order iterative functions given by Kansal et al. [5]. We have picked expression

(23) out of them, known as KM_8 . Finally, we contrast them with the optimal derivative free family of 8-order methods suggested by Soleymani and Vanani [14], out of which we have chosen the expression (21), denoted by SV_8 .

We compare our methods with existing methods on the basis of approximated zeros (x_j), absolute residual error ($|f(x_j)|$), error difference between two consecutive iterations $|x_{j+1} - x_j|$, $\left| \frac{e_{j+1}}{e_j^8} \right|$, asymptotic error constant $\eta = \lim_{n \rightarrow \infty} \left| \frac{e_{j+1}}{e_j^8} \right|$ and computational convergence order $\rho \approx \frac{\ln |\check{e}_{j+1}/\check{e}_j|}{\ln |\check{e}_j/\check{e}_{n-1}|}$, where $\check{e}_j = x_j - x_{n-1}$ (for the details, please see Cordero and Torregrosa [21]) and the results are mentioned in Tables 1–5.

The values of all above-mentioned parameters are available for many significant digits (with a minimum of a thousand digits), but, due to the page restrictions, results are displayed for some significant digits (for the details, please see Tables 1–5). The values of all these parameters have been calculated by adopting programming package *Mathematica* 9 for multiple precision arithmetic. Finally, the meaning of $a_1(\pm a_2)$ is $a_1 \times 10^{(\pm a_2)}$ in the following Tables 1–5.

Table 1. Convergence performance of distinct 8-order optimal derivative free methods for $f_1(x)$.

Cases	j	x_j	$ f(x_j) $	$ x_{j+1} - x_j $	ρ	$\left \frac{x_{j+1} - x_j}{(x_j - x_{n-1})^8} \right $	η
KT_8	1	0.75742117642117592668	2.0(−3)	2.5(−5)			
	2	0.75739624625375387946	1.0(−19)	1.3(−21)		8.387722076(+15)	8.409575862(+15)
	3	0.75739624625375387946	4.0(−150)	5.1(−152)	7.9999	8.409575862(+15)	
KM_8	1	0.75739472392262620965	1.2(−4)	1.5(−6)			
	2	0.75739624625375387946	1.2(−34)	1.5(−36)		5.252005934(+10)	2.765111335(+10)
	3	0.75739624625375387946	6.1(−275)	7.7(−277)	8.0093	2.765111335(+10)	
SV_8	1	0.75726839017571335554	1.0(−2)	1.3(−4)			
	2	0.75739624625375406009	1.4(−14)	1.8(−16)		2.529459671(+15)	1.540728199(+14)
	3	0.75739624625375387946	1.4(−110)	1.7(−112)	8.1026	1.540728199(+14)	
$PM1_8$	1	0.75739624679631343572	4.3(−8)	5.4(−10)			
	2	0.75739624625375387946	7.9(−60)	9.9(−62)		1.318011692(+13)	1.318013290(+13)
	3	0.75739624625375387946	9.7(−474)	1.2(−475)	8.0000	1.318013290(+13)	
$PM2_8$	1	0.75739624527627277118	7.8(−8)	9.8(−10)			
	2	0.75739624625375387946	5.3(−58)	6.7(−60)		8.002563231(+12)	8.002546457(+12)
	3	0.75739624625375387946	2.5(−459)	3.1(−461)	8.0000	8.002546457(+12)	
$PM3_8$	1	0.75739624669712714014	3.5(−8)	4.4(−10)			
	2	0.75739624625375387946	1.6(−60)	2.0(−62)		1.316590806(+13)	1.316592111(+13)
	3	0.75739624625375387946	2.3(−479)	2.9(−481)	8.0000	1.316592111(+13)	
$PM4_8$	1	0.75739625664695918279	8.3(−7)	1.0(−8)			
	2	0.75739624625375387946	1.7(−49)	2.1(−51)		1.522844707(+13)	1.522886893(+13)
	3	0.75739624625375387946	4.1(−391)	5.2(−393)	8.0000	1.522886893(+13)	

Table 2. Convergence performance of distinct 8-order optimal derivative free methods for $f_2(x)$.

