



# Article Hydrodynamical Study of Micropolar Fluid in a Porous-Walled Channel: Application to Flat Plate Dialyzer

# Dianchen Lu<sup>1</sup>, Muhammad Kahshan<sup>1,2,\*</sup> and A. M. Siddiqui<sup>3</sup>

- <sup>1</sup> Faculty of Science, Jiangsu University, Zhenjiang 212013, China; dclu@ujs.edu.cn
- <sup>2</sup> Department of Mathematics, COMSATS University Islamabad, Abbottabad 22060, Pakistan
- <sup>3</sup> Department of Mathematics, Pennsylvania State University, YorkCampus, 1031 Edgecomb Avenue, York, PA 17403, USA; ams5@psu.edu
- \* Correspondence: muhammadkahshan@gmail.com

Received: 3 April 2019; Accepted: 12 April 2019; Published: 15 April 2019



**Abstract:** This article investigates the two-dimensional creeping flow of a non-Newtonian micropolar fluid in a small width permeable channel. Fluid is absorbed through permeable walls at a variable rate. This situation arises in filtration and mass transfer phenomena in industrial and engineering processes. The exact solution of the equations of motion is obtained. Graphs of the velocity profiles and pressure drop reveal the significant impact of the non-Newtonian nature of the micropolar fluid on the flow. The obtained solutions are used to discuss the hydrodynamical aspects of the physiological phenomenon of blood filtration in an artificial kidney, the flat plate dialyzer (FPD). Expressions for finding the ultrafiltration rate and mean pressure drop in an FPD are derived. Ultrafiltration rate and the mean pressure difference in an FPD are computed using derived expressions. A comparison of these with the existing empirical and experimental results shows a good agreement. For certain values of parameters, the derived form of the flow rate reveals that the axial flow rate in an FPD decays exponentially along the membrane length. This is a well-established and admitted result used by several researchers for studying the hydrodynamics of blood flow in renal tubules of kidneys. It is concluded that the presented model can be used to study the hydrodynamical aspects of blood flow in an FPD.

Keywords: micropolar fluid; permeable channel; flat plate dialyzer; filtration rate

# 1. Introduction

Ultrafiltration and reverse osmosis are encountered in many industrial and biological processes. Reverse osmotic desalination, glomerular tubular ultrafiltration, transpiration cooling, proximal tubular reabsorption, and the process of blood filtration in an artificial kidney are some examples of these processes [1–7]. In these processes, the filtering fluid is normally pumped at an elevated pressure through porous-walled channels and tubes. For example, in the human body, the renal tubules of kidneys can be approximated by long narrow permeable tubes [2,3,5]. In the blood purification process in extra-corporeal circuits, the fluid commonly flows between flat parallel membranes [5,8].

In order to study fluid flows in the filtration process, one must characterize both the normal and tangential components of velocity to the porous wall since the usual Poiseuille law fails to describe such flow situations [9]. In the literature, Berman's work is highly cited in the study of laminar flow in porous channels and tubes [10,11]. In his studies, Berman obtained perturbation solutions for the velocity components and the pressure distribution in a porous-walled channel and annulus whose wall reabsorbs fluid at a constant rate. Steady motion of an incompressible Newtonian fluid was considered

in porous-walled ducts with constant suction/injection velocity at the walls. A two-dimensional steady-state and laminar flow of Newtonian fluid in a porous tube was studied by Yuan et al. [12]. They investigated the effect of injection and suction velocity on the flow in detail by solving the Navier–Stokes equations using the perturbation method.

A mathematical model for the hydrodynamics of the blood filtration in the renal tubule of human kidneys was studied in a couple of articles by Macey [2,3]. In his studies, Macey assumed the blood to be an incompressible Newtonian fluid and the renal tubule as a finite length porous-walled tube in which the flow rate decays linearly and exponentially, respectively. Low Reynolds number flow was assumed, and the exact solution for the velocity field and pressure distribution were obtained. Kozinski et al. [8] extended the work of Macey for porous-walled channels and tubes whose walls reabsorb fluid at an exponential rate. In recent years, Haroon et al. [13] proposed a mathematical model for fluid flow in renal tubules of kidneys. A two-dimensional model of creeping flow of Newtonian fluid in a permeable channel was proposed, where the fluid is absorbed through channel walls at a uniform rate. Siddiqui et al. [14] presented the creeping flow of an incompressible Newtonian fluid in a permeable channel with linear seepage velocity at the wall. An application to renal tubular flow was also furnished.

However, in all articles enlisted in this literature review,

- the fluid flowing in the channel was assumed to be Newtonian in nature,
- the no-slip condition was assumed to be held at the permeable wall,
- a seepage velocity of a constant, linear, or exponential type at the porous wall was assumed in advance.

Most of the industrial and biological fluids are admitted to be non-Newtonian [15], and the classical Newton's law of viscosity fails to describe the complex rheological properties of these fluids. Among many existing constitutive models representing non-Newtonian fluids, the micropolar fluid model is admitted to be a better and frequently-used model for physiological and biological fluids [16,17]. The no-slip condition is frequently used in the study of fluid mechanics problems. It states that the tangential velocity of the fluid layer in the region adjacent to boundaries has the same velocity as that of the boundary [9]. However, in many practical situations, this condition may fail to be valid, particularly when there are naturally permeable boundaries of the flow geometry [9,18,19]. A very thin layer of the fluid in the region adjacent to the permeable boundary slips, due to which a difference in the velocities of fluid layer and boundaries is encountered. Boundary conditions for a naturally-permeable wall, proposed by Beavers and Joseph [18] and slightly modified by Saffman [19], provide a mathematical form of the fluid slip phenomenon. In practical situations, seepage rates are normally determined by membrane characteristics and concentration polarization at the membrane surface and are not necessarily constant or known in advance.

