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On the Absolute Stable Difference Scheme for Third Order Delay Partial Differential Equations

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Abstract: The initial value problem for the third order delay differential equation in a Hilbert space with an unbounded operator is investigated. The absolute stable three-step difference scheme of a first order of accuracy is constructed and analyzed. This difference scheme is built on the Taylor's decomposition method on three and two points. The theorem on the stability of the presented difference scheme is proven. In practice, stability estimates for the solutions of three-step difference schemes for different types of delay partial differential equations are obtained. Finally, in order to ensure the coincidence between experimental and theoretical results and to clarify how efficient the proposed scheme is, some numerical experiments are tested.

Keywords: time delay; third order differential equations; difference scheme; stability

1. Introduction

Various problems in elasticity theory such as the problems of the longitudinal oscillations of a non-uniform viscoelastic rod, the problem of the longitudinal impact of a perfectly rigid body on a non-uniform finite-length viscoelastic rod with a variable cross-section, problems of wave propagation in a visco-elastic body, etc., lead to third order differential equations without the time delay term ([1–3]). Over the years, nonlocal and local boundary value problems have been of great interest due to their importance in the fields of engineering and science, especially in applied mathematics. Such problems have formed various research fields. Several nonlocal and local boundary value problems for differential equations have been investigated extensively in various works (for example, see [4–12] and the references given therein).

Differential equations having a delay term are used to model sociological, biological, as well as physical processes. They are used to model naturally occurring oscillation systems. A typical example of the occurrence of time delay can be seen in a sampled data control in control theory (see, for example, [13–17]). The presence of delay term in differential equations usually leads to difficulties in analyzing the differential equation. The boundedness and stability and the oscillation property of solutions for a third order delay ordinary differential and difference problems were widely studied (for example, see [18–25] and the references given therein).

Delay partial differential equations (DPDEs) arise in many applications such as control theory, climate models, medicine, biology, and much more (for example, see [26] and the references therein). The independent variables of partial differential equations having delay terms are time t together with one or more dimensional variable x , representing the position in space. It can also stand for the size of cells, relative DNA content, their level of mutation, as well as other parameters. The solutions of

partial differential equations having delay terms may stand for voltage, temperature, or densities or concentrations of various particles, for instance chemicals, cells, animals, bacteria, and so on. Numerical methods for partial differential equations with delay terms usually lead to specific difficulties, which are usually not present in equations without delay terms. The theory and applications of parabolic and hyperbolic partial differential equations having a time delay term were studied by numerous authors (for example, see [13,27–34] and the references given therein). Recent publications on third order DPDEs are not many.

Several physical models lead to initial-boundary value problems for third order DPDEs (see, e.g., [1,3,8]). It is known that such types of problems can be replaced with the initial value problem for a third order delay differential equation:

$$\begin{cases} u_{ttt}(t) + Au_t(t) = bAu(t-w) + f(t), & 0 < t < \infty, \\ u(t) = g(t), & -w \leq t \leq 0 \end{cases} \quad (1)$$

in a Hilbert space H with unbounded operator A . Here, $b \in R^1$. Assume that $f(t)$ is a continuous function on $[0, \infty)$ and $f(t) \in D(A^{1/2})$, $g(t)$ is a twice continuously differentiable function on $[-w, 0]$ and $g^{(k)}(t) \in D(A^{(3-k)/2})$ for $k = 0, 1, 2$.

Let us give the main theorem of paper [35].

Theorem 1. *The solution of Problem (1) satisfies the stability estimates:*

$$a_1 \leq da_0 + \int_0^w \|f(s)\|_{D(A^{1/2})} ds, d = 2 + |b|w, \quad (2)$$

$$a_{n+1} \leq da_n + \sum_{j=1}^{n+1} \int_{(j-1)w}^{jw} \|f(s)\|_{D(A^{1/2})} ds, n = 1, 2, \dots, \quad (3)$$

$$a_0 = \max \left\{ \max_{-w \leq t \leq 0} \|g_{tt}(t)\|_{D(A^{1/2})}, \max_{-w \leq t \leq 0} \|g_t(t)\|_{D(A)}, \max_{-w \leq t \leq 0} \|g(t)\|_{D(A^{3/2})} \right\},$$

$$a_n = \max \left\{ \max_{(n-1)w \leq t \leq nw} \|u_{tt}(t)\|_{D(A^{1/2})}, \max_{(n-1)w \leq t \leq nw} \|u_t(t)\|_{D(A)}, \max_{(n-1)w \leq t \leq nw} \|u(t)\|_{D(A^{3/2})} \right\}.$$

In practice, stability estimates for the solution of several problems for third order DPDEs were obtained.

Moreover, publications on the theory and applications of difference schemes (DSs) for third order DPDEs are not available. Thus, the construction and investigation of stable DSs for the approximate solutions of third order DPDEs is of great importance. Our aim in this paper is to construct the absolute stable three-step DS of the first order of accuracy of the third order DPDE for the approximate solution of the problem (1). We consider the uniform set of grid points:

$$[-w, \infty)_\tau = \{t_k : t_k = k\tau, -N \leq k < \infty, N\tau = \omega\}$$

with step $\tau > 0$. Applying Taylor's decomposition method on three and two points (see [36,37]), we present the DS of the first order of accuracy:

$$\begin{cases} \frac{u_{k+2} - 3u_{k+1} + 3u_k - u_{k-1}}{\tau^3} + A \frac{u_{k+2} - u_{k+1}}{\tau} = bAu_{k-N} + f(t_k), & k \geq 1, \\ u_k = g(t_k), & -N + 1 \leq k \leq 0, \\ (I + \tau^2 A) \frac{u_1 - u_0}{\tau} = g'(0), (I + \tau^2 A) \frac{u_2 - 2u_1 + u_0}{\tau^2} = g''(0), \\ (I + \tau^2 A) \frac{u_{mN+1} - u_{mN}}{\tau} = \frac{u_{mN} - u_{mN-1}}{\tau}, \\ (I + \tau^2 A) \frac{u_{mN+2} - 2u_{mN+1} + u_{mN}}{\tau^2} = \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}, & m = 1, 2, \dots \end{cases} \quad (4)$$

for the approximate solution of Problem (1).

The organization of this paper is as follows. In Section 2, the main theorem on the stability of DS (4) is established. In Section 3, stability estimates of DSs for the approximate solution of three problems for third order DPDEs are obtained. Numerical results are provided for one- and two-dimensional third order DPDEs in Section 4. Finally, Section 5 gives the conclusion and our future plans.

2. Stability of DS

All over the present paper, assume that H is a Hilbert space and A is a self-adjoint positive definite operator $A \geq \delta I$ in H and $R = (I - i\tau A^{\frac{1}{2}})^{-1}$, $\tilde{R} = (I + i\tau A^{\frac{1}{2}})^{-1}$.

Note that three-step DS (4) can obviously be rewritten as the system of single-step and two-step delay DS:

$$\left\{ \begin{array}{l} \frac{u_{k+1}-u_k}{\tau} = v_k, k \geq 0, \\ \frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2} + Av_{k+1} = p_k, p_k = bu_{k-N} + f(t_k), k \geq 1, \\ u_k = g(t_k), -N \leq k \leq 0, \\ (I + \tau^2 A)v_0 = g'(0), (I + \tau^2 A)\frac{v_1-v_0}{\tau} = g''(0), \\ (I + \tau^2 A)v_{mN} = \frac{u_{mN}-u_{mN-1}}{\tau}, \\ (I + \tau^2 A)\frac{v_{mN+1}-v_{mN}}{\tau} = \frac{u_{mN}-2u_{mN-1}+u_{mN-2}}{\tau^2}, m = 1, 2, \dots \end{array} \right. \quad (5)$$

for the solution of DS (4). Applying DC(5), we can obtain the formula for the solution of DS (4). For this, we will consider two cases $1 \leq k \leq N$ and $mN + 1 \leq k \leq (m + 1)N$, $m = 1, 2, \dots$, separately.

Let $1 \leq k \leq N$. Applying (5), we get the following DS:

$$\left\{ \begin{array}{l} \frac{u_k-u_{k-1}}{\tau} = v_{k-1}, 1 \leq k \leq N, \\ \frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2} + Av_{k+1} = p_k, p_k = bu_{k-N} + f(t_k), 1 \leq k \leq N-1, \\ u_k = g(t_k), -N \leq k \leq 0, \\ (I + \tau^2 A)v_0 = g'(0), (I + \tau^2 A)\frac{v_1-v_0}{\tau} = g''(0). \end{array} \right. \quad (6)$$

Therefore, we have that (see [38]):

$$u_0 = g(0), u_k = g(0) + \sum_{j=0}^{k-1} \tau v_j, 1 \leq k \leq N, \quad (7)$$

$$\begin{aligned} v_0 &= R\tilde{R}g'(0), v_1 = R\tilde{R}g'(0) + \tau R\tilde{R}g''(0), \\ v_k &= \frac{1}{2}[R^{k-1} + \tilde{R}^{k-1}]R\tilde{R}g'(0) + \frac{1}{2i}A^{-\frac{1}{2}}R(R^k - \tilde{R}^k)g''(0) \\ &\quad - \sum_{s=1}^{k-1} \frac{\tau}{2i}A^{-\frac{1}{2}}[R^{k-s} - \tilde{R}^{k-s}]p_s \\ &= \frac{1}{2}[R^{k-1} + \tilde{R}^{k-1}]R\tilde{R}g'(0) + \frac{1}{2i}A^{-\frac{1}{2}}R(R^k - \tilde{R}^k)g''(0) \\ &\quad + A^{-1} \left\{ \frac{1}{2} \sum_{s=2}^{k-1} [R^{k-s} + \tilde{R}^{k-s}](p_{s-1} - p_s) + 2p_{k-1} - [R^{k-1} + \tilde{R}^{k-1}]p_1 \right\}, 2 \leq k \leq N, \\ p_k &= bAg(t_{k-N}) + f(t_k), 1 \leq k \leq N-1. \end{aligned} \quad (8)$$