Cases	j	x_j	$ f(x_j) $	$ e_j $	ρ	$\left \frac{e_{j+1}}{e_j^8} \right $	η
KT_8	1	1.9299358075659180242	7.7(−6)	9.0(−5)	7.9999	2.570367432(+6)	2.580781373(+6)
	2	1.9298462428478622185	9.2(−28)	1.1(−26)			
	3	1.9298462428478622185	3.7(−203)	4.3(−202)			
KM_8	1	1.9300063313329939091	1.4(−5)	1.6(−4)	8.0002	1.872886840(+6)	1.859196359(+6)
	2	1.9298462428478622185	7.0(−26)	8.1(−25)			
	3	1.9298462428478622185	2.9(−188)	3.4(−187)			
SV_8	1	1.9299298655571245217	7.2(−6)	8.4(−5)	8.0154	1.272677056(+4)	5.345691399(+3)
	2	1.9298462428478622185	2.6(−30)	3.0(−29)			
	3	1.9298462428478622185	3.4(−226)	3.9(−225)			
$PM1_8$	1	1.9298703396056890283	2.1(−6)	2.4(−5)	8.0000	3.292189981(+5)	3.294743419(+5)
	2	1.9298462428478622185	3.2(−33)	3.7(−32)			
	3	1.9298462428478622185	1.1(−247)	1.3(−246)			
$PM2_8$	1	1.9299039277100182896	5.0(−6)	5.8(−5)	8.0000	1.415845181(+6)	1.419322205(+6)
	2	1.9298462428478622185	1.5(−29)	1.7(−28)			
	3	1.9298462428478622185	1.0(−217)	1.2(−216)			
$PM3_8$	1	1.9298835516272248348	3.2(−6)	3.7(−5)	8.0000	6.132728979(+5)	6.140666943(+5)
	2	1.9298462428478622185	2.0(−31)	2.3(−30)			
	3	1.9298462428478622185	4.2(−233)	4.8(−232)			
$PM4_8$	1	1.9298454768935056951	6.6(−8)	7.7(−7)	8.0000	1.600600022(+4)	1.600542542(+4)
	2	1.9298462428478622185	1.6(−46)	1.9(−45)			
	3	1.9298462428478622185	2.3(−355)	2.7(−354)			

Table 3. Convergence performance of distinct 8-order optimal derivative free methods for $f_3(x)$.

Cases	j	x_j	$ f(x_j) $	$ e_j $	ρ	$\left \frac{e_{j+1}}{e_j^8} \right $	η
KT_8	1	3.94856259325568 + 0.31584953607444i	2.8(−3)	2.7(−4)	7.999	3.278944412(+6)	3.291035449(+6)
	2	3.94854244556204 + 0.31612357089701i	1.1(−21)	1.1(−22)			
	3	3.94854244556204 + 0.31612357089701i	5.5(−169)	5.5(−170)			
KM_8	1	3.94541341953964 + 0.28830540896626i	2.7(−1)	2.8(−2)	8.3214	5.611004628(+5)	1.267588109(+4)
	2	3.94854253806613 + 0.31612376121596i	2.1(−6)	2.1(−7)			
	3	3.94854244556204 + 0.31612357089701i	5.2(−49)	5.1(−50)			
SV_8	1	3.94857741336794 + 0.31574108761478i	3.9(−3)	3.8(−4)	8.0000	1.895162520(+6)	1.896706799(+6)
	2	3.94854244556204 + 0.31612357089701i	9.1(−21)	9.0(−22)			
	3	3.94854244556204 + 0.31612357089701i	8.1(−162)	8.0(−163)			
$PM1_8$	1	3.94848048827814 + 0.31602117152370i	1.2(−3)	1.2(−4)	8.0001	5.923125406(+5)	5.903970786(+5)
	2	3.94854244556204 + 0.31612357089701i	2.5(−25)	2.5(−26)			
	3	3.94854244556204 + 0.31612357089701i	8.9(−199)	8.8(−200)			
$PM2_8$	1	3.94846874984553 + 0.31601667713734i	1.3(−3)	1.3(−4)	8.0001	6.241093912(+5)	6.214835024(+5)
	2	3.94854244556204 + 0.31612357089701i	5.1(−25)	5.0(−26)			
	3	3.94854244556204 + 0.31612357089701i	2.6(−196)	2.6(−197)			
$PM3_8$	1	3.94848290176499 + 0.31601668833975i	1.2(−3)	1.2(−4)	8.0001	6.078017700(+5)	6.059534898(+5)
	2	3.94854244556204 + 0.31612357089701i	3.1(−25)	3.1(−26)			
	3	3.94854244556204 + 0.31612357089701i	4.6(−198)	4.6(−199)			
$PM4_8$	1	3.94849208916059 + 0.31602400692668i	1.1(−3)	1.1(−4)	8.0000	5.704624073(+5)	5.691514905(+5)
	2	3.94854244556204 + 0.31612357089701i	1.4(−25)	1.4(−26)			
	3	3.94854244556204 + 0.31612357089701i	7.1(−201)	7.1(−202)			

Table 4. Convergence performance of distinct 8-order optimal derivative free methods for $f_4(x)$.

