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Distributional Chaoticity of C_0-Semigroup on a Frechet Space

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Abstract: This paper is mainly concerned with distributional chaos and the principal measure of C_0-semigroups on a Frechet space. New definitions of strong irregular (semi-irregular) vectors are given. It is proved that if C_0-semigroup $\mathcal{T}$ has strong irregular vectors, then $\mathcal{T}$ is distributional chaos in a sequence, and the principal measure $\mu_p(\mathcal{T})$ is 1. Moreover, $\mathcal{T}$ is distributional chaos equivalent to that operator T_t is distributional chaos for every $\forall t > 0$.

Keywords: Frechet spaces; C_0-semigroups; distributional chaos; strong irregular vector

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1. Introduction

Chaotic properties of dynamical systems have been ardently studied since the term chaos (namely, Li-Yorke chaos) was defined in 1975 by Li and Yorke [1]. To describe unpredictability in the evolution of dynamical systems, many properties related to chaos have been discussed (for example, References [2–13], where References [4–7] are some of our works done in recent years). In 1994, Schweizer and Smital in Reference [8] introduced a popular concept named distributional chaos for interval maps, by considering the dynamics of pairs with some statistical properties. The goal was to extend the definition of Li-Yorke chaos, and it was equivalent to positive topological entropy. Later, Reference [9] summarizes the connections between Li-Yorke, distributional, and ω-chaos. The notions of distributional chaos and principal measures were extended to general dynamical systems [10,11] and especially to the framework of linear dynamics in the last few years. It seems that the first example of a distributional chaotic operator on a Frechet space was given by Oprocha [14], whom investigated the annihilation operator of a quantum harmonic oscillator. Wu and Zhu [15] further proved that the principal measure of the annihilation operator studied in Reference [14] is 1. Since then, distributional chaos for linear operators has been studied by many authors, see for instance References [16–21].

The study of hypercyclicity and chaoticity for operators and C_0-semigroups has became a hot and active research area in the past two decades (such as References [22,23]). In Reference [24], Devaney chaos for C_0-semigroup of unbounded operators was discussed. The extension of distributional chaos to C_0-semigroup on weighted spaces of integrable functions was done in Reference [25]. Devaney chaos and distributional chaos are closely tied for the C_0-semigroup. Distributionally chaotic C_0-semigroups on Banach spaces were found in Reference [16]. A systematic investigation of distributional chaos for linear operators on Frechet space was given by Bernardes [17]. Recently, an extension of distributional chaos for a family of operators (including C_0-semigroups) on Frechet spaces were proposed by Conejero [26]. For other studies of C_0-semigroups or Frechet spaces see References [27–34] and others.

In the present work, in Section 2 we deal with the notion of strong irregular (semi-irregular) vectors for C_0-semigroups of operators on a Frechet spaces. It is proved that if a C_0-semigroup $\mathcal{T}$ on a Frechet space admits a strong irregular vector, then $\mathcal{T}$ is distributionally chaotic in a sequence, and the principal measure $\mu_p(\mathcal{T})$ is 1. In Section 3, using the properties of the upper density and lower density, we point out that the distributional chaoticity of $\mathcal{T}$ is equivalent to the distributional chaoticity of T_t $(\forall t > 0)$.

Throughout this paper, the set of natural numbers is denoted by $\mathbb{N} = \{1, 2, 3, \cdots\}$ and the set of positive real numbers is denoted by $\mathbb{R}^+ = (0, +\infty)$.

2. Preliminaries

The Frechet space in this paper is a vector space X, endowed with a separating increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms in the following metric.

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\|x - y\|_k}{1 + \|x - y\|_k}, \ \forall x, y \in X.$$

Throughout this paper, the Frechet space is denote by $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$ (or simply X) without otherwise statements and we let $\mathcal{L}(X)$ be the space of continuous linear operators on X.

One parameter family $\mathcal{T} = \{T_t\}_{t\geq 0} \subseteq \mathcal{L}(X)$ is called a C_0-semigroup of linear operators on X if:

(i) $T_0 = I$ (where I is the identity operator on X);
(ii) $T_t T_s = T_{t+s}$, $\forall s, t \geq 0$;
(iii) $\lim\limits_{s\to t} T_s(x) = T_t(x)$, $\forall x \in X$, $\forall s, t \geq 0$.