Applying Formulas (6)–(8), we obtain:

$$u_k = \begin{cases} g(0) + \tau R \tilde{R} g'(0), k = 0, \\ g(0) + \tau R \tilde{R} g'(0), k = 1, \\ g(0) + 2\tau R \tilde{R} g'(0) + \tau^2 R \tilde{R} g''(0), k = 2, \\ g(0) + \frac{1}{2i} A^{-\frac{1}{2}} \left(R^{k-2} - \tilde{R}^{k-2} \right) R \tilde{R} g'(0) - \frac{1}{2} A^{-1} R \left(2R \tilde{R} - \left(R^{k-1} + \tilde{R}^{k-1} \right) \right) g''(0) \\ + \sum_{j=2}^{k-1} \tau \sum_{s=1}^{j-1} \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{j-s} - \tilde{R}^{j-s}] p_s, 3 \leq k \leq N. \end{cases}$$

By an interchange of the order of summation, we get:

$$u_k = \begin{cases} g(0) + \tau R \tilde{R} g'(0), k = 0, \\ g(0) + \tau R \tilde{R} g'(0), k = 1, \\ g(0) + 2\tau R \tilde{R} g'(0) + \tau^2 R \tilde{R} g''(0), k = 2, \\ g(0) + \frac{1}{2i} A^{-\frac{1}{2}} \left(R^{k-2} - \tilde{R}^{k-2} \right) R \tilde{R} g'(0) - \frac{1}{2} A^{-1} R \left(2R \tilde{R} - \left(R^{k-1} + \tilde{R}^{k-1} \right) \right) g''(0) \\ + A^{-1} \sum_{s=1}^{k-2} \frac{\tau}{2} [2I - \left(R^{k-1-s} + \tilde{R}^{k-1-s} \right)] p_s, 3 \leq k \leq N \end{cases} \quad (9)$$

for the solution of DS (4).

Let $1 + mN \leq k \leq (m+1)N$, $m = 1, 2, \dots$. Applying (5), we can get the DS:

$$\begin{cases} \frac{u_k - u_{k-1}}{\tau} = v_{k-1}, mN + 1 \leq k \leq (m+1)N, \\ \frac{v_{k+1} - 2v_k + v_{k-1}}{\tau^2} + Av_{k+1} = p_k, p_k = bu_{k-N} + f(t_k), \\ mN + 1 \leq k \leq (m+1)N - 1, \\ u_{mN} \text{ is given, } (I + \tau^2 A)v_{mN} = \frac{u_{mN} - u_{mN-1}}{\tau}, \\ (I + \tau^2 A) \frac{v_{mN+1} - v_{mN}}{\tau} = \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}. \end{cases} \quad (10)$$

Therefore, we have that (see [38]):

$$u_k = u_{mN} + \sum_{j=mN+2}^{k-1} \tau v_j, mN + 1 \leq k \leq (m+1)N, \quad (11)$$

$$\begin{aligned} v_{mN} &= R \tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau}, v_{mN+1} = R \tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} + \tau R \tilde{R} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}, \\ v_k &= \frac{1}{2} [R^{k-mN-1} + \tilde{R}^{k-mN-1}] R \tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} \\ &\quad + \frac{1}{2i} A^{-\frac{1}{2}} R (R^{k-mN} - \tilde{R}^{k-mN}) \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\ &\quad - \sum_{s=mN+1}^{k-1} \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{k-s} - \tilde{R}^{k-s}] p_s \\ &= \frac{1}{2} [R^{k-1} + \tilde{R}^{k-1}] R \tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} + \frac{1}{2i} A^{-\frac{1}{2}} R (R^k - \tilde{R}^k) g''(0) \\ &\quad + A^{-1} \left\{ \frac{1}{2} \sum_{s=mN+2}^{k-1} [R^{k-s} + \tilde{R}^{k-s}] (p_{s-1} - p_s) \right. \\ &\quad \left. + 2p_{k-mN-1} - [R^{k-mN-1} + \tilde{R}^{k-mN-1}] p_{mN+1} \right\}, mN + 2 \leq k \leq (m+1)N. \end{aligned} \quad (12)$$

Applying Formulas (10)–(12), we can obtain:

$$u_k = \begin{cases} u_{mN} + \tau R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}}, k = mN + 1, \\ u_{mN} + 2\tau R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}} + \tau^2 R \tilde{R}^{\frac{u_{mN}-2u_{mN-1}+u_{mN-2}}{\tau^2}}, k = mN + 2, \\ u_{mN} + \frac{1}{2i} A^{-\frac{1}{2}} \left(R^{k-mN-2} - \tilde{R}^{k-mN-2} \right) R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}} \\ - \frac{1}{2} A^{-1} R \left(2R \tilde{R} - \left(R^{k-mN-1} + \tilde{R}^{k-mN-1} \right) \right) \frac{u_{mN}-2u_{mN-1}+u_{mN-2}}{\tau^2} \\ + \sum_{j=mN+2}^{k-1} \tau \sum_{s=mN+1}^{j-1} \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{j-s} - \tilde{R}^{j-s}] p_s, mN + 3 \leq k \leq (m+1)N. \end{cases}$$

By an interchange of the order of summation, we get the solution of DS (4):

$$u_k = \begin{cases} u_{mN} + \tau R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}}, k = mN + 1, \\ u_{mN} + 2\tau R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}} + \tau^2 R \tilde{R}^{\frac{u_{mN}-2u_{mN-1}+u_{mN-2}}{\tau^2}}, k = mN + 2, \\ u_{mN} + \frac{1}{2i} A^{-\frac{1}{2}} \left(R^{k-mN-2} - \tilde{R}^{k-mN-2} \right) R \tilde{R}^{\frac{u_{mN}-u_{mN-1}}{\tau}} \\ - \frac{1}{2} A^{-1} R \left(2R \tilde{R} - \left(R^{k-mN-1} + \tilde{R}^{k-mN-1} \right) \right) \frac{u_{mN}-2u_{mN-1}+u_{mN-2}}{\tau^2} \\ + A^{-1} \sum_{s=mN+1}^{k-2} \frac{\tau}{2} [2I - \left(R^{k-1-s} + \tilde{R}^{k-1-s} \right)] p_s, mN + 3 \leq k \leq (m+1)N. \end{cases} \quad (13)$$

The following lemma will be needed in the sequel.

Lemma 1. The following estimates are fulfilled:

$$\|R\|_{H \rightarrow H}, \|\tilde{R}\|_{H \rightarrow H} \leq 1, \quad (14)$$

$$\|R \tilde{R}^{-1}\|_{H \rightarrow H}, \|\tilde{R} R^{-1}\|_{H \rightarrow H} \leq 1, \quad (15)$$

$$\|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H}, \|\tau A^{\frac{1}{2}} \tilde{R}\|_{H \rightarrow H} \leq 1. \quad (16)$$

The proof of the estimates (14)–(16) is based on the spectral theory of a self-adjoint operator in a Hilbert space [39].

Now, let us study the stability of DS (4).

Theorem 2. The solution of DS (4) satisfies the following stability estimates:

$$b_1 \leq (2 + \tau|b|(N-2))b_0 + \tau \sum_{s=1}^{N-2} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad (17)$$

$$b_{m+1} \leq (2 + \tau|b|(N-2))b_m + \tau \sum_{s=mN+1}^{(m+1)N} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad (18)$$

$$b_0 = \max \left\{ \max_{-N \leq k \leq 0} \|A^{\frac{1}{2}} g_{tt}(t_k)\|_H, \max_{-N \leq k \leq 0} \|A g_t(t_k)\|_H, \max_{-N \leq k \leq 0} \|A^{\frac{3}{2}} g(t_k)\|_H \right\},$$

$$b_m = \max \left\{ \max_{(m-1)N \leq k \leq mN-2} \|A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2}\|_H, \right.$$

$$\left. \max_{(m-1)N+1 \leq k \leq mN} \|A \frac{u_k - u_{k-1}}{\tau}\|_H, \frac{1}{2} \max_{(m-1)N \leq k \leq mN} \|A^{\frac{3}{2}} u_k\|_H \right\}, m = 1, 2, \dots$$