Cases	j	x_j	$ f(x_j) $	$ e_j $	ρ	$\left \frac{e_{j+1}}{e_j^8} \right $	η
KT_8	1	4305.3099136661255630	3.3(−19)	1.5(−20)	8.0000	2.234387851(−22)	2.234387851(−22)
	2	4305.3099136661255630	1.1(−179)	4.8(−181)			
	3	4305.3099136661255630	1.4(−1463)	6.3(−1465)			
KM_8	1	4305.4966166546986926	4.2	1.9(−1)	9.5830	8.978735581(−4)	1.132645694(−16)
	2	4305.30991366647999238	3.0(−8)	1.3(−9)			
	3	4305.3099136661255630	2.4(−86)	1.1(−87)			
SV_8	1	4305.3099136661255630	1.5(−19)	6.9(−21)	8.0000	1.038308478(−22)	1.038308478(−22)
	2	4305.3099136661255630	1.2(−182)	5.4(−184)			
	3	4305.3099136661255630	1.6(−1487)	7.1(−1489)			
$PM1_8$	1	4305.3099136661255630	3.5(−20)	1.6(−21)	8.0000	2.393094045(−23)	2.393094045(−23)
	2	4305.3099136661255630	2.1(−188)	9.3(−190)			
	3	4305.3099136661255630	3.1(−1534)	1.4(−1535)			
$PM2_8$	1	4305.3099136661255630	4.0(−20)	1.8(−21)	8.0000	2.683028981(−23)	2.683028981(−23)
	2	4305.3099136661255630	5.8(−188)	2.6(−189)			
	3	4305.3099136661255630	1.3(−1530)	5.6(−1532)			
$PM3_8$	1	4305.3099136690636946	6.6(−8)	2.8(−9)	8.0000	7.055841652(−10)	7.055841652(−10)
	2	4305.3099136661255630	8.8(−77)	3.9(−78)			
	3	4305.3099136661255630	8.8(−628)	3.9(−629)			
$PM4_8$	1	4305.3099136661255630	4.0(−20)	1.8(−21)	8.0000	2.119306545(−23)	2.119306545(−23)
	2	4305.3099136661255630	5.8(−188)	2.6(−189)			
	3	4305.3099136661255630	1.3(−1530)	5.6(−1532)			

Table 5. Convergence performance of distinct 8-order optimal derivative free methods for $f_5(x)$.

Cases	j	x_j	$ f(x_j) $	$ e_j $	ρ	$\left \frac{e_{j+1}}{e_j^8} \right $	η
KT_8	1	1.4142135646255204265	6.4(−9)	2.3(−9)	8.0000	1.483428355	1.483428382
	2	1.4142135623730950488	2.8(−69)	9.8(−70)			
	3	1.4142135623730950488	3.7(−552)	1.3(−552)			
KM_8	1	1.4141886104951680577	7.1(−5)	2.5(−5)	14.972	1.171425936(+28)	0.1339769256
	2	1.4142135641342028617	5.0(−9)	1.8(−9)			
	3	1.4142135623730950488	3.5(−71)	1.2(−71)			
SV_8	1	1.4142135639458229191	4.4(−9)	1.6(−9)	8.0000	0.7923194647	0.7923194693
	2	1.4142135623730950488	8.4(−71)	3.0(−71)			
	3	1.4142135623730950488	1.3(−564)	4.7(−564)			
$PM1_8$	1	1.4142135629037874832	1.5(−9)	5.3(−10)	8.0000	0.2966856754	0.2966856763
	2	1.4142135623730950488	5.3(−75)	1.9(−75)			
	3	1.4142135623730950488	1.2(−598)	4.4(−599)			
$PM2_8$	1	1.4142135630941303743	2.0(−9)	7.2(−10)	8.0000	0.4230499025	0.4230499045
	2	1.4142135623730950488	8.7(−74)	3.1(−74)			
	3	1.4142135623730950488	1.0(−588)	3.5(−589)			
$PM3_8$	1	1.4142135672540404368	1.4(−8)	4.9(−9)	8.0000	2.742159025	2.742159103
	2	1.4142135623730950488	2.5(−66)	8.8(−67)			
	3	1.4142135623730950488	2.9(−528)	1.0(−528)			
$PM4_8$	1	1.4142135627314914846	1.0(−9)	3.6(−10)	8.0000	0.1905635592	0.1905635596
	2	1.4142135623730950488	1.5(−76)	5.2(−77)			
	3	1.4142135623730950488	2.8(−611)	1.0(−611)			

