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Difference of Some Positive Linear Approximation Operators for Higher-Order Derivatives

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Abstract: In the present paper, we deal with some general estimates for the difference of operators which are associated with different fundamental functions. In order to exemplify the theoretical results presented in (for example) Theorem 2, we provide the estimates of the differences between some of the most representative operators used in Approximation Theory in especially the difference between the Baskakov and the Szász–Mirakyan operators, the difference between the Baskakov and the Szász–Mirakyan–Baskakov operators, the difference of two genuine-Durrmeyer type operators, and the difference of the Durrmeyer operators and the Lupaş–Durrmeyer operators. By means of illustrative numerical examples, we show that, for particular cases, our result improves the estimates obtained by using the classical result of Shisha and Mond. We also provide the symmetry aspects of some of these approximations operators which we have studied in this paper.

Keywords: approximation operators; differences of operators; Szász–Mirakyan–Baskakov operators; Durrmeyer type operators; Bernstein polynomials; modulus of continuity

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1. Introduction, Definitions and Preliminary Results

Approximation by positive linear operators is a classical and important topic of research in Approximation Theory and Computer-Aided Geometric Design (CAGD). The basis of the familiar Bernstein operators is an important tool in Computer-Aided Geometric Design. This basis is used in order to construct Bézier curves, which have applications for designing curves for the cars industry and problems involving animations. In addition, the Bézier curves are used in order to control the velocity over time. A class of symmetric Beta-type distributions involving the symmetric Bernstein-type basis function was introduced and studied in [1]. In recent years, the quantum (or the q-) calculus and its variation, the so-called post-quantum or the (p,q)-calculus, which have many applications in quantum physics, attracted the attention of many researchers. For example, some variations of positive linear operators by using the (p,q)-calculus instead of their known forms involving the traditional q-calculus were, in fact, published recently in *Symmetry* itself (see [2]). In this connection, the readers are referred also to a subsequent survey-cum-expository review article by Srivastava [3] in which the above-mentioned variation aspect of the (p,q)-calculus was exposed. Several other applications of the positive linear operators in learning theory can also be found in the literature. For more details about this topic, the reader is referred to the applications of the Bernstein operators and the iterated Boolean sums of operators (see [4]) and the applications of the Durrmeyer operators (see [5]).

The attention of many researchers in the study of the differences of positive linear operators began with the question raised by Lupaş in regard with the possibility to give an estimate for the following commutator:

$$[B_n, \overline{\mathbb{B}}_n] := B_n \circ \overline{\mathbb{B}}_n - \overline{\mathbb{B}}_n \circ B_n,$$

where B_n are the Bernstein operators and $\overline{\mathbb{B}}_n$ are the Beta operators (see, for details, [6]).

In [7], an algebraic structure of positive linear operators, which map $C[0,1]$ into itself, was considered in order to give an inequality for the commutators of certain positive linear operators. In several sequels to this study, Gonska et al. (see, for example, [8–10]) considered an algebraic structure $(S, +, \circ, 0, I)$ which satisfies each of the following conditions:

(i) It is closed under both "$+$" and "$\circ$";
(ii) Both "$+$" and "$\circ$" are associative;
(iii) 0 is the identity for $+$ and I is the identity for "$\circ$";
(iv) 0 is an annihilator for "$\circ$", that is, $A \circ 0 = 0 \circ A = 0$;
(v) "+" is commutative;
(vi) "$\circ$" distributes over "+", that is, both of the distributive laws hold true.

The set

$$PLO = \{L : C[0,1] \to C[0,1] \quad \text{and} \quad L \ \text{ is linear and positive}\},$$

which is equipped with the canonical operations of addition and operator composition, is an algebraic structure defined above. The commutator given by

$$[A, B] := AB - BA \qquad (A, B \in PLO)$$

was studied from a quantitative point of view in [7].

A solution of the Lupaş problem was given by Gonska et al. [7] by using the Taylor expansion. The estimates for the differences of two positive linear operators, which have the same moments up to a certain order, were derived in [8–10]. In [11], the differences of certain positive linear operators, which have the same fundamental functions, were studied. These studies of the positive linear operators, which are defined on unbounded interval, become an interesting area of research in Approximation Theory (see [12–15]). Estimates for the differences of these operators in terms of weighted modulus of smoothness were obtained by Aral et al. [16]. The Bernstein polynomials are, by all means, the most investigated polynomials in Approximation Theory and were introduced by Bernstein in order to prove the Weierstrass Theorem. Various new generalizations of these operators were considered in, for example, [17,18]. In [19], estimates of the differences of the Bernstein operators and their derivatives were obtained. Recently, some interesting results on this topic were published in [20–25]. In the present paper, our approach involves positive linear operators which have substantially different fundamental functions. In fact, the results presented in this paper extend the earlier studies in [11] for more general classes of positive linear operators.

We denote by $E(I)$ the space of real-valued continuous functions defined on an interval $I \subseteq \mathbb{R}$, which contains the polynomials. Let

$$||f|| = \sup\{|f(x)| : x \in I\}$$

and

$$E_B(I) := \{f \in E(I) \quad \text{and} \quad \|f\| < \infty\}.$$

Let $e_j(t) := t^j$ $(j = 0, 1, 2, \cdots$. We consider the linear positive functional $F : E(I) \to R$ preserving constant function, namely, $F(e_0) = 1$. We also put

$$\mu_r^F = F\big((e_1 - \phi^F e_0)^r\big) := \sum_{i=0}^{r} \binom{r}{i} (-1)^i \, F(e_{r-i})[\phi^F]^i \quad (r \in \mathbb{N}),$$

where $\phi^F := F(e_1)$. For the functional F, the following basic result was obtained in [11].

