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Extended Degenerate r-Central Factorial Numbers of the Second Kind and Extended Degenerate r-Central Bell Polynomials

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Abstract: In this paper, we introduce the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. They are extended versions of the degenerate central factorial numbers of the second kind and the degenerate central Bell polynomials, and also degenerate versions of the extended r-central factorial numbers of the second kind and the extended r-central Bell polynomials, all of which have been studied by Kim and Kim. We study various properties and identities concerning those numbers and polynomials and also their connections.

Keywords: extended degenerate r-central factorial numbers of the second kind; extended degenerate r-central bell polynomials

1. Introduction

For $\lambda \in \mathbb{R}$, we recall that the degenerate exponential function $e_\lambda^x(t)$ is defined by (see [1–7])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \tag{1}$$

When $x = 1$, we let $e_\lambda(t) = e_\lambda^1(t)$. Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$.

We use the notation $(x)_n$ to denote the falling factorial sequence $(x)_n$, which is defined by (see [8–14])

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1) \tag{2}$$

More generally, for $\lambda \in \mathbb{R}$, the λ-falling factorial sequence $(x)_{n,\lambda}$ is given by (see [4])

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1) \tag{3}$$

Obviously, it is noted that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n, \quad \lim_{\lambda\to 0}(x)_{n,\lambda} = x^n, \quad (n \geq 0)$.

In Reference [4], the λ- binomial expansion is defined by

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \binom{x}{l}_\lambda t^l = \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}, \tag{4}$$

where
$$\binom{x}{l}_{\lambda}=\frac{(x)_{l,\lambda}}{l!}=\frac{x(x-\lambda)(x-2\lambda)\cdots(x-(l-1)\lambda)}{l!}.$$

The central factorial sequence is given by
$$x^{[0]}=1,\quad x^{[n]}=x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x-\frac{n}{2}+1),\quad (n\geq 1).$$

One can then easily show that the generating function of central factorial $x^{[n]}$, $(n\geq 0)$, is given by (see [3,15–20])
$$\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^{2x}=\sum_{n=0}^{\infty}x^{[n]}\frac{t^n}{n!}\tag{5}$$

As is defined in [18], for any non-negative integer n, the central factorial numbers of the first kind are given by
$$x^{[n]}=\sum_{k=0}^{n}t(n,k)x^k.\tag{6}$$

Then, from (5) and (6), we can show that the generating function of $t\,(n,k)$ satisfies the following equation:
$$\frac{1}{k!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)^k=\sum_{n=k}^{\infty}t\,(n,k)\,\frac{t^n}{n!}.$$

As the inverse to the central factorial numbers of the first kind, the central factorial numbers of the second kind are defined by (see [18,20–22])
$$x^n=\sum_{k=0}^{n}T_2(n,k)x^{[k]},\quad (n\geq 0)\tag{7}$$

The generating function of $T_2(n,k)$ can be easily derived from (7), which is given by (see [18])
$$\frac{1}{k!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k=\sum_{n=k}^{\infty}T_2(n,k)\frac{t^n}{n!},\quad (k\geq 0)\tag{8}$$

It can immediately be seen from (8) that
$$k!T_2(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\frac{1}{2}k-j)^n.\tag{9}$$

In Reference [22] were introduced the central Bell polynomials defined by
$$e^{x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}=\sum_{n=0}^{\infty}B_n^{(c)}(x)\frac{t^n}{n!}.\tag{10}$$

The Dobinski-like formula for $B_n^{(c)}(x)$ is given by (see [22])
$$B_n^{(c)}(x)=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}(-1)^k\frac{1}{(l+k)!}\left(\frac{l}{2}-\frac{k}{2}\right)^{l+1}\tag{11}$$

In Reference [3], the degenerate central factorial polynomials of the second kind are defined by

$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k e_\lambda^x(t)=\sum_{n=k}^{\infty}T_{2,\lambda}(n,k|x)\frac{t^n}{n!},\quad (k\geq 0). \tag{12}$$

When $x=0$, $T_{2,\lambda}(n,k)=T_{2,\lambda}(n,k|0)$, these are called degenerate central factorial numbers of the second kind.