Example 1. Chemical reactor problem:

In regard to fraction transformation in a chemical reactor, we consider

$$f_1(x) = \frac{x}{1-x} - 5 \log \left[\frac{0.4(1-x)}{0.4-0.5x} \right] + 4.45977, \tag{26}$$

where the variable x denotes a fractional transformation of a particular species A in the chemical reactor problem (for a detailed explanation, please have a look at [22]). It is important to note that, if $x \leq 0$, then the expression (26)

has no physical meaning. Hence, this expression has only a bounded region $0 \leq x \leq 1$, but its derivative is approaching zero in the vicinity of this region. Therefore, we have to take care of these facts while choosing required zero and initial approximations, which we consider as $\zeta = 0.757396246253753879459641297929$ and $x_0 = 0.76$, respectively:

Example 2. Van der Waal's equation:

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT. \quad (27)$$

The above expression interprets real and ideal gas behavior with variables a_1 and a_2 , respectively. For calculating the gas volume V , we can rewrite the above expression (27) in the following way:

$$PV^3 - (na_2P + nRT)V^2 + a_1n^2V - a_1a_2n^2 = 0. \quad (28)$$

By considering the particular values of parameters, namely a_1 and a_2 , n , P and T , we can easily get the following nonlinear function:

$$f_2(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289. \quad (29)$$

The function f_2 has three zeros and our required zero is $\zeta = 1.92984624284786221848752742787$. In addition, we consider the initial guess as $x_0 = 2$.

Example 3. If we convert the fraction of nitrogen–hydrogen to ammonia, then we obtain the following mathematical expression (for more details, please see [23,24])

$$f_3(z) = z^4 - 7.79075z^3 + 14.7445z^2 + 2.511z - 1.674. \quad (30)$$

The f_3 has four zeros and our required zero is $\zeta = 3.9485424455620457727 + 0.3161235708970163733i$. In addition, we consider the initial guess as $x_0 = 4 + 0.25i$.

Example 4. Let us assume an adiabatic flame temperature equation, which is given by

$$f_4(x) = \Delta H + \alpha(x - 298) + \frac{\beta}{2}(x^2 - 298^2) + \frac{\gamma}{3}(x^3 - 298^3), \quad (31)$$

where $\Delta H = -57798$, $\alpha = 7.256$, $\beta = 2.298 \times 10^{-3}$ and $\gamma = 0.283 \times 10^{-6}$. For the details of this function, please see the research articles [24,25]. This function has a simple zero $\zeta = 4305.30991366612556304019892945$ and assumes the initial approximation is $x_0 = 4307$ for this problem.

Example 5. Finally, we assume a piece-wise continuous function [5], which is defined as follows:

$$f_5(x) = \begin{cases} -(x^2 - 2), & \text{if } x < \sqrt{2}, \\ x^2 - 2, & \text{if } x \geq \sqrt{2}. \end{cases} \quad (32)$$

The above function has a simple zero $\zeta = 1.41421356237309504880168872421$ with an initial guess being $x_0 = 1.5$.

5. Graphical Comparison by Means of Attraction Basins

It is known that a good selection of initial guesses plays a definitive role in iterative methods—in other words, that all methods converge if the initial estimation is chosen suitably. We numerically approximate the domain of attraction of the zeros as a qualitative measure of how demanding the method on the initial approximation of the root is. In order to graphically compare

by means of attraction basins, we investigate the dynamics of the new methods $PM1_8$, $PM2_8$, $PM3_8$ and $PM4_8$ and compare them with available methods from the literature, namely SM_8 , KT_8 and KM_8 . For more details and many other examples of the study of the dynamic behavior for iterative methods, one can consult [26–29].

