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Review

The Pauli Exclusion Principle and the Problems of Its Experimental Verification

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Abstract: The modern state of the Pauli exclusion principle is shortly discussed. We describe the discovery by Pauli, his principle for electrons, and how it was generalized for all elementary particles in the framework of quantum mechanics. The motivations and theoretical conceptions that induced the experiments for verification of the Pauli exclusion principle are analyzed. The results and methodology of two different types of experiments are discussed: (1) the search of unusual atoms and nuclei in the stable non-Pauli states, and (2) the experiments in which the emitted radiation of non-Pauli transitions is measured. In conclusion, the comments on the discussed experiments that follow from the general quantum mechanical conceptions and group theory are formulated.

Keywords: pauli exclusion principle; spin-statistics connection; indistinguishability principle; permutation symmetry; boson and fermion particles

1. Introduction: Discovery of the Pauli Exclusion Principle and Its Generalized Formulation for all Elementary Particles

Wolfgang Pauli formulated his principle before the creation of quantum mechanics. As is well known, the conceptions of quantum mechanics were formulated in 1925 by Heisenberg, Born, and Jordan [1,2] in the matrix formalism. In 1926, Schrödinger, based on the wave-particle dualism, suggested by de Broglie [3], introduced the wave function ψ describing microparticles and formulated his famous wave equation [4,5].

Pauli arrived at the formulation of his principle trying to explain regularities in the anomalous Zeeman effect in strong magnetic fields. In his first studies of the Zeeman effect, Pauli was interested in the explanation of the simplest case, the doublet structure of the alkali spectra. In December of 1924, Pauli submitted a paper on the Zeeman effect [6], in which he showed that Bohr's theory of a doublet structure, which was based on the non-vanishing angular moment of a closed shell, such as K-shell of the alkali atoms, is incorrect and the closed shell has no angular and magnetic moments. Pauli came to the conclusion that, instead of the angular momentum of the closed shells of the atomic core, a new quantum property of the electron had to be introduced. In that paper, he wrote prophetic words, remarkable for that time:

> "According to this point of view, the doublet structure of alkali spectra ... is due to a particular two-valuedness of the quantum theoretic properties of the electron, which cannot be described from the classical point of view."

This non-classical two-valued quantum property of electron is now called *spin*. In anticipating the quantum nature of the magnetic moment of electron before the creation of modern quantum mechanics, Pauli exhibited a striking intuition.

Basing on his results on the classification of spectral terms in a strong magnetic field, Pauli came to conclusion that a single electron must occupy an entirely nondegenerate energy level. In the paper submitted for publication on 16 January 1925, Pauli formulated his principle as follows [7]:

In an atom there cannot be two or more equivalent electrons, for which in strong fields the values of all four quantum numbers coincide. If an electron exists in an atom for which all of these numbers have definite values, then this state is "occupied".

In this paper, Pauli introduced four quantum numbers of a single electron in atom, n, l, $j = l \pm 1/2$, and m_j (in the modern notations); by n and l, he denoted the well-known (at that time) principal and orbital angular momentum quantum numbers, where j and m_j are the total angular momentum and its projection, respectively. Thus, Pauli characterized the electron by some additional quantum number j, which in the case of $l = 0$ was equal to $\pm 1/2$. For this new quantum number j, Pauli did not give any physical interpretations, since he was sure that it could not be described in the terms of classical physics.

Pauli noted the following in his Nobel Prize lecture [8]:

> "Physicists found it difficult to understand the exclusion principle, since no meaning in terms of a model was given to the fourth degree of freedom of the *electron*."

Although not all physicist. Young scientist, firsts Ralph Kronig, then George Uhlenbeck and Samuel Goudsmith did not take into account the Pauli words that the fourth degree of freedom of electron cannot be described by classical physics and suggested the classical model of the spinning electron. In a book [9], I described in detail the discovery of spin using the reminiscences of main participants of this dramatic story.

The first studies devoted to the application of the newborn quantum mechanics to many-particle systems were performed independently by Heisenberg [10] and Dirac [11]. In these studies, the Pauli exclusion principle (PEP), originally formulated as the prohibition for two electrons to occupy the same quantum state, was obtained as a consequence of the antisymmetry of the Schrödinger wave function. In both papers [10,11], for the first time, the antisymmetric many-electron wave functions were constructed, and it was concluded that these functions cannot have two particles in the same state.

Heisenberg constructed the N-electron wave function using the antisymmetrization operator,

$$\hat{A}_N = \frac{1}{\sqrt{N!}} \sum_P (-1)^p P, \tag{1}$$

where P runs over all $N!$ permutations of N particles, and p is the parity of permutation P (number of transpositions of two particles that it contains) applied to the antisymmetrized product of one-electron wave functions.

