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Family Symmetries and Multi Higgs Doublet Models

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Abstract: Imposing a family symmetry on the Standard Model in order to reduce the number of its free parameters, due to the Schur's Lemma, requires an explicit breaking of this symmetry. To avoid the need for this symmetry to break, additional Higgs doublets can be introduced. In such an extension of the Standard Model, we investigate family symmetries of the Yukawa Lagrangian. We find that adding a second Higgs doublet (2HDM) does not help, at least for finite subgroups of the $U(3)$ group up to the order of 1025.

Keywords: lepton masses and mixing; family symmetry

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1. Introduction

At currently achievable energies, the Standard Model (SM) of fundamental interactions is a very good working theory. However, it is commonly believed that it is only an effective theory which at higher energies need to be modified. One of the signs of this state of things is a large number of free parameters (more than 20) which now need to be fitted from experiments. The main parameters are: masses, mixing angles and CP violating phases for quarks and leptons. The SM does not explain those parameters but introduces the mechanism by means of which all particles acquire masses by the so called Higgs mechanism. One of several proposals how to restrict number of a free parameters in the SM is to introduce symmetry between Yukawa constants in Yukawa SM interaction in such a way that after spontaneous symmetry breaking get masses and mixing matrix parameters for quarks and leptons which are consistent with experience. Such symmetry is known in the literature as a flavor symmetry [1] (but also family or horizontal symmetry). Lepton sector and especially neutrino physics is an attractive area to search for such a symmetry due to the so called lepton mixing matrix [2,3].

There exist direct links between the mixing and lepton masses. Charged lepton and neutrino mass matrices $M_{l(\nu)}$ are diagonalized by biunitary transformations (for Majorana neutrinos $(U_\nu)_R = (U_\nu)_L^*$):

$$(U_{l(\nu)})_L^\dagger \, M_{l(\nu)} \, (U_{l(\nu)})_R \, . \tag{1}$$

The lepton mixing matrix U_{PMNS} is composed from the charged lepton $(U_l)_L$ and neutrino $(U_\nu)_L$ diagonalizing matrices:

$$U_{PMNS} = (U_l)_L^\dagger \, (U_\nu)_L \, . \tag{2}$$

Elements of matrix (2) are determined in various of neutrino experiments. When we impose a family symmetry in the ordinary not extended SM, we obtain [4,5]:

$$A_L^{i\dagger}\,(M_l\,M_l^\dagger)\,A_L^i = (M_l\,M_l^\dagger)\,, \tag{3}$$
$$A_L^{i\dagger}\,(M_\nu\,M_\nu^\dagger)\,A_L^i = (M_\nu\,M_\nu^\dagger)\,, \tag{4}$$

where:

$$A_L^i = A_L(g_i)\,, \quad i = 1,2,\ldots,N\,, \tag{5}$$

are 3 dimensional representations matrices for the left handed lepton doublets for some N-order flavour symmetry group $\mathcal{G}$.

As a direct consequence of the Schur's Lemma—since $M_l\,M_l^\dagger$ and $M_\nu\,M_\nu^\dagger$ are proportional to the identity matrix, the lepton mixing matrix U_{PMNS} becomes trivial.

In the literature there are some ideas about how to escape from the trivialisation of a matrix (2). One approach is to break the family symmetry group by scalar singlet—so called "flavons" (e.g., [6,7]). Non trivial mixing can be also achieved by extending the Higgs sector by additional multiplets (e.g., [8,9]). A proposal for the two Higgs doublet model (2HDM) [10] was widely discussed in [11]. In the most general situation, the mass generation mechanism allows couplings with various Higgs fields. In this context, extensions of the SM assuming the existence of different numbers of doublets and Higgs triplets are allowed. Theoretical proposals assuming the existence of two Higgs fields are not the only possible and potentially experimentally verifiable space for applying the symmetry implementation. In this paper, the methodology proposed previously for 2HDM only [11] is extended to any number of additional Higgs fields.

