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Herglotz's Variational Problem for Non-Conservative System with Delayed Arguments under Lagrangian Framework and Its Noether's Theorem

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Abstract: Because Herglotz's variational problem achieves the variational representation of non-conservative dynamic processes, its research has attracted wide attention. The aim of this paper is to explore Herglotz's variational problem for a non-conservative system with delayed arguments under Lagrangian framework and its Noether's theorem. Firstly, we derive the non-isochronous variation formulas of Hamilton–Herglotz action containing delayed arguments. Secondly, for the Hamilton–Herglotz action case, we define the Noether symmetry and give the criterion of symmetry. Thirdly, we prove Herglotz type Noether's theorem for non-conservative system with delayed arguments. As a generalization, Birkhoff's version and Hamilton's version for Herglotz type Noether's theorems are presented. To illustrate the application of our Noether's theorems, we give two examples of damped oscillators.

Keywords: non-conservative system with delayed arguments; Noether's theorem; Herglotz generalized variational principle; Lagrangian framework

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1. Introduction

Time delay is a common phenomenon in nature and engineering. Although time delays have often been ignored in the past and many problems have been solved, with the increasingly precise requirements for the dynamical behavior and control of complex systems, the effects of time delays on the system need to be considered. It has been shown that even millisecond delay can lead to complex dynamical behavior of the system. In addition, for many delayed systems, if the time delay is ignored, it will lead to a completely wrong conclusion. Therefore, the study of the dynamical characteristics of time-delayed systems is not only extremely important to the understanding of these systems themselves, but also to the research of biology, ecology, neural network, physics, electronics and information science, mechanical engineering, and other research fields [1–4]. For the variational problem in the case of delay, El'sgol'c first mentioned its extremum characteristic in [5]. Hughes derived the necessary conditions for a time-delayed variational problem in 1968 [6], which is similar to the classical one. Frederico and Torres [7] were the first to propose and prove the extension of Noether's theorem to time-delay variational problems and optimal control. In 2013, in reference [8], we extended the results of [7] in three aspects: from Lagrange system to general non-conservative system; from a group of point transformations corresponding to generalized coordinates and time to a group of transformations that depend on generalized velocities; from Noether symmetry to Noether quasi-symmetry. In recent years, Noether's theorems with time delay have been extended to high-order variational problems [9], fractional systems [10], Hamilton systems [11], nonholonomic systems [12], Birkhoff systems [13,14], and dynamics on time scales [15,16], etc. Although some important results have been obtained in the dynamics modeling of time-delay systems and its Noether's theorems, in general, the research in this field is still in the preliminary stage and is still an open topic.

Recently, the Herglotz generalized variational principle (HGVP) with time delay was studied in [17]. Noether's theorem for Herglotz's problem with time delay was proved, in which Noether symmetry is defined by the invariant transformation of Lagrangian function in a one-parameter point transformation group. Refs. [18,19] extended the results of [17] to a high-order variational problem. HGVP refers to a kind of generalized variational principle proposed by Herglotz when he studied Hamilton system, contact transformation, and Poisson brackets, as shown in Herglotz [20,21] and Guenther et al. [22]. Different from the classical variational principle (CVP), the advantages of HGVP are as follows. First, it achieves a variational representation of the process of non-conservative dynamics. However, the CVP cannot represent a non-conservative system as an extremum of a functional. Second, the CVP can be used as its special case. Thus, HGVP may not only describe the physical processes described by CVP, but also some problems that CVP has difficultly applying. Third, HGVP unifies conservative and non-conservative processes into the same dynamics model, and thus can systematically deal with the actual dynamical problems. Noether's theorems [23,24] based on HGVP were extended to fractional order models [25–29], non-conservative Hamilton systems [30–32], non-holonomic systems [33], Birkhoff systems [34–37], non-conservative classical and quantum systems [38–40], and adiabatic invariants [41,42], etc. Although some advances have been made in the study of HGVP and Noether's theorems, but little work has been done on the HGVP with time delay and its symmetry and conservation laws.

Based on the two aspects as stated above, our motivation is to apply HGVP to the time-delay mechanical system and study Herglotz's variational problem for a non-conservative system with delayed arguments under Lagrangian framework and its Noether's theorem. The structure of this paper is arranged as follows. The HGVP with delayed arguments and its Euler–Lagrange equations are given in Section 2. In Section 3, the non-isochronous variation formulas of Hamilton–Herglotz action with time-delayed arguments are derived. In Section 4, the Noether symmetry is defined, and the criterion of symmetry is given for the Hamilton–Herglotz action case. The infinitesimal transformations we discussed depend on the generalized velocity. Herglotz type Noether's theorem for non-conservative systems with delayed arguments is proved. In Sections 5 and 6, Birkhoff and Hamilton generalization of Lagrange systems of Herglotz type with delayed arguments is given. To illustrate the application of our Noether's theorems, we give two examples of damped oscillators in Section 7. The conclusion of the paper is in Section 8.

2. HGVP for Non-Conservative Dynamics with Delayed Arguments

Considering a non-conservative mechanical system with delayed arguments, we assume that its configuration is described by $q_s\,(s = 1, 2, \cdots, n)$. We now define Herglotz's variational problem of the non-conservative system with delayed arguments as:

Suppose that functional z is determined by a first order differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = L\left(t, q_s\left(t\right), \dot{q}_s\left(t\right), q_s\left(t-\tau\right), \dot{q}_s\left(t-\tau\right), z\left(t\right)\right) \tag{1}$$

Determine the trajectory $q_s\left(t\right)$ that satisfy the boundary conditions

$$q_s\left(t_1\right) = q_{s1},\ q_s\left(t\right) = f_s\left(t\right) \quad t \in \left[t_0 - \tau, t_0\right] \tag{2}$$

and initial condition

$$z\left(t\right)\big|_{t=t_0} = z_0 \tag{3}$$

so as to extremize the value $z\left(t_1\right) \to$ extr. Here, $L = L\left(t, q_s, \dot{q}_s, q_{s\tau}, \dot{q}_{s\tau}, z\right)$ is the Lagrangian in the sense of Herglotz. $f_s\left(t\right)$ is a given function on $\left[t_0 - \tau, t_0\right]$, which is piecewise smooth. τ is the delay quantity, and $\tau < t_1 - t_0$, which is a given positive real number. Here, q_{s1} and z_0 are constants.

We call a functional z Hamilton–Herglotz action with delayed arguments. The Herglotz's variational problem above can be called the HGVP for non-conservative system with delayed arguments.