Dirac represents the many-electron antisymmetric function as a determinant constructed with one-electron wave functions ψ_{n_i} (see Equation (2). Let us stress that the determinantal representation of the electronic wave function, which at present is widely used in atomic and molecular calculations, was first introduced in general form by Dirac [11] in 1926. In 1929, Slater [12] introduced the spin functions into the determinant (so-called Slater's determinants) for calculations of the atomic multiplets.

$$\Psi_{n_1 n_2 \ldots n_r}(1,2,\ldots,r) = \begin{vmatrix} \psi_{n_1}(1) & \psi_{n_1}(2) & \ldots & \psi_{n_1}(r) \\ \psi_{n_2}(1) & \psi_{n_2}(2) & \ldots & \psi_{n_2}(r) \\ \ldots & \ldots & \ldots & \ldots \\ \psi_{n_r}(1) & \psi_{n_r}(2) & \ldots & \psi_{n_r}(r) \end{vmatrix}, \tag{2}$$

where the number of electrons $N = r$. After presenting the many-electron wave function in the determinantal form, Dirac [11] wrote the following:

> "An antisymmetric eigenfunction vanishes identically when two of the electrons are in the same orbit. This means that, in the solution of the problem with antisymmetric eigenfunctions,

there can be no stationary states with two or more electrons in the same orbit, which is just Pauli's exclusion principle."

Thus, with the creation of quantum mechanics, the prohibition on the occupation numbers of electron system states was supplemented with the prohibition of all types of permutation symmetry of electron wave functions except the antisymmetric ones.

However, the first application of PEP was performed in astrophysics by Fowler [13], in just the following year after Pauli suggested his principle. Fowler applied PEP for an explanation of the white-dwarf structure. The radius of white dwarfs is comparable with the earth's radius, while their mass is comparable with the solar mass. Consequently, the average density of white dwarfs is 10^6 times greater than the average density of the sun; it is approximately 10^6 g/sm^3. Fowler [13] resolved the paradox of why dense objects, such as white dwarfs, do not collapse at low temperature. He applied the Fermi–Dirac statistics to the electron gas in the white dwarfs, also introduced in 1926, and showed that, even at very low temperatures, the electron gas, called degenerate in these conditions, still possesses a high energy. Compression of a white dwarf leads to an increase of the inner electron pressure, which is due, according to Fowler, to the exclusion principle suggested by Pauli. Thus, the repulsion from PEP prevents the white dwarfs from undergoing gravitational collapse.

In 1928, Dirac [14] created the rigorous relativistic quantum theory of the electron, which naturally included the conception of spin. Some consequences of Dirac's relativistic theory were analyzed by Schrödinger in his remarkable paper [15]. In that paper, Schrödinger [15] revealed that the rapid oscillatory motion of a massless charge with a velocity c around a center of mass follows from the Dirac relativistic equation for the electron, which he named *Zitterbewegung*. This original picture developed by Schrödinger induced a broad discussion of the origin of spin, but it is beyond of the scope of this paper.

In 1932, Chadwick [16] discovered the neutron. In the same year, Heisenberg [17] considered consequences of the model, in which the nuclei are built from protons and neutrons, but not from electrons and protons, as was accepted at that time. Heisenberg assumed that the forces between all pairs of particles are equal and, in this sense, the proton and neutron can be considered as different states of one particle. He introduced a variable τ, where the value $\tau = -1$ was assigned to the proton state, and the value $\tau = 1$ was assigned to the neutron state. Wigner [18] called τ isotopic spin (at present, also named isobaric spin). The isotopic spin has only two values and, as in the fermion case, can be represented as $\tau = 1/2$. Taking into account that the nuclear spin (s) of protons and neutrons is also equal to $1/2$, Wigner studied the nuclear charge-spin supermultiplets for Hamiltonians, not depending on the isotope and nuclear spins.

Later, the analysis of experimental data for discovered elementary particles revealed that they obey only two types of permutation symmetry: symmetric and antisymmetric. This allowed formulating PEP not only for electrons, but for all elementary particle.

> *The only possible states of a system of identical particles possessing spin s are those for which the total wave function transforms upon interchange of any two particles as*
>
> $$P_{ij}\Psi(1,\ldots,i,\ldots j,\ldots,N) = (-1)^{2s}\Psi(1,\ldots,i,\ldots j,\ldots,N). \quad (3)$$
>
> *That is, it is symmetric for integer values of s (the Bose–Einstein statistics) and antisymmetric for half-integer values of s (the Fermi–Dirac statistics).*

This general formulation holds also for composite particles, which was first shown by Ehrenfest and Oppenheimer [19]. The authors considered some clusters of electrons and protons; they could be atoms, molecules, or nuclei (at that time the neutron was yet to be discovered). They formulated a rule, according to which the statistics of a cluster depends upon the number of particles from which they are built up. Systems with an odd number of particles obey the Fermi–Dirac statistics, while systems with an even number of particles obey the Bose–Einstein statistics. It was stressed that this rule is valid

if the interaction between composite particles does not change their internal states, i.e., the composite particle is stable enough to preserve its identity.

A good example of such a stable composite particle is the atomic nucleus. It consists of nucleons (protons and neutrons), which are fermions because they have spin s = 1/2. Depending on the value of the total nuclear spin, one can speak of boson nuclei or fermion nuclei (see Figure 1). The nuclei with an even number of nucleons have an integer value of the total spin *S* and are bosons; the nuclei with an odd number of nucleons have a half-integer value of the total spin *S* and are fermions. A well-known system, in which the validity of PEP for composite particles was precisely checked in an experiment, was the $^{16}O_2$ molecule (see a detailed discussion in Reference [20]).

