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A Nonlinear Five-Term System: Symmetry, Chaos, and Prediction

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Abstract: Chaotic systems have attracted considerable attention and been applied in various applications. Investigating simple systems and counterexamples with chaotic behaviors is still an important topic. The purpose of this work was to study a simple symmetrical system including only five nonlinear terms. We discovered the system's rich behavior such as chaos through phase portraits, bifurcation diagrams, Lyapunov exponents, and entropy. Interestingly, multi-stability was observed when changing system's initial conditions. Chaos of such a system was predicted by applying a machine learning approach based on a neural network.

Keywords: chaos; symmetry; entropy; prediction

1. Introduction

The study of chaos in nonlinear systems has attracted significant attention in recent research [1–4]. Interestingly, new chaotic systems have continued to be proposed. A memristor-based chaotic circuit showing multi-stability was constructed by Song et al. [1]. Azar and Serrano [2] designed port Hamiltonian systems with chaos while Askar and Al-khedhairi mentioned a chaos of duopoly game for a player's actions [5]. Chaos appeared in fractional systems, fractional-order maps, and discrete-time systems [6–8]. Chen et al. discovered entropy for indicating early-warning signals of zero-eigenvalue chaotic systems [9]. Disturbance observer control was applied to synchronize a chaotic system having one constant term and no equilibrium [10]. The special characteristics of chaos provide useful applications such as cryptography, transmission, security, and fractional chaotic memory [11–13]. Image encryption was developed by using a chaos of Farhan's system [11] while a combination of compressed sensing and chaos in encryption scheme was introduced in [14]. Parallel mode of chaotic cryptography provided a transmission efficiency and resisted dangerous attacks [15]. Xie et al. developed image restoration for chaos-based transmission, which is effective to reduce devices' consumption [12]. S-boxes were constructed with special systems' chaos [16,17]. Ouannas et al. investigated MIMO communications using chaos synchronization [18]. By applying constant phase elements, Petrzela implemented chaotic memory [13].

Increased interest in symmetry in chaotic system has been reported in recent works [19]. Zhu and Du presented a chaotic system with a symmetrical curve equilibrium [20]. Chaotic oscillators were built with asymmetrical logic functions, illustrating the feasibility for integrated circuits [21]. It is noted that a symmetrical hyperchaotic attractor was able to control [19]. Especially, Li et al. examined comprehensively the evolution of symmetry [22]. From the viewpoint of information security, the vital roles of symmetry are verified by the improvement of substitution box structures [23], image encryption [24], and symmetric key encryption [25]. Discovering symmetrical chaotic systems is still an open topic.

When studying chaotic systems, discovering simple systems and counterexamples with chaotic behaviors is a vital research topic [26]. Common three-dimensional chaotic systems often have more than six terms and five-term chaotic systems are the most elegant ones [26]. Moreover, chaotic systems without linear terms have rarely been reported [27]. The purpose of our work was to investigate a novel five-term chaotic system without a linear term. Table 1 is provided for comparison with the results of recent studies.

Table 1. Numbers of linear and nonlinear terms in some three-dimensional chaotic systems.

System	Linear Term	Nonlinear Term	Total Term
[17]	6	4	10
[1]	6	2	8
[20]	1	7	8
[27]	0	8	8
[16]	3	4	7
[28]	2	5	7
[11]	2	4	6
[9]	2	3	5
This work	0	5	5

This work focuses on the aim to study a symmetrical system with chaos. Its simple form and rich dynamics are presented in Section 2. Section 3 presents an entropy measurement of the system while chaos prediction using neural network is reported in Section 4. The last section concludes our work.

2. System without Linearity

We investigate a system with no linear terms:

$$\begin{aligned} \dot{x} &= ayz, \\ \dot{y} &= 1 - z^2, \\ \dot{z} &= bx^3 + yz. \end{aligned} \tag{1}$$

In system (1), a and b are positive parameters $(a, b > 0)$. Interestingly, five terms of the system (1) are nonlinear ones. Only few systems without linear terms have been studied [27]. Simple chaotic systems/circuits have attracted considerable attention because of their elegance [26]. From the viewpoint of terms, the simplest chaotic systems are five-term ones [26]. Therefore, we would like to consider system (1), which includes five nonlinear terms. In addition, the system can be implemented physically by using common electronic elements such as resistors, capacitors, operational amplifiers, and analog multipliers. A practical implementation of system (1) is illustrated in Figure 1. In the design, the circuit of system (1) includes five resistors, three capacitors, three operational amplifiers, and four analog multipliers. However, corresponding equations of the system do not describe certain events.