Proof. Let us estimate b_1 . Using Formula (9) and the estimates (14)–(16), we get that:

$$\|A^{\frac{3}{2}} u_1\|_H \leq \|A^{\frac{3}{2}} g(0)\|_H + \|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H} \|A g_t(0)\|_H \leq 2b_0,$$

$$\begin{aligned}
& \|A^{\frac{3}{2}}u_2\|_H \leq \|A^{\frac{3}{2}}g(0)\|_H + 2\|\tau A^{\frac{1}{2}}R\|_{H \rightarrow H}\|\tilde{R}\|_{H \rightarrow H}\|Ag_t(0)\|_H \\
& \quad + \|\tau A^{\frac{1}{2}}R\|_{H \rightarrow H}\|\tau A^{\frac{1}{2}}\tilde{R}\|_{H \rightarrow H}\|A^{\frac{1}{2}}g_{tt}(0)\|_H \leq 4b_0, \\
& \|A^{\frac{3}{2}}u_k\|_H \leq \|A^{\frac{3}{2}}g(0)\|_H + \frac{1}{2}[\|R\|_{H \rightarrow H}^{k-2} + \|\tilde{R}\|_{H \rightarrow H}^{k-2}]\|R\tilde{R}\|_{H \rightarrow H}\|Ag_t(0)\|_H \\
& \quad + \frac{1}{2}\|R\|_{H \rightarrow H}[2\|R\tilde{R}\|_{H \rightarrow H} + \|R\|_{H \rightarrow H}^{k-1} + \|\tilde{R}\|_{H \rightarrow H}^{k-1}]\|A^{\frac{1}{2}}g_{tt}(0)\|_H \\
& \quad + \frac{\tau}{2}|b|\sum_{s=1}^{k-2}[2 + \|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}]\|A^{\frac{3}{2}}g(t_{s-N})\|_H \\
& \quad + \frac{\tau}{2}\sum_{s=1}^{k-2}[2 + \|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}]\|A^{\frac{1}{2}}f(t_s)\|_H \leq \|A^{\frac{3}{2}}g(0)\|_H + \|Ag_t(0)\|_H \\
& \quad + 2\|A^{\frac{1}{2}}g_{tt}(0)\|_H + 2\tau|b|(N-2)\max_{-N \leq k \leq 0}\|A^{\frac{3}{2}}g(t_k)\|_H + 2\tau\sum_{s=1}^{k-2}\|A^{\frac{1}{2}}f(t_s)\|_H \\
& \leq 2\left(2 + \tau|b|(N-2)\right)b_0 + 2\tau\sum_{s=1}^{N-2}\|A^{\frac{1}{2}}f(t_s)\|_H, \quad 3 \leq k \leq N.
\end{aligned}$$

From that and $u_0 = g(0)$, it follows that:

$$\begin{aligned}
& \frac{1}{2}\max_{0 \leq k \leq N}\|A^{\frac{3}{2}}u_k\|_H \\
& \leq (2 + \tau|b|(N-2))b_0 + \tau\sum_{s=1}^{N-2}\|A^{\frac{1}{2}}f(t_s)\|_H
\end{aligned} \tag{19}$$

for the solution of DS (4). Applying Formulas (6)–(8), we can write:

$$\begin{aligned}
& \frac{u_1 - u_0}{\tau} = v_0 = R\tilde{R}g_t(0), \\
& \frac{u_2 - u_1}{\tau} = v_1 = R\tilde{R}g_t(0) + \tau R\tilde{R}g_{tt}(0), \\
& \frac{u_k - u_{k-1}}{\tau} = v_{k-1} = \frac{1}{2}[R^{k-2} + \tilde{R}^{k-2}]R\tilde{R}g_t(0) + \frac{1}{2i}R(R^{k-1} - \tilde{R}^{k-1})A^{-\frac{1}{2}}g_{tt}(0) \\
& \quad - \frac{\tau}{2i}\sum_{s=1}^{k-2}[R^{k-1-s} - \tilde{R}^{k-1-s}]bA^{\frac{1}{2}}g(t_{s-N}) - \frac{\tau}{2i}\sum_{s=1}^{k-2}[R^{k-1-s} - \tilde{R}^{k-1-s}]A^{-\frac{1}{2}}f(t_s), \quad 3 \leq k \leq N.
\end{aligned}$$

Using this formula and the estimates (14)–(16), we obtain:

$$\begin{aligned}
& \|A\frac{u_1 - u_0}{\tau}\|_H \leq \|R\tilde{R}\|_{H \rightarrow H}\|g_t(0)\|_H \leq b_0, \\
& \|A\frac{u_2 - u_1}{\tau}\|_H \leq \|R\tilde{R}\|_{H \rightarrow H}\|Ag_t(0)\|_H + \|\tau A^{\frac{1}{2}}R\|_{H \rightarrow H}\|\tilde{R}\|_{H \rightarrow H}\|A^{\frac{1}{2}}g_{tt}(0)\|_H \leq 2b_0, \\
& \|A\frac{u_k - u_{k-1}}{\tau}\|_H \leq \frac{1}{2}[\|R\|_{H \rightarrow H}^{k-2} + \|\tilde{R}\|_{H \rightarrow H}^{k-2}]\|R\tilde{R}\|_{H \rightarrow H}\|g_t(0)\|_H \\
& \quad + \frac{1}{2}[\|R\|_{H \rightarrow H}^{k-1} + \|\tilde{R}\|_{H \rightarrow H}^{k-1}]\|R\|_{H \rightarrow H}\|A^{\frac{1}{2}}g_{tt}(0)\|_H + \frac{\tau}{2}|b|\sum_{s=1}^{N-2}[\|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}]\|A^{\frac{3}{2}}g(t_{s-N})\|_H \\
& \quad + \frac{\tau}{2}\sum_{s=1}^{N-2}[\|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}]\|A^{\frac{1}{2}}f(t_s)\|_H
\end{aligned}$$

$$\leq (2 + \tau|b|(N-2))b_0 + \tau \sum_{s=1}^{N-2} \|A^{\frac{1}{2}}f(t_s)\|_H, 3 \leq k \leq N.$$

Combining these estimates, we obtain:

$$\max_{1 \leq k \leq N} \left\| A \frac{u_k - u_{k-1}}{\tau} \right\|_H \leq (2 + \tau|b|(N-2))b_0 + \tau \sum_{s=1}^{N-2} \|A^{\frac{1}{2}}f(t_s)\|_H \quad (20)$$

for the solution of DS (4). Applying Formulas (6)–(8), we can write:

$$\begin{aligned} \frac{u_2 - 2u_1 + u_0}{\tau^2} &= \tau^{-1}(v_1 - v_0) = \tilde{R}Rg_{tt}(0), \\ \frac{u_3 - 2u_2 + u_1}{\tau^2} &= \tau^{-1}(v_2 - v_1) \\ &= -\tau A(R\tilde{R})^2g_t(0) + (I - \tau^2A) (\tilde{R}R)^2g_{tt}(0) + \tau R\tilde{R}p_1 \\ &= -\tau A(R\tilde{R})^2g_t(0) + (I - \tau^2A) (\tilde{R}R)^2g_{tt}(0) + \tau R\tilde{R}[bAg(t_{1-N}) + f(t_1)], \\ \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} &= \tau^{-1}(v_{k+1} - v_k) \\ &= \tau^{-1} \left\{ \frac{1}{2}[R^k + \tilde{R}^k]R\tilde{R}g_t(0) + \frac{1}{2i}A^{-\frac{1}{2}}[R^{k+1} - \tilde{R}^{k+1}]Rg_{tt}(0) \right. \\ &\quad - \sum_{s=1}^k \frac{\tau}{2i}A^{-\frac{1}{2}}[R^{k+1-s} - \tilde{R}^{k+1-s}]p_s - \frac{1}{2}[R^{k-1} + \tilde{R}^{k-1}]R\tilde{R}g_t(0) \\ &\quad \left. - \frac{1}{2i}A^{-\frac{1}{2}}[R^k - \tilde{R}^k]Rg_{tt}(0) + \sum_{s=1}^k \frac{\tau}{2i}A^{-\frac{1}{2}}[R^{k-s} - \tilde{R}^{k-s}]p_s \right\} \\ &= \tau^{-1} \left\{ \frac{1}{2}[R^k + \tilde{R}^k - R^{k-1} - \tilde{R}^{k-1}]R\tilde{R}g_t(0) + \frac{1}{2i}A^{-1}[R^{k+1} - \tilde{R}^{k+1} - R^k + \tilde{R}^k]Rg_{tt}(0) \right. \\ &\quad \left. - \frac{\tau}{2i}A^{-\frac{1}{2}}[R - \tilde{R}]p_k + \sum_{s=1}^{k-1} \frac{\tau}{2i}A^{-\frac{1}{2}}[R^{k-s} - \tilde{R}^{k-s} - R^{k+1-s} + \tilde{R}^{k+1-s}]p_s \right\} \\ &= -\frac{i}{2}[R^k - \tilde{R}^k]R\tilde{R}A^{\frac{1}{2}}g_t(0) + \frac{1}{2}R[R^{k+1} + \tilde{R}^{k+1}]g_{tt}(0) - \tau R\tilde{R}Abg(t_{k-N}) - \tau R\tilde{R}f(t_k) \\ &\quad - \frac{\tau}{2} \sum_{s=1}^{k-1} [R^{k-s+1} + \tilde{R}^{k-s+1}]Abg(t_{s-N}) - \frac{\tau}{2} \sum_{s=1}^{k-1} [R^{k-s+1} + \tilde{R}^{k-s+1}]f(t_s), 2 \leq k \leq N-2. \end{aligned}$$

Using this formula and the estimates (14)–(16), we obtain:

$$\begin{aligned} \left\| A^{\frac{1}{2}} \frac{u_2 - 2u_1 + u_0}{\tau^2} \right\|_H &\leq \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}}g_{tt}(0)\|_H \leq b_0, \\ \left\| A^{\frac{1}{2}} \frac{u_3 - 2u_2 + u_1}{\tau^2} \right\|_H &\leq \|\tau A^{\frac{1}{2}}R\|_{H \rightarrow H} \|R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H}^2 \|Ag_t(0)\|_H + \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}}g_{tt}(0)\|_H \\ &\quad + \tau|b| \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}}g(t_{1-N})\|_H + \tau \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}}f(t_1)\|_H \\ &\leq (2 + \tau|b|)b_0 + \tau \|A^{\frac{1}{2}}f(t_1)\|_H, \\ \left\| A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} \right\|_H &\leq \frac{1}{2} [\|R\|_{H \rightarrow H}^k + \|\tilde{R}\|_{H \rightarrow H}^k] \|R\tilde{R}\|_{H \rightarrow H} \|Ag_t(0)\|_H \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \|R\|_{H \rightarrow H} [\|R\|_{H \rightarrow H}^{k+1} + \|\tilde{R}\|_{H \rightarrow H}^{k+1}] \|A^{\frac{1}{2}} g_{tt}(0)\|_H + \frac{\tau}{2} |b| \sum_{s=1}^{k-1} [\|R\|_{H \rightarrow H}^{k-s+1} + \|\tilde{R}\|_{H \rightarrow H}^{k-s+1}] \|A^{\frac{3}{2}} g(t_{s-N})\|_H \\
& + \frac{\tau}{2} \sum_{s=1}^{k-1} [\|R\|_{H \rightarrow H}^{k-s+1} + \|\tilde{R}\|_{H \rightarrow H}^{k-s+1}] \|A^{\frac{1}{2}} f(t_s)\|_H + \tau |b| \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}} g(t_{k-N})\|_H \\
& + \tau \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} f(t_k)\|_H \\
& \leq (2 + \tau |b| (N-2)) b_0 + \tau \sum_{s=1}^{N-2} \|A^{\frac{1}{2}} f(t_s)\|_H, 2 \leq k \leq N-2.
\end{aligned}$$