Additionally, new forms of results, equivalent to [11], for 2HDM are given. The obtained results depend on many phases and we present here a more detailed discussion concerning relations between them. We hope that it may help to determine the analytical origin of these solutions.

2. Multi Higgs Doublet Description

Discussion below stands for Dirac neutrinos. To describe the coupling between lepton fields and the Higgs field we take the n-Higgs doublet Yukawa interaction term of the form:

$$L_Y = -(h_i^l)_{\alpha\beta}\bar{L}_{\alpha L}\tilde{\Phi}_i l_{\beta R} - (h_i^\nu)_{\alpha\beta}\bar{L}_{\alpha L}\Phi_i \nu_{\beta R} + \text{H.c.}\,, \tag{6}$$

where $i = 1,2,\ldots,n$ and $\alpha,\beta = e,\mu,\tau$.

The charged lepton states $l_{\beta R}$ and neutrinos $\nu_{\beta R}$ are right-handed $SU(2)$ singlets and then the gauge doublets for the left-handed lepton and Higgs fields are:

$$L_{\alpha L} = \begin{pmatrix} \nu_{\alpha L} \\ l_{\alpha L} \end{pmatrix}\,, \tag{7}$$

$$\Phi_i = \begin{pmatrix} \varphi_i^0 \\ \varphi_i^- \end{pmatrix}\,, \quad \tilde{\Phi}_i = \begin{pmatrix} \varphi_i^{-*} \\ -\varphi_i^{0*} \end{pmatrix} = i\sigma_2\Phi_i^*\,, \tag{8}$$

where φ_i^0 and φ_i^- are complex scalar fields in spacetime for $i = 1,2,\ldots,n$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

The 3×3 Yukawa matrices h_i^l and h_i^ν each define the couplings of left-handed doublets with right-handed singlets via the i-th Higgs doublet. Due to the form of the Higgs potentials, ground states occur at non-zero φ, with the vacuum expectation values:

$$<\Phi_i>= \frac{1}{\sqrt{2}}\begin{pmatrix} v_i \\ 0 \end{pmatrix} \quad \text{and} \quad <\tilde{\Phi}_i>= \frac{1}{\sqrt{2}}\begin{pmatrix} -v_i^* \\ 0 \end{pmatrix}\,, \tag{9}$$

for some complex-valued v_i, where:

$$\sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2} = (\sqrt{2}G_F)^{-1/2} \sim 246\ \text{GeV}\,. \tag{10}$$

Mass matrices for charged leptons and neutrinos read as follows:

$$M_l = -\frac{1}{\sqrt{2}}(v_1^* h_1^l + \cdots + v_n^* h_n^l)\,, \tag{11}$$

$$M_\nu = \frac{1}{\sqrt{2}}(v_1 h_1^\nu + \cdots + v_n h_n^\nu)\,. \tag{12}$$

For some finite flavour group $\mathcal{G}$, the family symmetry means that after fields transformations (A_L, A_l^R and A_ν^R are 3 dimensional representations):

$$L_{\alpha L} \to L'_{\alpha L} = (A_L)_{\alpha,\chi}\, L_{\chi L}\,, \tag{13}$$

$$l_{\beta R} \to l'_{\beta R} = \left(A_l^R\right)_{\beta,\delta} l_{\delta R}\,, \tag{14}$$

$$\nu_{\beta R} \to \nu'_{\beta R} = \left(A_\nu^R\right)_{\beta,\delta} \nu_{\delta R}\,, \tag{15}$$

and (A_Φ is a n dimensional representation):

$$\Phi_i \to \Phi'_i = (A_\Phi)_{ik}\, \Phi_k\,, \tag{16}$$

the Lagrangian does not change:

$$\mathcal{L}(L_{\alpha L}, l_{\beta R}, \nu_{\beta R}, \Phi_i) = \mathcal{L}(L'_{\alpha L}, l'_{\beta R}, \nu'_{\beta R}, \Phi'_i)\,. \tag{17}$$