Having the importance of physical aspects described in the previous paragraph, this article is aimed at studying the hydrodynamical aspects of a non-Newtonian fluid, the micropolar fluid, in a porous-walled channel whose walls absorb the fluid at a variable rate in accordance with Darcy's law [20] and considering the wall slip effects. Thus, fluid seepage at the permeable wall of the channel is taken as a function of the difference of transmural pressure across the wall. The approach presented in this article is better than that of [2,3,8,13,14] because of the following reasons.

- The constitutive equation of the micropolar fluid model can be reduced to the Newtonian fluid model as a special case when certain parameters in this model are set to zero. Thus, a variety of industrial and biological non-Newtonian fluids along with the previously-studied Newtonian fluid can be investigated by the current results.
- Results for the no-slip flow can be recovered from our obtained solutions when the slip parameter approaches zero.
- The obtained solution also reveals that for particular values of parameters, a uniform, linear, and exponentially-decaying flow rate can be deduced from the results of the current article, which were assumed in advance in the previous studies.

#### 2. Basic Equations

Basic equations that govern the flow of an isotropic and incompressible micropolar fluid without any body forces and body couple are [16]:

$$\nabla \cdot \mathbf{U} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{U}}{Dt} = -\nabla p - (\mu + \mu_m) \nabla \times \nabla \times \mathbf{U} + \mu_m \nabla \times \mathbf{R}, \qquad (2)$$

$$\rho j_m \frac{D\mathbf{R}}{Dt} = \mu_m \nabla \times \mathbf{U} - 2\mu_m \mathbf{R} + (\alpha_m + \beta_m + \gamma_m) \nabla (\nabla \cdot \mathbf{R}) - \gamma_m (\nabla \times \nabla \times \mathbf{R}).$$
(3)

In these equations,  $\nabla$  is the gradient operator,  $D/Dt = \frac{\partial}{\partial t} + (\mathbf{U} \cdot \nabla)$ ,  $\mathbf{U}$  denotes the velocity vector,  $\mathbf{R}$  is the the micro-rotation vector,  $j_m$  is the micro-inertia coefficient,  $\mu$  is the coefficient of viscosity, and  $\mu_m$  represents the micro-rotation viscosity. Constants  $\alpha_m$ ,  $\beta_m$ , and  $\gamma_m$  are called the viscosity coefficients of the angular velocity. These coefficients satisfy the following constraints [16,17]:

$$\mu_m \ge 0, \qquad \mu \ge 0, 3\alpha_m + \beta_m + \gamma_m \ge 0, \quad |\beta_m| \le \gamma_m. \tag{4}$$

#### 3. Problem Statement

The schematic diagram of the considered system in this paper is described in Figure 1. A two-dimensional flow of an incompressible micropolar fluid between small width parallel plates is considered. We assume that the Reynolds numbers of the flow are small (of the order  $10^{-2}$ ) and all flow variables are independent of time, that is the motion is creeping. Both of these effects are of less importance in the flow situations arising in the mammalian body and also in the presently investigated FPD. The unimportance of these effects can be seen in the work presented by several authors in [5,8,10,14] for a porous-walled channel and in the work in [2,4,21,22] for a porous walled tube in the study of flow in renal tubules of kidneys. The unimportance of the time dependency and Reynolds number can also be seen in [23] for the study of blood flow in capillary with permeable walls. Therefore, it is stated that the analysis of this paper can be used to study reverse osmotic flows and ultrafiltration processes. Using the velocity and micro-rotation profiles given by:

$$\mathbf{U} = (\bar{u}(\bar{x}, \bar{y}), \bar{v}(\bar{x}, \bar{y})),$$
$$\mathbf{W} = (0, 0, \bar{\omega}(\bar{x}, \bar{y})),$$

and the assumption of steady and creeping motion, Equations (1)–(3) take the following form:

$$0 = \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}'},\tag{5}$$

$$\frac{\partial \bar{p}}{\partial \bar{y}} = (\mu + \mu_m) \left( \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right) + \mu_m \frac{\partial \bar{\omega}}{\partial \bar{x}}, \tag{6}$$

$$\frac{\partial \bar{p}}{\partial \bar{x}} = (\mu + \mu_m) \left( \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right) + \mu_m \frac{\partial \bar{\omega}}{\partial \bar{y}}, \tag{7}$$

$$0 = -2\mu_m \bar{w} + \mu_m \left(\frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial \bar{u}}{\partial \bar{y}}\right) + \gamma_m \left(\frac{\partial^2 \bar{\omega}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\omega}}{\partial \bar{y}^2}\right).$$
(8)