In References [17,33], Peris et al. introduced the notions of an irregular vector and a distributional irregular vector for operators in order to characterize distributional chaos. Similarly, we give notions of a strong irregular vector and strong semi-irregular vector.

$x \in X$ is called a strong irregular vector for a C_0-semigroup $\mathcal{T}$ on a Frechet space X if

$$\limsup_{t\to\infty} \|T_t x\|_k = \infty \quad and \quad \liminf_{t\to\infty} \|T_t x\|_k = 0$$

for every $k \in \mathbb{N}$.

$x \in X$ is called a strong semi-irregular vector, if

$$\limsup_{t\to\infty} \|T_t x\|_k = \infty \quad but \quad \liminf_{t\to\infty} \|T_t x\|_k \neq 0 \quad for\, some\, k \in \mathbb{N}$$

and there exists a sequence $\{T_{t_i}\}_{i\in\mathbb{N}}$ such that

$$\liminf_{i\to\infty} \|T_{t_i} x\|_k = 0$$

for every $k \in \mathbb{N}$.

3. Distributional Chaos in a Sequence of C_0-Semigroup

For any $x, y \in X$ and a sequence $\{p_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, we define the distributional function in a sequence of x and y with respect to $\mathcal{T} = \{T_t\}_{t\geq 0}$ as:

$$\Phi_{xy}^n : \mathbb{R}^+ \to [0, 1]$$
$$\Phi_{xy}^n(\varepsilon) = \frac{1}{n} card\{1 \leq i \leq n : \rho(T_{p_i}(x), T_{p_i}(y)) < \varepsilon\} \ (\forall \varepsilon \in \mathbb{R}^+)$$

where $card\{M\}$ denotes the cardinality of the set M (or denoted by $|M|$).

The upper and lower distributional functions of x and y are then defined by

$$\Phi_{xy}^*(\varepsilon, \{p_i\}_{i\in\mathbb{N}}) = \limsup_{n\to\infty} \Phi_{xy}^n(\varepsilon) \quad and \quad \Phi_{xy}(\varepsilon, \{p_i\}_{i\in\mathbb{N}}) = \liminf_{n\to\infty} \Phi_{xy}^n(\varepsilon)$$

respectively for $\forall\varepsilon > 0$.

Definition 1. *Let $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$ be a Frechet space. A C_0-semigroup of operators $\mathcal{T} = \{T_t\}_{t\geq 0}$ on X is said to be distributionally chaotic in a sequence if there exists a sequence $\{p_i\}_{i\in\mathbb{N}}$, an uncountable subset of $S \subset X$ and $\delta > 0$ such that for $\forall x, y \in S : x \neq y$ and $\forall\varepsilon > 0$, we have that:*

$$\Phi^*_{xy}(\varepsilon, \{p_i\}_{i\in\mathbb{N}}) = 1 \quad and \quad \Phi_{xy}(\delta, \{p_i\}_{i\in\mathbb{N}}) = 0.$$

In this case, S is called a distributionally δ-scrambled set in a sequence, and (x, y) is called a distributionally chaotic pair in a sequence.

To measure the degree of chaos for a given dynamical system, the concept of principal measure was introduced for general dynamical systems accompanying the appearance of distributional chaos [8,11]. For the study of principal measures of certain linear operators, we refer to References [14,15,35]. Naturally, the concept for the case of C_0-semigroup of operators on Frechet spaces can be extended.

Definition 2. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Frechet space X. The principal measure $\mu_p(\mathcal{T})$ of $\mathcal{T}$ is defined as follows:*

$$\mu_p(\mathcal{T}) = \sup_{x\in X} \frac{1}{diam(X)} \int_0^\infty (\Phi^*_{x,0}(s) - \Phi_{x,0}(s))ds\,,$$

where $\Phi^*_{x,0}(s)$ and $\Phi_{x,0}(s)$ are the upper and lower distributional functions of x and 0, and $diam(X)$ is the diameter of X.

Now we shall establish the relationship between strong irregular vectors and the distributional chaos of the C_0-semigroup of operators on Frechet spaces.