Let us recall that the degenerate central Bell polynomials are defined by (see [3])

$$e^{x\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)}=\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}, \tag{13}$$

In particular, $B_{n,\lambda}^{(c)}=B_{n,\lambda}^{(c)}(1)$ are called the degenerate central Bell numbers.

Note that $\lim_{\lambda\to 0}B_{n,\lambda}^{(c)}(x)=B_n^{(c)}(x),\quad (n\geq 0)$.

Carlitz [1] introduced the degenerate Stirling, Bernoulli, and Eulerian numbers as the first degenerate special numbers. Broder [23] investigated the r-Stirling numbers of the first and second kind as the numbers counting restricted permutations and restricted partitions, respectively. We recall here that the r-Stirling numbers of the second kind are given by (see [23])

$$\frac{1}{k!}e^{rt}(e^t-1)^k=\sum_{n=k}^{\infty}S_2^{(r)}(n+r,k+r)\frac{t^n}{n!}, \tag{14}$$

In this paper, we will introduce the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. Central analogues of Stirling numbers of the second kind and Bell polynomials are, respectively, the central factorial numbers of the second kind and the central Bell polynomials. Degenerate versions of the central factorial numbers of the second kind and the central Bell polynomials are, respectively, the degenerate central factorial numbers of the second kind and the degenerate central Bell polynomials. Extended versions of the degenerate central factorial numbers of the second kind and the degenerate central Bell polynomials are, respectively, the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. The central factorial numbers of the second kind have many applications in such diverse areas as approximation theory [21], finite difference calculus, spline theory, spectral theory of differential operators [24,25], and algebraic geometry [26,27]. For broad applications of the related complete and incomplete Bell polynomials, we let the reader consult the introduction in [11]. Here, we will study various properties and identities relating to those numbers and polynomials, and also their connections. Finally, we note that the present paper can be useful in the area of non-integer systems and let the reader refer to [28] for more research in this direction.

2. Extended Degenerate r-Central Factorial Numbers of the Second Kind and Extended Degenerate r-Central Bell Polynomials

From (12) and (13), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}&=\sum_{k=0}^{\infty}x^k\sum_{n=k}^{\infty}T_{2,\lambda}(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}x^kT_{2,\lambda}(n,k)\frac{t^n}{n!}.\end{aligned} \tag{15}$$

One can compare the coefficients on both sides of (15) to obtain

$$B_{n,\lambda}^{(c)}(x)=\sum_{k=0}^{n}T_{2,\lambda}(n,k)x^k,\quad (n\geq 0). \tag{16}$$

Throughout this paper, we assume that r is a nonnegative integer. The following definition is motivated by (14).

Definition 1. *The extended degenerate r-central factorial numbers of the second kind $T_\lambda^{(r)}(n+r,k+r)$ are defined as*

$$\frac{1}{k!}e_\lambda^r(t)\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k=\sum_{n=k}^{\infty}T_\lambda^{(r)}(n+r,k+r)\frac{t^n}{n!}. \tag{17}$$

Note that $\lim_{\lambda\to 0}T_\lambda^{(r)}(n+r,k+r)=T^{(r)}(n+r,k+r),\quad (n,k\geq 0),$
where $T^{(r)}(n+r,k+r)$ is the extended r-central factorial numbers of the second kind given by

$$\frac{1}{k!}e^{rt}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k=\sum_{n=k}^{\infty}T^{(r)}(n+r,k+r)\frac{t^n}{n!}. \tag{18}$$

Theorem 1. *For $n,\ k\in\mathbb{N}\cup\{0\}$, with $n\geq k$, we have*

$$T_\lambda^{(r)}(n+r,k+r)=\sum_{l=k}^{n}\binom{n}{l}T_{2,\lambda}(l,k)(r)_{n-l,\lambda}.$$

Proof. By (17), we get

$$\begin{aligned}\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k e_\lambda^r(t)&=\sum_{l=k}^{\infty}T_{2,\lambda}(l,k)\frac{t^l}{l!}\sum_{m=0}^{\infty}(r)_{m,\lambda}\frac{t^m}{m!}\\&=\sum_{n=k}^{\infty}\sum_{l=k}^{n}\binom{n}{l}T_{2,\lambda}(l,k)(r)_{n-l,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{19}$$

Therefore, by (17) and (19), we obtain the result. □

We note that by taking the limit as λ tends to 0, we get

$$T^{(r)}(n+r,k+r)=\sum_{l=k}^{n}\binom{n}{l}r^{n-l}T_2(l,k). \tag{20}$$

Theorem 2. *For $n,\ k\geq 0$, with $n\geq k$, we have*

$$T_\lambda^{(r)}(n+r,k+r)=\sum_{m=k}^{n}\sum_{l=k}^{m}\binom{m}{l}S_1(n,m)T_2(l,k)\lambda^{n-m}r^{m-l}, \tag{21}$$

where $S_1(n,m)$ are the signed Stirling numbers of the first kind.