For a non-conservative system with delayed arguments, it is easy from the above principle to obtain the Euler–Lagrange equations of Herglotz type, and we get

$$\begin{array}{l} \lambda\left(t\right)\left(\frac{\partial L}{\partial q_s}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)\left(t\right) \\ +\lambda\left(t+\tau\right)\left(\frac{\partial L}{\partial q_{s\tau}}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{s\tau}}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)\left(t+\tau\right)=0,\, t\in\left[t_0,t_1-\tau\right], \\ \lambda\left(t\right)\left(\frac{\partial L}{\partial q_s}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)\left(t\right)=0,\, t\in\left(t_1-\tau,t_1\right], \end{array} \tag{4}$$

where $\lambda\left(t\right)=\exp\left[-\int_{t_0}^{t}\frac{\partial L}{\partial z}\left(\theta\right)d\theta\right]$.

3. Non-Isochronous Variation of Hamilton–Herglotz Action with Delayed Arguments

Consider the infinitesimal transformations that depend not only on generalized coordinates, and time, but also on generalized velocities, that is,

$$\bar{t}=t+\Delta t,\, \bar{q}_s\left(\bar{t}\right)=q_s\left(t\right)+\Delta q_s \tag{5}$$

or their expansion

$$\bar{t}=t+\varepsilon_\sigma\xi_0^\sigma\left(t,q_k,\dot{q}_k,z\right),\bar{q}_s\left(\bar{t}\right)=q_s\left(t\right)+\varepsilon_\sigma\xi_s^\sigma\left(t,q_k,\dot{q}_k,z\right) \tag{6}$$

where ξ_0^σ and ξ_s^σ are the generators, and $\varepsilon_\sigma\ (\sigma=1,2,\cdots,r)$ are the infinitesimal parameters.

The function $z\left(t\right)$ is transformed by the infinitesimal transformation (5) into $\bar{z}\left(\bar{t}\right)$, and the relationship between them is as follows:

$$\bar{z}\left(\bar{t}\right)=z\left(t\right)+\Delta z\left(t\right) \tag{7}$$

where Δz is the non-isochronous variation. By calculating the non-isochronous variation of Equation (1), we have

$$\Delta\dot{z}=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\Delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\Delta\dot{q}_{s\tau}+\frac{\partial L}{\partial z}\Delta z \tag{8}$$

Note that, for any differentiable function F, the following formulae hold [43]:

$$\Delta F=\delta F+\dot{F}\Delta t,\, \frac{\mathrm{d}}{\mathrm{d}t}\delta F=\delta\dot{F},\, \Delta\dot{F}=\frac{\mathrm{d}}{\mathrm{d}t}\Delta F-\dot{F}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{9}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta z=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\Delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\Delta\dot{q}_{s\tau}+L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t+\frac{\partial L}{\partial z}\Delta z \tag{10}$$

From Equation (10), we get

$$\begin{array}{l} \Delta z\left(t\right)\lambda\left(t\right)-\Delta z\left(t_0\right) \\ =\int_{t_0}^{t}\lambda\left(t\right)\left(\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial q_{s\tau}}\Delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\Delta\dot{q}_{s\tau}+L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t \end{array} \tag{11}$$

where $\Delta z\left(t_0\right)=0$. By performing variable substitution operations $t=\theta+\tau$ for the fourth and fifth items in Equation (11), and noting the boundary condition (2), we have

$$\begin{aligned}
&\int_{t_0}^{t} \lambda(t)\left(\frac{\partial L}{\partial q_{s\tau}}(t)\,\Delta q_{s\tau} + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\,\Delta \dot{q}_{s\tau}\right)\mathrm{d}t \\
&= \int_{t_0-\tau}^{t-\tau} \lambda(\theta+\tau)\left\{\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\left[\delta q_{s\tau}(\theta+\tau) + \dot{q}_{s\tau}(\theta+\tau)\,\Delta\theta\right]\right. \\
&\left. + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\left[\delta \dot{q}_{s\tau}(\theta+\tau) + \ddot{q}_{s\tau}(\theta+\tau)\,\Delta\theta\right]\right\}\mathrm{d}\theta \\
&= \int_{t_0}^{t-\tau} \lambda(\theta+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\,\Delta q_{s\tau}(\theta+\tau) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\,\Delta \dot{q}_{s\tau}(\theta+\tau)\right]\mathrm{d}\theta \\
&+ \int_{t_0-\tau}^{t_0} \lambda(\theta+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\,\dot{q}_{s\tau}(\theta+\tau) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\,\ddot{q}_{s\tau}(\theta+\tau)\right]\Delta\theta\mathrm{d}\theta
\end{aligned} \tag{12}$$

Substituting Equation (12) into Equation (11), we get

$$\begin{aligned}
&\Delta z(t)\,\lambda(t) \\
&= \int_{t_0-\tau}^{t_0} \lambda(t+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\dot{q}_{s\tau}(t+\tau) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,\ddot{q}_{s\tau}(t+\tau)\right]\Delta t\mathrm{d}t \\
&+ \int_{t_0}^{t-\tau}\left\{\lambda(t)\left(\frac{\partial L}{\partial t}(t)\,\Delta t + \frac{\partial L}{\partial q_s}(t)\,\Delta q_s(t) + \frac{\partial L}{\partial \dot{q}_s}(t)\,\Delta \dot{q}_s(t) + L(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\right. \\
&\left. + \lambda(t+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\Delta q_{s\tau}(t+\tau) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,\Delta \dot{q}_{s\tau}(t+\tau)\right]\right\}\mathrm{d}t \\
&+ \int_{t-\tau}^{t} \lambda(t)\left(\frac{\partial L}{\partial t}(t)\,\Delta t + \frac{\partial L}{\partial q_s}(t)\,\Delta q_s(t) + \frac{\partial L}{\partial \dot{q}_s}(t)\,\Delta \dot{q}_s(t) + L(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t
\end{aligned} \tag{13}$$

Equation (11) can also be written as

$$\begin{aligned}
&\Delta z(t)\,\lambda(t) \\
&= \int_{t_0}^{t}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda(t)\left(\frac{\partial L}{\partial \dot{q}_s}(\Delta q_s - \dot{q}_s\Delta t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\Delta q_{s\tau} - \dot{q}_{s\tau}\Delta t) + L\Delta t\right)\right]\right. \\
&+ \lambda(t)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial q_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)(\Delta q_s - \dot{q}_s\Delta t) \\
&\left. + \lambda(t)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)(\Delta q_{s\tau} - \dot{q}_{s\tau}\Delta t)\right\}\mathrm{d}t
\end{aligned} \tag{14}$$