Theorem 1. *Let $\mathcal{T}$ is a C_0-semigroup on a Frechet space X. If $\mathcal{T}$ admits a strong irregular vector, then $\mathcal{T}$ is distributionally chaotic in a sequence.*

Proof. Let $x \in X$. Since $\mathcal{T}$ admits a strong irregular vector, there exists two increasing sequences $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\lim_{j\to\infty} \|T_{n_j}(x)\|_k = \infty \quad and \quad \lim_{j\to\infty} \|T_{m_j}(x)\|_k = 0$$

for every $k \in \mathbb{N}$.

Let

$b_1 = l_1 = 2$, $b_2 = 2^{b_1}$, $b_3 = 2^{b_1+b_2}, \cdots, b_i = 2^{b_1+\cdots+b_{i-1}} = 2^{\sum_{k=1}^{i-1} b_k}$ for all $i > 1$;

$l_2 = b_1 + b_2, l_3 = b_1 + b_2 + b_3, \cdots, l_i = \sum_{h=1}^{i} b_h$ for all $i > 1$.

$\{n_j{}'\}_{j\in\mathbb{N}}$ and $\{m_j{}'\}_{j\in\mathbb{N}}$ are, respectively, the subsequence of $\{n_j\}_{j\in\mathbb{N}}$ and $\{m_j\}_{j\in\mathbb{N}}$ such that $m_j{}' < n_j{}'$ when $j \leq b_1$ or $l_{2s} < j < l_{2s+1}$, and $n_j{}' < m_j{}'$ when $l_{2s-1} < j < l_{2s}$ for any $s \in \mathbb{N}$.

Let

$$p_j = \begin{cases} n_j{}' & j \leq b_1 \, or \, l_{2s} < j < l_{2s+1}, \quad s \in \mathbb{N} \\ m_j{}' & l_{2s-1} < j < l_{2s}, \quad s \in \mathbb{N} \end{cases}$$

then, $\{p_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ is an increasing sequence.

Denote $\Gamma = \{\alpha x : \alpha \in (0,1)\}$. The following prove that Γ is a distributional δ-scrambled set of $\mathcal{T}$ in $\{p_j\}_{j\in\mathbb{N}}$ for some $\delta > 0$.

In fact, for any pair $x, y \in \Gamma$ with $x \neq y$, it is clear that there exists $\alpha \in (0,1)$ such that $x - y = \alpha x$.

Since $\lim_{j\to\infty}\|T_{m_j}(x)\|_k = 0$ $(\forall k \in \mathbb{N})$, then, for $\forall\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|T_{m_j{}'}(\alpha x)\|_k < \varepsilon$ for each $j \geq N$. Then, $\forall k \in \mathbb{N}$,

$$\frac{\|T_{p_j}(x) - T_{p_j}(y)\|_k}{1+\|T_{p_j}(x) - T_{p_j}(y)\|_k} = \frac{\|T_{p_j}(x-y)\|_k}{1+\|T_{p_j}(x-y)\|_k} = \frac{\alpha\|T_{p_j}\|_k}{1+\alpha\|T_{p_j}\|_k} < \frac{\alpha\varepsilon}{1+\alpha\varepsilon}$$

So,

$$\begin{aligned}\Phi^*_{xy}(\varepsilon,\{p_j\}_{j\in\mathbb{N}}) &= \limsup_{n\to\infty}\tfrac{1}{n}card\{1\leq j\leq n:\rho(T_{p_j}(x),T_{p_j}(y))<\varepsilon\}\\ &= \limsup_{n\to\infty}\tfrac{1}{n}card\{1\leq j\leq n:\sum_{k=1}^{\infty}\tfrac{1}{2^k}\tfrac{\left\|T_{p_j}(x)-T_{p_j}(y)\right\|_k}{1+\left\|T_{p_j}(x)-T_{p_j}(y)\right\|_k}<\varepsilon\}\\ &\geq \limsup_{n\to\infty}\tfrac{1}{n}card\{1\leq j\leq n:\sum_{k=1}^{\infty}\tfrac{1}{2^k}\tfrac{\alpha x}{1+\alpha x}<\varepsilon\}\\ &= \limsup_{s\to\infty}\tfrac{1}{l_{2s}}card\{1\leq j\leq l_{2s}:\sum_{k=1}^{\infty}\tfrac{1}{2^k}\tfrac{\alpha x}{1+\alpha x}<\varepsilon\}\\ &\geq \limsup_{s\to\infty}\tfrac{b_{2s}}{l_{2s}}\\ &= \limsup_{s\to\infty}\tfrac{2^{b_1+b_2+\cdots+b_{2s-1}}}{\sum_{h=1}^{2s-1}b_h+2^{b_1+b_2+\cdots+b_{2s-1}}}\\ &= 1.\end{aligned} \tag{1}$$

Let $\delta = 1$.