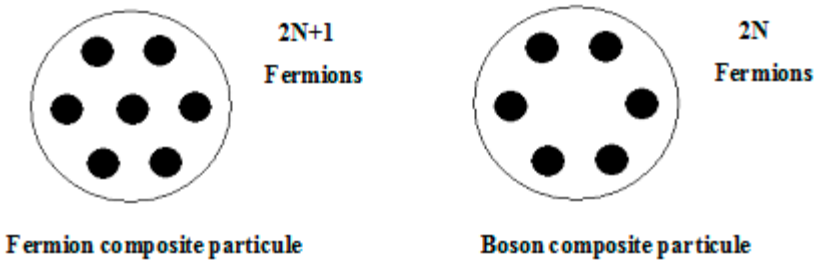

Figure 1. The statistics of composite particles.

As follows from the generalized formulation of PEP, particles with half-integer spin (fermions) are described by antisymmetric wave functions, and particles with integer spin (bosons) are described by symmetric wave functions. This is the so-called *spin–statistics connection* (SSC). However, PEP does not reduce only to SSC. It can be considered from another also very important aspect—the restrictions on the allowed symmetry types of many-particle wave functions. Specifically, only two types of permutation symmetry are allowed: symmetric and antisymmetric. Both belong to the one-dimensional representations of the permutation group, while all other types of permutation symmetry are forbidden. However, the Schrödinger equation is invariant under any permutation of identical particles. The Hamiltonian of an identical particle system commutes with the permutation operators,

$$[P, H]_{-} = 0. \tag{4}$$

From this, it follows that the solutions of the Schrödinger equation may belong to any representation of the permutation group, including multi-dimensional representations. The reasons for this paradox were revealed by the author (see References [20,21]), where the properties of identical particle systems, whose symmetry is described by multi-dimensional representation of the permutation group, were analyzed. In those studies, one of the aspects of PEP, according to which only particles with symmetric and antisymmetric permutation symmetry are allowed, was theoretically substantiated.

Section 2 is devoted to the experimental verifications of PEP. In Section 2.1, the motivation and theoretical models that induced the experiments for verification of PEP are analyzed. In Section 2.2, a comparative discussion of results and methodology of different types of experiments is presented. In conclusion, Section 2.3 presents some critical comments on the discussed experiments that follow from the general quantum mechanical conceptions and group theory. In Appendix A, the necessary mathematical apparatus for the permutation group is represented.

2. Experimental Verifications of the Pauli Exclusion Principle

2.1. Motivation: Theoretical Conceptions on the PEP Violations

All experimental data known to date agree with PEP. The question can be asked, why we need to make special experiments for finding its possible violations? In 1956, for the first time, it was shown that the fundamental laws can be violated. Wu et al. [22] discovered parity non-conservation in

β-decay. Then, in 1964, the violation of Charge-Parity (CP) invariance was discovered [23]. After that, all fundamental laws started to be tested.

Okun, in his reviews [24,25], presented a detailed discussion of the possible violations of the electric charge conservation law, due to a decay of the electron. This idea was first raised by Okun, Zeldovich, and Voloshin [26,27] and Ignat'ev et al. [28]. The probable decay of one electron on the $1s^2$ atomic energy level will induce the radiative transition $2p^6 \rightarrow 1s^1$ allowed in quantum mechanics (see Figure 2), while the transition to the filled $1s^2$ K-shell can only occur if PEP is violated. As was stressed in References [26,27], in contrast with the spontaneous violation of parity conservation in β-decay or violation of CP invariance, the non-conservation of electric charge requires a global reconstruction of the contemporary theory of electromagnetic processes. Nevertheless, in some publications, the authors discussed the conditions in which charge non-conservation can take place (see References [28–31]).

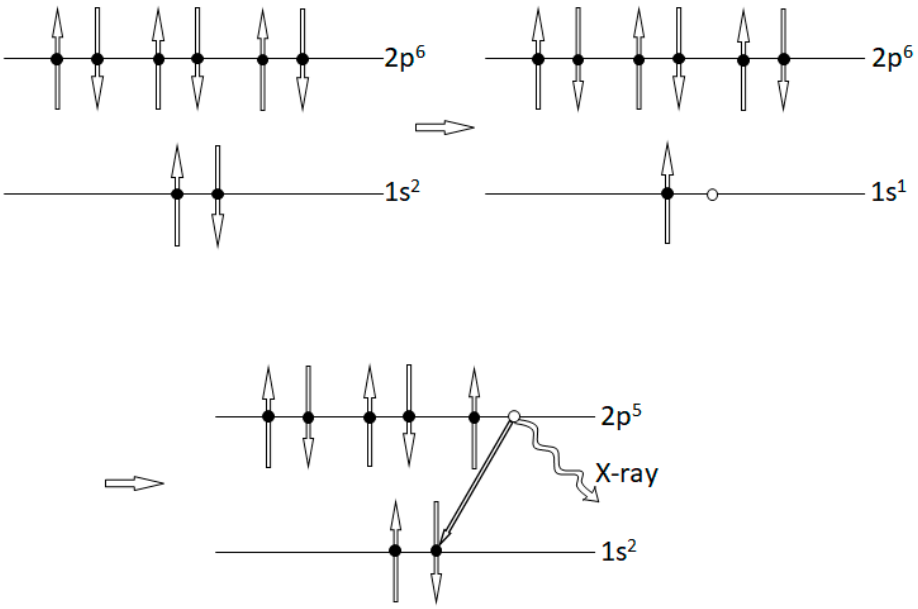

Figure 2. Scheme of permitted radiative transition to K-shell, if the 1*s* electron undergoes spontaneous decay.