Considering coordinate transformation (2)

$$(x, y, z) \rightarrow (-x, y, -z), \tag{2}$$

system (1) is invariant. Therefore, system (1) is symmetric. It is worth noting that symmetry in nonlinear systems has attracted interest in recent years [19–21].

By solving

$$\begin{cases} ayz = 0, \\ 1 - z^2 = 0, \\ bx^3 + yz = 0, \end{cases} \tag{3}$$

we get two equilibria of system (1)

$$E_1(0,0,1), \tag{4}$$

$$E_2(0,0,-1). \tag{5}$$

The Jacobian matrix of system (1) is given by

$$J = \begin{bmatrix} 0 & az & ay \\ 0 & 0 & -2z \\ 3bx^2 & z & y \end{bmatrix}. \tag{6}$$

Because of symmetry, by considering the Jacobian matrix at the equilibrium E_1, we get the characteristic equation

$$\lambda^3 + 2\lambda = 0. \tag{7}$$

and two eigenvalues

$$\lambda_1 = 0, \tag{8}$$

$$\lambda_{2,3} = \pm j\sqrt{2}. \tag{9}$$

Therefore, this calculation shows that the system (1) is at a *critical case* for E_1 and E_2.

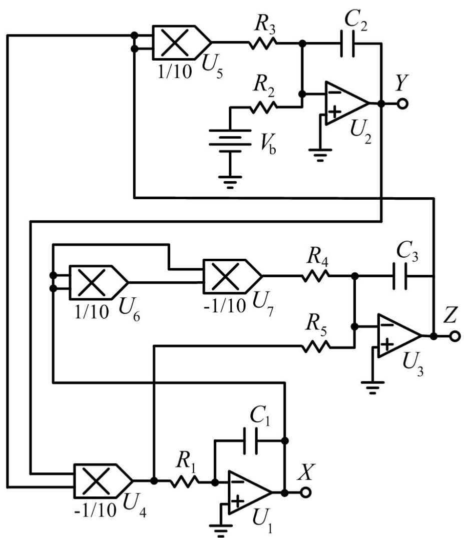


Figure 1. Illustration of a circuit, which is designed to realize system (1). Voltages at the outputs of three operational amplifiers X, Y, Z correspond to three state variables x, y, z of system (1).

System (1) displays rich dynamics when varying a. The bifurcation diagram in Figure 2 shows windows of chaos, which are also verified by maximum Lyapunov exponents (see Figure 3). Chaos can be found in ranges, for example $[1, 1.21]$, and $[1.903, 2.275]$. Illustration of chaos is presented in Figure 4 for $a = 1$, and $b = 0.05$. The maximum Lyapunov exponent equals to 0.02714. Chaotic dynamics is similar to the observed chaotic one of the Lorenz system [29]. The Lorenz system describes the atmospheric convection and includes seven terms (with five linear terms). Our system has five terms (without linear terms).The waveforms of the variables x and z in Figure 5 display slow–fast dynamics. Slow–fast dynamics are important to get the autowaves [30].

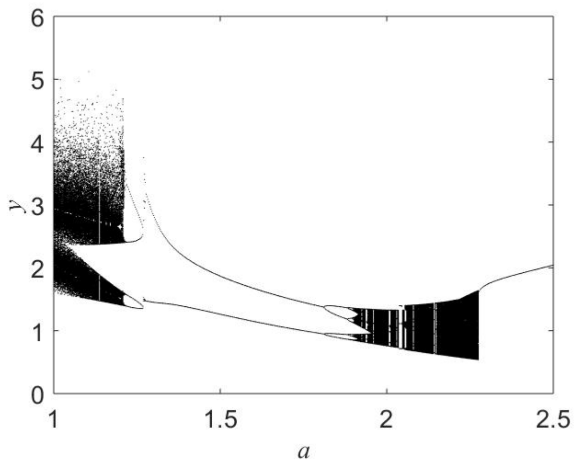


Figure 2. Bifurcation diagram. We change parameter a while keeping $b = 0.05$, and initial conditions $(0.5, 1, 0.5)$.