Proof. Replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (18), we obtain

$$\begin{aligned}\frac{1}{k!}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{k}e_{\lambda}^{r}(t)&=\sum_{m=k}^{\infty}\lambda^{-m}T^{(r)}(m+r,k+r)\frac{1}{m!}\Big(\log(1+\lambda t)\Big)^{m}\\&=\sum_{m=k}^{\infty}\lambda^{-m}T^{(r)}(m+r,k+r)\sum_{n=m}^{\infty}S_{1}(n,m)\frac{\lambda^{n}t^{n}}{n!}\\&=\sum_{n=k}^{\infty}\sum_{m=k}^{n}\lambda^{n-m}S_{1}(n,m)T^{(r)}(m+r,k+r)\frac{t^{n}}{n!}.\end{aligned}\tag{22}$$

Now, by substituting the expression of $T^{(r)}(m+r,k+r)$ in (20) into (22), we finally get

$$\frac{1}{k!}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{k}e_{\lambda}^{r}(t)=\sum_{n=k}^{\infty}\sum_{m=k}^{n}\sum_{l=k}^{m}\binom{m}{l}S_{1}(n,m)T_{2}(l,k)\lambda^{n-m}r^{m-l}\frac{t^{n}}{n!},$$

from which the result follows. □

Example 1. *Here, we will illustrate the formula (21) for small values of n. The following values of $T_2(n,k)$ can be determined, for example, from the formula in (9):*

$$T_2(n,n)=1,\ T_2(n,0)=\delta_{n,0},\ T_2(2,1)=T_2(3,2)=T_2(4,1)=T_2(4,3)=0, T_2(3,1)=\tfrac{1}{4},\ T(4,2)=1.\tag{23}$$

In addition, we recall the following values of $S_1(n,k)$:

$$\begin{aligned}&S_1(n,n)=1,\ S_1(n,0)=\delta_{n,0},\ S_1(2,1)=-1,\ S_1(3,1)=2,\\&S_1(3,2)=-3,\ S_1(4,1)=S_1(4,3)=-6,\ S_1(4,2)=11.\end{aligned}\tag{24}$$

Now, from (21), (23), and (24), we easily have

$$\begin{aligned}&T_{\lambda}^{(r)}(n+r,n+r)=1,\ T_{\lambda}^{(r)}(1+r,r)=r,\ T_{\lambda}^{(r)}(2+r,r)=-\lambda r+r^{2},\\&T_{\lambda}^{(r)}(3+r,r)=2\lambda^{2}r-3\lambda r^{2}+r^{3},\ T_{\lambda}^{(r)}(4+r,r)=-6\lambda^{3}r+11\lambda^{2}r^{2}-6\lambda r^{3}+r^{4},\\&T_{\lambda}^{(r)}(2+r,1+r)=-\lambda+2r,\ T_{\lambda}^{(r)}(3+r,1+r)=2\lambda^{2}-6\lambda r+3r^{2}+\frac{1}{4},\\&T_{\lambda}^{(r)}(3+r,2+r)=-3\lambda+3r,\ T_{\lambda}^{(r)}(4+r,1+r)=-6\lambda^{3}+22\lambda^{2}r-18\lambda r^{2}-\frac{3}{2}\lambda+4r^{3}+r,\\&T_{\lambda}^{(r)}(4+r,2+r)=11\lambda^{2}-18\lambda r+6r^{2}+1,\ T_{\lambda}^{(r)}(4+r,3+r)=-6\lambda+4r.\end{aligned}$$