The theoretical ideas for experimental searches of possible violations of PEP were discussed by many authors (References [24,25,32–35]). Okun [24,25] noted that the careful reader can make the conclusion from the famous Dirac book [36] that, in quantum mechanics with a permutation-invariant Hamiltonian (in the case of identical particles), the transitions to filled shells are forbidden regardless of whether PEP is violated. In 1980, Amado and Primakoff [37] used similar arguments discussing the experiments by Reines and Sobel [38] for electrons and by Logan and Ljubicic [39] for nuclei. They showed that, due to the identity of particles, these experiments cannot test PEP, since the transitions from the normal antisymmetric fermionic states are possible only to the states with the same antisymmetric character (superselection rules). The non-Pauli electrons, if they do exist, must have another symmetry and, according to the superselection rules, transitions are allowed only inside the same system of non-Pauli electrons.

The theoretical model of a small violation of PEP was first suggested by Ignat'ev and Kuz'min [40]. Greenberg and Mohapatra [41] and Okun [42] tried to generalize this model to quantum field theory. However, they recognized that, as follows from the Govorkov studies [43,44], even small violations of PEP lead to appearance of states with negative norm (negative probability) in quantum field theory. Greenberg and Mohapatra (see References [32,33,45]) discussed different modifications of their theory that were not connected to the Govorkov results [43,44]. One of the possible ways was to use the conception of infinite statistics [46]. At present, the Greenberg quon model [47] is widely applied in

experiments searching for small violations of PEP (see report by Hilborn [48]). Interesting information is contained in the original Greenberg's report [32].

2.2. Experimental Tests of the PEP Violations: What Is Measured

There are two basically different types of experiments to test the PEP violation. The first one is based on the search of atoms or nuclei that are in a stable non-Pauli state; the second one is based on the measurements of emitted radiation if non-Pauli transitions take place in atoms or nuclei.

The idea for the first type of experiment was suggested by Gavrin, Ignatiev, and Kuzmin [49] and then developed by Novikov and Pomanskii [50]. They suggested searching for anomalous atoms that could be created in our universe for billions of years during the stage of nucleosynthesis. In these elements, one of valence electrons of the upper shell "fell" as a consequence of violating the Pauli transitions on the lowest shell, forming the three-electron population on the K-shell forbidden by PEP. The valence shell of anomalous atoms of element Z should have one electron less, just as for the atoms of element $Z-1$. Thus, their chemical properties should be similar.

This was the main point of experiments made by Novikov, Pomanskii et al. [51], and Nolte et al. [52]. They tried detecting small concentrations of anomalous atoms of the element with atomic number Z within a very large natural sample of an element with atomic number $Z-1$. In their experiments [51,52], the accelerator mass spectrometry detection method with a time-of-flight set-up [53] was used. Their method allowed measuring the concentration of atoms different from the stable isotope (less than 10^{-15}). The search for non-Pauli atoms of $^{20}\widetilde{\mathrm{Ne}}$ and $^{36}\widetilde{\mathrm{Ar}}$ was performed. Negative F (fluorine) and Cl (chlorine) ions were extracted from natural fluorine and chlorine samples, respectively. The results of processing these measurements yielded the following upper limits:

$$\begin{aligned} \frac{^{20}\widetilde{\mathrm{Ne}}}{^{20}\mathrm{Ne}} &< 2\times 10^{-21}, \\ \frac{^{36}\widetilde{\mathrm{Ar}}}{^{36}\mathrm{Ar}} &< 4\times 10^{-17}. \end{aligned} \tag{5}$$

Nolte et al. [52] also performed experiments for the search of non-Pauli nuclei (see also References [54–57]).

The second type of experiment, in which the non-Pauli transitions are measured, is performed in nuclei and atoms. The violations of PEP in nuclei were studied by searching for different types of non-Pauli transitions with γ emission [58,59], p emission [60,61], n emission [62], non-Pauli β^+ and β^- decays [59,63], and nuclear reactions, for instance, on ^{12}C [64]. It should be mentioned that the prohibition of non-Pauli transitions [37], following from the superselection rules is not valid for transitions changing the number of identical fermions and in some other cases (see References [65,66]).

According to Bellini et al. [65], the following experimental bounds on PEP violations in ^{12}C nuclei for non-Pauli transitions were reported:

$$\begin{aligned} \tau\left(^{12}\mathrm{C} \to ^{12}\widetilde{\mathrm{C}} + \gamma\right) &\geq 5.0\times 10^{31}\ \mathrm{yr}, \\ \tau\left(^{12}\mathrm{C} \to ^{11}\widetilde{\mathrm{B}} + \mathrm{p}\right) &\geq 8.9\times 10^{29}\ \mathrm{yr}, \\ \tau\left(^{12}\mathrm{C} \to ^{11}\widetilde{\mathrm{C}} + \mathrm{n}\right) &\geq 3.4\times 10^{30}\ \mathrm{yr}, \\ \tau\left(^{12}\mathrm{C} \to ^{11}\widetilde{\mathrm{N}} + \mathrm{e}^- + \widetilde{\nu}_\mathrm{e}\right) &\geq 3.1\times 10^{30}\ \mathrm{yr}, \\ \tau\left(^{12}\mathrm{C} \to ^{12}\widetilde{\mathrm{B}} + \mathrm{e}^+ + \widetilde{\nu}_\mathrm{e}\right) &\geq 2.1\times 10^{30}\ \mathrm{yr}. \end{aligned} \tag{6}$$