Symmetry conditions can be written as an eigenproblem of a direct product of unitary group representations to the eigenvalue 1. For any group elements we have:

$$\left((A_\Phi)^\dagger \otimes (A_L)^\dagger \otimes (A_l^R)^T\right)_{k,\alpha,\delta;i,\beta,\gamma} (h_i^l)_{\beta,\gamma} = (h_k^l)_{\alpha,\delta}\,, \tag{18}$$

$$\left((A_\Phi)^T \otimes (A_L)^\dagger \otimes (A_\nu^R)^T\right)_{k,\alpha,\delta;i,\beta,\gamma} (h_i^\nu)_{\beta,\gamma} = (h_k^\nu)_{\alpha,\delta}\,. \tag{19}$$

It is sufficient to check the above equations for group generators only as then they will automatically be satisfied for all group elements.

In such a model, the invariance equations for the mass matrices are not trivial, so we avoid the consequences of Schur's Lemma:

$$A_L M^{l(\nu)} \left(A_{l(\nu)}^R\right)^\dagger = \frac{1}{\sqrt{2}} \sum_{i,k=1}^{n} h_i^{l(\nu)} (A_\Phi)_{i,k}\, v_k \neq M^{l(\nu)}\,. \tag{20}$$

The same conclusion is valid if one assumes that neutrinos have Majorana nature. In such a frame, the Yukawa interaction Lagrangian has to be rewritten in an appropriate way (see [11]) producing family symmetry condition in the form:

$$\left((A_\Phi)^T \otimes (A_\Phi)^T \otimes (A^L)^\dagger \otimes (A^L)^\dagger\right)_{k,m,\chi,\eta,i,j,\alpha,\beta} (h_{ij}^\nu)_{\alpha,\beta} = (h_{km}^\nu)_{\chi,\eta}\,. \tag{21}$$

3. Two Higgs Doublet Model (2HDM) Results

As a potential flavour symmetry group $\mathcal{G}$, we chose finite, non-abelian subgroups of $U(3)$, up to the order of 1025. This class of groups is very important in practice [12], even though there exist models in which the flavour symmetry group cannot be conceived as a subgroup of $U(3)$ (the upper limit on the group order was of course due to the calculation time). Using the GAP [13] system for computational discrete algebra, with the included SMALL GROUPS LIBRARY [14] and the REPSN [15] package for constructing representations of finite groups, we have found groups which fulfil the requirements of our model and impose flavour symmetry on the Yukawa Lagrangian. Next we have calculated the Yukawa matrices and created mass matrices and mixing matrices. The last step was to check agreement with experimental data. In total we have found 10862 groups with at least one 2 dimensional and at least one 3 dimensional irreducible representation. Only 413 of these groups are subgroups of $U(3)$. Either a group has at least one faithful 3 dimensional irreducible representation (there are 173 such groups) or it has at least one faithful 1+2 reducible representation (there are 240 such groups).

3.1. Results for Dirac Combinations

All obtained solutions for Yukawa matrices for Dirac neutrinos and charged leptons can be expressed through seven base forms. Putting $\omega = e^{2\pi i k / 3}$ and allowing any integer k value (note that $\omega^3 = 1$ and $\omega^2 = \omega^*$), the first three forms are:

$$h_1 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}. \tag{22}$$

The next three forms can be obtained from the above ones through a simple interchange $h_1 \rightleftarrows h_2$. The last, seventh form, valid only for $k = \ldots, -2, 1, 4, \ldots$, is (note the diagonality of these matrices):

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad h_2 = h_1^*. \tag{23}$$

Dirac neutrinos are always defined by "ordered" pairs:

$$\{ h_1^\nu , h_2^\nu \} = \{ h_1 , e^{i\phi} h_2 \}, \tag{24}$$

where ϕ are some real phases (which depend on the actual group and its representations' combinations). For each of such Dirac neutrinos' solutions, there always exist two different solutions for charged leptons, defined by the two corresponding "ordered" pairs:

$$\{ h_1^l , h_2^l \} = \{ h_2 , e^{-i(\delta_l + \phi)} h_1 \}, \tag{25}$$

where $\delta_l = 0, \pi$.