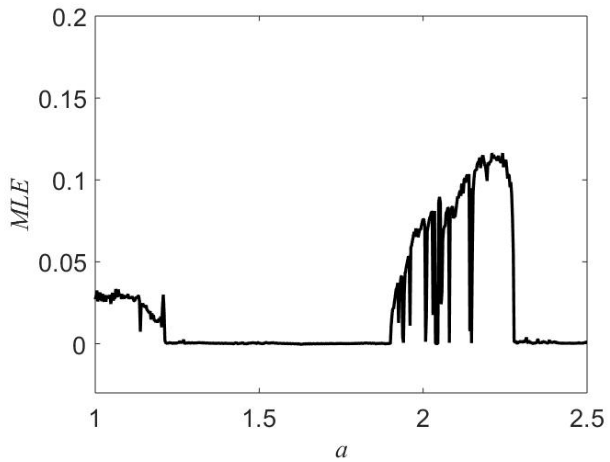


Figure 3. Maximum Lyapunov exponents. We change a while keeping $b = 0.05$, and initial conditions $(0.5, 1, 0.5)$.

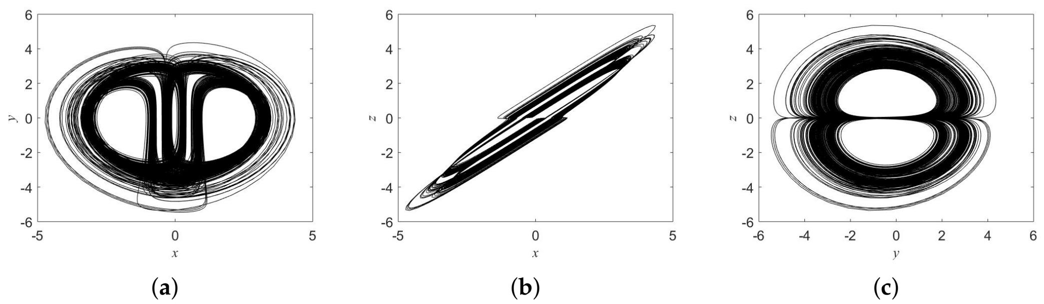


Figure 4. Attractors observed in three planes illustrating chaos in system (1) for $a = 1$: (**a**) $x - y$, (**b**) $x - z$, and (**c**) $y - z$.

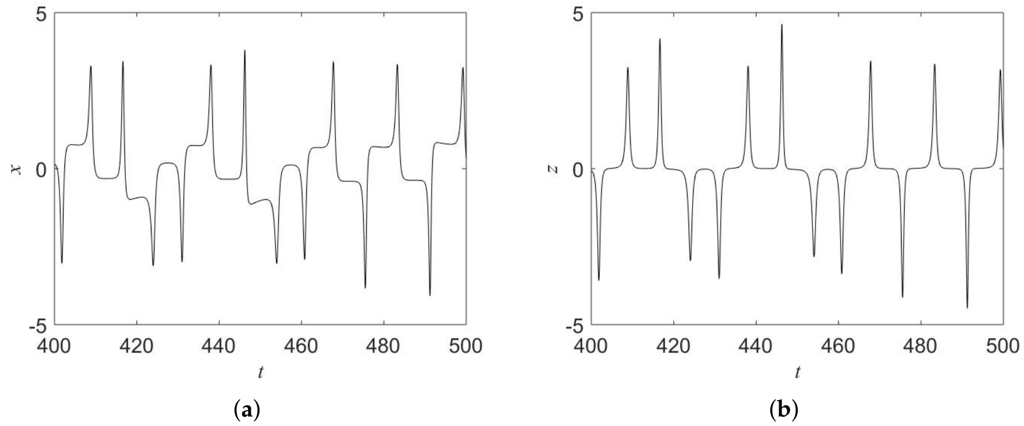


Figure 5. (**a**) Waveform of x, and (**b**) waveform of z observed in system (1) for $a = 1$, and $b = 0.05$.

The symmetrical property of system (1) leads to the appearance of multistabily, which has been investigated in Figure 6. We plot simultaneously two bifurcation diagrams for initial conditions $(\pm 0.5, 1, \pm 0.5)$. Different coexisting attractors are reported in Figure 7.

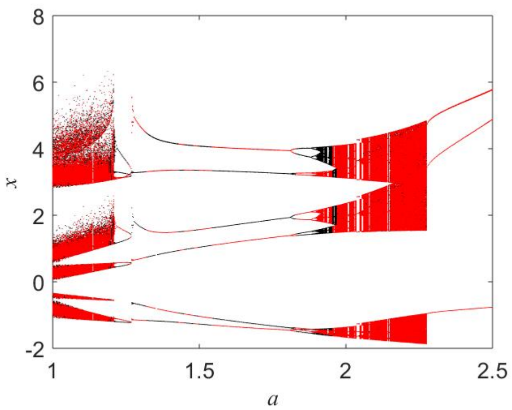


Figure 6. Bifurcation diagrams for initial conditions $(0.5, 1, 0.5)$ (black) and $(-0.5, 1, -0.5)$ (red) while keeping $b = 0.05$.