The experimental tests of the PEP violations in atoms were classified and discussed in reviews by Gillaspy [67], Ignatiev [34], and some others (see also Table 1 in Elliott et al. [68]). As we mentioned in Section 2.1, in 1980, when very few experimental tests of the PEP violations were performed, Amado and Primakoff [37] showed that, due to the superselection rules, the non-Pauli transitions are strictly forbidden regardless of PEP. Nevertheless, in subsequent experiments, it was accepted that there are

no prohibitions on the transition of non-Pauli electrons via the cascade process to the filled K-shell, although it can be expected that these transitions *already* occurred. This was the main idea behind the above-described experiments of the first type [49–52].

Ramberg and Show [69] suggested injecting electric current in a copper strip, assuming that, if these "new" electrons contain a non-Pauli electron, it can be captured by Cu atoms and then cascade down to the filled K-shell. They searched for X-rays of the PEP-violating transition from the $2p^7$ energy level to the $1s^2$ level, forming the $1s^3$ level forbidden by PEP. For the probability of PEP-violating transitions, which is usually denoted by the parameter $1/2\ \beta^2$ [40,70], the following quite impressive low value was obtained [69]:

$$\frac{1}{2}\beta^2 \leq 1.7 \times 10^{-26}. \quad (7)$$

The experiment of Ramberg and Show induced a great number of PEP experimental verifications made according to the scheme represented in Figure 3.

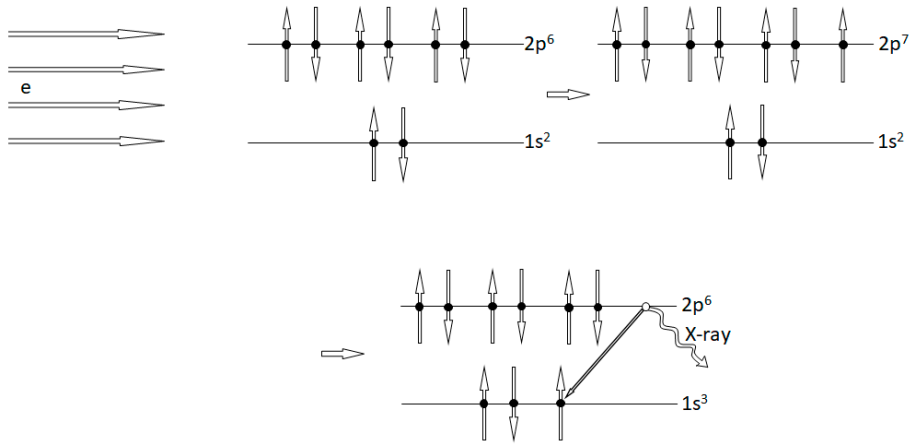

Figure 3. The schematic representation of the formation of the Pauli-forbidden K-shell population in experiments of the Ramberg and Snow type.

In the last few years, systematic tests of PEP for electrons were carried out by the Violation PEP (VIP) collaboration [71–74] using the experimental methodology of Ramberg and Show, but with essentially improved sensitivity. This was achieved by using more precise experimental apparatus, e.g., applying a special high-resolution X-ray detector and performing experiments deeply underground, which provided very low cosmic background. The limit obtained in VIP experiments for the probability of the Pauli-forbidden atomic transition from the $2p^7$ shell to the closed $1s^2$ shell of Cu atoms was as follows [72]:

$$\frac{1}{2}\beta^2 < 4.7 \times 10^{-29}. \quad (8)$$

In the experiments performed in the Los Alamos laboratory by Elliott et al. [68], Pb instead of Cu was used. Pb produces higher-energy X-rays and has several other properties that are beneficial for an experimental precision. They reported a much stronger limit on the violation of PEP for electrons, i.e.,

$$\frac{1}{2}\beta^2 < 2.6 \times 10^{-39}. \quad (9)$$

This limit was obtained by a modified procedure of the experimental data processing. The authors [68] did not divide electrons into "new" and "old"; they considered all free electrons, i.e., all electrons in the conductivity band. The application of this approach to the VIP data also changed their limit by 10 orders. However, if one accepts the possibility of cascade transitions of "old" non-Pauli electrons to the K-shell, which motivated Ramberg and Show to inject the electric current with "new" electrons in a copper strip, the idea of taking into account all electrons in a conductivity

band would become non-substantiated. However, the study by Elliott et al. [68] indicated that the result of such kinds of experiments strongly depends on an estimation of the number of free electrons that can interact with Cu atoms.

At present, the VIP collaboration is upgrading their set-up and is performing VIP-2 experiments [75–79]. They implemented different improvements in the experimental apparatus [75,76]. The first results obtained on their upgrade set-up, using a modified processing of experimental data, showed a small improvement in comparison with the VIP final result in Equation (8), i.e., $\frac{1}{2}\beta^2 < 4.2 \times 10^{-29}$ [76] and $\frac{1}{2}\beta^2 < 3.4 \times 10^{-29}$ [77]. These results were obtained assuming a straight path of injected electrons across the Cu strip, as was accepted by Ramberg and Show [69].