Assuming complex c_x, real v_1, ϕ_1, v_2, ϕ_2 and putting:

$$M = c_x \left[v_1 e^{i\phi_1} h_1 + v_2 e^{i\phi_2} h_2 \right], \tag{26}$$

we get exactly the same set of three eigenvalues of $M M^\dagger$ for any of the seven above Yukawa matrices forms (note that this set of eigenvalues is invariant with respect to $v_1 \rightleftarrows v_2$ and/or $\phi_1 \rightleftarrows \phi_2$, hence we can safely use them for all cases $h_1 \rightleftarrows h_2$, and that if $v_1 v_2 \geq 0$ then $m_1^2 \leq m_2^2 \leq m_3^2$ when $\pi \leq \phi_2 - \phi_1 \leq 4\pi/3$ and $m_1^2 \geq m_2^2 \geq m_3^2$ when $0 \leq \phi_2 - \phi_1 \leq \pi/3$):

$$eigenvalues\ \left(M\, M^\dagger \right) \;=\; \begin{pmatrix} m_1^2 \\ \\ m_2^2 \\ \\ m_3^2 \end{pmatrix} \;=\; |c_x|^2 \begin{pmatrix} v_1^2 + v_2^2 + 2\, v_1\, v_2\, \cos\left(\phi_2 - \phi_1\right) \\ \\ v_1^2 + v_2^2 + 2\, v_1\, v_2\, \cos\left(\phi_2 - \phi_1 - 2\,\pi/3\right) \\ \\ v_1^2 + v_2^2 + 2\, v_1\, v_2\, \cos\left(\phi_2 - \phi_1 + 2\,\pi/3\right) \end{pmatrix}. \quad (27)$$

The mass squared differences are (note that if $v_1 v_2 \geq 0$ then $\Delta m_{31}^2 \geq \Delta m_{32}^2 \geq \Delta m_{21}^2 \geq 0$ when $7\pi/6 \leq \phi_2 - \phi_1 \leq 4\pi/3$ and $\Delta m_{31}^2 \leq \Delta m_{32}^2 \leq \Delta m_{21}^2 \leq 0$ when $\pi/6 \leq \phi_2 - \phi_1 \leq \pi/3$):

$$\begin{aligned} \Delta m_{21}^2 &= +2\,\sqrt{3}\,|c_x|^2\, v_1\, v_2\, \sin\left(\phi_2 - \phi_1 - \pi/3\right), \\ \Delta m_{31}^2 &= -2\,\sqrt{3}\,|c_x|^2\, v_1\, v_2\, \sin\left(\phi_2 - \phi_1 + \pi/3\right), \\ \Delta m_{32}^2 &= -2\,\sqrt{3}\,|c_x|^2\, v_1\, v_2\, \sin\left(\phi_2 - \phi_1\right). \end{aligned} \quad (28)$$

In all cases $M\, M^\dagger = M^\dagger\, M$ and then for the seventh form $M\, M^\dagger = diag\ \left(m_1^2,\, m_2^2,\, m_3^2 \right)$ (so, no neutrino mixing is possible at all), while for the first six forms it is:

$$M\, M^\dagger \;=\; |c_x|^2 \begin{pmatrix} v_1^2 + v_2^2 & v_1\, v_2\, e^{-i\,(\phi_2 - \phi_1 + 2\pi k/3)} & v_1\, v_2\, e^{i\,(\phi_2 - \phi_1 - 2\pi k/3)} \\ \\ v_1\, v_2\, e^{i\,(\phi_2 - \phi_1 + 2\pi k/3)} & v_1^2 + v_2^2 & v_1\, v_2\, e^{-i\,(\phi_2 - \phi_1)} \\ \\ v_1\, v_2\, e^{-i\,(\phi_2 - \phi_1 - 2\pi k/3)} & v_1\, v_2\, e^{i\,(\phi_2 - \phi_1)} & v_1^2 + v_2^2 \end{pmatrix}, \quad (29)$$

and the unitary matrix that diagonalizes it (so that $U^\dagger\, M\, M^\dagger\, U = diag\ \left(m_1^2,\, m_2^2,\, m_3^2 \right)$) is:

$$U \;=\; \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2\pi\, i\, k\,/\,3} & e^{-2\pi\, i\,(k-1)\,/\,3} & e^{-2\pi\, i\,(k+1)\,/\,3} \\ \\ 1 & e^{-2\pi\, i\,/\,3} & e^{2\pi\, i\,/\,3} \\ \\ 1 & 1 & 1 \end{pmatrix}. \quad (30)$$