Due to the symmetry of the geometry, we have considered the upper half of the channel. The corresponding boundary conditions are:

$$\bar{v}(\bar{x},0) = 0, \tag{9}$$

$$\partial \bar{u}(\bar{x},0)/\partial \bar{y} = 0, \tag{10}$$

$$\bar{v}(\bar{x},a) = \frac{L_p}{\mu t} \left[ \bar{p}(\bar{x},a) - P_T \right], \tag{11}$$

$$\bar{u}(\bar{x},a) = -\bar{\phi}\partial\bar{u}(\bar{x},a)/\partial\bar{y}, \qquad (12)$$

$$\bar{\omega}(\bar{x},0) = 0, \tag{13}$$

$$\tilde{\omega}(\tilde{x},a) = 0, \tag{14}$$

$$\frac{1}{a} \int_{0}^{a} \bar{p}(0, \bar{y}) d\bar{y} = \bar{p}_{i}, \tag{15}$$

$$2w \int_0^a \bar{u}(0,\bar{y}) d\bar{y} = \bar{Q}_0.$$
 (16)

Equations (9) and (10) are the symmetry conditions at the center line of the channel. Equation (11)is the consequence of Darcy's law at the permeable wall of the channel, where  $L_p$  is the mechanical filtration coefficient of the channel wall, which is usually measured in units of  $cm^2 (L_p/\mu t)$  is called the hydraulic permeability of the channel wall), t is the wall thickness, and  $P_T$  can be visualized as the back pressure that opposes the fluid leakage and is equal to the difference of hydrostatic and the osmotic pressures outside the channel wall. Equation (12) is the well-known Beavers and Joseph slip condition [18] at a permeable wall, modified by Saffmann et al. [19], where  $\bar{\phi}$  is the slip coefficient of the permeable wall. Equations (13) and (14) show the vanishing of microrotation at the center line and boundary [16,22] whereas, Equations (15) and (16) are the inlet conditions. In these equations,  $\bar{p}_i$ is the mean pressure and  $\bar{Q}_0$  is the flow rate at the inlet of the channel at  $\bar{x} = 0$ .

1



Figure 1. Geometry of the problem.

## 4. Dimensionless Formulation and Solution

The following parameters are used to transform equations into dimensionless form:

$$x = \frac{\bar{x}}{L}; \quad y = \frac{\bar{y}}{a}; \quad u(x,y) = \frac{a^2\bar{u}}{\bar{Q}_i}; \quad v(x,y) = \frac{aL\bar{v}}{\bar{Q}_i}; \quad \omega(x,y) = \frac{a^3\bar{\omega}}{\bar{Q}_i}.$$
(17)

After substitution of dimensionless parameters, Equations (5)-(8) reduce to the following form:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y'}, \tag{18}$$

$$\frac{\partial p}{\partial y} = \frac{\lambda^2}{1 - N^2} \left[ \lambda^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + N^2 \frac{\partial \omega}{\partial x} \right], \tag{19}$$

$$\frac{\partial p}{\partial x} = \frac{1}{1 - N^2} \left[ \lambda^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + N^2 \frac{\partial \omega}{\partial y} \right]$$
(20)

$$0 = -2\omega + \left(\lambda^2 \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{2 - N^2}{M^2} \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right).$$
(21)

where  $\lambda = a/L$  is a small number describing the ratio of the channel width to its length,  $p(x,y) = [\bar{p}(\bar{x},\bar{y}) - P_T] a^4/\mu L \bar{Q}_i$  is the dimensionless pressure,  $N^2 = \mu_m/(\mu + \mu_m)$ ,  $(0 \le N < 1)$  is called the coupling number, and  $M^2 = a^2 \mu_m (2\mu + \mu_m)/\gamma_m (\mu + \mu_m)$  is called the micropolar parameter [17]. In the limiting case, when  $\mu_m \to 0$ , implying  $N \to 0$ , Equations (19) and (20) are uncoupled with Equation (21), and they reduce to classical Navier–Stokes equations for creeping flow of a Newtonian fluid in a channel.

Boundary conditions (9) and (16) take the following dimensionless form:

$$v(x,0) = 0, \qquad (22)$$

$$\partial u(x,0)/\partial y = 0,$$
 (23)

$$v(x,1) = K p(x,1),$$
 (24)

$$u(x,1) = -\phi \,\partial u(x,1)/\partial y, \tag{25}$$

$$\omega(x,0) = 0, \tag{26}$$

$$\omega(x,1) = 0, \qquad (27)$$

$$\int_{0}^{1} p(0,y) \, dy = p_i, \tag{28}$$

$$2W \int_0^1 u(0,y) \, dy = 1. \tag{29}$$

where  $K = \frac{L_p L^2}{a^3 t}$  is the dimensionless wall filtration parameter,  $\phi = \frac{\bar{\phi}}{a}$  is the dimensionless wall slip coefficient,  $W = \frac{w}{a}$  is the ratio of channel width to height, and  $Q(x) = \frac{\bar{Q}(\bar{x})}{\bar{Q}_i}$ .