Milloti et al. [78] essentially reconsidered a very simplified physical picture accepted by Ramberg and Show for processing their experiment. They stressed that the path of electrons in metals is not straight; electrons diffuse through the metal and perform complex random walks. The one-dimensional (1D) diffusion transport model with the classical Einstein expression for the diffusion constant was used. The authors [78] took also into account that the real process of the electron scattering occurs in impurities, lattice imperfections, phonons, etc. Using the experimental data of the last VIP-2 experiment [77] and considering their accepted physical process of electron scattering, they obtained an improved, very high limit on the PEP violation probability, i.e.,

$$\frac{1}{2}\beta^2 < 2.6 \times 10^{-40}. \tag{10}$$

Finally, Milloti et al. [78] noted that their analysis of the random walk that electrons undergo in copper material is mostly classical, and they planned on performing it at the quantum-mechanical level. In the next VIP-2 publication [79], it was confirmed that, at present, they are working to extend their theoretical analysis to the quantum domain.

Thus, the treatment of VIP-2 experiments is still in the development stage. It can be expected that some comments, outlined below, following from the general quantum-mechanical conceptions and group theory, can be useful for the treatment of the VIP-2 experimental data, as well as for other types of the experimental tests of PEP.

2.3. Comments on the Experimental Tests of the Pauli-Forbidden Electron Transitions

A. Let us discuss the theoretical problems that can arise in applications of the experimental scheme depicted in Figure 3. Usually, experimenters consider the violation of PEP as a small admixture of the symmetric wave functions to the antisymmetric ones. As we mentioned in previous subsections, they start from the Ignatiev–Kuzmin [40] and Greenberg–Mohapatra [41] theoretical suggestions, while the Greenberg quon model [46] was applied in recent years. These theoretical approaches are based on the second quantization formalism created only for the symmetric and antisymmetric states. In general, this limitation on the permutation symmetry of the possible states is not valid. As we noted in Section 1, the Hamiltonian of an identical particle system commutes with the permutation operators (see Equation (4)). From this, it follows that the solutions of the Schrödinger equation may belong to any irreducible representations of the permutation group, including multi-dimensional representations. The violation of PEP means that there are some electrons described by wave functions with arbitrary permutation symmetry, not necessarily with the symmetric one. It is only evident that these electrons are not described by the antisymmetric wave functions; otherwise, they would be identical to the "normal" Pauli electrons.

Thus, the states of non-Pauli electrons are characterized by another permutation symmetry than the "normal" Pauli electrons. The non-Pauli electrons can possess all possible types of permutation symmetry. Next, I consider this in more detail.

B. Let us consider a quantum system of N identical particles that does not obey PEP. For this system, all possible permutation symmetries can be realized. The permutation symmetry of the N identical particle system is characterized by the irreducible representations of the permutation group

π_N, which are labeled by the symbol $[\lambda]$ of the corresponding Young diagram with N boxes and denoted by $\Gamma^{[\lambda]}$ or simply by $[\lambda]$ (see Appendix A). The construction of the basis functions of irreducible representations of the permutation group π_N can be performed by applying the Young operators $\omega_{rt}^{[\lambda]}$ (Equation (A5)) to the non-symmetrized product of one-electron orthonormal functions (Equation (A2)). The obtained function $\Psi_{rt}^{[\lambda]}$ (Equation (A6)) is transformed under permutations according to the representation $\Gamma^{[\lambda]}$. From the generalized formulation of PEP, see p.3, it follows that only irreducible one-dimensional representations, either $\Gamma^{[N]}$ or $\Gamma^{[1^N]}$, are realized in our nature; all other irreducible representations are forbidden. However, for non-Pauli electrons, this restriction is not valid. It is instructive to analyze properties of such unusual electrons.

One of the consequences of the different permutation symmetry of wave functions for bosons and fermions is the dependence of their energy on the particle statistics. For the same law of dynamic interaction, the so-called exchange terms enter the expression for the energy of fermion and boson systems with opposite signs. The expression for the energy in the state described by an arbitrary Young diagram $[\lambda]$ with N boxes was derived in Reference [80] in a general case of non-orthogonal one-particle functions. In the case where all functions are different and orthogonal, one gets

$$E_t^{[\lambda]} = \sum_a \langle \psi_a | h | \psi_a \rangle + \sum_{a<b} \left[\langle \psi_a \psi_b | g | \psi_a \psi_b \rangle + \Gamma_{tt}^{[\lambda]}(P_{ab}) \langle \psi_a \psi_b | g | \psi_b \psi_a \rangle \right], \quad (11)$$

where $\Gamma_{tt}^{[\lambda]}(P_{ab})$ is the diagonal matrix element of the transposition P_{ab} of functions ψ_a and ψ_b in Equation (A2), and h and g are one- and two-particle operators, respectively. Only exchange terms in Equation (11) depend upon the symmetry of the state. For one-dimensional representations, $\Gamma_{tt}^{[\lambda]}(P_{ab})$ does not depend on the number of particles and permutations, i.e., $\Gamma^{[N]}(P_{ab}) = 1$ and $\Gamma^{[1^N]}(P_{ab}) = -1$ for all P_{ab} and N. For multi-dimensional representations, the matrix elements $\Gamma_{tt}^{[\lambda]}(P_{ab})$ depend on $[\lambda]$ and P_{ab}; in general, they are different for different pairs of identical particles (the matrices of transpositions for all irreducible representations of groups π_2–π_6 are presented in Reference [81], Appendix 5).