Theorem 3. *For $n,\ k\geq 0$, with $n\geq k$, we have*

$$T_{\lambda}^{(r)}(n+r,k+r)=\sum_{m=0}^{n-k}\binom{m+k}{m}m!\binom{r}{m}T_{2,\lambda}(n,m+k|\frac{m}{2}).$$

Proof. Now, we observe that

$$\begin{aligned}\frac{1}{k!}e_\lambda^r(t)\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k &= \frac{1}{k!}e_\lambda^{\frac{r}{2}}(t)\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)\right)^r\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k\\ &= \frac{1}{k!}\sum_{m=0}^{\infty}\binom{r}{m}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^{m+k}e_\lambda^{\frac{m}{2}}(t)\\ &= \sum_{m=0}^{\infty}\binom{r}{m}\frac{(m+k)!}{k!}\frac{1}{(m+k)!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^{m+k}e_\lambda^{\frac{m}{2}}(t)\\ &= \sum_{m=0}^{\infty}\binom{r}{m}m!\binom{m+k}{m}\sum_{n=m+k}^{\infty}T_{2,\lambda}(n,m+k|\frac{m}{2})\frac{t^n}{n!}\\ &= \sum_{n=k}^{\infty}\sum_{m=0}^{n-k}\binom{r}{m}m!\binom{m+k}{m}T_{2,\lambda}(n,m+k|\frac{m}{2})\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Therefore, by (17) and (25), we obtain the theorem. □

One can easily show that the inverse function of $e_\lambda(t)$ is given by

$$\log_\lambda(t) = \frac{t^\lambda - 1}{\lambda}, \quad (t > 0),$$

so that $e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t$, $\lim_{\lambda\to 0}\log_\lambda(t) = \log(t)$.
If $g(t) = e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)$, then one can see that

$$g^{-1}(t) = \log_\lambda\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^2, \tag{26}$$

where $g \circ g^{-1}(t) = g^{-1}\circ g(t) = t$.

Theorem 4. *For $n \geq 0$, we have*

$$\begin{aligned}(x+r)_{n,\lambda} &= \sum_{k=0}^{n} T_\lambda^{(r)}(n+r,k+r)x^{[k]}\\ &= \sum_{k=0}^{n} T_{2,\lambda}(n,k|\frac{k}{2}+r)(x)_k.\end{aligned}$$

Proof. By (1) and (4), we get

$$\begin{aligned} e_\lambda^{x+r}(t) &= e_\lambda^r(t)\left(e_\lambda(t)-1+1\right)^x \\ &= e_\lambda^r(t)\sum_{k=0}^{\infty}(x)_k\frac{1}{k!}\left(e_\lambda(t)-1\right)^k \\ &= \sum_{k=0}^{\infty}(x)_k\frac{1}{k!}e_\lambda^{\frac{k}{2}+r}(t)\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k \\ &= \sum_{k=0}^{\infty}(x)_k\sum_{n=k}^{\infty}T_{2,\lambda}(n,k|\frac{k}{2}+r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(x)_kT_{2,\lambda}(n,k|\frac{k}{2}+r)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

Now, from the observations in (26) and (5), we have

$$\begin{aligned} e_\lambda^{x+r}(t) &= e_\lambda^r(t)e_\lambda^x(t) \\ &= e_\lambda^r(t)\left(e_\lambda\left(\log_\lambda\left(\frac{g(t)}{2}+\sqrt{1+\frac{g(t)^2}{4}}\right)^2\right)\right)^x \\ &= e_\lambda^r(t)\left(\frac{g(t)}{2}+\sqrt{1+\frac{g(t)^2}{4}}\right)^{2x} \\ &= \sum_{k=0}^{\infty}x^{[k]}\frac{1}{k!}e_\lambda^r(t)\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k \\ &= \sum_{k=0}^{\infty}x^{[k]}\sum_{n=k}^{\infty}T_\lambda^{(r)}(n+r,k+r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}x^{[k]}T_\lambda^{(r)}(n+r,k+r)\frac{t^n}{n!}. \end{aligned} \tag{28}$$

From (4), we note also that

$$e_\lambda^{x+r}(t) = \sum_{n=0}^{\infty}(x+r)_{n,\lambda}\frac{t^n}{n!}. \tag{29}$$