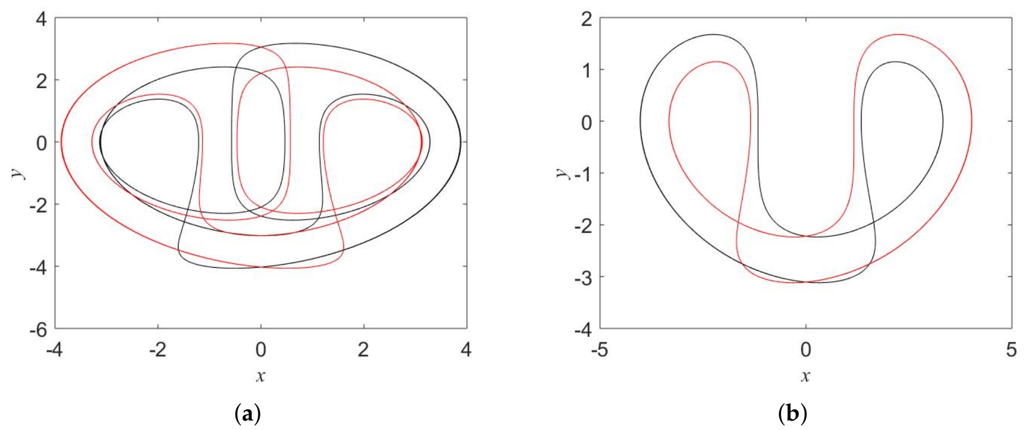


Figure 7. *Cont*.

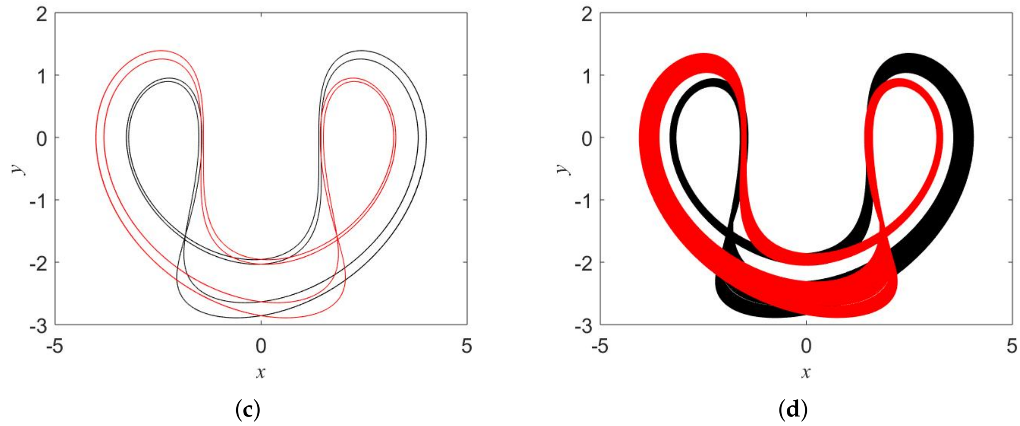


Figure 7. Coexisting attractors in system (1) for $(x(0), y(0), z(0) = (0.5, 1, 0.5)$ (black) and $(x(0), y(0), z(0) = (-0.5, 1, -0.5)$ (red): **(a)** $a = 1.23$, **(b)** $b = 1.6$, **(c)** $b = 1.845$, and **(d)** $a = 1.93$.

3. System's Entropy

Entropy is an important tool not only in information theory but also in nonlinear works [31]. Entropy represents quantifies of information in a particular information system. It is useful for researches to describe nonlinear system's complexity with entropy. Interestingly, entropy measurement has been witnessed for chaotic systems for last years [28,32]. Memristor-based chaotic oscillator has been developed by Liu et al. to achieve a high spectral entropy [3]. Chen et al. has indicated the usage of entropy as early warning indexes of chaotic signals [9].

We calculate entropy of system (1) to consider its complexity. The approximate entropy (ApEn) is measured for x variable. ApEn highlights advantages such as small samples demand, simple computation, and noise reduction [33].