If we take into account that (a) the states of N particle system with different $[\lambda_N]$ have different analytical formulae for its energy, and (b) transitions between states with different symmetry $[\lambda_N]$ are strictly forbidden (superselection rule), then we must conclude that each type of symmetry $[\lambda_N]$ corresponds to a definite kind of particle with statistics determined by this permutation symmetry.

On the other hand, the classification of state with respect to the Young diagrams $[\lambda_N]$ is connected exclusively to the identity of particles. Therefore, there must be some additional *inherent particle characteristics,* which establishes the N particle system in a state with definite permutation symmetry, such as integer and half-integer values of particle spin for bosons and fermions, and this inherent characteristic must be different for different $[\lambda_N]$.

Thus, the non-Pauli particles, characterized by different types of permutation symmetry $[\lambda_N]$, *are not identical* and may not be mixed not only with "normal" electrons with symmetry $[1^N]$, but also among themselves.

C. One of the most important problems in the experimental testing of PEP violations is an estimation of the energy of the Pauli-forbidden transitions, i.e., how to define the energy range of the anomalous X-rays or gamma radiation that is expected to be observed. As it was discussed above, the energy of an identical particle system depends upon its permutation symmetry $[\lambda_N]$ and, for $N \geq 3$, several possible symmetries, corresponding to different energies, must be considered. However, at present, we do not know any quantum-mechanical phenomena where PEP is not satisfied. Therefore, the probability of the existence of non-Pauli electrons must be very small, and it is reasonable to assume that, in the electronic systems studied in experiments, no more than one non-Pauli electron can exist. This was accepted in the VIP experiments.

As described by Shi et al. [77], the calculation of the Pauli-forbidden radiative transition energy was performed using the relativistic multiconfiguration Dirac–Fock method developed by Desclaux [82].

In this method, the multiconfiguration Dirac–Fock equations were self-consistently solved, and most of relativistic effects such as the Breit–Pauli operators, the Lamb shift, radiation corrections, and others were considered. Mallow et al. [83] applied this method to study muonic atoms. Muons (μ-meson) have the same negative charge as an electron, but they are heavier ($m_\mu \cong 200 m_e$). As detected in different experiments, muons can be attached to atoms in molecules and solids, forming muonic atoms (see Reference [83]). Since a non-Pauli electron, as with a muon, is also non-identical to normal electrons and differs from muons only by mass, the VIP group assumed that a non-Pauli electron can also be attached to atoms in solids; in particular, it can be captured by Cu atoms in metallic strips. The capture probability into the $2p$-shell was assumed to be the same as in the Ramberg–Snow experiment [69], as 0.1. In this connection, I would like to mention that the authors [69] made their estimation of the capture probability from rather qualitative considerations. It is desirable to define this parameter more accurately (if it is possible). Several comments on the calculation method [77] of the Pauli-forbidden radiative transition energy are presented below.

(a) Mallon et al. [83] applied the Desclaux method for heavy atoms. For these atoms, the relativistic approach is appropriate; however, in this case, the j–j coupling should be used (as they did), and spin ceases to be a good quantum number. On the other hand, the elements of the fourth period of the periodical table, with $3d$-electrons, can be calculated with good precision at the non-relativistic level (see very precise MR CI calculations in References [84,85]). Thus, Cu can be calculated at the non-relativistic level using the Russel-Saunders (L–S) coupling and without the Breit–Pauli, Lamb shift, and others relativistic corrections. It is worthwhile to mention that, in this case, the non-relativistic Greenberg quon model [46] is quite justified.

(b) If, in analogy with muons, it can be assumed that a non-Pauli electron can be attached to atoms in solids, it is quite doubtful that it will be located on the same atomic shells as the "normal" Pauli electrons and then to the K-shell. This is a crucial point in the treatment of experiments for detecting the Pauli forbidden transition, and it must be argued more carefully.

3. Concluding Remarks

In this review, I presented a comparative discussion of different types of PEP verification experiments. I analyzed the methodology and results of these experiments and formulated some critical comments following from the general quantum mechanical conceptions and group theory.

The most precise results in experiments for the measurement of the Pauli-forbidden transitions in different nuclear reactions were observed for the nucleus ^{12}C (see Bellini et al. [65] and Equation (6) in Section 2). It is important to mention that the prohibition of non-Pauli transitions following from the superselection rules is not valid for transitions with a changing number of identical fermions.

The most precise results in experiments for the measurement of the Pauli-forbidden radiation for electrons were obtained recently in the VIP-2 experiments (see Equation (10)). These experiments are still in the development stage. Thus, we can expect more precise results in the near future. However, in Section 2.3, we mentioned several physical problems in the treatment of VIP experiments that must be substantiated more rigorously.