In order to find exact solution of Equations (18)–(21), we use the fact that since the parameter  $\lambda$  is very small as compared to unity, therefore terms of the order  $\lambda^2$  can be ignored in these equations [14,22–24]. Note that this fact will be justified for the FPD in a later section. The exact solution of the reduced system subject to the boundary conditions (22)–(29) is then readily obtained as follows:

$$p(x) = p_i \cosh \eta \, x - \frac{\eta}{2WK} \sinh \eta \, x \tag{30}$$

$$u(x,y) = \frac{1}{2} \left( 1 - N^2 \right) \left[ y^2 - 1 - 2\phi \right]$$
(31)  
$$- \frac{N^2}{2 - N^2} \left\{ 2 \frac{\cosh My - \cosh M}{M \sinh M} - y^2 + 1 \right\} \frac{dp}{dx},$$
  
$$v(x,y) = -\frac{1}{6} \left( 1 - N^2 \right) \left[ y^3 - 3y - 6\phi y \right]$$
(32)  
$$- \frac{N^2}{2 - N^2} \left\{ 6 \frac{\sinh My - My \cosh M}{M^2 \sinh M} - y^3 + 3y \right\} \frac{d^2p}{dx^2},$$

Symmetry 2019, 11, 541

$$w(x,y) = \frac{1-N^2}{2-N^2} \left(\frac{\sinh My}{\sinh M} - y\right), \tag{33}$$

where:

$$\eta^2 = \frac{3K}{\xi},\tag{34}$$

$$\xi = \left(1 - N^2\right) \left[1 + 3\phi + \frac{N^2}{2 - N^2} \left\{3\frac{\sinh M - M \cosh M}{M^2 \sinh M} + 1\right\}\right].$$
 (35)

In the limiting case, when *N* and  $\phi$  approach zero (and also *m* approaches zero), the presented solutions reduce to the solution for Newtonian fluid flow in a permeable channel [23].

The dimensionless mean pressure P(x) taken over any cross-section of the channel is defined as:

$$P(x) = \frac{1}{2} \int_{-1}^{1} p(x, y) dy$$
  
=  $p(x)$  (36)

Hence, the difference of mean pressure  $\triangle p(x)$  can be obtained as:

$$\Delta P(x) = P(0) - P(x),$$
  
=  $p_i (1 - \cosh \eta x) - \frac{\eta}{2WK} \sinh \eta x$  (37)

Unlike ordinary fluids, the stress tensor for a micropolar fluid is not symmetric. Thus, the shear stresses  $\tau_{xy}$  and  $\tau_{yx}$  are not equal for a micropolar fluid. These stresses can be computed using the following dimensionless expressions:

$$\tau_{xy} = \frac{\partial u}{\partial y} - \frac{N^2}{1 - N^2} \omega, \qquad (38)$$

$$\tau_{yx} = \frac{1}{1 - N^2} \frac{\partial u}{\partial y} + \frac{N^2}{1 - N^2} \omega.$$
(39)

The dimensionless volume flow rate at any cross-section of the channel can be computed as:

$$Q(x) = W \int_{-1}^{1} u(x, y) dy$$
  
=  $\cosh \eta x - \frac{2 p_i W K}{\eta} \sinh \eta x.$  (40)

The expression for leakage flux q(x) through channel walls is given by:

$$q(x) = -\frac{dQ}{dx},$$
  
= 2 p<sub>i</sub> W K cosh η x - η sinh η x (41)

The fractional re-absorption (FR) is the amount of fluid that has been reabsorbed through the channel walls. It can be computed using the following expression:

$$FR = \frac{Q(0) - Q(1)}{Q(0)}.$$
(42)

6 of 16

The streamlines can be determined by solving the following equation:

$$\frac{dx}{u} = \frac{dy}{v}.$$

This results in the following equation of streamlines:

$$\left[y^{3} - 3y - 6\phi y - \frac{N^{2}}{2 - N^{2}} \left\{ 6 \frac{\sinh My - My \cosh M}{M^{2} \sinh M} - y^{3} + 3y \right\} \right] \frac{dp}{dx} = C,$$
(43)

where *C* is a constant due to integration.

#### 5. Numerical Results and Discussion

This section contains numerical calculations and their graphical interpretations performed on the computational software Maple [25]. In order to study the effects of the wall permeability parameter K, the coupling number N, the micropolar parameter M, and the wall slip parameter  $\phi$  on the flow patterns, velocity profile, and the hydrostatic pressure, graphs of the numerical calculations are plotted.

Figures 2–5 are plotted to present the flow pattern of the micropolar fluid in a permeable channel. It was observed that the flow was positive axial through the channel, and no reverse flow and reverse leakage were seen. However, as the wall permeability of the channel was increased to a certain value, a reverse flow happened, and a stagnation point flow can be seen in Figure 3. A slight decrease (about 10%) in the fluid seepage also happened as the coupling number *N* increased. This is observed in Figure 4. However, we see that in the case of micropolar fluid flow in a permeable channel, the wall slip parameter did not affect the fluid seepage significantly in contrast to the Newtonian fluid flow in a permeable channel [24]. This observation can be seen in Figure 5.

A three-dimensional view of the tangential and normal velocity profiles is plotted in Figures 6 and 7. These figures present variations of u(x, y) and v(x, y) with the wall permeability parameter *K*. A parabolic velocity profile is observed in Figure 6, which had its maximum value at the center line of the channel and minimum at the walls. The tangential velocity was found to decrease with the increasing values of K, whereas the opposite effects can be seen in Figure 7, in which the normal velocity increased as the wall permeability increased. Thus, fluid seepage through walls was enhanced by increasing *K*. This is due to the fact that enhancement of *K* resulted in the increase of the wall permeability of the channel. Thus, more fluid was allowed to pass through the walls, which increased the magnitude of seepage velocity. This in turn reduced the magnitude of tangential velocity, which is observed in Figures 6 and 7.