Therefore, by (27), (28), and (29), we have the desired result. □

Note that, taking the limit as λ tends to 0, we have

$$(x+r)^n = \sum_{k=0}^{n}T^{(r)}(n+r,k+r)x^{[k]} = \sum_{k=0}^{n}T_2(n,k|\frac{k}{2}+r)(x)_k\,.$$

Definition 2. *The extended degenerate r-central Bell polynomials $B_{n,\lambda}^{(c,r)}(x)$ are defined by*

$$e_\lambda^r(t)e^{x\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)} = \sum_{n=0}^{\infty}B_{n,\lambda}^{(c,r)}(x)\frac{t^n}{n!}. \tag{30}$$

Specifically, $B_{n,\lambda}^{(c,r)}(1) = B_{n,\lambda}^{(c,r)}$ are called the extended degenerate r-central Bell numbers.

Theorem 5. *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(c,r)}(x) = \sum_{k=0}^{n} x^k T_\lambda^{(r)}(n+r, k+r).$$

Proof. From (30), we note that

$$\begin{aligned} e_\lambda^r(t) e^{x\left(e_\lambda^{\frac{1}{2}}(t) - e^{-\frac{1}{2}}(t)\right)} &= \sum_{k=0}^{\infty} x^k \frac{1}{k!} \left(e_\lambda^{\frac{1}{2}}(t) - e^{-\frac{1}{2}}(t)\right)^k e_\lambda^r(t) \\ &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} T_\lambda^{(r)}(n+r, k+r) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} x^k T_\lambda^{(r)}(n+r, k+r) \frac{t^n}{n!}. \end{aligned} \tag{31}$$

Therefore, from (30) and (31), the theorem follows. □

The central difference operator δ for a given function f is given by

$$\delta f(x) = f(x + \frac{1}{2}) - f(x - \frac{1}{2}),$$

and by induction we can show

$$\delta^k f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(x + l - \frac{k}{2}), \quad (k \geq 0). \tag{32}$$

Theorem 6. *Let $n, \ k$ be nonnegative integers. Then, we have*

$$\frac{1}{k!} \delta^k (r)_{n,\lambda} = \begin{cases} 0, & \text{if } n < k, \\ T_\lambda^{(r)}(n+r, k+r), & \text{if } n \geq k. \end{cases}$$

Proof. By the binomial theorem, we have

$$\begin{aligned} \frac{1}{k!} e_\lambda^r(t) \left(e_\lambda^{\frac{1}{2}}(t) - e^{-\frac{1}{2}}(t)\right)^k &= \frac{1}{k!} e_\lambda^{r-\frac{k}{2}}(t) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e_\lambda^l(t) \\ &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e_\lambda^{r-\frac{k}{2}+l}(t) \\ &= \sum_{n=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (r - \frac{k}{2} + l)_{n,\lambda} \frac{t^n}{n!}. \end{aligned} \tag{33}$$

If we choose $f(x) = (x)_{n,\lambda}, \ (n \geq 0)$ in (32), then we have

$$\delta^k (r)_{n,\lambda} = \sum_{l=0}^{k} \binom{k}{l} (r + l - \frac{k}{2})_{n,\lambda} (-1)^{k-l}. \tag{34}$$

From (33) and (34), the following equation is obtained.

$$\frac{1}{k!}e_{\lambda}^{r}(t)\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{k}=\sum_{n=0}^{\infty}\frac{1}{k!}\delta^{k}(r)_{n,\lambda}\frac{t^{n}}{n!}. \tag{35}$$

Therefore, by (17) and (35), we have the result. □

From Theorem 4 and Theorem 5, we have

$$\begin{aligned} B_{n,\lambda}^{(c,r)}(x) &= \sum_{k=0}^{n} T_{\lambda}^{(r)}(n+r,k+r)x^{k} \\ &= \sum_{k=0}^{n} x^{k}\frac{1}{k!}\delta^{k}(r)_{n,\lambda}, \quad (n\geq 0). \end{aligned} \tag{36}$$

Theorem 7. *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(c,r)}(x)=\sum_{m=0}^{n}\binom{n}{m}(r)_{n-m,\lambda}B_{m,\lambda}^{(c)}(x).$$