Combining these estimates, we can get:

$$\begin{aligned}
& \max_{0 \leq k \leq N-2} \left\| A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} \right\|_H \\
& \leq (2 + \tau |b| (N-2)) b_0 + \tau \sum_{s=1}^{N-2} \|A^{\frac{1}{2}} f(t_s)\|_H
\end{aligned} \tag{21}$$

for the solution of DS (4). Estimate (17) follows from (19)–(21).

Now, let us estimate b_{m+1} . Using Formula (13) and the estimates (14)–(16), we can obtain:

$$\begin{aligned}
& \|A^{\frac{3}{2}} u_{mN+1}\|_H \leq \|A^{\frac{3}{2}} u_{mN}\|_H + \|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \leq 2b_m, \\
& \|A^{\frac{3}{2}} u_{mN+2}\|_H \leq \|A^{\frac{3}{2}} u_{mN}\|_H + 2\|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \\
& + \|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|\tau A^{\frac{1}{2}} \tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \leq 4b_m, \\
& \|A^{\frac{3}{2}} u_k\|_H \leq \|A^{\frac{3}{2}} u_{mN}\|_H + \frac{1}{2} [\|R\|_{H \rightarrow H}^{k-mN-2} + \|\tilde{R}\|_{H \rightarrow H}^{k-mN-2}] \|R\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \\
& + \frac{1}{2} \|R\|_{H \rightarrow H} [2\|R\tilde{R}\|_{H \rightarrow H} + \|R\|_{H \rightarrow H}^{k-mN-1} + \|\tilde{R}\|_{H \rightarrow H}^{k-mN-1}] \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \\
& + \frac{\tau}{2} |b| \sum_{s=mN+1}^{k-2} [2 + \|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}] \|A^{\frac{3}{2}} u(t_{s-N})\|_H \\
& + \frac{\tau}{2} \sum_{s=mN+1}^{k-2} [2 + \|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}] \|A^{\frac{1}{2}} f(t_s)\|_H \\
& \leq 2 \left(2 + \tau |b| (N-2) \right) b_m + 2\tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad mN+3 \leq k \leq (m+1)N.
\end{aligned}$$

Combining these estimates, we obtain:

$$\begin{aligned}
& \frac{1}{2} \max_{mN+1 \leq k \leq (m+1)N} \|A^{\frac{3}{2}} u_k\|_H \\
& \leq (2 + \tau |b| (N-2)) b_m + \tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H
\end{aligned} \tag{22}$$

for the solution of DS (4). Applying Formulas (10)–(12), we can write:

$$\frac{u_{mN+1} - u_{mN}}{\tau} = v_{mN} = R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau},$$

$$\begin{aligned}
\frac{u_{mN+2} - u_{mN+1}}{\tau} &= v_{mN+1} = R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} + \tau R\tilde{R} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}, \\
\frac{u_k - u_{k-1}}{\tau} &= v_{k-1} = \frac{1}{2} [R^{k-mN-2} + \tilde{R}^{k-mN-2}] R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} \\
&\quad + \frac{1}{2i} (R^{k-mN-1} - \tilde{R}^{k-mN-1}) A^{-\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
&\quad - \frac{\tau}{2i} \sum_{s=mN+1}^{k-2} [R^{k-1-s} - \tilde{R}^{k-1-s}] b A^{\frac{1}{2}} u(t_{s-N}) \\
&\quad - \frac{\tau}{2i} \sum_{s=mN+1}^{k-2} [R^{k-1-s} - \tilde{R}^{k-1-s}] A^{-\frac{1}{2}} f(t_s), \quad mN+3 \leq k \leq (m+1)N.
\end{aligned}$$

Using this formula and the estimates (14)–(16), we can obtain:

$$\begin{aligned}
\|A \frac{u_{mN+1} - u_{mN}}{\tau}\|_H &\leq \|R\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \leq b_m, \\
\|A \frac{u_{mN+2} - u_{mN+1}}{\tau}\|_H &\leq \|R\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \\
&\quad + \|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \leq 2b_m, \\
\|A \frac{u_k - u_{k-1}}{\tau}\|_H &\leq \frac{1}{2} [\|R\|_{H \rightarrow H}^{k-mN-2} + \|\tilde{R}\|_{H \rightarrow H}^{k-mN-2}] \|R\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \\
&\quad + \frac{1}{2} [\|R\|_{H \rightarrow H}^{k-1} + \|\tilde{R}\|_{H \rightarrow H}^{k-1}] \|R\|_{H \rightarrow H} \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \\
&\quad + \frac{\tau}{2} |b| \sum_{s=mN+1}^{(m+1)N-2} [\|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}] \|A^{\frac{3}{2}} u(t_{s-N})\|_H \\
&\quad + \frac{\tau}{2} \sum_{s=mN+1}^{(m+1)N-2} [\|R\|_{H \rightarrow H}^{k-1-s} + \|\tilde{R}\|_{H \rightarrow H}^{k-1-s}] \|A^{\frac{1}{2}} f(t_s)\|_H \\
&\leq (2 + \tau |b| (N-2)) b_m + \tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad mN+3 \leq k \leq (m+1)N.
\end{aligned}$$

Combining these estimates, we obtain:

$$\max_{mN+1 \leq k \leq (m+1)N} \left\| A \frac{u_k - u_{k-1}}{\tau} \right\|_H \leq (2 + \tau |b| (N-2)) b_m + \tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H \quad (23)$$

for the solution of DS (4). Applying Formulas (10)–(12), we can write:

$$\begin{aligned}
\frac{u_{mN+2} - 2u_{mN+1} + u_{mN}}{\tau^2} &= \tau^{-1} (v_{mN+1} - v_{mN}) = \tilde{R}R \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}, \\
\frac{u_{mN+3} - 2u_{mN+2} + u_{mN+1}}{\tau^2} &= \tau^{-1} (v_{mN+2} - v_{mN+1}) \\
&= -\tau A (R\tilde{R})^2 \frac{u_{mN} - u_{mN-1}}{\tau} + (I - \tau^2 A) (\tilde{R}R)^2 \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} + \tau R\tilde{R} p_{mN+1} \\
&= -\tau A (R\tilde{R})^2 \frac{u_{mN} - u_{mN-1}}{\tau} + (I - \tau^2 A) (\tilde{R}R)^2 \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
\frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} &= \tau^{-1} (v_{k+1} - v_k)
\end{aligned}$$

$$\begin{aligned}
& +\tau R\tilde{R}[bAu(t_{1-N}) + f(t_{mN+1})], \\
& = \tau^{-1} \left\{ \frac{1}{2} [R^{k-mN} + \tilde{R}^{k-mN}] R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} \right. \\
& \quad + \frac{1}{2i} A^{-\frac{1}{2}} [R^{k-Mn+1} - \tilde{R}^{k-Mn+1}] R \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
& \quad - \sum_{s=Mn+1}^k \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{k+1-s} - \tilde{R}^{k+1-s}] p_s - \frac{1}{2} [R^{k-Mn-1} + \tilde{R}^{k-Mn-1}] R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} \\
& \quad - \frac{1}{2i} A^{-\frac{1}{2}} [R^{k-mN} - \tilde{R}^{k-mN}] R \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
& \quad \left. + \sum_{s=mN+1}^k \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{k-s} - \tilde{R}^{k-s}] p_s \right\} \\
& = \tau^{-1} \left\{ \frac{1}{2} [R^{k-mN} + \tilde{R}^{k-mN} - R^{k-mN-1} - \tilde{R}^{k-mN-1}] R\tilde{R} \frac{u_{mN} - u_{mN-1}}{\tau} \right. \\
& \quad + \frac{1}{2i} A^{-1} [R^{k+1} - \tilde{R}^{k+1} - R^k + \tilde{R}^k] R \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
& \quad - \frac{\tau}{2i} A^{-\frac{1}{2}} [R - \tilde{R}] p_{k-mN} + \sum_{s=mN+1}^{k-1} \frac{\tau}{2i} A^{-\frac{1}{2}} [R^{k-s} - \tilde{R}^{k-s} - R^{k+1-s} + \tilde{R}^{k+1-s}] p_s \Big\} \\
& = -\frac{i}{2} [R^{k-mN} - \tilde{R}^{k-mN}] R\tilde{R} A^{\frac{1}{2}} \frac{u_{mN} - u_{mN-1}}{\tau} \\
& \quad + \frac{1}{2} R [R^{k+1} + \tilde{R}^{k+1}] \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2} \\
& \quad - \tau R\tilde{R} A b u(t_{k-N}) - \tau R\tilde{R} f(t_{k-mN}) - \frac{\tau}{2} \sum_{s=mN+1}^{k-1} [R^{k-s+1} + \tilde{R}^{k-s+1}] A b u(t_{s-N}) \\
& \quad - \frac{\tau}{2} \sum_{s=mN+1}^{k-1} [R^{k-s+1} + \tilde{R}^{k-s+1}] f(t_s), \quad mN+2 \leq k \leq (m+1)N-2.
\end{aligned}$$