The impact of coupling number N on u and v at any cross-sectional element x = 0.3 of the channel is plotted in Figures 8 and 9. In Figure 8, it is observed that initially, u increased by increasing the coupling number N, then a point  $y = y_0$  can be seen in the figure at which u is independent of N, and after that, u started decreasing by increasing the coupling number. Thus, a circle centered at  $(x_0, y_0)$ can be drawn at which the tangential velocity  $u(x_0, y)$  is invariant under the effect of coupling number. Figure 9 shows the effect of N on the normal velocity. It shows that by increasing the magnitude of N, the normal velocity v decreased.

Variations in the micro-rotation velocity at the cross-section x = 0.3 are sketched in Figures 10 and 11. These figure reveal that the micro-rotation velocity was enhanced by strengthening the coupling number *N*. A damping in the magnitude of *w* was observed as the wall permeability parameter was increased.

Variations in the mean pressure difference  $\triangle P(x)$  in the axial direction are sketched in Figures 12 and 13. The magnitude of the mean pressure difference was decreased by increasing the permeability coefficient of the wall, whereas  $\triangle P(x)$  increased when the coupling number was increased.



**Figure 2.** Streamline pattern of the flow for  $p_i = 0.02$ , K = 0.004,  $\phi = 0.5$ , N = 0.5, M = 5, W = 1288.



**Figure 3.** Streamline pattern of the flow for  $p_i = 0.02$ , K = 0.04,  $\phi = 0.5$ , N = 0.5, M = 5, W = 1288.



**Figure 4.** Streamline pattern of the flow for  $p_i = 0.02$ , K = 0.004,  $\phi = 0.5$ , N = 0.98, M = 5, W = 1288.



**Figure 5.** Streamline pattern of the flow for  $p_i = 0.02$ , K = 0.004,  $\phi = 20$ , N = 0.5, M = 5, W = 1288.



**Figure 6.** Variation of tangential velocity with *K* for  $p_i = 0.02$ ,  $\phi = 0.5$ , N = 0.5, M = 5, W = 1288.



**Figure 7.** Variation of normal velocity with *K* for  $p_i = 0.02$ ,  $\phi = 0.5$ , N = 0.5, M = 5, W = 1288.



**Figure 8.** Variation of tangential velocity with *N* at x = 0.3 for  $p_i = 0.02, K = 0.004, \phi = 0.5, M = 5, W = 1288.$ 



**Figure 9.** Variation of normal velocity with *N* at x = 0.3 for  $p_i = 0.02$ , K = 0.004,  $\phi = 0.5$ , M = 5, W = 1288.



**Figure 10.** Variation of the micro-rotation with *N* at x = 0.3 for  $p_i = 0.02, K = 0.004, \phi = 0.5, M = 5, W = 1288$ .



**Figure 11.** Variation of the micro-rotation with *K* at x = 0.3 for  $p_i = 0.02, \phi = 0.5, N = 0.5, M = 5$ , W = 1288.



**Figure 12.** Variation of mean pressure drop with *K* for  $p_i = 0.02$ ,  $\phi = 0.5$ , N = 0.5, M = 5, W = 1288.



**Figure 13.** Variation of mean pressure drop with *N* for  $p_i = 0.02$ , K = 0.004,  $\phi = 0.5$ , M = 5, W = 1288.

# 6. Application to a Flat Plate Dialyzer

In this section, theoretical expressions for the calculation of the ultrafiltration rate and the mean axial pressure drop in a flat plate hemodialyzer (FPH) are presented using the results of the preceding section. An FPH consists of several blood compartments. Each compartment comprises a pair of rectangular sheets made up of regenerated cellulose. The edges of each sheet are fastened by a pair of rectangular grooved plastic boards. The blood flows between the cellulose sheets, whereas the dialyzing fluid passes in a counter-current or a cross-current flow along the grooves in the hemodialyzer board [5,26,27]. The volume of blood lost by the seepage through the cellulose in a given time, from a known recirculating volume, is the ultrafiltration rate.

If *L* is the length of cellulose, then the ultrafiltration rate for the presented model is given as:

$$\bar{Q}_A = \bar{Q}(0) - \bar{Q}(L),$$
 (44)

where the  $\bar{}$  sign denotes the dimensional quantity. The non-dimensional expression for  $\bar{Q}_A$  can be obtained using the dimensionless parameters defined in Section 4 and Equation (40). Thus, we obtain the ultrafiltration rate as:

$$Q_{A} = Q(0) - Q(1),$$
  
=  $1 - \cosh \eta + \frac{2p_{i}WK}{\eta} \sinh \eta.$  (45)

The expression for mean pressure drop in the axial direction between x = 0 and x = L can also be obtained in a similar manner. By using Equation (37), we have the following expression for the mean pressure drop in a flat plate hemodialyzer:

$$\Delta P(1) = P(0) - P(1),$$
  
=  $p_i \left[ 1 - \cosh \eta + \frac{\eta}{2p_i W K \sinh \eta} \right].$  (46)