Lemma 1 (see [11]). *Let $f \in E(I)$ with $f^{(4)} \in E_B(I)$. Then*

$$\left| F(f) - f(\phi^F) - \frac{\mu_2^F}{2!} f^{(2)}(\phi^F) - \frac{\mu_3^F}{3!} f^{(3)}(\phi^F) \right| \le \frac{\mu_4^F}{4!} \|f^{(4)}\|.$$

Let us now consider the fundamental functions $p_{m,k}, b_{m,k} \ge 0$, $k \in K$, and $p_{m,k}, b_{m,k} \in C(I)$ such that

$$\sum_{k \in K} p_{m,k}(x) = \sum_{k \in K} b_{m,k}(x) = e_0,$$

where K is a set of non-negative integers, that is,

$$K = \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Suppose also that $F_{m,k}, G_{m,k} : E(I) \to R$ are the linear positive functionals such that

$$F_{m,k}(e_0) = G_{m,k}(e_0) = 1$$

and denote

$$D(I) := \left\{ f \in E(I) \,\middle|\, \sum_{k \in K} p_{m,k} F_{m,k}(f) \in C(I) \text{ and } \sum_{k \in K} b_{m,k} G_{m,k}(f) \in C(I) \right\}.$$

Define the positive linear operators $U_m, V_m : D(I) \to C(I)$ as follows:

$$U_m(f,x) := \sum_{k \in K} p_{m,k}(x) F_{m,k}(f) \quad \text{and} \quad V_m(f,x) := \sum_{k \in K} b_{m,k}(x) G_{m,k}(f).$$

In [11], the following result concerning the difference of the operators U_m and V_m was proved.

Theorem 1 (see [11]). *Suppose that*

$$p_{m,k} = b_{m,k} \quad \text{and} \quad \phi^{F_{m,k}} = \phi^{G_{m,k}} \quad k \in K;\ m \in \mathbb{N}.$$

Let $f \in D(I)$ with $f^{(i)} \in E_B(I)$ $(i = 2,3,4)$. Then

$$|(U_m - V_m)(f,x)| \le \|f^{(2)}\| \gamma(x) + \|f^{(3)}\| \beta(x) + \|f^{(4)}\| \alpha(x) \quad (x \in I),$$

where

$$\gamma(x) := \sum_{k \in K} |\mu_2^{F_{m,k}} - \mu_2^{G_{m,k}}| p_{m,k}(x),$$

$$\beta(x) := \sum_{k \in K} |\mu_3^{F_{m,k}} - \mu_3^{G_{m,k}}| p_{m,k}(x)$$

and

$$\alpha(x) := \sum_{k\in K} (\mu_4^{F_{m,k}} + \mu_4^{G_{m,k}}) p_{m,k}(x).$$

In the series of papers [8–10], the results concerning the estimations of the differences of certain positive linear operators were based upon the fact that the positive linear operators have the same moments up to a certain order. In the recent paper [11], the approach involved the positive linear operators which have the same fundamental functions. The main goal of this paper is to extend the above result for the positive linear operators that have different fundamental functions. Furthermore, the condition $\phi^{F_{m,k}} = \phi^{G_{m,k}}$ of ([11], Theorem 4) is shown to be not necessary in order to obtain an estimate of the differences of the positive linear operators V_m and U_m.

Theorem 2. *Let $f \in D(I)$. If $f^{(i)} \in E_B(I)\ \ (i = 2, 3, 4)$, then*

$$\begin{aligned}|(U_m - V_m)(f, x)| \le{}& A(x)\|f^{(4)}\| + B(x)\|f^{(3)}\| + C(x)\|f^{(2)}\| \\ &+ 2\omega_1(f, \delta_1(x)) + 2\omega_1(f, \delta_2(x)) \quad (x \in I),\end{aligned}$$

where $\omega_1(f, \cdot)$ is the usual modulus of continuity,

$$A(x) = \frac{1}{4!} \sum_{k\in K} (p_{m,k}(x)\mu_4^{F_{m,k}} + b_{m,k}(x)\mu_4^{G_{m,k}}),$$

$$B(x) = \frac{1}{3!} \left| \sum_{k\in K} p_{m,k}(x)\mu_3^{F_{m,k}} - \sum_{k\in K} b_{m,k}(x)\mu_3^{G_{m,k}} \right|,$$

$$C(x) = \frac{1}{2!} \left| \sum_{k\in K} p_{m,k}(x)\mu_2^{F_{m,k}} - \sum_{k\in K} b_{m,k}(x)\mu_2^{G_{m,k}} \right|,$$

$$\delta_1(x) = \left(\sum_{k\in K} p_{m,k}(x) \left(\phi^{F_{m,k}} - x\right)^2 \right)^{1/2}$$

and

$$\delta_2(x) = \left(\sum_{k\in K} b_{m,k}(x) \left(\phi^{G_{m,k}} - x\right)^2 \right)^{1/2}.$$

Proof. First of all, by using Lemma 1, we get

$$\begin{aligned}|(U_m - V_m)(f, x)| \le{}& \left| \sum_{k\in K} p_{m,k}(x)F_{m,k}(f) - \sum_{k\in K} b_{m,k}(x)G_{m,k}(f) \right| \\ \le{}& \sum_{k\in K} p_{m,k}(x) \left| F_{m,k}(f) - f(\phi^{F_{m,k}}) - \frac{\mu_2^{F_{m,k}}}{2!} f''(\phi^{F_{m,k}}) - \frac{\mu_3^{F_{m,k}}}{3!} f'''(\phi^{F_{m,k}}) \right| \\ &+ \sum_{k\in K} b_{m,k}(x) \left| G_{m,k}(f) - f(\phi^{G_{m,k}}) - \frac{\mu_2^{G_{m,k}}}{2!} f''(\phi^{G_{m,k}}) - \frac{\mu_3^{G_{m,k}}}{3!} f'''(\phi^{G_{m,k}}) \right| \\ &+ \left| \sum_{k\in K} p_{m,k}(x) \frac{\mu_2^{F_{m,k}}}{2!} - \sum_{k\in K} b_{m,k}(x) \frac{\mu_2^{G_{m,k}}}{2!} \right| \cdot ||f''||\end{aligned}$$