Approximate entropy calculation [33] is presented briefly as follows. Firstly, we take a set of data $x(1)$, $x(2)$, ..., $x(n)$ from system (1). Vectors $X(j)$ for $j = 1, \ldots, n - m + 1$ are constructed by $X(j) = (x(j), ..., x(j + m - 1))$ with a given m. The distance between vectors $X(i)$ and $X(j)$ is given by $d(X(i), X(j))$. As a result, we get the relative frequency of $X(i)$ being similar to $X(j)$:

$$C_i^m(r) = \frac{K}{n - m + 1}, \tag{10}$$

where K is the number of j satisfying $d(X(i), X(j)) \le r$ for a given $X(i)$.

We obtain the approximate entropy

$$ApEn = \phi^m(r) - \phi^{m+1}(r), \tag{11}$$

in which

$$\phi^m(r) = \frac{1}{n - m - 1} \sum_{i=1}^{n-m+1} \log C_i^m(r). \tag{12}$$

Figure 8 depicts the approximate entropy [33] for parameter a. ApEn measures regularity and unpredictability of x. Significantly small values of ApEn indicate regular signals. As shown in Figure 8, system's complex behavior can be witnessed for two ranges of a ($[1, 1.21]$, and $[1.903, 2.275]$). Table 2 reports three examples of calculated ApEn values for a. The values of ApEn in the case 1 (0.2162), and the case 3 (0.3526) verify the complexity of system (1). Tiny value of ApEn ($7.418 \times 10^{-7} \approx 0$) confirms the periodical behavior of the system in the case 2.

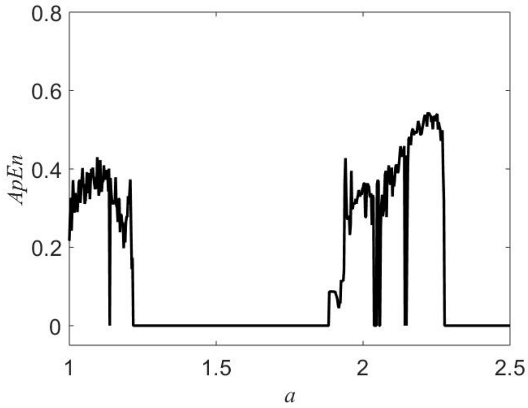


Figure 8. Approximate entropy (ApEn) of system (1) calculated for $a \in [1, 2.5]$.

Table 2. Examples of calculated ApEn values.

Cases	$\boldsymbol{a}$	**ApEn**
1	1	0.2162
2	1.5	7.418×10^{-7}
3	2	0.3526

4. Chaos Prediction

Artificial neuron network is constructed by connecting many neurons [34,35]. Numerous applications of neural networks in practice have been found in computer vision, pattern recognition, natural language processing, and robotics [36,37]. Ability of neuron network to represent nonlinear system has been investigated and attracted considerable interest [38–40]. Predicting chaotic system is challenge due to its sensitivity with initial conditions.

In this section, we build a simple feed-forward neural network (see Figure 9) to predict signals of system (1). As illustrated in Figure 9, the artificial neuron network includes four layers: input layer, two hidden layers, and output layer. It is considered as a deep neural network because there are multiple layers before the output layer [41]. The input layer includes three neurons. Each hidden layer is composed of ten neurons while there are only three neurons in the output layer. The numbers of hidden neurons and hidden layers are selected by considering the specific dynamics of the system such as multistability, and slow-fast dynamics. The computational roles of hidden layers are similar in order to model the dynamical system from its time series. It is worth noting that the hardest task of machine learning, choosing the suitable balance between model complexity and simplicity, must be considered seriously. The effective and robust architecture of the neural network as well as the optimization of network's parameters guarantee the good performance of the network. In this work, we construct a simple feed-forward neural network. Compared with advanced networks, for example convolutional neural network, recurrent neural network, liquid state machine, and echo state network, the proposed architecture is effective and robust when being applied to system (1).

A dataset is generated by running system (1) with different initial conditions. The proposed neural network is trained with the data set (x, y, z) by applying the Levenberg–Marquardt algorithm. It is noted that there are different training algorithms such as the Levenberg–Marquardt algorithm, Bayesian Regularization algorithm, Scaled Conjugate Gradient algorithm, and the Fletcher–Powell Conjugate Gradient [37,42]. In this work, we use the Levenberg–Marquardt algorithm because of its good convergence and robustness. The obtained performance is 2.3051×10^{-9}. After the training process, we achieve a network matching with the data set. Outputs of the network present expected signals. Figure 10 illustrates the prediction results (X, Y, Z) compared with actual data (x, y, z). The agreement of the prediction results with the actual data shows the capability of the network for predicting chaos of system (1) in short term. In comparison to other works [43–45], the neural network is simple and displays good performance.