By performing variable substitution operations $t = \theta + \tau$ for the terms in Equation (14) with delay τ, and using condition (2), we get

$$\begin{aligned}
&\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda(t)\,\frac{\partial L}{\partial \dot{q}_{s\tau}}(\Delta q_{s\tau} - \dot{q}_{s\tau}\Delta t)\right]\mathrm{d}t \\
&= \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda(t)\,\frac{\partial L}{\partial \dot{q}_{s\tau}}\delta q_{s\tau}\right]\mathrm{d}t \\
&= \int_{t_0-\tau}^{t-\tau} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\lambda(\theta+\tau)\,\frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\,\delta q_{s\tau}(\theta+\tau)\right]\mathrm{d}\theta \\
&= \int_{t_0}^{t-\tau} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\lambda(\theta+\tau)\,\frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\,\delta q_s(\theta)\right]\mathrm{d}\theta
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\int_{t_0}^{t} \lambda(t)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)(\Delta q_{s\tau} - \dot{q}_{s\tau}\Delta t)\,\mathrm{d}t \\
&= \int_{t_0-\tau}^{t-\tau} \lambda(\theta+\tau)\left(-\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)(\theta+\tau)\,\delta q_{s\tau}(\theta+\tau)\,\mathrm{d}\theta \\
&= \int_{t_0}^{t-\tau} \lambda(\theta+\tau)\left(-\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)(\theta+\tau)\,\delta q_s(\theta)\,\mathrm{d}\theta
\end{aligned} \tag{16}$$

From Equations (15) and (16), we can rewrite Equation (14) as

$$\begin{aligned}
&\Delta z(t)\,\lambda(t) = \int_{t_0}^{t-\tau}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda(t)\left(\frac{\partial L}{\partial \dot{q}_s}(t)\,(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t) + L(t)\,\Delta t\right)\right.\right. \\
&\left. + \lambda(t+\tau)\,\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t)\right] \\
&+ \lambda(t)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial z}(t)\,\frac{\partial L}{\partial \dot{q}_s}(t)\right)(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t) \\
&\left. + \lambda(t+\tau)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)(t+\tau)\,(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t)\right\}\mathrm{d}t \\
&+ \int_{t-\tau}^{t}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda(t)\left(\frac{\partial L}{\partial \dot{q}_s}(t)\,(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t) + L(t)\,\Delta t\right)\right]\right. \\
&\left. + \lambda(t)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial z}(t)\,\frac{\partial L}{\partial \dot{q}_s}(t)\right)(\Delta q_s(t) - \dot{q}_s(t)\,\Delta t)\right\}\mathrm{d}t,
\end{aligned} \tag{17}$$

Since

$$\Delta t = \varepsilon_\sigma \xi_0^\sigma,\ \Delta q_s = \varepsilon_\sigma \xi_s^\sigma \quad (s = 1, 2, \cdots, n) \tag{18}$$

Substituting Equation (18) into Equations (13) and (17), we get

$$\begin{aligned} &\Delta z(t)\,\lambda(t) \\ &= \int_{t_0-\tau}^{t_0} \left\{ \lambda(t+\tau) \left[\tfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\dot{q}_{s\tau}(t+\tau)\,\xi_0^\sigma + \tfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,\ddot{q}_{s\tau}(t+\tau)\,\xi_0^\sigma \right] \right\} \varepsilon_\sigma \mathrm{d}t \\ &+ \int_{t_0}^{t-\tau} \left\{ \lambda(t) \left[\tfrac{\partial L}{\partial t}(t)\,\xi_0^\sigma + \tfrac{\partial L}{\partial q_s}(t)\,\xi_s^\sigma + \tfrac{\partial L}{\partial \dot{q}_s}(t) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) + L(t)\,\dot{\xi}_0^\sigma \right] \right. \\ &\left. + \lambda(t+\tau) \left[\tfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\xi_s^\sigma + \tfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) \right] \right\} \varepsilon_\sigma \mathrm{d}t \\ &+ \int_{t-\tau}^{t} \left\{ \lambda(t) \left[\tfrac{\partial L}{\partial t}(t)\,\xi_0^\sigma + \tfrac{\partial L}{\partial q_s}(t)\,\xi_s^\sigma + \tfrac{\partial L}{\partial \dot{q}_s}(t) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) + L(t)\,\dot{\xi}_0^\sigma \right] \right\} \varepsilon_\sigma \mathrm{d}t \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\Delta z(t)\,\lambda(t) = \int_{t_0}^{t-\tau} \left\{ \tfrac{\mathrm{d}}{\mathrm{d}t} \left[\lambda(t) \left(\tfrac{\partial L}{\partial \dot{q}_s}(t) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) + L(t)\,\xi_0^\sigma \right) \right. \right. \\ &\left. + \lambda(t+\tau)\,\tfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) \right] \\ &+ \lambda(t) \left(-\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\partial L}{\partial \dot{q}_s}(t) + \tfrac{\partial L}{\partial q_s}(t) + \tfrac{\partial L}{\partial z}(t)\,\tfrac{\partial L}{\partial \dot{q}_s}(t) \right) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) \\ &\left. + \lambda(t+\tau) \left(-\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\partial L}{\partial \dot{q}_{s\tau}} + \tfrac{\partial L}{\partial q_{s\tau}} + \tfrac{\partial L}{\partial z} \tfrac{\partial L}{\partial \dot{q}_{s\tau}} \right)(t+\tau) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) \right\} \varepsilon_\sigma \mathrm{d}t \\ &+ \int_{t-\tau}^{t} \left\{ \tfrac{\mathrm{d}}{\mathrm{d}t} \left[\lambda(t) \left(\tfrac{\partial L}{\partial \dot{q}_s}(t) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) + L(t)\,\xi_0^\sigma \right) \right] \right. \\ &\left. + \lambda(t) \left(-\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\partial L}{\partial \dot{q}_s}(t) + \tfrac{\partial L}{\partial q_s}(t) + \tfrac{\partial L}{\partial z}(t)\,\tfrac{\partial L}{\partial \dot{q}_s}(t) \right) \left(\xi_s^\sigma - \dot{q}_s(t)\,\xi_0^\sigma \right) \right\} \varepsilon_\sigma \mathrm{d}t \end{aligned} \tag{20}$$

Equations (19) and (20) are the non-isochronous variation formulas of Hamilton–Herglotz action with delayed arguments.

4. Herglotz Type Noether's Theorem for Non-Conservative Systems with Delayed Arguments

If Hamilton–Herglotz action remains unchanged through the infinitesimal transformation of the group, namely $\Delta z(t_1) = 0$, it is known as Noether symmetry for non-conservative mechanical system with delayed arguments.