$$
\begin{aligned}
&+\left|\sum_{k\in K} p_{m,k}(x)\frac{\mu_3^{F_{m,k}}}{3!}-\sum_{k\in K} b_{m,k}(x)\frac{\mu_3^{G_{m,k}}}{3!}\right|\cdot||f'''||\\
&+\sum_{k\in K} p_{m,k}(x)|f(\phi^{F_{m,k}})-f(x)|+\sum_{k\in K} b_{m,k}(x)|f(\phi^{G_{m,k}})-f(x)|\\
\le{}& \frac{1}{4!}\left(\sum_{k\in K} p_{m,k}(x)\mu_4^{F_{m,k}}+\sum_{k\in K} b_{m,k}(x)\mu_4^{G_{m,k}}\right)||f^{(iv)}||\\
&+\left|\sum_{k\in K} p_{m,k}(x)\frac{\mu_2^{F_{m,k}}}{2!}-\sum_{k\in K} b_{m,k}(x)\frac{\mu_2^{G_{m,k}}}{2!}\right|\cdot||f''||\\
&\quad+\left|\sum_{k\in K} p_{m,k}(x)\frac{\mu_3^{F_{m,k}}}{3!}-\sum_{k\in K} b_{m,k}(x)\frac{\mu_3^{G_{m,k}}}{3!}\right|\cdot||f'''||\\
&+\sum_{k\in K} p_{m,k}(x)|f(\phi^{F_{m,k}})-f(x)|+\sum_{k\in K} b_{m,k}(x)|f(\phi^{G_{m,k}})-f(x)|\\
={}& A(x)\|f^{(4)}\|+B(x)\|f^{(3)}\|+C(x)\|f^{(2)}\|\\
&+\left(1+\frac{\sum_{k\in K} p_{m,k}(x)\left(\phi^{F_{m,k}}-x\right)^2}{\delta_1^2(x)}\right)\omega_1(f,\delta_1(x))\\
&+\left(1+\frac{\sum_{k\in K} b_{m,k}(x)\left(\phi^{G_{m,k}}-x\right)^2}{\delta_2^2(x)}\right)\omega_1(f,\delta_2(x))\\
={}& A(x)\|f^{(4)}\|+B(x)\|f^{(3)}\|+C(x)\|f^{(2)}\|+2\omega_1(f,\delta_1(x))+2\omega_1(f,\delta_2(x)).
\end{aligned}
$$

This completes the proof of Theorem 2. □

Remark 1. *Let*

$$\nu_1(x)=\left(U_m\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}$$

and

$$\nu_2(x)=\left(V_m\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

Then, by using the result of Shisha and Mond [26], we find that

$$
\begin{aligned}
|(U_m-V_m)(f;x)|&\le|U_m(f;x)-f(x)|+|V_m(f;x)-f(x)|\\
&\le 2\omega_1(f,\nu_1(x))+2\omega_1(f,\nu_2(x)).
\end{aligned}
$$

Since

$$F_{m,k}^2(e_1)\le F_{m,k}(e_1^2)$$

and

$$G_{m,k}^2(e_1)\le G_{m,k}(e_1^2),$$

it follows that

$$\delta_i(x)\le\nu_i(x)\qquad(i=1,2).$$

2. Applications of Theorem 2

As applications of the Theorem 2, in this section, we give estimates of the differences between some of the most used positive linear operators in Approximation Theory. The considered examples involve the Baskakov type operators, the Szász–Mirakyan type operators, and the Durrmeyer type operators. We also show for the Durrmeyer type operators that, in some particular cases, our result improves the estimates obtained by using the classical result of Shisha and Mond [26].

2.1. Difference Between the Baskakov and the Szász–Mirakyan Operators

The Szász–Mirakyan operators are defined by

$$S_m(f,x) = \sum_{k=0}^{\infty} p_{m,k}(x) F_{m,k}(f), \tag{1}$$

where

$$p_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!} \quad \text{and} \quad F_{m,k}(f) = f\left(\frac{k}{m}\right).$$

Lemma 2. *The moments of S_m satisfy the following relation*:

$$S_m(e_{n+1},x) = \frac{x}{m} S_m'(e_n,x) + x S_m(e_n,x).$$

In particular,

$$S_m(e_0,x) = 1, \quad S_m(e_1,x) = x \quad \textit{and} \quad S_m(e_2,x) = x^2 + \frac{x}{m}$$

and

$$S_m(e_3,x) = x^3 + \frac{3x^2}{m} + \frac{x}{m^2} \quad \textit{and} \quad S_m(e_4,x) = x^4 + \frac{6x^3}{m} + \frac{7x^2}{m^2} + \frac{x}{m^3}.$$

Remark 2. *We have*

$$\phi^{F_{m,k}} = F_{m,k}(e_1) = \frac{k}{m}$$

and, for $r \in \mathbb{N}$, we get

$$\mu_r^{F_{m,k}} := F_{m,k}(e_1 - \phi^{F_{m,k}} e_0)^r = 0.$$

The Baskakov operators are defined by

$$V_m(f;x) = \sum_{k=0}^{\infty} v_{m,k}(x) G_{m,k}(f), \tag{2}$$

where

$$v_{m,k}(x) = \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}} \quad \text{and} \quad G_{m,k}(f) = f\left(\frac{k}{m}\right).$$