Let $Q : \mathbb{C} \rightarrow \mathbb{C}$ be a rational map on the complex plane. For $z \in \mathbb{C}$, we define its orbit as the set $\text{orb}(z) = \{z, Q(z), Q^2(z), \dots\}$. A point $z_0 \in \mathbb{C}$ is called a periodic point with minimal period m if $Q^m(z_0) = z_0$, where m is the smallest positive integer with this property (and thus $\{z_0, Q(z_0), \dots, Q^{m-1}(z_0)\}$ is a cycle). The point having minimal period 1 is known as a fixed point. In addition, the point z_0 is called repelling if $|Q'(z_0)| > 1$, attracting if $|Q'(z_0)| < 1$, and neutral otherwise. The Julia set of a nonlinear map $Q(z)$, denoted by $J(Q)$, is the closure of the set of its repelling periodic points. The complement of $J(Q)$ is the Fatou set $F(Q)$.

In our case, the methods $PM1_8$, $PM2_8$, $PM3_8$ and $PM4_8$ and SM_8 , KT_8 and KM_8 provide the iterative rational maps $Q(z)$ when they are applied to find the roots of complex polynomials $p(z)$. In particular, we are interested in the basins of attraction of the roots of the polynomials where the basin of attraction of a root z^* is the complex set $\{z_0 \in \mathbb{C} : \text{orb}(z_0) \rightarrow z^*\}$. It is well known that the basins of attraction of the different roots lie in the Fatou set $F(Q)$. The Julia set $J(Q)$ is, in general, a fractal and, in it, the rational map Q is unstable.

For a graphical point of view, we take a 512×512 grid of the square $[-3, 3] \times [-3, 3] \subset \mathbb{C}$ and assign a color to each point $z_0 \in D$ according to the simple root to which the corresponding orbit of the iterative method starting from z_0 converges, and we mark the point as black if the orbit does not converge to a root in the sense that, after at most 15 iterations, it has a distance to any of the roots that is larger than 10^{-3} . We have used only 15 iterations because we are using eighth-order methods. Therefore, if the method converges, it is usually very fast. In this way, we distinguish the attraction basins by their color.

Different colors are used for different roots. In the basins of attraction, the number of iterations needed to achieve the root is shown by the brightness. Brighter color means less iteration steps. Note that black color denotes lack of convergence to any of the roots. This happens, in particular, when the method converges to a fixed point that is not a root or if it ends in a periodic cycle or at infinity. Actually and although we have not done it in this paper, infinity can be considered an ordinary point if we consider the Riemann sphere instead of the complex plane. In this case, we can assign a new “ordinary color” for the basin of attraction of infinity. Details for this idea can be found in [30].

We have tested several different examples, and the results on the performance of the tested methods were similar. Therefore, we merely report the general observation here for two test problems in the following Table 6.

Table 6. Test problems p_1 and p_2 and their roots.

Test Problem	Roots
$p_1(z) = z^2 - 1$	1, -1
$p_2(z) = z^2 - z - 1/z$	1.46557, $-0.232786 \pm 0.792552i$

From Figures 1 and 2, we conclude that our methods, namely, $PM1_8$, $PM3_8$ and $PM4_8$, are showing less chaotic behavior and have less non-convergent points as compared to the existing methods, namely SM_8 and KM_8 . In addition, our methods, namely, $PM1_8$, $PM3_8$ and $PM4_8$, have almost similar basins of attraction to KT_8 . On the other hand, Figures 3 and 4 confirm that our methods, namely, $PM1_8$, $PM2_8$, $PM3_8$ and $PM4_8$, have less divergent points as compared to the existing methods, namely KT_8 and KM_8 . There is no doubt that the SM_8 behavior is better than all other mentioned methods, namely, $PM1_8$, $PM3_8$ and $PM4_8$ in problem $p_2(z)$ in terms of chaos.

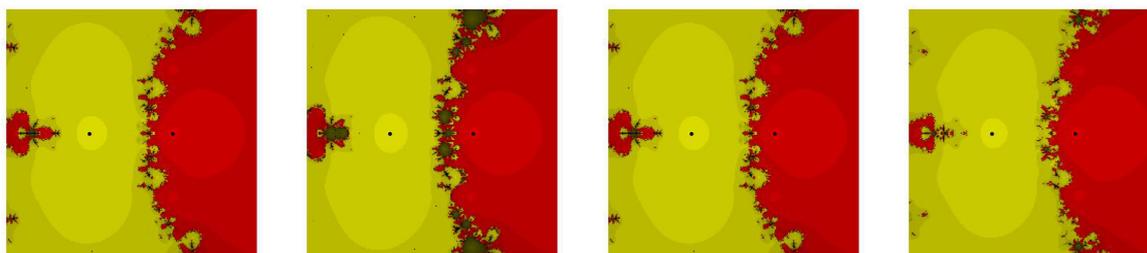


Figure 1. The dynamical behavior of our methods namely, PM_{18} , PM_{28} , PM_{38} and PM_{48} , respectively, from left to right for test problem $p_1(z)$.