Since $\lim_{j\to\infty}\|T_{n_j}(x)\|_k = \infty$ $(\forall k \in \mathbb{N})$, there exists $M \in \mathbb{N}$ such that $\|T_{n_j{}'}(\alpha x)\|_k > \delta(\forall k \in \mathbb{N})$ for each $j \geq M$.

Thus,

$$\begin{aligned}\Phi_{xy}(\delta,\{p_j\}_{j\in\mathbb{N}}) &= \liminf_{n\to\infty}\tfrac{1}{n}card\{1\leq j\leq n:\rho(T_{p_j}(x),T_{p_j}(y))<\delta\}\\ &= \liminf_{s\to\infty}\tfrac{1}{l_{2s+1}}card\{1\leq j\leq l_{2s+1}:\sum_{k=1}^{\infty}\tfrac{1}{2^k}\tfrac{\alpha\delta}{1+\alpha\delta}<\delta\}\\ &\leq \liminf_{s\to\infty}\tfrac{l_{2s}}{l_{2s+1}}\\ &= \liminf_{s\to\infty}\tfrac{\sum_{h=1}^{2s}b_h}{\sum_{h=1}^{2s}b_h+2^{b_1+b_2+\cdots+b_{2s}}}\\ &= 0.\end{aligned} \tag{2}$$

By (1) and (2), $\Gamma = \{\alpha x : \alpha \in (0,1)\}$ is a distributionally δ-scrambled set of Γ in $\{p_j\}_{j\in\mathbb{N}}$. So, Γ is distributionally chaotic in a sequence as Γ is uncountable.

This completes the proof. □

As an important class of operators in linear dynamics, the backward shift [35,36] admits principal measure 1 if it is distributionally chaotic. In addition, it is easy to see that every distributionally chaotic operator on a Banach space (as a special Frechet space) has a principal measure of 1. So we wonder whether the C_0-semigroup on the Frechet space above with a principal measure of 1 is distributionally chaotic. The answer is positive.

Theorem 2. *Let $\mathcal{T}$ be a C_0-semigroup of operators on a Frechet space X. Assume that $\mathcal{T}$ admits a strong irregular vector X_0, then the principal measure $\mu_p(\mathcal{T}) = 1$.*

Proof. From the definition of a strong irregular vector, for every $k \in \mathbb{N}$, one has:

$$\liminf_{t\to\infty}\|T_t(x_0)\|_k = 0 \text{ and } \limsup_{t\to\infty}\|T_t(x_0)\|_k = \infty.$$

Given arbitrary $\varepsilon \in (0,1)$, one can find a sequence $\{t_i^\varepsilon\}_{i\in\mathbb{N}} \in \mathbb{R}^+$ and a positive number N_1 such that $\rho(T_{t_i^\varepsilon}(x_0), 0) < \varepsilon$ for all $\{t_i^\varepsilon : t_i^\varepsilon \in \{t_1^\varepsilon, t_2^\varepsilon, \cdots\}, t_i^\varepsilon > N_1\}$.

$$\begin{aligned}\Phi^*_{x_0,0}(\varepsilon,\{t_1^\varepsilon\}_{i\in\mathbb{N}}) &= \limsup_{n\to\infty}\tfrac{1}{n}card\{1\le i\le n:\rho(T_{t_i^\varepsilon}(x_0),0)<\varepsilon\}\\ &\ge \limsup_{n\to\infty}\tfrac{1}{n}(n-N_1)\\ &=1.\end{aligned}$$

On the other hand, we show that $\forall k\in\mathbb{N}$, $\Phi_{x_0,0}(\varepsilon)=0$ for every $\varepsilon\in(0,diam(X))$.