Lemma 3. *The moments satisfy the following relation*:

$$V_m(e_{n+1},x) = \frac{x(1+x)}{m} V_m'(e_n,x) + x V_m(e_n,x).$$

The moments of the Baskakov operators up to order 4 are listed below:

$$\begin{aligned}
V_m(e_0,x) &= 1\\
V_m(e_1,x) &= x\\
V_m(e_2,x) &= \frac{x^2(m+1)+x}{m}\\
V_m(e_3,x) &= \frac{x^3(m+1)(m+2)+3x^2(m+1)+x}{m^2}\\
V_m(e_4,x) &= \frac{x^4(m+1)(m+2)(m+3)+6x^3(m+1)(m+2)+7x^2(m+1)+x}{m^3}.
\end{aligned}$$

Remark 3. *We have*

$$\phi^{G_{m,k}} = G_{m,k}(e_1) = \frac{k}{m},$$

and, for $r \in N$, we get

$$\mu_r^{G_{m,k}} := G_{m,k}(e_1 - \phi^{G_{m,k}} e_0)^r = 0.$$

Now, as an application of Theorem 2, the difference of V_m and S_m defined, respectively, by Equations (1) and (2), can be given as Proposition 1 below.

Proposition 1. *Let $I = [0,\infty)$, $f \in D(I)$ and $f^{(s)} \in E_B(I)$ $(s = 1,2,3,4)$. Then, for each $x \in [0,\infty)$, it is asserted that*

$$|(V_m - S_m)(f,x)| \leq 2\omega_1\left(f, \sqrt{\frac{x(1+x)}{m}}\right) + 2\omega_1\left(f, \sqrt{\frac{x}{m}}\right).$$

The proof of Proposition 1 follows from Remarks 2 and 3, Lemmas 2 and 3, and Theorem 2. We, therefore, omit the details involved.

2.2. Difference Between the Baskakov and the Szász–Mirakyan–Baskakov Operators

In the year 1983, Prasad et al. [27] introduced a class of the Szász–Mirakyan–Baskakov type operators. These operators were subsequently improved by Gupta [28] as follows:

$$M_m(f,x) = \sum_{k=0}^{\infty} p_{m,k}(x) H_{m,k}(f), \tag{3}$$

where

$$H_{m,k}(f) = (m-1)\int_0^{\infty} v_{m,k}(t) f(t)dt.$$

Here $p_{m,k}$ and $v_{m,k}$ are defined in Equations (1) and (2), respectively.

Remark 4. *Since*

$$H_{m,k}(e_r) = (m-1)\int_0^{\infty} \binom{m+k-1}{k} \frac{t^k}{(1+t)^{m+k}} t^r dt = \frac{(k+r)!(m-r-2)!}{k!(m-2)!},$$

we get

$$\phi^{H_{m,k}} = H_{m,k}(e_1) = \frac{k+1}{m-2}$$

and

$$\begin{aligned}
\mu_2^{H_{m,k}} &= H_{m,k}(e_1 - \phi^{H_{m,k}} e_0)^2 \\
&= H_{m,k}(e_2) + \left(\frac{k+1}{m-2}\right)^2 - 2H_{m,k}(e_1)\left(\frac{k+1}{m-2}\right) \\
&= \frac{(k+2)(k+1)}{(m-2)(m-3)} - \left(\frac{k+1}{m-2}\right)^2 \\
&= \frac{k^2 + mk + m - 1}{(m-2)^2(m-3)},
\end{aligned}$$

$$\begin{aligned}
\mu_3^{H_{m,k}} &= H_{m,k}(e_1 - \phi^{H_{m,k}} e_0)^3 \\
&= H_{m,k}(e_3) - 3H_{m,k}(e_2)\left(\frac{k+1}{m-2}\right) \\
&\quad + 3H_{m,k}(e_1)\left(\frac{k+1}{m-2}\right)^2 - H_{m,k}(e_0)\left(\frac{k+1}{m-2}\right)^3 \\
&= \frac{4k^3 + 6mk^2 + (2m^2 + 4m - 4)k + 2m(m-1)}{(m-2)^3(m-3)(m-4)}
\end{aligned}$$

and

$$\begin{aligned}
\mu_4^{H_{m,k}} &= H_{m,k}(e_1 - \phi^{H_{m,k}} e_0)^4 \\
&= H_{m,k}(e_4) - 4H_{m,k}(e_3)\left(\frac{k+1}{m-2}\right) + 6H_{m,k}(e_2)\left(\frac{k+1}{m-2}\right)^2 \\
&\quad - 4H_{m,k}(e_1)\left(\frac{k+1}{m-2}\right)^3 + H_{m,k}(e_0)\left(\frac{k+1}{m-2}\right)^4 \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{(m-5)(m-4)(m-3)(m-2)} - 4\frac{(k+1)(k+2)(k+3)}{(m-4)(m-3)(m-2)}\left(\frac{k+1}{m-2}\right) \\
&\quad + 6\frac{(k+1)(k+2)}{(m-3)(m-2)}\left(\frac{k+1}{m-2}\right)^2 - 4\frac{(k+1)}{(m-2)}\left(\frac{k+1}{m-2}\right)^3 + \left(\frac{k+1}{m-2}\right)^4.
\end{aligned}$$

In Proposition 2 below, a quantitative result concerning the estimate of the difference between M_m and V_m is proved.