Using this formula and the estimates (14)–(16), we obtain that:

$$\begin{aligned}
& \left\| A^{\frac{1}{2}} \frac{u_{mN+2} - 2u_{mN+1} + u_{mN}}{\tau^2} \right\|_H \leq \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \leq b_m, \\
& \left\| A^{\frac{1}{2}} \frac{u_{mN+3} - 2u_{mN+2} + u_{mN+1}}{\tau^2} \right\|_H \\
& \leq \|\tau A^{\frac{1}{2}} R\|_{H \rightarrow H} \|R\|_{H \rightarrow H} \|\tilde{R}\|_{H \rightarrow H}^2 \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H \\
& \quad + \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \\
& \quad + \tau \|b\| \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}} u(t_{1-N})\|_H + \tau \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}} f(t_{mN+1})\|_H \\
& \leq (2 + \tau \|b\|) b_m + \tau \|A^{\frac{1}{2}} f(t_{mN+1})\|_H, \\
& \left\| A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} \right\|_H \\
& \leq \frac{1}{2} [\|R\|_{H \rightarrow H}^{k-mN} + \|\tilde{R}\|_{H \rightarrow H}^{k-mN}] \|R\tilde{R}\|_{H \rightarrow H} \|A \frac{u_{mN} - u_{mN-1}}{\tau}\|_H
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \|R\|_{H \rightarrow H} [\|R\|_{H \rightarrow H}^{k+1} + \|\tilde{R}\|_{H \rightarrow H}^{k+1}] \|A^{\frac{1}{2}} \frac{u_{mN} - 2u_{mN-1} + u_{mN-2}}{\tau^2}\|_H \\
& + \frac{\tau}{2} |b| \sum_{s=mN+1}^{k-1} [\|R\|_{H \rightarrow H}^{k-s+1} + \|\tilde{R}\|_{H \rightarrow H}^{k-s+1}] \|A^{\frac{3}{2}} u(t_{s-N})\|_H \\
& + \frac{\tau}{2} \sum_{s=mN+1}^{k-1} [\|R\|_{H \rightarrow H}^{k-s+1} + \|\tilde{R}\|_{H \rightarrow H}^{k-s+1}] \|A^{\frac{1}{2}} f(t_s)\|_H \\
& + \tau |b| \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{3}{2}} u(t_{k-(m+1)N})\|_H + \tau \|R\tilde{R}\|_{H \rightarrow H} \|A^{\frac{1}{2}} f(t_{k-mN})\|_H \\
& \leq (2 + \tau |b| (N-2)) b_m + \tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad mN+2 \leq k \leq (m+1)N-2.
\end{aligned}$$

Combining these estimates, we obtain:

$$\begin{aligned}
& \max_{mN \leq k \leq (m+1)N-2} \left\| A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} \right\|_H \\
& \leq (2 + \tau |b| (N-2)) b_m + \tau \sum_{s=mN+1}^{(m+1)N-2} \|A^{\frac{1}{2}} f(t_s)\|_H
\end{aligned} \quad (24)$$

for the solution of DS (4). Estimate (18) follows from (22)–(24). Theorem 2 is proven.

Note that applying Theorem 2, we can obtain the stability estimate:

$$\begin{aligned}
& \max_{mN \leq k \leq (m+1)N-2} \left\| A^{\frac{1}{2}} \frac{u_{k+2} - 2u_{k+1} + u_k}{\tau^2} \right\|_H + \max_{mN+1 \leq k \leq (m+1)N} \left\| A \frac{u_k - u_{k-1}}{\tau} \right\|_H \\
& + \frac{1}{2} \max_{mN+1 \leq k \leq (m+1)N} \|A^{\frac{3}{2}} u_k\|_H \leq (2 + \tau |b| (N-2))^m b_0 \\
& + \sum_{j=1}^m (2 + \tau |b| (N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|A^{\frac{1}{2}} f(t_s)\|_H, \quad m = 0, 1, \dots
\end{aligned} \quad (25)$$

for the solution of DS (4). $\square$

3. Applications

Note that the generality of this approach permits studying of a general class of DPDEs. We consider the applications of Theorem 2 for three types of problems. First, the mixed problem for the one-dimensional DPDE with nonlocal conditions:

$$\begin{cases} \frac{\partial^3 u(t, x)}{\partial t^3} - (a(x) u_{tx}(t, x))_x + \delta u_t(t, x) \\ = b(- (a(x) u_x(t - w, x))_x + \delta u(t - w, x)) + f(t, x), \\ 0 < t < \infty, \quad 0 < x < l, \\ u(t, 0) = u(t, l), \quad u_x(t, 0) = u_x(t, l), \quad 0 \leq t < \infty, \\ u(t, x) = g(t, x), \quad -w \leq t \leq 0, \quad 0 \leq x \leq l \end{cases} \quad (26)$$

is studied. Under compatibility conditions, Problem (26) has a unique solution $u(t, x)$ for the given smooth functions $a(x) \geq a > 0$, $x \in (0, l)$, $\delta > 0$, $a(l) = a(0)$, $g(t, x)$, $-w \leq t \leq 0$, $0 \leq x \leq l$, $f(t, x)$, $0 < t < \infty$, $0 < x < l$, and $b \in R^1$.

The construction of full discretization to Problem (26) is completed in two stages. In the first stage, we consider the uniform grid space:

$$[0, l]_h = \{x = x_n : x_n = nh, \quad 0 \leq n \leq M, \quad Mh = l\}$$

with step $h > 0$. Let $L_{2h} = L_2([0, l]_h)$ be a Hilbert space of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0, l]_h$, equipped with the norm:

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [0, l]_h} |\varphi(x)|^2 h \right)^{1/2}.$$

Let A_h^x be the second order difference operator defined by:

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta \varphi_n\}_1^{M-1} \quad (27)$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well known that A_h^x is a self-adjoint positive definite operator in L_{2h} . Applying A_h^x in (26), we can obtain the initial value problem for an infinite system of third order differential equations:

$$\begin{cases} u_{ttt}^h(t, x) + A_h^x u_t^h(t, x) = b A_h^x u(t - w, x) + f^h(t, x), \\ 0 < t < \infty, x \in [0, l]_h, \\ u^h(t, x) = g^h(t, x), -w \leq t \leq 0, x \in [0, l]_h. \end{cases} \quad (28)$$

In the second stage, we use DS (4) for (28):

$$\begin{cases} \frac{u_{k+2}^h(x) - 3u_{k+1}^h(x) + 3u_k^h(x) - u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x) - u_{k+1}^h(x)}{\tau} \\ = b A_h^x u_{k-N}^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k, x), k \geq 1, x \in [0, l]_h, \\ u_k^h(x) = g^h(t_k, x), -N \leq k \leq 0, \\ (I_h + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = g_t^h(0, x), \\ (I_h + \tau^2 A_h^x) \frac{u_2^h(x) - 2u_1^h(x) + u_0^h(x)}{\tau^2} = g_{tt}^h(0, x), x \in [0, l]_h, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau}, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+2}^h(x) - 2u_{mN+1}^h(x) + u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x) - 2u_{mN-1}^h(x) + u_{mN-2}^h(x)}{\tau^2}, m = 1, 2, \dots \end{cases} \quad (29)$$

Theorem 3. The solutions of DS (29) obey the stability estimates:

$$\begin{aligned} & \max_{mN \leq k \leq (m+1)N-2} \left\| \frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\tau^2} \right\|_{W_{2h}^1} + \max_{mN+1 \leq k \leq (m+1)N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{W_{2h}^2} \\ & + \frac{1}{2} \max_{mN+1 \leq k \leq (m+1)N} \|u_k^h\|_{W_{2h}^3} \leq C_1 \left[(2 + \tau|b|(N-2))^m b_0^h \right. \\ & \left. + \sum_{j=1}^m (2 + \tau|b|(N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|A^{\frac{1}{2}} f(t_s)\|_{W_{2h}^1} \right], m = 0, 1, \dots, \\ & b_0^h = \max \left\{ \max_{-N \leq k \leq 0} \|A^{\frac{1}{2}} g_{tt}^h(t_k)\|_{W_{2h}^1}, \max_{-N \leq k \leq 0} \|g_t^h(t_k)\|_{W_{2h}^2}, \max_{-N \leq k \leq 0} \|g^h(t_k)\|_{W_{2h}^3} \right\}. \end{aligned}$$

hold, where C_1 does not depend on $\tau, h, g^h(t_k)$, and $f_k^h(x)$.