In order to check the accuracy of the presented formulas, exact values of the involved parameters corresponding to the flat-plate hemodialyzer are needed. For this purpose, we have also used the experimental data provided in [5,27] corresponding to a flat-plate disposable artificial kidney. It is also referred to as RPkidney data and is presented in Table 1. From the data in Table 1, we found that the parameter  $\lambda$  was of order  $10^{-8}$ . Therefore, ignoring terms of the order  $\lambda^2$  is justified for the flow in a flat plate dialyzer. By making use of these parameters along with N = 0.5, M = 5,  $\phi = 0.5$  [22,24,28–30] in Equation (44), we obtain the equation in one variable *K*. By expanding the hyperbolic functions in this equation in the power series of *K* up to  $O(K^5)$ , a real solution of this equation was found to be K = 0.00049. The magnitude of the ultrafiltration coefficient  $L_p$  was then calculated from  $K = \frac{L_p L^2}{a^3 t}$ . This resulted in  $L_p = 5.24 \times 10^{-16}$  cm<sup>2</sup>. In a similar way and adopting the same steps, Equation (46) results in the value of mean pressure drop in a flat plate hemodialyzer as P(0) - P(L) = 6.58 mm Hg.

Parameter	Abbreviation	Numerical Value
Number of blood		8
compartments		
Membrane length	L	42 cm
Membrane width	w	11.6 cm
Membrane thickness	t	$2.59 imes10^{-3}~\mathrm{cm}$
Blood half channel	а	$9 imes 10^{-3}~{ m cm}$
height		
Fluid viscosity	μ	$6.9 \times 10^{-3} \text{ dynes-s/cm}^2$
Transmembrane		-
pressure difference	$\bar{p}_i - P_T$	150 mm Hg
at the entrance		_
Total ultrafiltration	$8\bar{Q}_w$	200 mL/h
rate		
Total entrance	$8ar{Q}_0$	160 mL/min
volume flow rate		

Table 1. Physiological data related to the RPkidney [4,27].

The value of the filtration coefficient  $L_p$  is usually not given in the data for membranes of hemodialyzers. The result of experiments performed by Kaufmann et al. [31] show that at the normal body temperature, regenerated cellulose has hydraulic permeability as  $2.41 \times 10^{-11}$  cm<sup>3</sup>/dynes-s for the membrane having a thickness of  $7.5 \times 10^{-3}$  cm. When the viscosity of fluid was taken as  $6.9 \times 10^{-3}$  dynes-s/cm<sup>2</sup> from Table 1, this yielded the value of  $L_p$  as  $1.25 \times 10^{-15}$  cm<sup>2</sup>. The value of  $L_p$  calculated by the empirical results of Marshall et al. [5] by using the experimental data of Kaufmann et al. revealed that  $Lp = 6.36 \times 10^{-16}$  cm<sup>2</sup>. The experiments also showed that the mean axial pressure drop in the artificial kidney was about 15 mm Hg [5,27]. Thus, a good agreement in the order of magnitudes of the ultrafiltration coefficient and mean pressure drop can be observed between the presented results and the earlier computed experimental and empirical results. This builds confidence in stating that the presented model can be used to obtain theoretical results in advance to study the hydrodynamical aspects of the flow in a flat plate hemodialyzer.

Data presented in Table 1 and the estimated value of  $L_p$  reveal the dimensionless wall permeability (or filtration) parameter  $K \ll 1$ . Expanding (45) in power series of K, we have:

$$Q_A = -\frac{1}{2} \eta^2 + 2 p_i W K + O(K^2).$$
(47)

In terms of dimensional variables, we have:

$$\bar{Q}_A \approx \frac{A L_p}{\mu t} \left( \bar{p}_i - P_T \right) \left[ 1 - \frac{3 \mu L^2 \bar{Q}_0}{2 A \xi a^3 (\bar{p}_i - P_T)} \right],$$
(48)

where A = 2wL is the area of the membrane. This equation reveals that for hemodialyzers in which:

$$\frac{3\,\mu\,L^2\,\bar{Q}_0}{2\,A\,\xi\,a^3\,(\bar{p}_i - P_T)} << 1,\tag{49}$$

the ultrafiltration rate is given by:

$$\bar{Q}_A \approx \frac{A L_p}{\mu t} \left( \bar{p}_i - P_T \right).$$
(50)

An inspection of Equation (50) suggests that the ultrafiltration rate is directly proportional to the mechanical filtration coefficient  $L_p$  and membrane area and is inversely proportional to the membrane thickness and channel half width. The linear dependence of  $\bar{Q}_A$  on  $\bar{p}_i - P_T$  has been found experimentally by Malino et al. [32] and Mcdonald [33] and was suggested empirically by Marshall et al. [5]. A series of experiments performed by Brown et al. [34] also highlighted the dependence of  $\bar{Q}_A$  on the mechanical filtration coefficient, membrane thickness, and the membrane area.

Another important fact related to the renal tubular flow in kidneys can be observed from the expressions describing the mean pressure, flow rate, and leakage flux, respectively in (36), (40) and (41). These equations can be rewritten as:

$$\frac{P(x)}{p_i} = \cosh \eta \, x - \frac{1}{\zeta} \sinh \eta \, x, \tag{51}$$

$$Q(x) = \cosh \eta \, x - \zeta \, \sinh \eta \, x, \tag{52}$$

$$\frac{q(x)}{\eta} = \zeta \cosh \eta \, x - \sinh \eta \, x \tag{53}$$

where  $\zeta = \frac{2 p_i WK}{\eta}$ . A consideration of the right-hand sides of these equations suggests that the parameter  $\zeta$  influences the behavior of mean pressure, flow rate, and the leakage flux strongly. Figure 14 is drawn to explore this fact for some values of  $\zeta$ . This figure reveals that when  $\zeta > 1$ , the flow rate started decaying from its maximum value o one ( $\bar{Q}_0$  in dimensional form), became zero at certain point, and then became negative as  $\eta x \to \infty$ . Thus, a reverse flow situation arose when  $\zeta > 1$ . For  $\zeta < 1$ , Q(x) decayed initially from Q(0) = 1, attained its minimum value at a point, and then started increasing. This caused the reverse leakage phenomena for  $\zeta < 1$ . For  $\zeta = 1$ , the graph of flow rate behaved as an exponentially-decaying function. A similar discussion can also be made for the behavior of P(x) and q(x).