Proposition 2. *If $f \in D([0,\infty))$ with $f^{(i)} \in C_B[0,\infty)$ $(i = 2,3,4)$, then, for each $x \in [0,\infty)$, it is asserted that*

$$|(M_m - V_m)(f,x)| \le A(x)\|f^{(4)}\| + B(x)\|f^{(3)}\| + C(x)\|f^{(2)}\| + 2\omega_1(f,\delta_1(x)) + 2\omega_1(f,\delta_2(x)),$$

where

$$\begin{aligned}
A(x) = \frac{1}{8(m-5)(m-4)(m-3)(m-2)^4}\Big\{&x^2(x+1)^2 m^5 \\
&+ x(4x^3 + 14x^2 + 14x + 5)m^4 \\
&+ (x+1)(24x^2 + 5x + 3)m^3 + 28x^2 + 7x - 8)m^2\Big\},
\end{aligned}$$

$$B(x) = \frac{x(x+1)(2x+1)m^3 + (2x+1)(3x+1)m^2 - m}{3(m-2)^2(m-3)(m-4)},$$

$$C(x) = \frac{x(1+x)m^2 + (x+1)m - 1}{2(m-2)^2(m-3)},$$

$$\delta_1(x) = \sqrt{\frac{x(1+x)}{m}}$$

and

$$\delta_2(x) = \frac{\sqrt{4x^2 + (4+m)x + 1}}{(m-2)}.$$

Proof. Applying Remarks 3 and 4, together with Lemma 2, we find that

$$
\begin{aligned}
A(x) &= \frac{1}{4!}\sum_{k\in K}\left(p_{m,k}(x)\mu_4^{H_{m,k}} + v_{m,k}(x)\mu_4^{G_{m,k}}\right) \\
&= \frac{1}{4!}\sum_{k=0}^{\infty} p_{m,k}(x)\left[\frac{(k+1)(k+2)(k+3)(k+4)}{(m-5)(m-4)(m-3)(m-2)}\right. \\
&- 4\frac{(k+1)(k+2)(k+3)}{(m-4)(m-3)(m-2)}\left(\frac{k+1}{m-2}\right) \\
&\left.+ 6\frac{(k+1)(k+2)}{(m-3)(m-2)}\left(\frac{k+1}{m-2}\right)^2 - 4\frac{(k+1)}{(m-2)}\left(\frac{k+1}{m-2}\right)^3 + \left(\frac{k+1}{m-2}\right)^4\right] \\
&= \frac{1}{8(m-5)(m-4)(m-3)(m-2)^4}\left\{x^2(x+1)^2m^5\right. \\
&+ x(4x^3+14x^2+14x+5)m^4 \\
&\left.+\ (x+1)(24x^2+5x+3)m^3 + 28x^2+7x-8)m^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(x) &= \frac{1}{3!}\left|\sum_{k=0}^{\infty} p_{m,k}(x)\mu_3^{H_{m,k}}\right. \\
-\sum_{k=0}^{\infty} v_{m,k}(x)\mu_3^{G_{m,k}} &= \frac{x(x+1)(2x+1)m^3 + (2x+1)(3x+1)m^2 - m}{3(m-2)^2(m-3)(m-4)}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
C(x) &= \frac{1}{2}\left|\sum_{k=0}^{\infty} p_{m,k}(x)\mu_2^{H_{m,k}} - \sum_{k=0}^{\infty} v_{m,k}(x)\mu_2^{G_{m,k}}\right| \\
&= \frac{x(1+x)m^2 + (x+1)m - 1}{2(m-2)^2(m-3)},
\end{aligned}
$$

$$
\begin{aligned}
\delta_1(x) &= \left(\sum_{k=0}^{\infty} v_{m,k}(x)(\phi^{G_{m,k}} - x)^2\right)^{1/2} \\
&= \left(\sum_{k=0}^{\infty} v_{m,k}(x)\left(\frac{k}{m} - x\right)^2\right)^{1/2} \\
&= \sqrt{\frac{x(1+x)}{m}}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2(x) &= \left(\sum_{k=0}^{\infty} p_{m,k}(x)(\phi^{H_{m,k}} - x)^2\right)^{1/2} \\
&= \left(\sum_{k=0}^{\infty} p_{m,k}(x)\left(\frac{k+1}{m-2} - x\right)^2\right)^{1/2} \\
&= \frac{\sqrt{4x^2 + (4+m)x + 1}}{(m-2)}.
\end{aligned}
$$

Now, by using Theorem 2, Proposition 2 is proved. □

2.3. Difference Between the Baskakov and the Szász–Mirakyan–Kantorovich Operators

Let $p_{m,k}$ be the Szász–Mirakyan basis function defined in Equation (1). In addition, let

$$J_{m,k}(f) = m\int_{k/m}^{(k+1)/m} f(t)dt.$$

The Szász–Mirakyan–Kantorovich operators are defined by

$$K_m(f;x) = \sum_{k=0}^{\infty} p_{m,k}(x)J_{m,k}(f). \tag{4}$$

Remark 5. *The following result can be obtained by simple computation:*

$$\phi^{J_{m,k}} = J_{m,k}(e_1) = \frac{k}{m} + \frac{1}{2m}.$$

Moreover, we have

$$\begin{aligned}
\mu_2^{J_{m,k}} &= J_{m,k}(e_1 - \phi^{J_{m,k}}e_0)^2 \\
&= J_{m,k}(e_2) - 2\left(\frac{k}{m} + \frac{1}{2m}\right)^2\left(\frac{k}{m} + \frac{1}{2m}\right)^2 \\
&= \frac{1}{12m^2},
\end{aligned}$$

$$\begin{aligned}
\mu_3^{J_{m,k}} &:= J_{m,k}(e_1 - \phi^{J_{m,k}}e_0)^3 \\
&= J_{m,k}(e_3) - 3J_{m,k}(e_2)\left(\frac{k}{m} + \frac{1}{2m}\right) \\
&\quad + 3J_{m,k}(e_1)\left(\frac{k}{m} + \frac{1}{2m}\right)^2 - J_{m,k}(e_0)\left(\frac{k}{m} + \frac{1}{2m}\right)^3 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mu_4^{J_{m,k}} &:= J_{m,k}(e_1 - \phi^{J_{m,k}}e_0)^4 \\
&= J_{m,k}(e_4) - 4J_{m,k}(e_3)\left(\frac{k}{m} + \frac{1}{2m}\right) + 6J_{m,k}(e_2)\left(\frac{k}{m} + \frac{1}{2m}\right)^2 \\
&\quad - 4J_{m,k}(e_1)\left(\frac{k}{m} + \frac{1}{2m}\right)^3 + J_{m,k}(e_0)\left(\frac{k}{m} + \frac{1}{2m}\right)^4 \\
&= \frac{1}{80m^4}.
\end{aligned}$$

The following quantitative result concerning the difference between K_m and V_m is proved next.