In fact, given $\forall\varepsilon\in(0,diam(X))$. Since $\limsup\limits_{t\to\infty}\|T_t(x_0)\|_k=\infty$ for every $k\in\mathbb{N}$, then for any sequence $\{t_i\}_{i\in\mathbb{N}}\in\mathbb{N}^+$, there exists a positive number N_2 such that $\rho(T_{t_i}(x_0),0)>\varepsilon$ for $\{t_i : t_i\in\{t_i\}_{i\in\mathbb{N}}, t_i>N_2\}$. So

$$\Phi_{x_0,0}(\varepsilon,\{t_i^\varepsilon\}_{i\in\mathbb{N}})=\liminf_{n\to\infty}\frac{1}{n}card\{1\le i\le n:\rho(T_{t_i}(x_0),0)<\varepsilon\}\le\liminf_{n\to\infty}\frac{N_2}{n}=0.$$

Hence,

$$\begin{aligned}\mu_n(\mathcal{T}) &= \sup_{x\in X}\tfrac{1}{diam(X)}\int_0^\infty(\Phi^*_{x,0}(\varepsilon)-\Phi_{x,0}(\varepsilon))d_\varepsilon\\ &= \sup_{x\in X}\tfrac{1}{diam(X)}\int_0^{diam(X)}(\Phi^*_{x,0}(\varepsilon)-\Phi_{x,0}(\varepsilon))d_\varepsilon\\ &\ge \tfrac{1}{diam(X)}\int_0^{diam(X)}(\Phi^*_{x_0,0}(\varepsilon,\{t_i^\varepsilon\}_{i\in\mathbb{N}})-\Phi_{x_0,0}(\varepsilon,\{t_i^\varepsilon\}_{i\in\mathbb{N}}))d_\varepsilon\\ &=1.\end{aligned}$$

This completes the proof. □

4. Distributionally Chaotic C_0-Semigroup

For any $x,y\in X$ and any $t>0$, the distributional function of x and y with respect to $\mathcal{T}=\{T_t\}_{t\ge0}$ is defined as follows:

$\Phi^t_{x,y}:\mathbb{R}^+\to[0,1]$
$\Phi^t_{x,y}(\varepsilon)=\frac{1}{t}\mu(\{0\le i\le t:\rho(T_s(x),T_s(y))<\varepsilon\}).\quad\forall\varepsilon>0$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

The upper and lower distributional functions of x and y are then defined by:

$$\Phi^*_{x,y}(\varepsilon)=\limsup_{t\to\infty}\Phi^t_{x,y}(\varepsilon)\text{ and }\Phi_{x,y}(\varepsilon)=\liminf_{t\to\infty}\Phi^t_{x,y}(\varepsilon),\ \forall\varepsilon>0$$

respectively.

Definition 3. *Let $(X,(\|\cdot\|_k)_{k\in\mathbb{N}},\rho)$ be a Frechet space. A C_0-semigroup of operators $\mathcal{T}=\{T_t\}_{t\ge0}$ on X is said to be distributionally chaotic if one can find an uncountable subset $S\in X$ and $\delta>0$ such that, for $\forall x,y\in S: x\ne y$ and for $\forall\varepsilon>0$, we have:*

$$\Phi^*_{x,y}(\varepsilon)=1\text{ and }\Phi_{x,y}(\delta)=0.$$

In this case, S is called a distributionally δ-scrambled set and (x,y) a distributionally chaotic pair.

Let $E\in\mathbb{R}^+$ be a Lebesgue measurable set; the upper density and lower density of E are defined as:

$$\overline{Dens}(E)=\limsup_{t\to\infty}\frac{\mu(E\cap[0,t])}{t}\text{ and }\underline{Dens}(E)=\liminf_{t\to\infty}\frac{\mu(E\cap[0,t])}{t}$$

respectively. Then, the conditions $\Phi^*_{x,y}(\varepsilon)=1$, $\Phi_{x,y}(\delta)=0$ in Definition 3 are equivalent to:

$$\overline{Dens}(\{t\ge0:\rho(T_t(x),T_t(y))<\varepsilon\})=1\text{ and }\underline{Dens}(\{t\ge0:\rho(T_t(x),T_t(y))<\delta\})=0$$

respectively.