Note that the U matrix does not depend on the phase difference $\phi_2 - \phi_1$ at all so, it will be exactly the same for neutrinos $(U_\nu)_L$ and charged leptons $(U_l)_L$. Hence, the $U_{PMNS} = (U_l)_L^\dagger\, (U_\nu)_L = I$ (so again, no neutrino mixing is possible at all).

In order to directly apply the above equations to Dirac neutrinos, one should put $c_x \to c_\nu$, $v_1 \to v_1$, $\phi_1 \to \phi_1$, $v_2 \to v_2$, $\phi_2 \to \phi + \phi_2$ (so $\phi_2 - \phi_1 \to \phi + \phi_2 - \phi_1$), while for charged leptons, one should put $c_x \to c_l$, $v_1 \to v_2$, $\phi_1 \to -\,(\delta_l + \phi + \phi_2)$, $v_2 \to v_1$, $\phi_2 \to -\phi_1$ (so $\phi_2 - \phi_1 \to \delta_l + \phi + \phi_2 - \phi_1$), where we assume that the vacuum expectation values are $v_1\, e^{i\,\phi_1}$ and $v_2\, e^{i\,\phi_2}$ (the same for neutrinos and charged leptons, of course). When moving between Dirac neutrinos and charged leptons, we can easily notice that all equations are invariant with respect to $v_1 \rightleftarrows v_2$ and the phase difference $\phi_2 - \phi_1$ is simply shifted by $\delta_l = 0, \pi$. That means that, for $\delta_l = \pi$, all mass squared differences Δm_{ij}^2 will change signs, so that their mass ordering schemes will be reversed, while for $\delta_l = 0$ there will be no change at all.

The ratios of experimental mass squared differences for neutrinos and charged leptons are:

$$\begin{aligned} |\Delta m_{atm}^2 / \Delta m_{sol}^2| \quad &\approx \quad (2.4\text{–}2.6) \times 10^{-3} / \,(7.4\text{–}7.7) \times 10^{-5} \quad \approx \quad 33 \pm 3, \\ \left(m_\tau^2 - m_e^2 \right) \,/\, \left(m_\mu^2 - m_e^2 \right) \quad &\approx \quad 282.8, \end{aligned} \quad (31)$$

and so they cannot be reproduced in this theoretical frame (they would need to be exactly equal while they differ by a factor of about 8 to 9).

Moreover, Equation (27) are not able to reproduce experimental masses of charged leptons at all. They return, in all cases, $1 \leq m_\tau / m_\mu \leq 2$ and requiring $m_\mu / m_e \approx 206.8$ or $m_\tau / m_e \approx 3477$ returns $m_\tau \approx m_\mu$.

3.2. Results for Majorana Combinations

All obtained solutions for Majorana neutrinos, which do not produce a scalar mass matrix M_ν, are defined by "ordered" quadruples:

$$\{\, h^\nu_{11}\,,\, h^\nu_{12}\,,\, h^\nu_{21}\,,\, h^\nu_{22}\,\} = \{\, h_2\,,\, v_0\, e^{i\,(\phi+\phi_0)}\, I_3\,,\, v_0\, e^{i\,(\delta+\phi+\phi_0)}\, I_3\,,\, e^{i\,(\delta+2\phi)}\, h_1\,\}, \tag{32}$$

where h_1 and h_2 are defined by Equation (23), ϕ are some real phases (which depend on the actual group and its representations' combinations), $\delta = 0, \pi$ and $v_0\, e^{i\,\phi_0}$ is a free complex parameter. For each of such Majorana neutrinos' solutions, there always exist two different solutions for charged leptons, defined by the two corresponding "ordered" pairs $\{\, h^l_1\,,\, h^l_2\,\} = \{\, h_2\,,\, e^{-i\,(\delta_l+\phi)}\, h_1\,\}$, where $\delta_l = 0, \pi$ (note that δ_l is independent of δ so, there always exist four different combinations for each ϕ).