Proposition 3. *Let $I = [0, \infty)$. If $f \in D(I)$ with $f^{(i)} \in E_B(I) \quad (i = 2, 3, 4)$, then, for each $x \in [0, \infty)$, it is asserted that*

$$|(K_m - V_m)(f, x)| \le A(x)\|f^{(4)}\| + C(x)\|f^{(2)}\| + 2\omega_1(f, \delta_1) + 2\omega_1(f, \delta_2),$$

where

$$A(x) = \frac{1}{1920m^4} \quad \textit{and} \quad C(x) = \frac{1}{24m^2}$$

and

$$\delta_1(x) = \sqrt{\frac{x(1+x)}{m}} \quad \textit{and} \quad \delta_2(x) = \frac{\sqrt{4mx+1}}{2m}.$$

Proof. Applying Remarks 3 to 5 and Lemma 2, we get

$$\begin{aligned} A(x) &:= \frac{1}{4!}\sum_{k=0}^{\infty}(p_{m,k}(x)\mu_4^{J_{m,k}} + v_{m,k}(x)\mu_4^{G_{m,k}}) \\ &= \frac{1}{4!}\sum_{k=0}^{\infty} p_{m,k}(x)\frac{1}{80m^4} \\ &= \frac{1}{1920m^4} \end{aligned}$$

and

$$\begin{aligned} B(x) &= \frac{1}{3!}\left|\sum_{k=0}^{\infty} p_{m,k}(x)\mu_3^{J_{m,k}} - \sum_{k=0}^{\infty} v_{m,k}(x)\mu_3^{G_{m,k}}\right| \\ &= 0. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} C(x) &= \frac{1}{2!}\left|\sum_{k=0}^{\infty} p_{m,k}(x)\mu_2^{J_{m,k}} - \sum_{k=0}^{\infty} v_{m,k}(x)\mu_2^{G_{m,k}}\right| \\ &= \frac{1}{24m^2}, \end{aligned}$$

$$\begin{aligned} \delta_1(x) &= \left(\sum_{k=0}^{\infty} v_{m,k}(x)(\phi^{G_{m,k}} - x)^2\right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} v_{m,k}(x)\left(\frac{k}{m} - x\right)^2\right)^{1/2} \\ &= \sqrt{\frac{x(1+x)}{m}} \end{aligned}$$

and

$$\begin{aligned} \delta_2(x) &= \left(\sum_{k=0}^{\infty} p_{m,k}(x)(\phi^{J_{m,k}} - x)^2\right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} p_{m,k}(x)\left(\frac{k}{m} + \frac{1}{2m} - x\right)^2\right)^{1/2} \\ &= \frac{\sqrt{4mx+1}}{2m}. \end{aligned}$$

Upon collecting the above estimates and by using Theorem 2, the proof of Proposition 3 is completed. □

2.4. Difference of Two Genuine-Durrmeyer Type Operators

Let $\rho > 0$ and $f \in C[0,1]$. Suppose also that

$$F^{\rho}_{m,k}(f) = \begin{cases} f(0) & (k=0) \\ \displaystyle\int_0^1 \frac{t^{k\rho-1}(1-t)^{(m-k)\rho-1}}{B(k\rho,(m-k)\rho)} f(t)dt & (k \neq 0,1) \\ f(1) & (k=1). \end{cases}$$

Păltănea and Gonska (see [29–31]) introduced and studied a new class of the Bernstein–Durrmeyer type operators defined by

$$U^{\rho}_m : C[0,1] \to \Pi_m \quad \text{and} \quad U^{\rho}_m(f;x) := \sum_{k=0}^{m} F^{\rho}_{m,k}(f) p_{m,k}(x),$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}.$$

Neer and Agrawal [32] introduced a class of the genuine-Durrmeyer type operators as follows:

$$\tilde{U}^{\rho}_m(f;x) = \sum_{k=0}^{m} F^{\rho}_{m,k}(f) p^{<\frac{1}{m}>}_{m,k}(x),$$

where

$$p^{<\frac{1}{m}>}_{m}(x) = \frac{2 \cdot m!}{(2m)!} \binom{m}{k} (mx)_k (m-mx)_{m-k}.$$

Proposition 4 below provides an estimate of the difference between U^{ρ}_m and $\tilde{U}^{\rho}_m$.