Given $M \subset \mathbb{N}^+$, the upper density and lower density of M are defined as

$$\overline{dens}(M) = \limsup_{n\to\infty} \frac{card(M \cap [0, n-1])}{t} \text{ and } \underline{dens}(M) = \liminf_{n\to\infty} \frac{card(M \cap [0, n-1])}{t}$$

respectively. The conditions in the definition of the distributional chaos for operator T are equivalent to

$$\overline{dens}(\{n \in \mathbb{N} : \|T_n(x) - T_n(y)\|_k < \varepsilon\}) = 1,\ \underline{dens}(\{n \in \mathbb{N} : \|T_n(x) - T_n(y)\|_k < \delta\}) = 0.$$

Theorem 3. *Let $\mathcal{T}$ be a C_0-semigroup of operators on a Frechet space X. $x \in X$, $t_0 > 0$, $\forall k \in \mathbb{N}$, let $C^k_{t_0} = \sup_{0\le t\le t_0} \|T(x)_t\|_k$. Then for every $\varepsilon, \delta > 0$ and all $N > 0$:*

(i) $\mu(\{t \in [0, N] : \|T_t(x)\|_k > \delta\}) \le t_0 \left|\{s \in \mathbb{N} : s \le \frac{N}{t_0} + 1,\ \|T^{s-1}_{t_0}(x)\|_k > \frac{\delta}{C^k_{t_0}}\}\right|$;

(ii) $t_0 \left|\{s \in \mathbb{N} : s \le N,\ \|T^s_{t_0}(x)\|_k > \delta\}\right| \le \mu(\{t \in [0, Nt_0] : \|T_t(x)\|_k > \frac{\delta}{C^k_{t_0}}\})$;

(iii) $\mu(\{t \in [0, N] : \|T_t(x)\|_k < \varepsilon\}) \le t_0 \left|\{s \in \mathbb{N} : s \le \frac{N}{t_0} + 1,\ \|T^s_{t_0}(x)\|_k < \varepsilon C^k_{t_0}\}\right|$;

(iv) $t_0 \left|\{s \in \mathbb{N} : s \le N,\ \|T^s_{t_0}(x)\|_k < \varepsilon\}\right| \le \mu(\{t \in [0, (N+1)t_0] : \|T_t(x)\|_k < \varepsilon C^k_{t_0}\})$.

Proof. (i) Let $A = \{t \le N : \|T_t(x)\|_k > \delta\}$, $B = \{s \in \mathbb{N} : \exists t^* \in A \cap [(s-1)t_0, st_0]\}$, then,

$$B \subseteq \{s \in \mathbb{N} : 1 \le s \le \frac{N}{t_0} + 1, \|T^{s-1}_{t_0}(x)\|_k > \frac{\delta}{C^k_{t_0}}\}$$

Indeed, if there exists $t^* \in [(s-1)t_0, st_0]$ such that $t^* \le N$ and $\|T_{t^*}(x)\|_k > \delta$, then

$$1 \le \frac{t^*}{t_0} \le s \le \frac{t^*}{t_0} + 1 \le \frac{N}{t_0} + 1.$$

and because $t^* - (s-1)t_0 \le t_0$, then

$$\begin{aligned} \delta < \|T_{t^*}(x)\|_k = \|T_{t^*-(s-1)t_0} T_{(s-1)t_0}(x)\|_k &\le (\sup_{0\le t\le t_0} \|T_t(x)\|_k) \|T_{(s-1)t_0}(x)\|_k \\ &= C^k_{t_0} \|T_{(s-1)t_0}(x)\|_k = C^k_{t_0} \|T^{s-1}_{t_0}(x)\|_k. \end{aligned}$$

That is,

$$\|T^{s-1}_{t_0}(x)\|_k > \frac{\delta}{C^k_{t_0}}.$$

Therefore,

$$\mu(A) \le \sum_{s\in B} \mu([(s-1)t_0, st_0]).$$

(ii) Let $M = \{s \in \mathbb{N} : s \le N, \|T^s_{t_0}(x)\|_k > \delta\}$. Then, for every $t \in [(s-1)t_0, st_0]$, we have that

$$\delta < \|T^s_{t_0}(x)\|_k = \|T_{st_0}(x)\|_k = \|T_{st_0-t} T_t(x)\|_k \le C^k_{t_0} \|T_t(x)\|_k.$$

(The last inequality is right for the reason that $st_0 - t \le t_0$).