According to Equation (19), we can obtain the criterion of Noether symmetry for the non-conservative system. That is,

Criterion 1. *If the generators ξ_0^σ and ξ_s^σ of infinitesimal transformation (5) make the following conditions true, when $t \in [t_0 - \tau, t_0)$, there is*

$$\lambda(t+\tau) \left[\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\dot{q}_{s\tau}(t+\tau)\,\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,\ddot{q}_{s\tau}(t+\tau)\,\xi_0^\sigma \right] = 0 \tag{21}$$

when $t \in [t_0, t_1 - \tau]$, there is

$$\begin{aligned} &\lambda(t) \left[\tfrac{\partial L}{\partial t}(t)\,\xi_0^\sigma + \tfrac{\partial L}{\partial q_s}(t)\,\xi_s^\sigma + \tfrac{\partial L}{\partial \dot{q}_s}(t) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) + L(t)\,\dot{\xi}_0^\sigma \right] \\ &+ \lambda(t+\tau) \left[\tfrac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\xi_s^\sigma + \tfrac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) \right] = 0 \end{aligned} \tag{22}$$

when $t \in (t_1 - \tau, t_1]$, there is

$$\lambda(t) \left[\frac{\partial L}{\partial t}(t)\,\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\,\xi_s^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t) \left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\,\dot{\xi}_0^\sigma \right) + L(t)\,\dot{\xi}_0^\sigma \right] = 0 \tag{23}$$

where $s = 1, 2, \cdots, n$ and $\sigma = 1, 2, \cdots, r$, then the transformation corresponds to the Noether symmetry of non-conservative system with delayed arguments.

By Noether symmetry, we can find the conserved quantity, and we have the following results.

Theorem 1. *For non-conservative system (4) with delayed arguments, if the infinitesimal transformation (5) corresponds to its Noether symmetry, then r linearly independent conserved quantities of Herglotz type exist, such as*

$$\begin{array}{l} I^{\sigma} = \lambda\left(t\right)\left[\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right] \\ +\lambda\left(t+\tau\right)\frac{\partial L}{\partial \dot{q}_{s\tau}}\left(t+\tau\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right), t \in \left[t_0, t_1-\tau\right] \end{array} \tag{24}$$

and

$$I^{\sigma} = \lambda\left(t\right)\left[\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right], t \in \left(t_1-\tau, t_1\right] \tag{25}$$

where $\sigma = 1, 2, \cdots, r$ *and* $\lambda\left(t\right) = \exp\left[-\int_{t_0}^{t}\frac{\partial L}{\partial z}\left(\theta\right)d\theta\right]$.

Proof. Considering that the transformation (5) is Noether symmetric, then $\Delta z\left(t_1\right) = 0$, and from Formula (20), we have

$$\begin{array}{l} \int_{t_0}^{t_1-\tau}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right) + \lambda\left(t+\tau\right)\frac{\partial L}{\partial \dot{q}_{s\tau}}\left(t+\tau\right)\right.\right. \\ \times\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right)\Big] + \lambda\left(t\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}\left(t\right) + \frac{\partial L}{\partial q_s}\left(t\right) + \frac{\partial L}{\partial z}\left(t\right)\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) \\ \left.+\lambda\left(t+\tau\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}} + \frac{\partial L}{\partial q_{s\tau}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_{s\tau}}\right)\left(t+\tau\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right)\right\}\varepsilon_{\sigma}\mathrm{d}t \\ +\int_{t_1-\tau}^{t_1}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right)\right]\right. \\ \left.+\lambda\left(t\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}\left(t\right) + \frac{\partial L}{\partial q_s}\left(t\right) + \frac{\partial L}{\partial z}\left(t\right)\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right)\right\}\varepsilon_{\sigma}\mathrm{d}t = 0 \end{array} \tag{26}$$

Substituting the Euler–Lagrange Equation (4) into Equation (26), we get

$$\begin{array}{l} \int_{t_0}^{t_1-\tau}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right)\right.\right. \\ \left.\left.+\lambda\left(t+\tau\right)\frac{\partial L}{\partial \dot{q}_{s\tau}}\left(t+\tau\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right)\right]\right\}\varepsilon_{\sigma}\mathrm{d}t \\ +\int_{t_1-\tau}^{t_1}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right)\right]\right\}\varepsilon_{\sigma}\mathrm{d}t = 0 \end{array} \tag{27}$$

Since the infinitesimal parameters $\varepsilon_{\sigma}\left(\sigma = 1, 2, \cdots, r\right)$ are independent and the interval $\left[t_0, t_1\right]$ is arbitrary, we get

$$\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right)\right. \\ \left.+\lambda\left(t+\tau\right)\frac{\partial L}{\partial \dot{q}_{s\tau}}\left(t+\tau\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right)\right] = 0,\ t \in \left[t_0, t_1-\tau\right] \end{array} \tag{28}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\left(t\right)\left(\frac{\partial L}{\partial \dot{q}_s}\left(t\right)\left(\xi_s^{\sigma} - \dot{q}_s\left(t\right)\xi_0^{\sigma}\right) + L\left(t\right)\xi_0^{\sigma}\right)\right] = 0, t \in \left(t_1-\tau, t_1\right]. \tag{29}$$

Thus, the theorem holds. □

Theorem 1 is Herglotz type Noether's theorem for non-conservative system with delayed arguments, and the conserved quantity (24) given by the theorem can be called Herglotz type Noether conserved quantity.

5. Birkhoff Generalization of Herglotz Type Noether's Theorem

For the Birkhoff system with delayed arguments, the functional z can be defined by the differential Equation [35]:

$$\begin{array}{l} \frac{\mathrm{d}z}{\mathrm{d}t} = R_{\mu}\left(t, a^{\nu}\left(t\right), z\left(t\right)\right)\dot{a}^{\mu}\left(t\right) + R_{\mu}\left(t, a^{\nu}\left(t-\tau\right), z\left(t\right)\right)\dot{a}^{\mu}\left(t-\tau\right) \\ -B\left(t, a^{\nu}\left(t\right), a^{\nu}\left(t-\tau\right), z\left(t\right)\right) \end{array} \tag{30}$$