Proposition 4. *Let $f \in C^4[0,1]$. Then the following inequality holds true:*

$$\begin{aligned} \left| \left(U^{\rho}_m - \tilde{U}^{\rho}_m \right)(f;x) \right| \leq\ & A(x)\|f^{(4)}\| + B(x)\|f^{(3)}\| + C(x)\|f^{(2)}\| \\ & + 2\omega_1\left(f, \delta_1(x)\right) + 2\omega_1\left(f, \delta_2(x)\right), \end{aligned}$$

where

$$\begin{aligned} A(x) :=\ & \frac{x(1-x)(n-1)}{8m^3(m\rho+1)(m\rho+2)(m\rho+3)(m+1)(m+2)(m+3)} \\ & \cdot \Big\{ m\rho(3m^4+5m^3+7m^2-5m-6) + 4m^5 + 4m^4 + 4m^3 - 30m^2 + 30m \\ & + 36 + x(1-x)(m-2)(m-3)(m\rho-6)(2m^3+6m^2+11m+6), \end{aligned}$$

$$B(x) := \frac{x(1-x)|1-2x|(m-2)(m-1)(3m+2)}{3(m\rho+1)(m\rho+2)m^2(m+1)(m+2)},$$

$$C(x) := \frac{x(1-x)(m-1)}{2(m\rho+1)m(m+1)},$$

$$\delta_1(x) := \sqrt{\frac{x(1-x)}{m}}$$

and

$$\delta_2(x) := \sqrt{\frac{2x(1-x)}{m+1}}.$$

Proof. In Theorem 2, we set

$$F_{m,k}(f) = G_{m,k}(f) = F^{\rho}_{m,k}(f),$$

so that we have

$$\phi^{F_{m,k}} = \phi^{G_{m,k}} = \frac{k}{m};$$

$$\mu_2^{F_{m,k}} = \mu_2^{G_{m,k}} = F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^2 = \frac{k(m-k)}{m^2(m\rho+1)};$$

$$\begin{aligned}\mu_3^{F_{m,k}} = \mu_3^{G_{m,k}} &= F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^3 \\ &= \frac{2k(2k^2-3km+m^2)}{m^3(m\rho+1)(m\rho+2)}\end{aligned}$$

and

$$\begin{aligned}\mu_4^{F_{m,k}} &= \mu_4^{G_{m,k}} = F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^4 \\ &= \frac{3k(k^3m\rho - 2k^2m^2\rho + km^3\rho - 6k^3 + 12k^2m - 8km^2 + 2m^3)}{m^4(m\rho+1)(m\rho+2)(m\rho+3)}.\end{aligned}$$

Now, by considering the following relations:

$$\sum_{k=0}^{n} p_{m,k}(x) = 1,$$

$$\sum_{k=0}^{n} \frac{k}{m} p_{m,k}(x) = x,$$

$$\sum_{k=0}^{m} \left(\frac{k}{m}\right)^2 p_{m,k}(x) = \frac{x(mx-x+1)}{m},$$

$$\sum_{k=0}^{m} \left(\frac{k}{m}\right)^3 p_{m,k}(x) = \frac{x(m^2x^2-3mx^2+3mx+2x^2-3x+1)}{m^2},$$

$$\begin{aligned}\sum_{k=0}^{m} \left(\frac{k}{m}\right)^4 p_{m,k}(x) = \frac{x}{m^3}\Big(&m^3x^3 - 6m^2x^3 + 6m^2x^2 + 11mx^3 \\ &-18mx^2 - 6x^3 + 7mx + 12x^2 - 7x + 1\Big),\end{aligned}$$

$$\sum_{k=0}^{m} p_{m,k}^{<\frac{1}{m}>}(x) = 1,$$

$$\sum_{k=0}^{m} \left(\frac{k}{m}\right) p_{m,k}^{<\frac{1}{m}>}(x) = x,$$

$$\sum_{k=0}^{m}\left(\frac{k}{m}\right)^2 p_{m,k}^{<\frac{1}{m}>}(x) = x^2 + \frac{2x(1-x)}{m+1},$$

$$\sum_{k=0}^{m}\left(\frac{k}{m}\right)^3 p_{m,k}^{<\frac{1}{m}>}(x) = x^3 + \frac{6mx^2(1-x)}{(m+1)(m+2)} + \frac{6x(1-x)}{(m+1)(m+2)}$$

and

$$\sum_{k=0}^{m}\left(\frac{k}{m}\right)^4 p_{m,k}^{<\frac{1}{m}>}(x) = x^4 + \frac{12(m^2+1)x^3(1-x)}{(m+1)(m+2)(m+3)} + \frac{12(3m-1)x^2(1-x)}{(m+1)(m+2)(m+3)} + \frac{2(13m-1)x(1-x)}{m(m+1)(m+2)(m+3)},$$

the proof of Proposition 4 is completed. □

Example 1. *Applying Proposition 2 for $f(x) = \frac{x}{x^2+1}$, $x \in [0,1]$ and $\rho = 2$, we get the following estimate:*

$$|(U_m^{\rho} - \tilde{U}_m^{\rho})(f;x)| \le E_m(f), \tag{5}$$

where

$$E_m(f) = K_1\|f^{(4)}\| + K_2\|f^{(3)}\| + K_3\|f^{(2)}\| + 2(\delta_1+\delta_2)\|f'\| \quad \text{and} \quad f \in C^4[0,1]$$

and

$$K_1 := \frac{(m-1)}{64m^3(2m+1)(m+1)^2(2m+3)(m+2)(m+3)} \cdot \left(\frac{1}{2}(m-2)(m-3)^2(2m^3+6m^2+11m+6) + 10m^5 + 14m^4 + 18m^3 - 40m^2 + 18m + 36\right),$$

$$K_2 := \frac{(m-2)(m-1)(3m+2)}{24(2m+1)(m+1)^2m^2(m+2)},$$

$$K_3 := \frac{m-1}{8(2m+1)m(m+1)},$$

$$\delta_1 := \frac{1}{2\sqrt{m}}$$

and

$$\delta_2 := \frac{1}{\sqrt{2(m+1)}}.$$

Now, by using the result of Shisha and Mond (see [26]; see also Remark 1), we get the following estimate:

$$|(U_m^{\rho} - \tilde{U}_m^{\rho})(f;x)| \le E_m^{(SM)}(f), \tag{6}$$

where

$$E_m^{(SM)}(f) = \left(\sqrt{\frac{3}{2m+1}} + \sqrt{\frac{5m+1}{(m+1)(2m+1)}}\right)\|f'\|, \; f \in C^1[0,1].$$

Table 1 below contains the values of $E_m(f)$ and $E_m^{(SM)}(f)$ for certain given values of n. We note here that, for this particular case, the estimate in Equation (5) is better than the estimate given by the Shisha–Mond result in Equation (6).