Proof. DS (29) can be written in abstract form:

$$\left\{ \begin{array}{l} \frac{u_{k+2}^h - 3u_{k+1}^h + 3u_k^h - u_{k-1}^h}{\tau^3} + A_h \frac{u_{k+2}^h - u_{k+1}^h}{\tau} = bA_h u_{k-N}^h + f_k^h, k \geq 1, \\ u_k^h = g_k^h, -N \leq k \leq 0, \\ (I_h + \tau^2 A_h) \frac{u_1^h - u_0^h}{\tau} = g_t^h(0), (I_h + \tau^2 A_h) \frac{u_2^h - 2u_1^h + u_0^h}{\tau^2} = g_{tt}^h(0), \\ (I_h + \tau^2 A_h) \frac{u_{mN+1}^h - u_{mN}^h}{\tau} = \frac{u_{mN}^h - u_{mN-1}^h}{\tau}, \\ (I_h + \tau^2 A_h) \frac{u_{mN+2}^h - 2u_{mN+1}^h + u_{mN}^h}{\tau^2} = \frac{u_{mN}^h - 2u_{mN-1}^h + u_{mN-2}^h}{\tau^2}, m = 1, 2, \dots \end{array} \right. \quad (30)$$

in a Hilbert space L_{2h} with a self-adjoint positive definite operator $A_h = A_h^x$. Here, $g_k^h = g_k^h(x)$, $f_k^h = f_k^h(x)$, and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions defined on $[0, l]_h$. Therefore, the estimate of Theorem 3 follows from the estimate (25). Theorem 3 is proven. $\square$

Second, let Ω be the unit open cube in the m -dimensional Euclidean space.

$\mathbb{R}^n(x = (x_1, \dots, x_m) : 0 < x_k < 1, k = 1, \dots, n)$ with boundary $S, \overline{\Omega} = \Omega \cup S$. In $[0, \infty) \times \Omega$, the mixed problem for the DPDE with the Dirichlet condition:

$$\left\{ \begin{array}{l} u_{ttt}(t, x) - \sum_{r=1}^n (a_r(x) u_{tx_r}(t, x))_{x_r} = -b \sum_{r=1}^n (a_r(x) u_{x_r}(t - w, x))_{x_r}, \\ 0 < t < \infty, x \in \Omega, \\ u(t, x) = 0, x \in S, 0 \leq t < \infty, \\ u(t, x) = g(t, x), -w \leq t \leq 0, x \in \overline{\Omega} \end{array} \right. \quad (31)$$

is investigated. Under compatibility conditions, Problem (31) has a unique solution $u(t, x)$ for the given smooth functions $a_r(x) \geq a > 0, (x \in \Omega), g(t, x), -w \leq t \leq 0, x \in \overline{\Omega}, f(t, x), 0 < t < \infty, x \in \Omega$, and $b \in R^1$.

The construction of full discretization to Problem (31) is completed in two stages. In the first stage, we consider the uniform grid space:

$$\overline{\Omega}_h = \{x = x_r = (h_1 j_1, \dots, h_n j_n), j = (j_1, \dots, j_n), 0 \leq j_r \leq N_r, \\ N_r h_r = 1, r = 1, \dots, n\}, \Omega_h = \overline{\Omega}_h \cap \Omega, S_h = \overline{\Omega}_h \cap S$$

and introduce the Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 j_1, \dots, h_n j_n)\}$ defined on $\overline{\Omega}_h$ equipped with the norm:

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in \Omega_h} |\varphi^h(x)|^2 h_1 \dots h_n \right)^{\frac{1}{2}}.$$

We consider the difference operator A_h^x defined by the formula:

$$A_h^x u^h = - \sum_{r=1}^n \left(\alpha_r(x) u_{x_r}^h \right)_{x_r, j_r}, \quad (32)$$

acting in the space of grid functions $u^h(x)$, which satisfy the conditions $u^h(x) = 0$ for all $x \in S_h$. It is well known that A_h^x is a self-adjoint positive definite operator in L_{2h} . Applying A_h^x in (31), we can obtain that:

$$\begin{cases} u_{ttt}^h(t, x) + A_h^x u_t^h(t, x) = b A_h^x u(t - w, x) + f^h(t, x), \\ 0 < t < \infty, x \in \Omega_h, \\ u^h(t, x) = g^h(t, x), -w \leq t \leq 0, x \in \overline{\Omega}_h. \end{cases} \quad (33)$$

In the second stage, we also get the difference scheme as the one-dimensional problem case:

$$\begin{cases} \frac{u_{k+2}^h(x) - 3u_{k+1}^h(x) + 3u_k^h(x) - u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x) - u_{k+1}^h(x)}{\tau} \\ = b A_h^x u_{k-N}^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k, x), k \geq 1, x \in \Omega_h, \\ u_k^h(x) = g^h(t_k, x), -N \leq k \leq 0, \\ (I_h + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = g_t^h(0, x), \\ (I_h + \tau^2 A_h^x) \frac{u_2^h(x) - 2u_1^h(x) + u_0^h(x)}{\tau^2} = g_{tt}^h(0, x), x \in \overline{\Omega}_h, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau}, x \in \overline{\Omega}_h \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+2}^h(x) - 2u_{mN+1}^h(x) + u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x) - 2u_{mN-1}^h(x) + u_{mN-2}^h(x)}{\tau^2}, x \in \overline{\Omega}_h, m = 1, 2, \dots \end{cases} \quad (34)$$

Theorem 4. The solution of DS (34) obeys the following stability estimates:

$$\begin{aligned} & \max_{mN \leq k \leq (m+1)N-2} \left\| \frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\tau^2} \right\|_{W_{2h}^1} + \max_{mN+1 \leq k \leq (m+1)N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{W_{2h}^2} \\ & + \frac{1}{2} \max_{mN+1 \leq k \leq (m+1)N} \|u_k^h\|_{W_{2h}^3} \leq C_2 \left[(2 + \tau|b|(N-2))^m b_0^h \right. \\ & \left. + \sum_{j=1}^m (2 + \tau|b|(N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|A^{\frac{1}{2}} f(t_s)\|_{W_{2h}^1} \right], m = 0, 1, \dots, \\ & b_0^h = \max \left\{ \max_{-N \leq k \leq 0} \|A^{\frac{1}{2}} g_{tt}^h(t_k)\|_{W_{2h}^1}, \max_{-N \leq k \leq 0} \|g_t^h(t_k)\|_{W_{2h}^2}, \max_{-N \leq k \leq 0} \|g^h(t_k)\|_{W_{2h}^3} \right\}. \end{aligned}$$

where C_2 does not depend on $\tau, h, g^h(t_k)$, and $f_k^h(x)$.

Proof. DS (34) can be written in abstract form:

$$\begin{cases} \frac{u_{k+2}^h - 3u_{k+1}^h + 3u_k^h - u_{k-1}^h}{\tau^3} + A_h \frac{u_{k+2}^h - u_{k+1}^h}{\tau} = b A_h u_{k-N}^h + f_k^h, k \geq 1, \\ u_k^h = g_k^h, -N \leq k \leq 0, \\ (I_h + \tau^2 A_h) \frac{u_1^h - u_0^h}{\tau} = g_t^h(0), (I_h + \tau^2 A_h) \frac{u_2^h - 2u_1^h + u_0^h}{\tau^2} = g_{tt}^h(0), \\ (I_h + \tau^2 A_h) \frac{u_{mN+1}^h - u_{mN}^h}{\tau} = \frac{u_{mN}^h - u_{mN-1}^h}{\tau}, \\ (I_h + \tau^2 A_h) \frac{u_{mN+2}^h - 2u_{mN+1}^h + u_{mN}^h}{\tau^2} = \frac{u_{mN}^h - 2u_{mN-1}^h + u_{mN-2}^h}{\tau^2}, m = 1, 2, \dots \end{cases}$$

in a Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h = A_h^x$ by Formula (32). Here, $g_k^h = g_k^h(x)$, $f_k^h = f_k^h(x)$, and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions defined on $\overline{\Omega}_h$ with the values in L_{2h} . Then, the estimate of Theorem 4 follows from Estimate (25) and the following theorem. $\square$

Theorem 5. The solution of the difference elliptic problem: [40]

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h; u^h(x) = 0, x \in S_h$$

obeys the estimate:

$$\sum_{r=1}^n \|u^h_{x_r, x_{\bar{r}}}\|_{L_{2h}} \leq C_3 \|\omega^h\|_{L_{2h}},$$

where C_3 does not depend on h and ω^h .

Third, in $[0, \infty) \times \Omega$, the mixed problem for DPDE with the Neumann boundary condition:

$$\begin{cases} u_{ttt}(t, x) - \sum_{r=1}^n (a_r(x) u_{tx_r}(t, x))_{x_r} + \delta u_t(t, x) \\ = b \left(- \sum_{r=1}^n (a_r(x) u_{x_r}(t - w, x))_{x_r} + \delta u(t - w, x) \right), \\ 0 < t < \infty, x \in \Omega, \\ \frac{\partial u(t, x)}{\partial p} = 0, x \in S, 0 \leq t < \infty \\ u(t, x) = g(t, x), -w \leq t \leq 0, x \in \overline{\Omega} \end{cases} \quad (35)$$

is investigated. Here, $\vec{p}$ is the normal vector to S . Under compatibility conditions, Problem (31) has a unique solution $u(t, x)$ for the given smooth functions $a_r(x) \geq a > 0$, $(x \in \Omega)$, $g(t, x)$, $-w \leq t \leq 0$, $x \in \overline{\Omega}$, $f(t, x)$, $0 < t < \infty$, $x \in \Omega$ and $b \in R^1$.