The corresponding Birkhoff's equations with delayed arguments of Herglotz type are

$$\begin{aligned}
&-\lambda(t)\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\nu}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\nu\right)\dot a^\nu\right.\\
&\left.+\left(R_\mu\frac{\partial R_{\nu\tau}}{\partial z}-\frac{\partial R_\mu}{\partial z}R_{\nu\tau}\right)\dot a^\nu_\tau+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}\right](t)\\
&-\lambda(t+\tau)\left[\left(\frac{\partial R_{\nu\tau}}{\partial a^\mu_\tau}-\frac{\partial R_{\mu\tau}}{\partial a^\nu_\tau}\right)\dot a^\nu_\tau-\frac{\partial B}{\partial a^\mu_\tau}-\frac{\partial R_{\mu\tau}}{\partial t}+\left(R_{\mu\tau}\frac{\partial R_\nu}{\partial z}-\frac{\partial R_{\mu\tau}}{\partial z}R_\nu\right)\dot a^\nu\right.\\
&\left.+\left(R_{\mu\tau}\frac{\partial R_{\nu\tau}}{\partial z}-\frac{\partial R_{\mu\tau}}{\partial z}R_{\nu\tau}\right)\dot a^\nu_\tau+\frac{\partial R_{\mu\tau}}{\partial z}B-R_{\mu\tau}\frac{\partial B}{\partial z}\right](t+\tau)=0,\ t\in[t_0,t_1-\tau],\\
&-\lambda(t)\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\nu}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\nu\right)\dot a^\nu\right.\\
&\left.+\left(R_\mu\frac{\partial R_{\nu\tau}}{\partial z}-\frac{\partial R_\mu}{\partial z}R_{\nu\tau}\right)\dot a^\nu_\tau+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}\right](t)=0,\ t\in(t_1-\tau,t_1]
\end{aligned}\tag{31}$$

We take the infinitesimal transformation of the group as follows:

$$\bar t=t+\varepsilon_\sigma\xi_0^\sigma(t,a^\nu,z),\ \bar a^\mu(\bar t)=a^\mu(t)+\varepsilon_\sigma\xi_\mu^\sigma(t,a^\nu,z)\ (\mu=1,2,\cdots,2n).\tag{32}$$

Then, the criterion of Noether symmetry for the Birkhoff system (31) can be expressed as

Criterion 2. *If the generators ξ_0^σ and ξ_μ^σ of infinitesimal transformation (32) make the following conditions true, when $t\in[t_0-\tau,t_0)$, there is*

$$\begin{aligned}
&\lambda(t+\tau)\left[R_{\mu\tau}(t+\tau)\ddot a^\mu_\tau(t+\tau)\xi_0^\sigma+R_{\mu\tau}(t+\tau)\dot a^\mu_\tau(t+\tau)\dot\xi_0^\sigma\right.\\
&+\frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\dot a^\mu_\tau(t+\tau)\xi_0^\sigma\\
&\left.+\frac{\partial R_{\mu\tau}}{\partial a^\nu_\tau}(t+\tau)\dot a^\nu_\tau(t+\tau)\dot a^\mu_\tau(t+\tau)\xi_0^\sigma-\frac{\partial B}{\partial a^\nu_\tau}(t+\tau)\dot a^\nu_\tau(t+\tau)\xi_0^\sigma\right]=0
\end{aligned}\tag{33}$$

When $t\in[t_0,t_1-\tau]$, there is

$$\begin{aligned}
&\lambda(t)\left[\left(\frac{\partial R_\mu}{\partial t}(t)\xi_0^\sigma+\frac{\partial R_\mu}{\partial a^\nu}(t)\xi_\nu^\sigma\right)\dot a^\mu(t)+R_\mu(t)\dot\xi_\mu^\sigma-B(t)\dot\xi_0^\sigma\right.\\
&\left.-\frac{\partial B}{\partial a^\nu}(t)\xi_\nu^\sigma-\frac{\partial B}{\partial t}(t)\xi_0^\sigma\right]+\lambda(t+\tau)\left[\left(\frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\xi_0^\sigma\right.\right.\\
&\left.\left.+\frac{\partial R_{\mu\tau}}{\partial a^\nu_\tau}(t+\tau)\xi_\nu^\sigma\right)\dot a^\mu_\tau(t+\tau)+R_{\mu\tau}(t+\tau)\dot\xi_\mu^\sigma-\frac{\partial B}{\partial a^\nu_\tau}(t+\tau)\xi_\nu^\sigma\right]=0
\end{aligned}\tag{34}$$

When $t\in(t_1-\tau,t_1]$, there is

$$\begin{aligned}
&\lambda(t)\left[\left(\frac{\partial R_\mu}{\partial t}(t)\xi_0^\sigma+\frac{\partial R_\mu}{\partial a^\nu}(t)\xi_\nu^\sigma\right)\dot a^\mu(t)+R_\mu(t)\dot\xi_\mu^\sigma-B(t)\dot\xi_0^\sigma\right.\\
&\left.-\frac{\partial B}{\partial a^\nu}(t)\xi_\nu^\sigma-\frac{\partial B}{\partial t}(t)\xi_0^\sigma\right]=0
\end{aligned}\tag{35}$$

Then, the transformation corresponds to the Noether symmetry of Birkhoff system with delayed arguments.

Theorem 2. *For Birkhoff system (31) with delayed arguments, if the infinitesimal transformation (32) corresponds to its Noether symmetry, then r linearly independent conserved quantities of Herglotz type exist, such as*

$$I^\sigma=\lambda(t+\tau)R_{\mu\tau}(t+\tau)\dot a^\mu_\tau(t+\tau)\xi_0^\sigma,\quad t\in[t_0-\tau,t_0)\tag{36}$$

and

$$I^\sigma=\lambda(t)\left(R_\mu(t)\xi_\mu^\sigma-B(t)\xi_0^\sigma\right)+\lambda(t+\tau)R_{\mu\tau}(t+\tau)\xi_\mu^\sigma,\ t\in[t_0,t_1-\tau]\tag{37}$$

and

$$I^\sigma=\lambda(t)\left(R_\mu(t)\xi_\mu^\sigma-B(t)\xi_0^\sigma\right),\quad t\in(t_1-\tau,t_1]\tag{38}$$

where $\sigma=1,2,\cdots,r$ and $\lambda(t)=\exp\left[-\int_{t_0}^t\left(\frac{\partial R_\mu}{\partial z}\dot a^\mu+\frac{\partial R_{\mu\tau}}{\partial z}\dot a^\mu_\tau-\frac{\partial B}{\partial z}\right)(\theta)\,d\theta\right]$.

In Reference [35], Herglotz type Noether's theorem for Birkhoff systems with delayed arguments was studied. However, the above Equations (33) and (36) were not obtained in [35] due to an error in calculating the non-isochronous variation in the interval $t \in [t_0 - \tau, t_0]$.