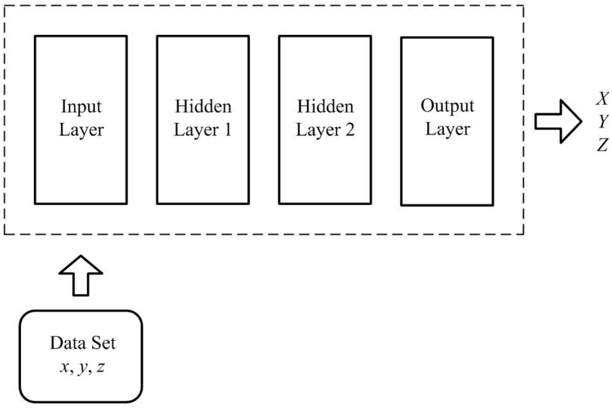
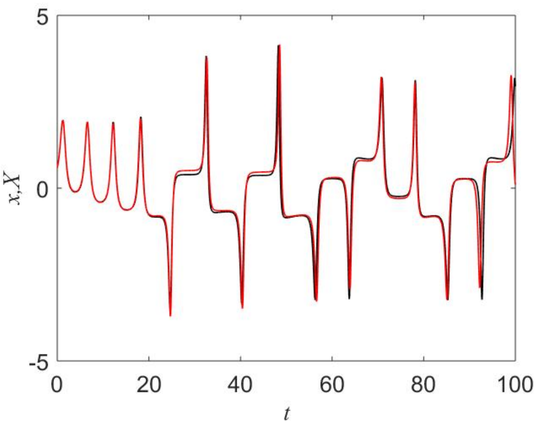
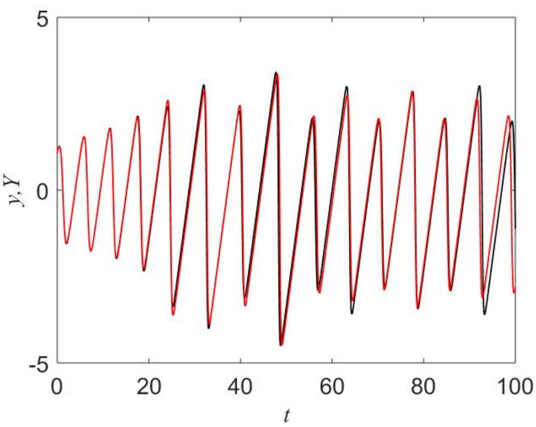


Figure 9. Neuron network includes four layers. Data set is provided by system (1) and is used for training.

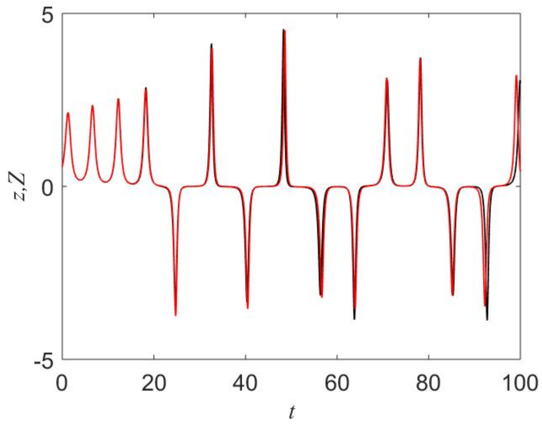


(**a**)



(**b**)

Figure 10. *Cont.*



(**c**)

Figure 10. Signals x, y, z of system (1) (black color) and desired signals X, Y, Z at the output of the neural network (red color): (**a**) x and X, (**b**) y and Y, (**c**) z and Z.

5. Conclusions

Our work introduces a symmetry nonlinear system with remarkable dynamics. There are only five nonlinear terms in the system, which generates chaos. By considering the initial conditions, we find coexisting attractors in the system verifying its multistability feature. Entropy measurement also indicates the system's complexity. We believe that our work contributes to the known list of chaotic systems with algebraic simplicity. It is possible for us to apply a modified version of such a system to describe turbulent flows [46,47]. We implemented a neural-based approach to predict a system's chaos in short-term. Long-term prediction of such chaotic signals should be considered. In addition, prediction results will be applied to control chaos in our future investigation. In addition, realization of the system for practical chaos-based applications will be studied in our future works.

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Conflicts of Interest: The authors declare no conflict of interest.

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