Proof. From (30), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda}^{(c,r)}(x)\frac{t^{n}}{n!} &= e_{\lambda}^{r}(t)e^{x\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)} \\ &= \sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^{l}}{l!}\sum_{m=0}^{\infty}B_{m,\lambda}^{(c)}\frac{t^{m}}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(r)_{n-m,\lambda}B_{m,\lambda}^{(c)}(x)\frac{t^{n}}{n!}. \end{aligned} \tag{37}$$

Therefore, by comparing the coefficients on both sides of (37), the desired result is achieved. □

Theorem 8. *For $m,\ n,\ k \geq 0$, with $n \geq m+k$, we have*

$$\binom{m+k}{m}T_{\lambda}^{(r)}(n+r,m+k+r)=\sum_{l=m}^{n-k}\binom{n}{l}T_{\lambda}^{(r)}(l+r,m+r)T_{2,\lambda}(n-l,k).$$

Proof. We further observe that

$$\begin{aligned} \frac{1}{m!}e_{\lambda}^{r}(t)\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{m}\frac{1}{k!}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{k} &= \frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}e_{\lambda}^{r}(t)\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{m+k} \\ &= \binom{m+k}{m}\sum_{n=m+k}^{\infty}T_{\lambda}^{(r)}(n+r,m+k+r)\frac{t^{n}}{n!}, \end{aligned} \tag{38}$$

where $m,\ k$ are nonnegative integers. Alternatively, the left-hand side of (38) can be expressed by

$$\begin{aligned} \frac{1}{m!}e_{\lambda}^{r}(t)\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{m}\frac{1}{k!}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{k} &= \sum_{l=m}^{\infty}T_{\lambda}^{(r)}(l+r,m+r)\frac{t^{l}}{l!}\sum_{j=k}^{\infty}T_{2,\lambda}(j,k)\frac{t^{j}}{j!} \\ &= \sum_{n=m+k}^{\infty}\sum_{l=m}^{n-k}\binom{n}{l}T_{\lambda}^{(r)}(l+r,m+r)T_{2,\lambda}(n-l,k)\frac{t^{n}}{n!}. \end{aligned} \tag{39}$$

Therefore, by (38) and (39), the desired identity is obtained. □

3. Conclusions

In recent years, many researchers have studied a lot of old and new special numbers and polynomials by means of generating functions, through combinatorial methods, umbral calculus, differential equations, p-adic integrals, p-adic q-integrals, special functions, complex analyses, and so on.

The study of degenerate versions of special numbers and polynomials began with Carlitz [1]. Kim and his colleagues have been studying degenerate versions of various special numbers and polynomials by making use of the same methods. Studying degenerate versions of known special numbers and polynomials can be very a fruitful research and is highly rewarding. For example, this line of study led even to the introduction of degenerate Laplace transforms and degenerate gamma functions (see [4]).

In this paper, we introduced the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. We studied various properties and identities relating to those numbers and polynomials and also their connections. This study was done by using generating function techniques.

Central analogues of Stirling numbers of the second kind and Bell polynomials are, respectively, the central factorial numbers of the second kind and the central Bell polynomials. Degenerate versions of the central factorial numbers of the second kind and the central Bell polynomials are, respectively, the degenerate central factorial numbers of the second kind and the degenerate central Bell polynomials. Extended versions of the degenerate central factorial numbers of the second kind and the degenerate central Bell polynomials are, respectively, the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. The central factorial numbers of the second kind have many applications in diverse areas such as approximation theory [21], finite difference calculus, spline theory, spectral theory of differential operators [24,25], and algebraic geometry [26,27].

For future research projects, we would like to continue to work on some special numbers and polynomials and their degenerate versions, as well as try to explore their applications not only in mathematics but also in the sciences and engineering [29].

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References

1. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Util. Math.* **1979**, *15*, 51–88.
2. Jeong, J.; Rim, S.-H.; Kim, B.M. On finite-times degenerate Cauchy numbers and polynomials. *Adv. Differ. Equ.* **2015**, *2015*, 321. [CrossRef]
3. Kim, T.; Kim, D.S. Degenerate central Bell numbers and polynomials. *Rev. R. Acad. Clenc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **2019**. [CrossRef]
4. Kim, T.; Kim, D.S. Degenerate Laplace transform and degenerate gamma function. *Russ. J. Math. Phys.* **2017**, *24*, 241–248. [CrossRef]
5. Kim, Y.; Kim, B.M.; Jang, L.-C.; Kwon, J. A note on modified degenerate gamma and Laplace transformation. *Symmetry* **2018**, *10*, 471. [CrossRef]
6. Pyo, S.-S. Degenerate Cauchy numbers and polynomials of the fourth kind. *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, *28*, 127–138.