6. Hamilton Generalization of Herglotz Type Noether's Theorem

For the Hamilton system with delayed arguments, the functional z can be defined by the differential equation [30]

$$\begin{array}{l}\frac{\mathrm{d}z}{\mathrm{d}t} = p_s(t)\dot{q}_s(t) + p_s(t-\tau)\dot{q}_s(t-\tau) \\ -H(t, q_s(t), p_s(t), q_s(t-\tau), p_s(t-\tau), z(t))\end{array} \tag{39}$$

The corresponding Hamilton's equations with delayed arguments of Herglotz type are

$$\begin{array}{l}\lambda(t)\left[\dot{p}_s(t) + \frac{\partial H}{\partial q_s}(t) + p_s(t)\frac{\partial H}{\partial z}(t)\right] \\ +\lambda(t+\tau)\left[\dot{p}_{s\tau}(t+\tau) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau) + p_{s\tau}(t+\tau)\frac{\partial H}{\partial z}(t+\tau)\right] = 0 \\ \lambda(t)\left[-\dot{q}_s(t) + \frac{\partial H}{\partial p_s}(t)\right] + \lambda(t+\tau)\left[-\dot{q}_{s\tau}(t+\tau) + \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right] = 0, \\ t \in [t_0, t_1 - \tau], \\ \lambda(t)\left[\dot{p}_s(t) + \frac{\partial H}{\partial q_s}(t) + p_s(t)\frac{\partial H}{\partial z}(t)\right] = 0, \\ \lambda(t)\left[-\dot{q}_s(t) + \frac{\partial H}{\partial p_s}(t)\right] = 0, \quad t \in (t_1 - \tau, t_1]\end{array} \tag{40}$$

Let the infinitesimal transformation be

$$\begin{array}{l}\bar{t} = t + \varepsilon_\sigma \xi_0^\sigma(t, q_k, p_k, z), \\ \bar{q}_s(\bar{t}) = q_s(t) + \varepsilon_\sigma \xi_s^\sigma(t, q_k, p_k, z), \\ \bar{p}_s(\bar{t}) = p_s(t) + \varepsilon_\sigma \eta_s^\sigma(t, q_k, p_k, z), \ (s = 1, 2, \cdots, n).\end{array} \tag{41}$$

Then, the criterion of Noether symmetry for the Hamilton system (40) can be expressed as

Criterion 3. *If the generators ξ_0^σ, ξ_s^σ and η_s^σ of infinitesimal transformation (41) make the following conditions true, when $t \in [t_0 - \tau, t_0)$, there is*

$$\begin{array}{l}\lambda(t+\tau)\left[p_{s\tau}(t+\tau)\ddot{q}_{s\tau}(t+\tau)\xi_0^\sigma + p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)\dot{\xi}_0^\sigma + \dot{q}_{s\tau}(t+\tau)\times\right. \\ \left.\times\dot{p}_{s\tau}(t+\tau)\xi_0^\sigma - \frac{\partial H}{\partial q_{s\tau}}(t+\tau)\dot{q}_{s\tau}(t+\tau)\xi_0^\sigma - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\dot{p}_{s\tau}(t+\tau)\xi_0^\sigma\right] = 0.\end{array} \tag{42}$$

When $t \in [t_0, t_1 - \tau]$, there is

$$\begin{array}{l}\lambda(t)\left[\dot{q}_s(t)\eta_s^\sigma + p_s(t)\dot{\xi}_s^\sigma - H(t)\dot{\xi}_0^\sigma - \frac{\partial H}{\partial t}(t)\xi_0^\sigma\right. \\ \left.-\frac{\partial H}{\partial q_s}(t)\xi_s^\sigma - \frac{\partial H}{\partial p_s}(t)\eta_s^\sigma\right] + \lambda(t+\tau)\left[\dot{q}_{s\tau}(t+\tau)\eta_s^\sigma + p_{s\tau}(t+\tau)\dot{\xi}_s^\sigma\right. \\ \left.-\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\xi_s^\sigma - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\eta_s^\sigma\right] = 0\end{array} \tag{43}$$

When $t \in (t_1 - \tau, t_1]$, there is

$$\begin{array}{l}\lambda(t)\left[\dot{q}_s(t)\eta_s^\sigma + p_s(t)\dot{\xi}_s^\sigma - H(t)\dot{\xi}_0^\sigma - \frac{\partial H}{\partial t}(t)\xi_0^\sigma\right. \\ \left.-\frac{\partial H}{\partial q_s}(t)\xi_s^\sigma - \frac{\partial H}{\partial p_s}(t)\eta_s^\sigma\right] = 0\end{array} \tag{44}$$

Then, the transformation corresponds to the Noether symmetry of Hamilton system with delayed arguments.

Theorem 3. *For the Hamilton system (40) with delayed arguments, if the infinitesimal transformation (41) corresponds to its Noether symmetry, then r linearly independent conserved quantities of Herglotz type exist, such as*

$$I^{\sigma} = \lambda\left(t+\tau\right) p_{s\tau}\left(t+\tau\right) \dot{q}_{s\tau}\left(t+\tau\right) \xi_0^{\sigma}, \quad t \in \left[t_0-\tau, t_0\right) \tag{45}$$

and

$$I^{\sigma} = \lambda\left(t\right)\left(p_s\left(t\right)\xi_s^{\sigma} - H\left(t\right)\xi_0^{\sigma}\right) + \lambda\left(t+\tau\right) p_{s\tau}\left(t+\tau\right)\xi_s^{\sigma}, \quad t \in \left[t_0, t_1-\tau\right] \tag{46}$$

and

$$I^{\sigma} = \lambda\left(t\right)\left(p_s\left(t\right)\xi_s^{\sigma} - H\left(t\right)\xi_0^{\sigma}\right), \quad t \in \left(t_1-\tau, t_1\right] \tag{47}$$

where $\sigma = 1, 2, \cdots, r$ *and* $\lambda\left(t\right) = \exp\left[\int_{t_0}^{t} \frac{\partial H}{\partial z}\left(\theta\right) d\theta\right]$.

In Reference [30], Herglotz type Noether's theorem for the Hamilton system with delayed arguments was studied. However, the above Equations (42) and (45) were not obtained in [30] due to an error in calculating the non-isochronous variation in the interval $t \in \left[t_0-\tau, t_0\right]$.