**Figure 14.** Variation of the axial flow rate with  $\eta x$ .

The above discussion together with Figure 14 reveal that for the creeping motion of a micropolar fluid in a porous-walled channel, in order to have no reverse flow and no reverse leakage, the value of

parameter  $\zeta$  must be approximately one. Substituting  $\zeta = 1$  in Equation (52) yields  $Q(x) = \exp(-\eta x)$ . This is a well-accepted result regarding the flow rate of filtrate in the renal tubule of kidneys, which was empirically proven by Kellman [35] and used by many researchers in the study of fluid flow in a permeable tube with application to renal tubules of kidneys [3,21,24,36].

## 7. Conclusions

The micropolar fluid model is admitted to have a generalized constitutive relation that describes the physiological flows of non-Newtonian fluids well. Therefore, the results of this article are applicable to study a vast family of physiological fluids in a permeable channel of small width. The obtained results can be reduced to the usual Newtonian models' result as a special case. The derived equations for the ultrafiltration rate and the mean pressure difference can be confidently used in studying the flow in a flat plate hemodialyzer. In applying the current results to study the problem of flow in a flat plate hemodialyzer, one should not overlook the physical aspects of the flow phenomenon. It is also concluded that the presented results are theoretical at their heart, therefore one should perform more experimental and theoretical investigation in order to have a complete understanding of the flow in a flat plate hemodialyzer.

**Author Contributions:** Conceptualization, M.K.; funding acquisition, D.L.; investigation, M.K.; methodology, M.K. and A.M.S.; project administration, D.L.; resources, D.L.; writing, original draft, M.K.; writing, review and editing, A.M.S.

Funding: This research was funded by China Postdoctoral Science Foundation grant number 2018M632237.

**Conflicts of Interest:** The authors declare no conflict of interest.

# Abbreviations

The following abbreviations are used in this manuscript:

FPD	Flat plate dialyzer
ρ	Fluid density
μ	Coefficient of viscosity
$\mu_m$	Coefficient of the micro-rotation viscosity
$\alpha_m, \beta_m, \gamma_m$	Viscosity coefficients of the angular velocity
$j_m$	Micro-inertia coefficient
и	Dimensionless tangential velocity component
υ	Dimensionless transverse velocity component
ω	Dimensionless microrotation velocity
$\bar{Q}_i$	Inlet flow rate
$\bar{p}_i$	Inlet pressure
λ	Ratio of the channel width to its length
$N^2$	Coupling number
$M^2$	Micropolar fluid parameter
Κ	Dimensionless wall filtration coefficient
W	Channel width to height ratio
$\phi$	Dimensionless wall slip parameter
$Q_A$	Dimensionless ultrafiltration rate
$\triangle P$	Dimensionless mean pressure drop

# References

- 1. Voutchkov, N. Desalination Engineering: Planning and Design; McGraw Hill: New York, NY, USA, 2012.
- 2. Macey, R.I. Pressure flow patterns in a cylinder with reabsorbing walls. *Bull. Math. Biophys.* **1963**, 25, 303–312. [CrossRef]
- 3. Macey, R.I. Hydrodynamics in the renal tubule. Bull. Math. Biophys. 1965, 27, 117–124. [CrossRef] [PubMed]
- 4. Marshall, E.A.; Trowbridge, E.A. Flow of a Newtonian fluid through a permeable tube: The application to the proximal renal tubule. *Bull. Math. Biophys.* **1974**, 25, 457–476. [CrossRef]