In general, we should state that, although the experiments on the verification of PEP were performed for more than 50 years, we still do not have a satisfactory and robust model for describing the possible violation of PEP. This is a consequence of the exclusive complexity of the problems arising in these experiments. For instance, in the measurements of the Pauli-forbidden radiation for electrons, we should detect particles for which we do not know anything, except that they are charged as electrons and their wave function is not antisymmetric.

Nevertheless, despite the above-discussed problems, I agree with the general statement made by Okun [25]: "in fundamental physics, if something could be tested, it should be tested."

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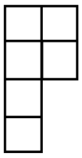
Appendix A. Some Necessary Knowledge on the Permutation Group

The permutation symmetry is classified according to the irreducible representations of the permutation group π_N. They are labeled by Young diagrams.

$$\begin{gathered}[\lambda] = [\lambda_1 \lambda_2 \ldots \lambda_k], \\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k, \quad \sum_{i=1}^{k} \lambda_i = N.\end{gathered} \tag{A1}$$

Here, λ_i is represented by a row of λ_i cells. The presence of several rows of identical length λ_i is convenient to indicate by the power of λ_i. For example,



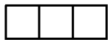
$[\lambda] = [2^2\ 1^2]$.

For π_2, one can form only two Young diagrams from two cells.

[2]

$[1^2]$

For the permutation group of three elements, π_3, one can form three Young diagrams from three cells.

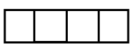


[3]

[2.1]

$[1^3]$

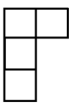
Group π_4 has five Young diagrams.



[4]

[3 1]

$[2^2]$



$[2\ 1^2]$

$[1^4]$

Each Young diagram $[\lambda]$ uniquely corresponds to a specific irreducible representation $\Gamma^{[\lambda]}$ of the group π_N. The assignment of a Young diagram determines the permutation symmetry of the basis functions for an irreducible representation, i.e., it determines the behavior of the basis functions under permutations of their arguments. The diagram with only one row corresponds to a symmetric function in all its arguments, and with one column corresponds to a completely antisymmetric function, both correspond to one-dimensional representations. All other types of diagrams correspond to intermediate types of symmetry and describe multi-dimensional representations.

There are certain rules that enable one to find the matrices of irreducible representations of the permutations group from the form of the corresponding Young diagram. Such rules are especially simple in the case of the so-called standard orthogonal representation, or the Young–Yamanouchi representation.

Let us consider a non-symmetrized product of one-electron functions.

$$\Psi_0 = \psi_{n_1\sigma_1}(1)\psi_{n_2\sigma_2}(2)\ldots\psi_{n_N\sigma_N}(N), \tag{A2}$$

where i in the argument of the function denotes the set of particle spin and space coordinates. For simplicity, we consider the case when all one-particle functions are different. There will be no qualitative changes in the results if some of them coincide. The antisymmetric many-electron function can be constructed acting on Ψ_0 (Equation (2)) by the antisymmetrization operator

$$\hat{A}_N = \frac{1}{\sqrt{N!}}\sum_P (-1)^P P, \tag{A3}$$

where P runs over all $N!$ permutations of N particles, and p is the *parity* of permutation P (number of transpositions of two particles in P).

$$\begin{array}{rcl} \Psi^{[1^N]} & = & \frac{1}{\sqrt{N!}}\sum_P (-1)^P P\psi_{n_1\sigma_1}(1)\psi_{n_2\sigma_2}(2)\ldots\psi_{n_N\sigma_N}(N) \\ & \equiv & \frac{1}{\sqrt{N!}}\det\left|\psi_{n_1\sigma_1}(1)\psi_{n_2\sigma_2}(2)\ldots\psi_{n_N\sigma_N}(N)\right|. \end{array} \tag{A4}$$

The basis functions for an irreducible representation $\Gamma^{[\lambda]}$ can be constructed similarly by means of the so-called normalized Young operators,

$$\omega_{rt}^{[\lambda]} = \sqrt{\frac{f_\lambda}{N!}}\sum_P \Gamma_{rt}^{[\lambda]}(P)P, \tag{A5}$$

where the summation over P runs over all $N!$ permutations of the group $\pi_{\mathbf{N}}$, $\Gamma^{[\lambda]}$ the matrix elements, and f_λ is the dimension of the irreducible representation $\Gamma^{[\lambda]}$. For $[\lambda] = [1^N]$, the Young operator coincides with the presented above antisymmetrization operator.

The application of the Young operator to a non-symmetrized product of orthonormal one-particle functions produces a normalized function,

$$\Psi_{rt}^{[\lambda]} = \omega_{rt}^{[\lambda]}\Psi_0 = \sqrt{\frac{f_\lambda}{N!}}\sum_P \Gamma_{rt}^{[\lambda]}(P)P\Psi_0, \tag{A6}$$

transforming under permutations as the r-th column of the representation $\Gamma^{[\lambda]}$,

$$P\Psi_{rt}^{[\lambda]} = \sum_u \Gamma_{ut}^{[\lambda]}(P)\Psi_{ut}. \tag{A7}$$

The set of f_λ functions $\Psi_{rt}^{[\lambda]}$ with fixed second index t forms a basis for the representation $\Gamma^{[\lambda]}$, and t enumerates f_λ different bases for $\Gamma^{[\lambda]}$. The second index t describes the symmetry under permutation of the one-electron functions in the non-symmetrized product (Equation (A2)).

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