The construction of full discretization to Problem (35) is completed in two stages. In the first stage, we introduce the second order difference operator A_h^x defined by:

$$A_h^x u^h = - \sum_{r=1}^n \left(\alpha_r(x) u_{x_r}^h \right)_{x_r j_r} + \delta u^h, \quad (36)$$

acting in the space of grid functions $u^h(x)$ that satisfy the conditions $D^h u^h(x) = 0$ for all $x \in S_h$. Here, D^h is the approximation of operator $\frac{\partial}{\partial \vec{p}}$. It is known that A_h^x is the self-adjoint positive definite operator in L_{2h} . Using the difference operator A_h^x , we get the initial value problem (33). Therefore, in the second stage, we use DS (4) for Problem (33):

$$\begin{cases} \frac{u_{k+2}^h(x) - 3u_{k+1}^h(x) + 3u_k^h(x) - u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x) - u_{k+1}^h(x)}{\tau} \\ = b A_h^x u_{k-N}^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k, x), k \geq 1, x \in \Omega_h, \\ u_k^h(x) = g^h(t_k, x), -N \leq k \leq 0, \\ (I_h + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = g_t^h(0, x), \\ (I_h + \tau^2 A_h^x) \frac{u_2^h(x) - 2u_1^h(x) + u_0^h(x)}{\tau^2} = g_{tt}^h(0, x), x \in \overline{\Omega}_h, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau}, x \in \overline{\Omega}_h \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+2}^h(x) - 2u_{mN+1}^h(x) + u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x) - 2u_{mN-1}^h(x) + u_{mN-2}^h(x)}{\tau^2}, x \in \overline{\Omega}_h, m = 1, 2, \dots \end{cases} \quad (37)$$

Theorem 6. The solution of the difference scheme (37) obeys the stability estimates in Theorem 4.

Proof. DS (37) can be written in abstract form:

$$\begin{cases} \frac{u_{k+2}^h - 3u_{k+1}^h + 3u_k^h - u_{k-1}^h}{\tau^3} + A_h \frac{u_{k+2}^h - u_{k+1}^h}{\tau} = b A_h u_{k-N}^h + f_k^h, k \geq 1, \\ u_k^h = g_k^h, -N \leq k \leq 0, \\ (I_h + \tau^2 A_h) \frac{u_1^h - u_0^h}{\tau} = g_t^h(0), (I_h + \tau^2 A_h) \frac{u_2^h - 2u_1^h + u_0^h}{\tau^2} = g_{tt}^h(0), \\ (I_h + \tau^2 A_h) \frac{u_{mN+1}^h - u_{mN}^h}{\tau} = \frac{u_{mN}^h - u_{mN-1}^h}{\tau}, \\ (I_h + \tau^2 A_h) \frac{u_{mN+2}^h - 2u_{mN+1}^h + u_{mN}^h}{\tau^2} = \frac{u_{mN}^h - 2u_{mN-1}^h + u_{mN-2}^h}{\tau^2}, m = 1, 2, \dots \end{cases}$$

in a Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h = A_h^x$ by Formula (36). Here, $g_k^h = g_k^h(x)$, $f_k^h = f_k^h(x)$, and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions

defined on $\overline{\Omega}_h$ with the values in L_{2h} . Therefore, the estimate of Theorem 6 follows from the estimate (25) and the following theorem. $\square$

Theorem 7. *The solution of the elliptic difference problem: [40]*

$$A_h^x u^h(x) = \omega^h(x), \quad x \in \Omega_h; \quad D^h u^h(x) = 0, \quad x \in S_h$$

satisfies the estimate:

$$\sum_{r=1}^n \|u^h_{x_r x_r}\|_{L_{2h}} \leq C_4 \|\omega^h\|_{L_{2h}},$$

where C_4 is independent of h and ω^h .

4. Numerical Results

It is well known that when the analytical methods fail to work properly, the numerical methods for getting partial differential equations' approximate solutions play a vital role in applied mathematics. In the operator approach, constants in theorems can be large; therefore, in this case a nice stability result must be supported numerically. For this reason, it is important to see that for such a type of theoretical result, we need numerical applications when one cannot know concrete values of constants in stability estimates. Therefore, the first order of accuracy DSs for the solution of one- and two-dimensional DPDEs are presented. To solve this problem, a procedure of modified Gauss elimination is applied. The result of the numerical experiment supports the theoretical statements for the solution of these DSs.

4.1. One-Dimensional Problem

First, we consider the mixed problem, with the exact solution $u(t, x) = e^{-t} \cos x$,

$$\begin{cases} u_{ttt}(t, x) - u_{txx}(t, x) = -0.1u_{xx}(t-1, x) \\ -2e^{-t} \cos x - 0.1e^{-(t-1)} \cos x, \quad t > 0, \quad 0 < x < \pi, \\ u_x(t, 0) = u_x(t, \pi) = 0, \quad 0 \leq t < \infty, \\ u(t, x) = e^{-t} \cos x, \quad -1 \leq t \leq 0, \quad 0 \leq x \leq \pi \end{cases} \quad (38)$$

for the one-dimensional DPDE.

Applying DS (4), we get the following DS:

$$\begin{cases} \frac{u_n^{k+2} - 3u_n^{k+1} + 3u_n^k - u_n^{k-1}}{\tau^3} \\ - \frac{u_{n+1}^{k+2} - u_{n+1}^{k+1} - 2(u_n^{k+2} - u_n^{k+1}) + u_{n-1}^{k+2} - u_{n-1}^{k+1}}{h^2} \\ = -0.1 \frac{u_{n+1}^{k-N} - 2u_n^{k-N} - u_{n-1}^{k-N}}{h^2} - 2e^{-t_k} \cos x_n - 0.1e^{-(t_k-N)} \cos x_n, \\ t_k = k\tau, \quad lN + 1 \leq k \leq (l+1)N - 2, \quad l = 0, 1, \dots, \quad 1 \leq k \leq N - 1, \\ N\tau = 1, \quad x_n = nh, \quad 1 \leq n \leq M - 1, \quad Mh = \pi, \\ u_n^k = e^{-t_k} \cos x_n, \quad -N \leq k \leq 0, \quad 0 \leq n \leq M, \\ \frac{u_n^{k+1} - u_n^k}{\tau} = -e^{-t_k} \cos x_n, \quad -N \leq k \leq 0, \quad 0 \leq n \leq M, \\ \frac{u_n^{k+2} - 2u_n^{k+1} + u_n^k}{\tau^2} = e^{-t_k} \cos x_n, \quad -N \leq k \leq 0, \quad 0 \leq n \leq M, \\ u_1^k - u_0^k = u_M^k - u_{M-1}^k = 0, \quad 0 \leq k \leq N. \end{cases} \quad (39)$$

It can be written as the second order difference problem with matrix coefficients:

$$A u_{n+1} + B u_n + C u_{n-1} = D \varphi_n, \quad 1 \leq n \leq M - 1; \quad u_0 = u_1, \quad u_M = u_{M-1}. \quad (40)$$

Here and in the future, we put:

$$u_s = \begin{bmatrix} u_s^{lN} \\ \vdots \\ u_s^{(l+1)N} \end{bmatrix}_{(N+1) \times 1}, \quad s = n, n \pm 1.$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & -a & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & a & -a \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1) \times (N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ b & -3b & 3b-c & b-c & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & b & -3b & 3b-c & c-b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -3b & 3b-c & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & 3b-c & c-b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & -3b & 3b-c & c-b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & b & -3b & 3b-c & c-b \\ -\frac{1}{\tau^2} & \frac{1}{\tau} & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ \frac{1}{\tau^2} & -\frac{2}{\tau^2} & \frac{1}{\tau^2} & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1) \times (N+1)},$$

where:

$$a = \frac{1}{\tau h^2}, \quad b = -\frac{1}{\tau^3}, \quad c = \frac{2}{\tau h^2},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{lN} \\ \vdots \\ \varphi_n^{(l+1)N} \end{bmatrix}_{(N+1) \times 1}, \quad \left\{ \begin{array}{l} \varphi_n^{lN} = \cos x_n, \quad -M \leq k \leq 0, \quad 0 \leq n \leq M, \\ \varphi_n^k = f(t_k, x_n) = -0.1 \frac{u_{n+1}^{k-N} - 2u_n^{k-N} - u_{n-1}^{k-N}}{h^2} \\ \quad - 2e^{-t_k} \cos x_n - 0.1e^{-(t_k-N)} \cos x_n, \\ t_k = k\tau, \quad lN + 1 \leq k \leq (l+1)N - 2, \\ l = 0, 1, \dots, \quad 1 \leq n \leq M - 1, \\ \varphi_n^{(l+1)N-1} = -\cos x_n, \quad -M \leq k \leq 0, \quad 0 \leq n \leq M, \\ \varphi_n^{(l+1)N} = \cos x_n, \quad -M \leq k \leq 0, \quad 0 \leq n \leq M, \end{array} \right.$$

and $D = I_{N+1}$ is the identity matrix.

To solve this second order difference problem, we use the following formula:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 1, 0, \quad (41)$$

where $u_M = (I - \alpha_M)^{-1} \beta_M$, α_j ($j = 1, \dots, M-1$) are $(N+1) \times (N+1)$ square matrices, β_j ($j = 1, \dots, M-1$) are $(N+1) \times 1$ column matrices, α_1 is the identity, and β_1 is zero matrices, and:

$$\begin{aligned}\alpha_{n+1} &= -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} &= (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), n = 1, \dots, M-1.\end{aligned}\quad (42)$$

The errors are computed by:

$$E_M^N = \max_{lN+1 \leq k \leq (l+1)N-1, 1 \leq n \leq M-1} |u(t_k, x_n) - u_n^k| \quad (43)$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) , and the results are given in Table 1.