7. Examples

Example 1. *Study the Noether symmetry and conserved quantity of a non-conservative system with delayed arguments. The Lagrangian of the system in the sense of Herglotz is*

$$L = \frac{1}{2}\left[\dot{q}^2\left(t\right) + \dot{q}^2\left(t-\tau\right)\right] - \frac{1}{2}q^2\left(t\right) - z\left(t\right) \tag{48}$$

Functional z satisfies the equation

$$\frac{dz}{dt}\left(t\right) = \frac{1}{2}\left[\dot{q}^2\left(t\right) + \dot{q}^2\left(t-\tau\right)\right] - \frac{1}{2}q^2\left(t\right) - z\left(t\right) \tag{49}$$

Equation (4) gives

$$\begin{aligned} &e^{t}\left[q\left(t\right)+\ddot{q}\left(t\right)+\dot{q}\left(t\right)\right]+e^{t+\tau}\left[\ddot{q}_{\tau}\left(t+\tau\right)+\dot{q}_{\tau}\left(t+\tau\right)\right]=0, t\in\left[t_0,\, t_1-\tau\right], \\ &e^{t}\left[q\left(t\right)+\ddot{q}\left(t\right)+\dot{q}\left(t\right)\right]=0,\; t\in\left(t_1-\tau, t_1\right] \end{aligned} \tag{50}$$

According to Criterion 1, when $t \in \left[t_0, t_1-\tau\right]$, the criterion equation is

$$\begin{aligned} &e^{t}\Big[-q\left(t\right)\xi_1+\dot{q}\left(t\right)\left(\dot{\xi}_1-\dot{q}\left(t\right)\dot{\xi}_0\right)+\tfrac{1}{2}\left(\dot{q}^2\left(t\right)+\dot{q}_{\tau}^2\left(t\right)\right)\dot{\xi}_0 \\ &-\tfrac{1}{2}q^2\left(t\right)\dot{\xi}_0-z\left(t\right)\dot{\xi}_0\Big]+e^{t+\tau}\dot{q}_{\tau}\left(t+\tau\right)\left(\dot{\xi}_1-\dot{q}\left(t\right)\dot{\xi}_0\right)=0 \end{aligned} \tag{51}$$

There is a solution to Equation (51), which is

$$\xi_0 = 0, \xi_1 = q\left(t\right) + \dot{q}\left(t\right) + \frac{q^2\left(t\right)}{\left(1+e^{\tau}\right)\dot{q}\left(t\right)} \tag{52}$$

when $t \in \left(t_1-\tau, t_1\right]$, the criterion equation is

$$\begin{aligned} &e^{t}\Big[-q\left(t\right)\xi_1+\dot{q}\left(t\right)\left(\dot{\xi}_1-\dot{q}\left(t\right)\dot{\xi}_0\right)+\tfrac{1}{2}\dot{q}^2\left(t\right)\dot{\xi}_0 \\ &+\tfrac{1}{2}\dot{q}_{\tau}^2\left(t\right)\dot{\xi}_0-\tfrac{1}{2}q^2\left(t\right)\dot{\xi}_0-z\left(t\right)\dot{\xi}_0\Big]=0 \end{aligned} \tag{53}$$

There is a solution to Equation (53), which is

$$\xi_0 = 0,\; \xi_1 = q\left(t\right) + \dot{q}\left(t\right) + \frac{q^2\left(t\right)}{\dot{q}\left(t\right)} \tag{54}$$

when $t \in [t_0 - \tau, t_0)$, from Equation (21), we have

$$e^{t+\tau}\dot{q}_\tau (t+\tau)\, \ddot{q}_\tau (t+\tau)\, \xi_0 = 0 \tag{55}$$

Obviously, $\xi_0 = 0$ satisfies Equation (55). The generators (52) and (54) are associated with the Noether symmetry of the current system. According to Theorem 1, when $t \in [t_0, t_1 - \tau]$, we have

$$I = e^t \left[q^2 (t) + (1 + e^\tau) \left(q (t)\, \dot{q} (t) + \dot{q}^2 (t) \right) \right] \tag{56}$$

when $t \in (t_1 - \tau, t_1]$, we have

$$I = e^t \left[q^2 (t) + q (t)\, \dot{q} (t) + \dot{q}^2 (t) \right] \tag{57}$$

Equations (56) and (57) are the conserved quantities of the system.

Example 2. *Consider a damped two-degree-of-freedom oscillator with time delay. The Lagrangian of Herglotz type is*

$$\begin{aligned} L = \tfrac{1}{2} m \left\{ [\dot{q}_1 (t) + \dot{q}_1 (t-\tau)]^2 + [\dot{q}_2 (t) + \dot{q}_2 (t-\tau)]^2 \right\} \\ -\tfrac{1}{2} k \left\{ [q_1 (t) + q_1 (t-\tau)]^2 + [q_2 (t) + q_2 (t-\tau)]^2 \right\} - \tfrac{c}{m} z (t) \end{aligned} \tag{58}$$

where m is the mass of the particle, k is the stiffness coefficient, and c the damping coefficient, and m, k, c are constants.

The differential equations of motion of the system are

$$\begin{aligned} & e^{ct/m} \left\{ -k [q_1 (t) + q_1 (t-\tau)] - m [\ddot{q}_1 (t) + \ddot{q}_1 (t-\tau)] - c [\dot{q}_1 (t) + \dot{q}_1 (t-\tau)] \right\} \\ & + e^{c(t+\tau)/m} \left\{ -k [q_1 (t+\tau) + q_1 (t)] - m [\ddot{q}_1 (t+\tau) + \ddot{q}_1 (t)] - c [\dot{q}_1 (t+\tau) + \dot{q}_1 (t)] \right\} = 0, \\ & e^{ct/m} \left\{ -k [q_2 (t) + q_2 (t-\tau)] - m [\ddot{q}_2 (t) + \ddot{q}_2 (t-\tau)] - c [\dot{q}_2 (t) + \dot{q}_2 (t-\tau)] \right\} \\ & + e^{c(t+\tau)/m} \left\{ -k [q_2 (t+\tau) + q_2 (t)] - m [\ddot{q}_2 (t+\tau) + \ddot{q}_2 (t)] - c [\dot{q}_2 (t+\tau) + \dot{q}_2 (t)] \right\} = 0 \end{aligned} \tag{59}$$

for $t \in [t_0, t_1 - \tau]$, and

$$\begin{aligned} & e^{ct/m} \left\{ -k [q_1 (t) + q_1 (t-\tau)] - m [\ddot{q}_1 (t) + \ddot{q}_1 (t-\tau)] - c [\dot{q}_1 (t) + \dot{q}_1 (t-\tau)] \right\} = 0, \\ & e^{ct/m} \left\{ -k [q_2 (t) + q_2 (t-\tau)] - m [\ddot{q}_2 (t) + \ddot{q}_2 (t-\tau)] - c [\dot{q}_2 (t) + \dot{q}_2 (t-\tau)] \right\} = 0 \end{aligned} \tag{60}$$