Hence,

$$\bigcup_{s\in M} [(s-1)t_0, st_0] \subseteq \{t \in [0, Nt_0] : \|T_t(x)\|_k > \frac{\delta}{C^k_{t_0}}\}.$$

Thus,

$$t_0|M| \le \mu(\{t \in [0, Nt_0] : \|T_t(x)\|_k > \frac{\delta}{C^k_{t_0}}\}).$$

(iii) and (iv) can be obtained with analogous considerations.
This completes the proof. □

Theorem 4. *Let $\mathcal{T}$ be a C_0-semigroup of operators on a Frechet space X. $x \in X$, $t_0 > 0$, $\forall k \in \mathbb{N}$, let $C^k_{t_0} = \sup\limits_{0 \le t \le t_0} \|T(x)_t\|_k$. Then for $\forall \varepsilon, \delta > 0$ and all $N > 0$:*

(i) $\overline{Dens}(\{t \ge 0 : \|T_t(x)\|_k > \delta\}) \le \overline{dens}(\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k > \frac{\delta}{C^k_{t_0}}\})$;

(ii) $\overline{dens}\left|\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k > \delta\}\right| \le \overline{Dens}(\{t \ge 0 : \|T_t(x)\|_k > \frac{\delta}{C^k_{t_0}}\})$;

(iii) $\overline{Dens}(\{t \ge 0 : \|T_t(x)\|_k < \varepsilon\}) \le \overline{dens}\left|\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k < \varepsilon C^k_{t_0}\}\right|$;

(iv) $\overline{dens}(\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k < \varepsilon\}) \le \overline{Dens}(\{t \ge 0 : \|T_t(x)\|_k < \varepsilon C^k_{t_0}\})$.

Proof. (i)′ By (i) of Theorem 3,

$$\begin{aligned}
\overline{Dens}(\{t \ge 0 : \|T_t(x)\|_k > \delta\}) &= \limsup_{t\to\infty} \tfrac{1}{t}\mu(\{[0,t] \cap \{t \ge 0 : \|T_t(x)\|_k > \delta\}\}) \\
&= \limsup_{N\to\infty} \tfrac{1}{N}\mu(\{t \in [0,N] : \|T_t(x)\|_k > \delta\}) \\
&\le \limsup_{N\to\infty} \tfrac{t_0}{N}\mu(\{s \in \mathbb{N} : s \le \tfrac{N}{t_0} + 1, \|T^{s-1}_{t_0}(x)\|_k > \tfrac{\delta}{C^k_{t_0}}\}) \\
&= \limsup_{N\to\infty} \tfrac{1}{N}\left|\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k > \tfrac{\delta}{C^k_{t_0}}\} \cap [0,N]\right| \\
&= \overline{dens}(\{s \in \mathbb{N} : \|T^s_{t_0}(x)\|_k > \tfrac{\delta}{C^k_{t_0}}\}).
\end{aligned}$$

(ii)′, (iii)′ and (iv)′ can be obtained with analogous considerations.
This completes the proof. □

Theorem 5. *Let $\mathcal{T} = \{T_t\}_{t \ge 0}$ be a C_0-semigroup of operators on a Frechet space X. Then the following properties are equivalent.*

(i) *$\mathcal{T}$ is distributionally chaotic;*

(ii) *$\forall t > 0$, T_t is distributionally chaotic;*

(iii) *There exists $t_0 > 0$ such that T_{t_0} is distributionally chaotic.*

Proof. Let $S \subset X$ be a distributionally δ-scrambled set for $\mathcal{T}$. Then, for $\forall x, y \in S : x \neq y$, there exists a $0 < \delta < 1$ such that

$$\underline{Dens}(\{s \ge 0 : \rho(T_s(x), T_s(y)) < \delta\}) = 0.$$

It means that

$$\liminf_{t\to\infty} \frac{\mu(\{s \ge 0 : \rho(T_s(x), T_s(y)) < \delta\} \cap [0,t])}{t} = 0.$$

i.e.,

$$\limsup_{t\to\infty} \frac{\mu(\{s \ge 0 : \rho(T_s(x), T_s(y)) > \delta\} \cap [0,t])}{t} = 1.$$