Assuming complex c_ν (usually denoted as $g/(2M)$), real v_0, ϕ_0, v_1, ϕ_1, v_2, ϕ_2 and putting (where we assume that the vacuum expectation values are $v_1\, e^{i\,\phi_1}$ and $v_2\, e^{i\,\phi_2}$):

$$\begin{aligned} M_\nu &= c_\nu \left[v_1^2\, e^{2\,i\,\phi_1}\, h_{11} + v_1\, v_2\, e^{i\,(\phi_1+\phi_2)}\, (h_{12} + h_{21}) + v_2^2\, e^{2\,i\,\phi_2}\, h_{22} \right] \\ &= c_\nu \left[v_2^2\, e^{i\,[\delta+2(\phi+\phi_2)]}\, h_1 + v_1^2\, e^{2\,i\,\phi_1}\, h_2 + v_0\, v_1\, v_2 \left(1 + e^{i\,\delta}\right) e^{i\,(\phi+\phi_0+\phi_1+\phi_2)}\, I_3 \right], \end{aligned} \tag{33}$$

we find that the neutrino mass matrix M_ν is a diagonal matrix and $M_\nu\, M_\nu^\dagger = M_\nu^\dagger\, M_\nu = diag\,\left(m_1^2,\, m_2^2,\, m_3^2\right)$, where:

$$\begin{pmatrix} m_1^2 \\ \\ \\ m_2^2 \\ \\ \\ m_3^2 \end{pmatrix} = |c_\nu|^2 \begin{pmatrix} v_1^4 + v_2^4 + 2\,v_1^2\,v_2^2\,\cos\left[\delta + 2\,(\phi+\phi_2-\phi_1)\right] \\ +4\,v_0\,v_1\,v_2\,\left[\,v_1^2\,\cos\left(\phi+\phi_2-\phi_1+\phi_0\right) \right. \\ \left. +v_2^2\,\cos\left(\phi+\phi_2-\phi_1-\phi_0\right) + v_0\,v_1\,v_2\,\right]\,\cos\left(\delta/2\right) \\ \\ v_1^4 + v_2^4 + 2\,v_1^2\,v_2^2\,\cos\left[\delta + 2\,(\phi+\phi_2-\phi_1) + 2\,\pi/3\right] \\ +4\,v_0\,v_1\,v_2\,\left[\,v_1^2\,\cos\left(\phi+\phi_2-\phi_1+\phi_0-2\,\pi/3\right) \right. \\ \left. +v_2^2\,\cos\left(\phi+\phi_2-\phi_1-\phi_0-2\,\pi/3\right) + v_0\,v_1\,v_2\,\right]\,\cos\left(\delta/2\right) \\ \\ v_1^4 + v_2^4 + 2\,v_1^2\,v_2^2\,\cos\left[\delta + 2\,(\phi+\phi_2-\phi_1) - 2\,\pi/3\right] \\ +4\,v_0\,v_1\,v_2\,\left[\,v_1^2\,\cos\left(\phi+\phi_2-\phi_1+\phi_0+2\,\pi/3\right) \right. \\ \left. +v_2^2\,\cos\left(\phi+\phi_2-\phi_1-\phi_0+2\,\pi/3\right) + v_0\,v_1\,v_2\,\right]\,\cos\left(\delta/2\right) \end{pmatrix}. \tag{34}$$