7. Upadhyaya, L.M. On the degenerate Laplace transform IV. *Int. J. Eng. Sci. Res.* **2018**, *6*, 198–209.
8. Carlitz, L. Some remarks on the Bell numbers. *Fibonacci Quart.* **1980**, *18*, 66–73.
9. Duran, U.; Acikgoz, M.; Araci, S. On (q, r, w)-Stirling numbers of the second kind. *J. Inequal. Spec. Funct.* **2018**, *9*, 9–16.
10. Kim, T.; Yao, Y.; Kim, D.S.; Jang, G.-W. Degenerate r-Stirling numbers and r-Bell polynomials. *Russ. J. Math. Phys.* **2018**, *25*, 44–58. [CrossRef]
11. Kim, T.; Kim, D.S.; Kim, G.-W. On central complete and incomplete Bell polynomials I. *Symmetry* **2019**, *11*, 288. [CrossRef]
12. Roman, S. *The Umbral Calculus*; Pure and Applied Mathematics 111; Academic Press Inc. [Harcourt Brace Jovanovich, Publishers]: New York, NY, USA, 1984.
13. Simsek, Y. Identities and relations related to combinatorial numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 127–135.
14. Simsek, Y. Identities on the Changhee numbers and Apostol-type Daehee polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2017**, *27*, 199–212.
15. Carlitz, L.; Riordan, J. The divided central differences of zero. *Can. J. Math.* **1963**, *15*, 94–100. [CrossRef]
16. Charalambides, C.A. Central factorial numbers and related expansions. *Fibonacci Quart.* **1981**, *19*, 451–456.
17. Kim, D.S.; Dolgy, D.V.; Kim, D.; Kim, T. Some identities on r-central factorial numbers and r-central Bell polynomials. *arXiv* **2019**, arXiv:1903.11689.
18. Kim, T. A note on central factorial numbers. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 575–588.
19. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Extended central factorial polynomials of the second kind. *Adv. Differ. Equ.* **2019**, *2019*, 24. [CrossRef]
20. Zhang, W. Some identities involving the Euler and the central factorinal numbers. *Fibonacci Quart.* **1998**, *36*, 154–157.
21. Butzer, P.L.; Schmidt, M.; Stark, E.L.; Vogt, L. Central factorial numbers; their main properties and some applications. *Numer. Funct. Anal. Optim.* **1989**, *10*, 419–488. [CrossRef]
22. Kim, T.; Kim, D.S. A note on central Bell numbers and polynomials. *Russ. J. Math. Phys.* **2019**, to appear.
23. Broder, A.Z. The r-Stirling numbers. *Discret. Math.* **1984**, *49*, 241–259. [CrossRef]
24. Everitt, W.N.; Kwon, K.H.; Littlejohn, L.L.; Wellman, R.; Yoon, G.J. JacobiStirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression. *J. Comput. Appl. Math.* **2007**, *208*, 29–56. [CrossRef]
25. Loureiro, A.F. New results on the Bochner condition about classical orthogonal polynomials. *J. Math. Anal. Appl.* **2010**, *364*, 307–323. [CrossRef]
26. Eastwood, M.; Goldschmidt, H. Zero-energy fields on complex projective space. *J. Differ. Geom.* **2013**, *94*, 129–157. [CrossRef]
27. Shadrin, S.; Spitz, L.; Zvonkine, D. On double Hurwitz numbers with completed cycles. *J. Lond. Math. Soc.* **2012**, *86*, 407–432. [CrossRef]
28. Caponetto, R.; Dongola, G.; Fortuna, L.; Gallo, A. New results on the synthesis of FO-PID controllers. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 997–1007. [CrossRef]
29. Kim, D.S.; Kim, T. A Note on Polyexponential and Unipoly Functions. *Russ. J. Math. Phys.* 2019, *94*, 40–49. [CrossRef]