Table 1. Errors of difference scheme (DS) (39).

lN, M	20,20	40,40	80,80
$0, t \in [0, 1]$	0.0141	0.0068	0.0040
$1, t \in [1, 2]$	0.0559	0.0322	0.0172
$2, t \in [2, 3]$	0.1346	0.0746	0.0392
$3, t \in [3, 4]$	0.2011	0.1011	0.0561

4.2. Two-Dimensional Problem

Second, the mixed problem with the Dirichlet condition:

$$\begin{cases} u_{ttt}(t, x, y) - u_{txx}(t, x, y) - u_{tyy}(t, x, y) = -0.1u_{xx}(t-1, x) - 0.1u_{yy}(t-1, x, y) \\ -3e^{-t} \sin x \sin y - 0.2e^{-(t-1)} \sin x \sin y, t > 0, 0 < x, y < \pi, \\ u(t, 0, y) = u(t, \pi, y) = 0, 0 \leq t < \infty, 0 \leq y \leq \pi, \\ u(t, x, 0) = u(t, x, \pi) = 0, 0 \leq t < \infty, 0 \leq x \leq \pi \\ u(t, x, y) = e^{-t} \sin x \sin y, -1 \leq t \leq 0, 0 \leq x, y \leq \pi \end{cases} \quad (44)$$

for the two-dimensional DPDE is considered. The exact solution of Problem (44) is $u(t, x, y) = e^{-t} \sin x \sin y$.

Applying DS (4) to the problem (44), we get the following DS of the first order of accuracy in t :

$$\begin{cases} \frac{u_{n,m}^{k+2} - 3u_{n,m}^{k+1} + 3u_{n,m}^k - u_{n,m}^{k-1}}{\tau^3} \\ - \frac{u_{n+1,m}^{k+2} - u_{n+1,m}^{k+1} - 2(u_{n,m}^{k+2} - u_{n,m}^{k+1}) + u_{n-1,m}^{k+2} - u_{n-1,m}^{k+1}}{\tau h^2} \\ - \frac{u_{n,m+1}^{k+2} - u_{n,m+1}^{k+1} - 2(u_{n,m}^{k+2} - u_{n,m}^{k+1}) + u_{n,m-1}^{k+2} - u_{n,m-1}^{k+1}}{\tau h^2} \\ = -0.1 \frac{u_{n+1,m}^{k-N} - 2u_{n,m}^{k-N} - u_{n-1,m}^{k-N}}{h^2} - \frac{u_{n,m+1}^{k-N} - 2u_{n,m}^{k-N} - u_{n,m-1}^{k-N}}{h^2} \\ - 3e^{-t_k} \sin x_n \sin y_m - 0.2e^{-t_{k-N}} \sin x_n \sin y_m, \\ t_k = k\tau, lN+1 \leq k \leq (l+1)N-2, \\ l = 0, 1, \dots, 1 \leq k \leq N-1, \\ N\tau = 1, x_n = nh, y_m = mh, 1 \leq n, m \leq M-1, Mh = \pi, \\ u_{n,m}^k = \sin x_n \sin y_m, -N \leq k \leq 0, 0 \leq n, m \leq M, \\ \frac{u_{n,m}^{k+1} - u_{n,m}^k}{\tau} = \sin x_n \sin y_m, \\ -N \leq k \leq 0, 0 \leq n, m \leq M, \\ \frac{u_{n,m}^{k+2} - 2u_{n,m}^{k+1} + u_{n,m}^k}{\tau^2} = \sin x_n \sin y_m, \\ -N \leq k \leq 0, 0 \leq n, m \leq M, \\ u_{0,m}^k = u_{M,m}^k = 0, 0 \leq k \leq N, 0 \leq m \leq M, \\ u_{n,0}^k = u_{n,M}^k = 0, 0 \leq k \leq N, 0 \leq n \leq M. \end{cases} \quad (45)$$

It can be written as the second order difference problem with the matrix coefficients' form:

$$A u_{n+1} + B u_n + C u_{n-1} = D \varphi_n, \quad 1 \leq n \leq M-1; u_0 = 0, \quad u_M = 0. \quad (46)$$

Here and in the future, we put:

$$u_s = \left[u_{0,s}^{IN}, \dots, u_{0,s}^{(l+1)N}, u_{1,s}^{IN}, \dots, u_{1,s}^{(l+1)N}, \dots, u_{M,s}^{IN}, \dots, u_{M,s}^{(l+1)N} \right]_{(N+1)(M+1) \times 1}^T, \quad s = n, n \pm 1.$$

A, B, C, I are $(N+1)(M+1) \times (N+1)(M+1)$ square matrices, and I, R are identity matrices.

Here,

$$a = \frac{1}{\tau h^2}, \quad b = -\frac{1}{\tau^3}, \quad c = \frac{2}{\tau h^2},$$

$$A = C = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 \\ 0 & E & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & E & 0 \\ 0 & 0 & \cdot & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} Q & O & O & O & \cdot & O & O & O & O \\ E & D & E & O & \cdot & O & O & O & O \\ O & E & D & E & \cdot & O & O & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & O & O & \cdot & 0 & E & D & E \\ O & O & O & O & O & O & O & O & Q \end{bmatrix},$$

where:

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & -a & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & -a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & -3b & 3b-c & b-c & \cdots & 0 & 0 & 0 \\ 0 & b & -3b & 3b-c & \cdots & 0 & 0 & 0 \\ 0 & 0 & b & -3b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c-b & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 3b-c & c-b & 0 \\ 0 & 0 & 0 & 0 & \cdots & -3b & 3b-c & c-b \\ -\frac{1}{\tau} & \frac{1}{\tau} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{\tau^2} & -\frac{2}{\tau^2} & \frac{1}{\tau^2} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$Q = I_{(N+1) \times (N+1)}, O = O_{(N+1) \times (N+1)},$$

$$\varphi_n = \begin{bmatrix} \varphi_{0,n}^{lN} \\ \vdots \\ \varphi_{0,n}^{(l+1)N} \\ \varphi_{1,n}^{lN} \\ \vdots \\ \varphi_{1,n}^{(l+1)N} \\ \vdots \\ \varphi_{M,n}^{lN} \\ \vdots \\ \varphi_{M,n}^{(l+1)N} \end{bmatrix}_{(M+1)(N+1) \times 1}, \left\{ \begin{array}{l} \varphi_{m,n}^{lN} = \sin x_n \sin y_m, \\ -M \leq k \leq 0, 0 \leq n, m \leq M, \\ \varphi_{m,n}^k = -0.1 \frac{u_{n+1,m}^{k-N} - 2u_{n,m}^{k-N} - u_{n-1,m}^{k-N}}{h^2} \\ - \frac{u_{n,m+1}^{k-N} - 2u_{n,m}^{k-N} - u_{n,m-1}^{k-N}}{h^2} \\ - 3e^{-t_k} \sin x_n \sin y_m \\ - 0.2e^{-t_{k-N}} \sin x_n \sin y_m, \\ t_k = k\tau, lN + 1 \leq k \leq (l+1)N - 2, \\ l = 0, 1, \dots, 1 \leq n, m \leq M - 1, \\ \varphi_{m,n}^{(l+1)N-1} = -\sin x_n \sin y_m, \\ -M \leq k \leq 0, 0 \leq n, m \leq M, \\ \varphi_{m,n}^{(l+1)N} = \sin x_n \sin y_m, \\ -M \leq k \leq 0, 0 \leq n, m \leq M, \end{array} \right.$$

To solve this second order difference problem, we use the following formula:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 1, \quad u_M = 0, \quad (47)$$

where α_j ($j = 1, \dots, M-1$) are $(N+1)(M+1) \times (N+1)(M+1)$ square matrices, β_j ($j = 1, \dots, M-1$) are $(N+1)(M+1) \times 1$ column matrices, and α_1 and β_1 are zero matrices and:

$$\begin{aligned} \alpha_{n+1} &= -(B + C\alpha_n)^{-1} A_n, \\ \beta_{n+1} &= (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), \quad n = 1, \dots, M-1. \end{aligned} \quad (48)$$

The errors are computed by:

$$lE_M^N = \max_{lN+1 \leq k \leq (l+1)N-1, 1 \leq n, m \leq M-1} |u(t_k, x_n, y_m) - u_{n,m}^k| \quad (49)$$

of the numerical solutions, where $u(t_k, x_n, y_m)$ represents the exact solution and $u_{n,m}^k$ represents the numerical solution at (t_k, x_n, y_m) , and the results are given in Table 2.

Table 2. Errors of the difference scheme (45).

$l/N, M$	10,10	20,20	40,40
$0, t \in [0, 1]$	0.0370	0.0162	0.0083
$1, t \in [1, 2]$	0.0840	0.0456	0.0236
$2, t \in [2, 3]$	0.1028	0.0543	0.0276
$3, t \in [3, 4]$	0.1008	0.0521	0.0261

As seen in Tables 1 and 2, we obtained some numerical results. If M and N are doubled, the values of the errors decrease by a factor of approximately 1/2 for DS (39) and (45), respectively.

5. Conclusions

1. In this paper, the absolutely stable DS of a first order of accuracy for the approximate solution of the DPDE in a Hilbert space was presented. The theorem on the stability of this difference scheme was proven. In practice, stability estimates for the solutions of three-step difference schemes for different types of delay partial differential equations were obtained. Numerical results were given.
2. The mixed problem for the one-dimensional DPDE with the Dirichlet condition was studied in [41]. The first and second order of accuracy DSs for the numerical solution of this problem were presented. The illustrative numerical results were provided. We are interested in studying absolutely stable DSs of a high order of accuracy of the approximate solution of the initial value problem (1) for the DPDE in a Hilbert space.
3. Applying this approach and the method [30], we could study the existence and uniqueness of a bounded solution of the initial value problem for the semilinear DPDE:

$$\begin{cases} u_{ttt}(t) + Au_t(t) = f(t, u(t-w)), & 0 < t < \infty, \\ u(t) = g(t), & -w \leq t \leq 0 \end{cases} \quad (50)$$

in a Hilbert space H with an unbounded operator A . Moreover, applying the method of [28], we can investigate the convergence of DSs for the numerical solution of Problem (50).

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Sample Availability: Samples of the compounds are available from the authors.



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