If $\|T_s(x) - T_s(y)\|_k > \frac{2^k\delta}{1-\delta} (\forall k \in \mathbb{N})$, then

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{1}{1 + \|T_s(x) - T_s(y)\|_k} < 1 - \delta.$$

So,

$$\rho(T_s(x), T_s(y)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\|T_s(x) - T_s(y)\|_k}{1 + \|T_s(x) - T_s(y)\|_k} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{1}{1 + \|T_s(x) - T_s(y)\|_k} > \delta.$$

Thus,

$$\overline{Dens}(\{s \geq 0 : \|T_s(x) - T_s(y)\|_k > \frac{2^k \delta}{1-\delta}\}) = 1.$$

By (i)′ of Theorem 4, for every $t_0 > 0$, one has

$$\overline{dens}(\{s \in \mathbb{N} : \|T_{t_0}^k(T_s(x) - T_s(y))\|_k > \frac{2^k \delta}{C_{t_0}^k (1-\delta)}\}) = 1.$$

That is,

$$\underline{dens}(\{s \in \mathbb{N} : \|T_{kt_0+s}(x) - T_{kt_0+s}(y)\|_k < \frac{2^k \delta}{C_{t_0}^k (1-\delta)}\}) = 0.$$

On the other hand, for $\forall x, y \in S : x \neq y$ and every $0 < \varepsilon < 1$, since $\overline{Dens}(\{s \geq 0 : \rho(T_s(x), T_s(y)) < \varepsilon\}) = 1$, i.e.,

$$\limsup_{t \to \infty} \frac{\mu(\{s \geq 0 : \rho(T_s(x), T_s(y)) < \varepsilon\} \cap [0, t])}{t} = 1,$$

and

$$\rho(T_s(x), T_s(y)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\|T_s(x) - T_s(y)\|_k}{1 + \|T_s(x) - T_s(y)\|_k} < \varepsilon,$$

then

$$\overline{Dens}(\{s \geq 0 : \|T_s(x) - T_s(y)\|_k < \frac{\varepsilon}{2^k(1-\varepsilon)}\}) = 1$$

when $\|T_s(x) - T_s(y)\|_k < \frac{\varepsilon}{2^k(1-\varepsilon)}$ $(\forall k \in \mathbb{N})$.

By (iii)′ of Theorem 4, for every $t_0 > 0$, one has

$$\overline{dens}(\{s \in \mathbb{N} : \|T_{t_0}^k(T_s(x) - T_s(y))\|_k < \frac{\varepsilon C_{t_0}^k}{2^k(1-\varepsilon)}\}) = 1.$$

For the arbitrariness of $\varepsilon > 0$, we have

$$\overline{dens}(\{s \in \mathbb{N} : \|T_{kt_0+s}(x) - T_{kt_0+s}(y)\|_k < \varepsilon\}) = 1.$$

Thus, S is a δ'-scrambled set for T_t, where $\delta' = \frac{\delta}{C_{t_0}^k}$, $t = kt_0 + s$ $(\forall t_0 > 0)$, i.e., for all $t > 0$, T_t is distributionally chaotic.

(ii) implies (iii). It is trivial.

(iii) implies (i). The proof is analogous to the first implication.

This completes the proof. □

5. Discussion

Inspired by the definition of an irregular vector given by N.C. Bernardes Jr in Reference [17], this paper defines the strong irregular vector. In particular, it is proved that a C_0-semigroup on a Frechet space is distributionally chaotic in a sequence if it admits a strong irregular vector. In addition, the principal measure $\mu_p(\mathcal{T}) = 1$. These results extend the corresponding results in References [16,17,31,35]. In Section 4, using upper density and lower density, it is showed that the distributional chaoticity of $\mu_p(\mathcal{T}) = \{T_t\}_{t \geq 0}$ is equivalent to the distributional chaoticity of some T_{t_0} $(t_0 > 0)$. This result is consistent with the similar conclusion in Banach space or other Frechet spaces (see References [17,27,29,31,33] and others). Then, some further results regarding C_0–semigroups or Frechet spaces may be obtained in the future.

Since Li-Yorke chaos is a special case of distributional chaos, therefore, the conclusions of this paper are also correct for Li-Yorke chaos.

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