- 5. Marshall, E.A.; Trowbridge, E.A.; Aplin, A.J. Flow of a Newtonian fluid between parallel flat permeable plates—The application to a flat-plate hemodialyzer. *Math. Biophys.* **1975**, *27*, 119–139. [CrossRef]
- Sadeghi, R.; Shadloo, M.S.; Hirschler, M.; Hadjadj, H.A.; Nieken, U. Three-dimensional lattice Boltzmann simulations of high density ratio two-phase flows in porous media. *Comput. Math. Appl.* 2018, 75, 2445–2465. [CrossRef]
- Hirschler, M.H.; Shadloo, M.S.; Nieken, U. Viscous fingering phenomena in the early stage of polymer membrane formation. *J. Fluid Mech.* 2019, 864, 97–140. [CrossRef]
- 8. Kozinski, A.A.; Schmidt, F.P.; Lightfoot, E.N. Velocity profiles in porous-walled ducts. *Ind. Eng. Chem. Fundam.* **1970**, *9*, 502–505. [CrossRef]
- 9. Papanastasiou, T.C. Viscous Fluid Flow; CRC Press LLC: Florida, FA, USA, 2000.
- 10. Berman, A.S. Laminar flow in channels with porous walls. J. Appl. Phys. 1953, 24, 1232–1235. [CrossRef]
- 11. Berman, A.S. Laminar flow in an annulus with porous walls. J. Appl. Phys. 1958, 29, 71--75. [CrossRef]
- 12. Yuan, S.W.; Finkelstein, A.B. Laminar Pipe Flow With Injection and Suction Through a Porous Wall. *Trans. ASME* **1956**, *78*, 719–724.
- 13. Haroon, T.; Siddiqui, A.M.; Shahzad, A. Creeping Flow of Viscous Fluid through a Proximal Tubule with Uniform Reabsorption: A Mathematical Study. *Appl. Math. Sci.* **2016**, *10*, 795–807. [CrossRef]
- 14. Siddiqui, A.M.; Haroon, T.; Shahzad, A. Hydrodynamics of viscous fluid through porous slit with linear absorption. *Appl. Math. and Mech.* **2016**, *37*, 361–378. [CrossRef]
- 15. Byron, B.R. Dynamics of Polymeric Liquids; John Wiley & Sons: New York, NY, USA, 1987.
- 16. Eringen, A.C. Theory of micropolar fluids. J. Math. Mech. 1965, 1, 1–18.
- 17. Cowin, S.C. Polar fluids. Phys. Fluids 1968, 35, 1919–1927. [CrossRef]
- 18. Beavers, G.S.; Joseph, D.D. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.* **1967**, *30*, 197–207. [CrossRef]
- 19. Saffman, P.G. On the boundary condition at the surface of a porous medium. *Stud. Appl. Math.* **1971**, *50*, 93–101. [CrossRef]
- 20. Darcy, H.P.G. Dètermination des Lois D'ècoulement de L'eau à Travers le Sable; Victor Dalamont: Paris, France, 1856.
- 21. Siddiqui, A.M.; Haroon, T.; Kahshan, M. MHD flow of newtonian fluid in a permeable tubule. *Magnetohydrodynamics* **2015**, *51*, 655–672.
- 22. Kahshan, M.; Siddiqui, A.M.; Haroon, T. A micropolar fluid model for hydrodynamics in the renal tubule. *Eur. Phys. J. Plus* **2018**, 133, 546. [CrossRef]
- 23. Oka, S.; Murata, T. A theoretical study of the flow of blood in a capillary with permeable wall. *Jpn. J. Appl. Phys.* **1970**, *9*, 345. [CrossRef]
- 24. Siddiqui, A.M.; Haroon, T.; Kahshan, M.; Iqbal, M.Z. Slip Effects on the Flow of Newtonian Fluid in Renal Tubule. *J. Comput. Theor. Nanosci.* 2015, *12*, 4319–4328. [CrossRef]
- 25. Maple 18; Waterloo Maple Inc.: Waterloo, ON, Canada, 1981–2014.
- 26. Drukker, W.; Parsons, F.M.; Maher, J.F. *Replacement of Renal Function by Dialysis: A Textbook of Dialysis;* Springer Science & Business Media: Hingham, MA, USA, 2012.
- 27. Funck-Brentano, J.L.; Sausse, A.; Vantelon, J.; Granger, A.; Zingraff, J.; Man, N.K. A new disposable plate-kidney. *ASAIO J.* **1969**, *15*, 127–130.
- 28. Muthu, P.; Ratish, R.K.; Chandra, P. A study of micropolar fluid in an annular tube with application to blood flow. *J. Mech. Med. Biol.* **2008**, *8*, 561–576. [CrossRef]
- 29. Kiran, G.R.; Radhakrishnamcharya, G.; Bég, O.A. Peristaltic flow and hydrodynamic dispersion of a reactive micropolar fluid-simulation of chemical effects in the digestive process. *J. Mech. Med. Biol.* **2017**, *17*, 1750013. [CrossRef]
- 30. Shadloo, M.S.; Kimiaeifar, A.; Bagheri, D. Series solution for heat transfer of continuous stretching sheet immersed in a micropolar fluid in the existence of radiation. *Int. J. Numer. Methods Heat Fluid Flow* **2013**, *23*, 289–304. [CrossRef]
- 31. Funck-Brentano, J.L. Studies of intramembrane transport: A phenomenological approach. *AIChE J.* **1968**, *14*, 110–117.
- 32. Malinow, M.R.; Korzon, W. An experimental method for obtaining an ultrafiltrate of the blood. *Transl. Res.* **1947**, *12*, 461–471.

- 33. McDonald, J.R.; Harold, P. An automatic peritoneal dialysis machine: Preliminary report. *J. Urol.* **1966**, *96*, 397–401. [CrossRef]
- Brown, H.W.; Schreiner, G.E. Prolonged hemodialysis with bath refrigeration: The influence of dialyzer membrane thickness, temperature and other variables on performance. *Trans Am. Soc. Artif. Intern. Organs* 1962, *8*, 187–194. [CrossRef] [PubMed]
- 35. Kelman, R.B. A theoretical note on exponential flow in the proximal part of the mammalian nephron. *Bull. Math. Biol.* **1962**, *24*, 303–317. [CrossRef]
- 36. Radhakrishnamacharya, G.; Chandra, P; Kaimal, M.R. A hydrodynamical study of the flow in renal tubules. *Bull. Math. Biol.* **1981**, *43*, 151–163. [CrossRef] [PubMed]



 $\odot$  2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).