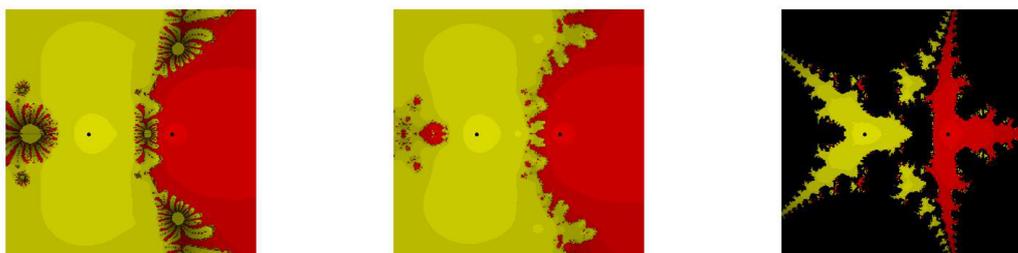


Figure 2. The dynamical behavior of methods SM_8 , KT_8 and KM_8 , respectively, from left to right for test problem $p_1(z)$.

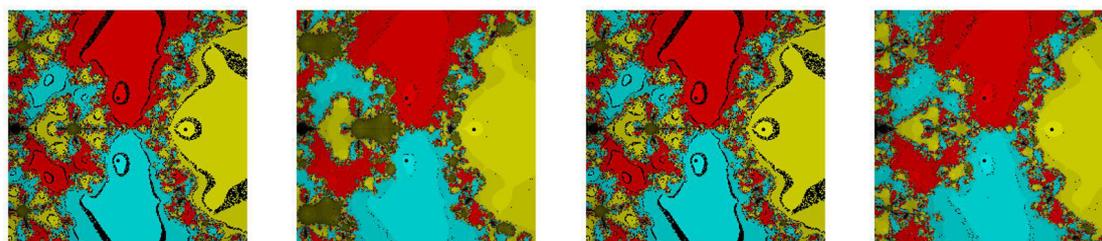


Figure 3. The dynamical behavior of our methods namely, PM_{18} , PM_{28} , PM_{38} and PM_{48} , respectively, from left to right for test problem $p_2(z)$.

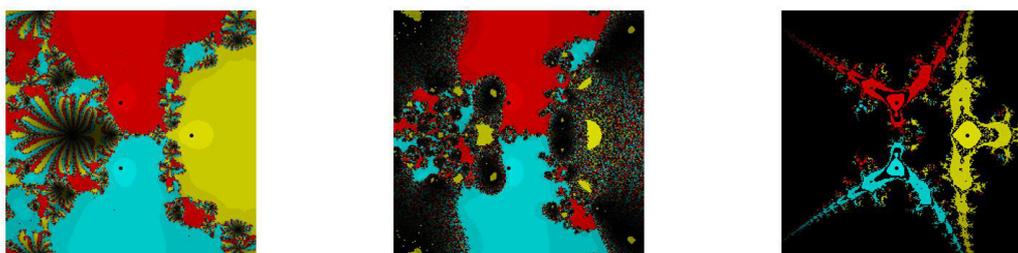


Figure 4. The dynamical behavior of methods SM_8 , KT_8 and KM_8 , respectively from left to right for test problem $p_2(z)$.

6. Conclusions

In this study, we present a new technique of eighth-order in a general way. The main advantages of our technique are that is a derivative free scheme, there is a choice of flexibility at the second substep, and it is capable of generating new 8-order derivative free schemes from every optimal 4-order method employing Steffensen's or Steffensen-type methods. Every member of (9) is an optimal method according to Kung–Traub conjecture. It is clear from the obtained results in Tables 1–5 that our methods have minimum residual error $|f(x_j)|$, the difference between two consecutive iterations $|x_{j+1} - x_j|$, and stable computational convergence order as compared to existing methods, namely, SM_8 , KT_8 and KM_8 . The dynamic study of our methods also confirms that they perform better than existing ones of similar order.

It is important to note that we are not claiming that our methods will always be superior to these methods. One may obtain different results when they rest them on distinct nonlinear functions because the computational results depend on several constraints, including initial approximation, body structure of the iterative method, the considered test problem, configuration of the used system and programming softwares, etc. In future work, we will try to obtain a new family of high-order optimal derivative free iteration functions that depend on the rational functional approach.

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