for $t \in (t_1 - \tau, t_1]$. According to Criterion 1, the criterion equation of the system is

$$\begin{aligned} & e^{ct/m} \left\{ -k [q_1 (t) + q_1 (t-\tau)]\, \xi_1 - m [\dot{q}_1 (t) + \dot{q}_1 (t-\tau)] \left(\dot{\xi}_1 - \dot{q}_1 (t)\, \dot{\xi}_0 \right) \right\} \\ & + e^{ct/m} \left\{ -k [q_2 (t) + q_2 (t-\tau)]\, \xi_2 - m [\dot{q}_2 (t) + \dot{q}_2 (t-\tau)] \left(\dot{\xi}_2 - \dot{q}_2 (t)\, \dot{\xi}_0 \right) + L (t)\, \dot{\xi}_0 \right\} \\ & + e^{c(t+\tau)/m} \left\{ -k [q_1 (t+\tau) + q_1 (t)]\, \xi_1 - m [\dot{q}_1 (t+\tau) + \dot{q}_1 (t)] \left(\dot{\xi}_1 - \dot{q}_1 (t)\, \dot{\xi}_0 \right) \right\} \\ & + e^{c(t+\tau)/m} \left\{ -k [q_2 (t+\tau) + q_2 (t)]\, \xi_2 - m [\dot{q}_2 (t+\tau) + \dot{q}_2 (t)] \left(\dot{\xi}_2 - \dot{q}_2 (t)\, \dot{\xi}_0 \right) \right\} = 0 \end{aligned} \tag{61}$$

for $t \in [t_0, t_1 - \tau]$. Equation (61) has a solution

$$\xi_0 = 0,\ \ \xi_1 = \dot{Q}_1 + \frac{kQ_1^2}{m\dot{Q}_1} + \frac{c}{m} Q_1,\ \ \xi_2 = \dot{Q}_2 + \frac{kQ_2^2}{m\dot{Q}_2} + \frac{c}{m} Q_2 \tag{62}$$

where $Q_s = q_s (t-\tau) + \left(1 + e^{c\tau/m} \right) q_s (t) + e^{c\tau/m}\, q_s (t+\tau), s = 1, 2$. The generator (62) is associated with the Noether symmetry of the current system. By Theorem 1, we obtain the conserved quantity as follows:

$$I = e^{ct/m} \left(m\dot{Q}_1^2 + kQ_1^2 + cQ_1\dot{Q}_1 + m\dot{Q}_2^2 + kQ_2^2 + cQ_2\dot{Q}_2 \right) = \text{const.} \tag{63}$$

When $t \in (t_1 - \tau, t_1]$, the criterion equation of the system is

$$e^{ct/m}\left\{-k\left[q_1(t)+q_1(t-\tau)\right]\xi_1 - m\left[\dot{q}_1(t)+\dot{q}_1(t-\tau)\right]\left(\dot{\xi}_1 - \dot{q}_1(t)\dot{\xi}_0\right)\right\} \\ +e^{ct/m}\left\{-k\left[q_2(t)+q_2(t-\tau)\right]\xi_2 - m\left[\dot{q}_2(t)+\dot{q}_2(t-\tau)\right]\left(\dot{\xi}_2 - \dot{q}_2(t)\dot{\xi}_0\right) + L(t)\dot{\xi}_0\right\} = 0 \tag{64}$$

Equation (64) has a solution

$$\xi_0 = 0,\ \xi_1 = \dot{q}_1(t) + \dot{q}_1(t-\tau) + \frac{k[q_1(t)+q_1(t-\tau)]^2}{m[\dot{q}_1(t)+\dot{q}_1(t-\tau)]} + \frac{c}{m}\left[q_1(t)+q_1(t-\tau)\right], \\ \xi_2 = \dot{q}_2(t) + \dot{q}_2(t-\tau) + \frac{k[q_2(t)+q_2(t-\tau)]^2}{m[\dot{q}_2(t)+\dot{q}_2(t-\tau)]} + \frac{c}{m}\left[q_2(t)+q_2(t-\tau)\right] \tag{65}$$

According to Theorem 1, we obtain the conserved quantity as follows:

$$I = me^{ct/m}\left\{\left[\dot{q}_1(t)+\dot{q}_1(t-\tau)\right]^2 + \left[\dot{q}_2(t)+\dot{q}_2(t-\tau)\right]^2\right\} \\ +ke^{ct/m}\left\{\left[q_1(t)+q_1(t-\tau)\right]^2 + \left[q_2(t)+q_2(t-\tau)\right]^2\right\} \\ +ce^{ct/m}\left\{\left[q_1(t)+q_1(t-\tau)\right]\left[\dot{q}_1(t)+\dot{q}_1(t-\tau)\right]\right\} \\ +ce^{ct/m}\left\{\left[q_2(t)+q_2(t-\tau)\right]\left[\dot{q}_2(t)+\dot{q}_2(t-\tau)\right]\right\} = \text{const.} \tag{66}$$

When $t \in [t_0 - \tau, t_0)$, from Equation (21), we have $\xi_0 = 0$. Therefore, Formulas (63) and (66) are conserved quantities led by Noether symmetry of the system.

8. Conclusions

Based on the HGVP, we studied the Noether symmetry and conserved quantities in the dynamics of non-conservative systems with delayed arguments. The Euler–Lagrange equations for the time-delayed non-conservative systems were presented. Non-isochronous variation Formulas (19) and (20) for Hamilton–Herglotz action with delayed arguments were derived. The infinitesimal transformation (6) depends not only on the generalized coordinates and time, but also on the generalized velocity. Based on the non-isochronous variational formulas, the Noether symmetry criteria for non-conservative systems with delayed arguments were established. Noether's theorem of Herglotz type for non-conservative systems with delayed arguments was proved, and was extended to the Birkhoff system and to the Hamilton system.

Recently, in Reference [44], we summarized some advances in the study of HGVP and its Noether's theorems, and put forward some views on future research. In addition, the spontaneous symmetry breaking is an important field concerned by physicists [45,46]. How can we apply Herglotz's generalized variational principle to non-conservative quantum systems and study the simultaneous symmetry breaking? This is also a topic worthy of further research. In short, there are still many problems worth exploring in the study of HGVP and its symmetry of time-delay non-conservative systems, etc.

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