The mass squared differences are:

$$\begin{aligned} \Delta m_{21}^2 &= 2\sqrt{3}\,|c_\nu|^2\,v_1\,v_2\,\{\,-v_1\,v_2\,\sin\left[\delta + 2\,(\phi+\phi_2-\phi_1) + \pi/3\right] \\ &\quad +2\,v_0\,\left[v_1^2\,\sin\left(\phi+\phi_2-\phi_1+\phi_0-\pi/3\right) + v_2^2\,\sin\left(\phi+\phi_2-\phi_1-\phi_0-\pi/3\right)\right]\,\cos\left(\delta/2\right)\,\}, \\ \Delta m_{31}^2 &= 2\sqrt{3}\,|c_\nu|^2\,v_1\,v_2\,\{\,+v_1\,v_2\,\sin\left[\delta + 2\,(\phi+\phi_2-\phi_1) - \pi/3\right] \\ &\quad -2\,v_0\,\left[v_1^2\,\sin\left(\phi+\phi_2-\phi_1+\phi_0+\pi/3\right) + v_2^2\,\sin\left(\phi+\phi_2-\phi_1-\phi_0+\pi/3\right)\right]\,\cos\left(\delta/2\right)\,\}, \\ \Delta m_{32}^2 &= 2\sqrt{3}\,|c_\nu|^2\,v_1\,v_2\,\{\,+v_1\,v_2\,\sin\left[\delta + 2\,(\phi+\phi_2-\phi_1)\right] \\ &\quad -2\,v_0\,\left[v_1^2\,\sin\left(\phi+\phi_2-\phi_1+\phi_0\right) + v_2^2\,\sin\left(\phi+\phi_2-\phi_1-\phi_0\right)\right]\,\cos\left(\delta/2\right)\,\}. \end{aligned} \tag{35}$$

For $\delta = \pi$ ($e^{i\,\delta} = -1$), we can also reuse Equations (27) and (28), when we put $c_x \to c_\nu$, $v_1 \to v_2^2$, $\phi_1 \to \pi + 2\,(\phi + \phi_2)$, $v_2 \to v_1^2$, $\phi_2 \to 2\phi_1$ (so $\phi_2 - \phi_1 \to \pi - 2\,(\phi + \phi_2 - \phi_1)$). Note that the mass ordering schemes will be reversed between $\delta = 0$ and $\delta = \pi$, which can easily be seen if one puts $v_0 = 0$ in Equations (34) and (35). For the corresponding charged leptons, we can also simply reuse Equations (27) and (28), where we put $c_x \to c_l$, $v_1 \to v_2$, $\phi_1 \to -\,(\delta_l + \phi + \phi_2)$, $v_2 \to v_1$, $\phi_2 \to -\phi_1$ (so $\phi_2 - \phi_1 \to \delta_l + \phi + \phi_2 - \phi_1$). As already mentioned, Equation (27) is not able to reproduce experimental masses of charged leptons at all. Moreover, as all relevant matrices are diagonal (for both neutrinos and charged leptons), no neutrino mixing is possible at all.

4. Conclusions

Multiple Higgs doublet models are in general promising that, in order to get a non trivial lepton mixing matrix, one will not need to explicitly break the family symmetry. However, our results for the 2HDM are utterly negative. The big open question is why. First, we select finite, non-abelian subgroups of $U(3)$ which are provided by the SMALL GROUPS LIBRARY [14] in GAP [13]. Is it possible that we miss some vital groups (due to the constrains that we impose when selecting groups or due to the used library itself)? Then, is it possible that non trivial solutions can only be obtained for models with more than two Higgs doublets? Finally, is it possible that the Equations (18), (19) and (21) will always lead to only trivial solutions, even though these models seem to be free from the consequences of the Schur's Lemma (20)? In order to address, at least partially, the last two questions, we are currently working on processing groups which are not subgroups of $U(3)$ for 2HDM and on a family symmetry approach with three Higgs doublets, hoping that it will give some positive outcome.

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Abbreviations

The following abbreviations are used in this manuscript:

SM	Standard Model
PMNS	Pontecorvo–Maki–Nakagawa–Sakata
2HDM	Two Higgs doublet model

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