Table 1. Estimates for the difference of $U_m^{\rho} f$ and $\tilde{U}_m^{\rho} f$.

m	$E_m(f)$	$E_m^{(SM)}(f)$
10	0.74402776700	0.84783596730
10^2	0.24073382150	0.27926364330
10^3	0.07632064786	0.08868768037
10^4	0.02414170100	0.02805750320
10^5	0.00763444237	0.00887294116
10^6	0.00241421554	0.00280588236
10^7	0.00076343666	0.00088729829
10^8	0.00024142458	0.00028058836

2.5. Difference of the Durrmeyer Operators and the Lupaş–Durrmeyer Operators

Durrmeyer [33] and, independently, Lupaş [34] defined the Durrmeyer operators by

$$M_m(f,x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t)\, f(t)\, dt \qquad (x \in [0,1]). \tag{7}$$

Gupta et al. [35] introduced a modification of the operator in Equation (7) as follows:

$$D_m^{<\frac{1}{m}>}(f;x) = (m+1)\sum_{k=0}^{m} p_{m,k}^{<\frac{1}{m}>}\int_0^1 p_{m,k}(t) f(t) dt \qquad (f \in C[0,1]). \tag{8}$$

Finally, the difference between M_m and $D_m^{<\frac{1}{m}>}$ is provided in the estimate asserted by Proposition 5 below.

Proposition 5. *Let $f \in C^4[0,1]$. Then the following inequality holds true*:

$$\begin{aligned}\left|\left(M_m - D_m^{<\frac{1}{m}>}\right)(f;x)\right| &\leq A(x)\|f^{(4)}\| + B(x)\|f^{(3)}\| + C(x)\|f^{(2)}\| \\ &\quad + 2\omega_1\left(f,\delta_1(x)\right) + 2\omega_1\left(f,\delta_2(x)\right),\end{aligned}$$

where

$$\begin{aligned}A(x) := \;&\frac{1}{8(m+1)(m+2)^5(m+3)^2(m+4)(m+5)} \\ &\cdot \{x(1-x)m(m-1)\,[x(1-x)(m-2)(m-3)(m-4) \\ &\quad\cdot (2m^3+6m^2+11m+6) + 11m^5 + 41m^4 + 77m^3 + 25m^2 + 26m + 24\Big] \\ &+ (m+1)^2(m+2)(m+3)(3m^2+5m+4)\Big\},\end{aligned}$$

$$B(x) := \frac{m|x(1-x)(1-2x)(m-1)(m-2)(3m+2) + m^3 + 4m^2 + 5m + 2|}{3(m+1)(m+2)^3(m+3)(m+4)},$$

$$C(x) := \frac{m(m-1)x(1-x) + (m+1)^2}{2(m+1)(m+2)^2(m+3)},$$

$$\delta_1(x) := \frac{\sqrt{x(1-x)m + (2x-1)^2}}{m+2}$$

and

$$\delta_2(x) := \frac{\sqrt{2x(1-x)m^2 + (1-2x)^2(m+1)}}{(m+2)\sqrt{m+1}}.$$

Proof. In Theorem 2, we let

$$F_{m,k}(f) = G_{m,k}(f) = (m+1)\int_0^1 p_{m,k}(t)f(t)dt,$$

so that we have

$$\phi^{F_{m,k}} = \phi^{G_{m,k}} = \frac{k+1}{m+2},$$

$$\begin{aligned}\mu_2^{F_{m,k}} = \mu_2^{G_{m,k}} &= F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^2 \\ &= \frac{(k+1)(m-k+1)}{(m+2)^2(m+3)},\end{aligned}$$

$$\begin{aligned}\mu_3^{F_{m,k}} = \mu_3^{G_{m,k}} &= F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^3 \\ &= \frac{2(k+1)(2k^2-3km+m^2-2k+m)}{(m+2)^3(m+3)(m+4)}\end{aligned}$$

and

$$\begin{aligned}\mu_4^{F_{m,k}} = \mu_4^{G_{m,k}} &= F_{m,k}\left(e_1 - \phi^{F_{m,k}}\right)^4 \\ &= \frac{3(k+1)(k-m-1)(k^2m-km^2-4k^2+4km-3m^2-5m-4)}{(m+2)^4(m+3)(m+4)(m+5)}.\end{aligned}$$

Now, by applying the relations from the proof of Proposition 2, the resulting estimate of the difference of the Durrmeyer operator and the Lupaş–Durrmeyer operator is as asserted by Proposition 5. □

Example 2. *By pplying Proposition 5 for $f(x) = \cos(2\pi x)$ for $x \in [0,1]$, we get the following estimate:*

$$\left|\left(M_m - D_m^{<\frac{1}{m}>}\right)(f;x)\right| \le E_m(f), \tag{9}$$

where

$$E_m(f) = K_1\|f^{(4)}\| + K_2\|f^{(3)}\| + K_3\|f^{(2)}\| + 2(\delta_1+\delta_2)\|f'\|,\ \ f \in C^4[0,1]$$

and

$$\begin{aligned}K_1 := &\frac{1}{8(m+1)(m+2)^5(m+3)^2(m+4)(m+5)} \\ &\cdot\left\{\frac{1}{4}m(m-1)\left[\frac{1}{4}(m-2)(m-3)(m-4)(2m^3+6m^2+11m+6)\right.\right. \\ &\left.+\ 11m^5+41m^4+77m^3+25m^2+26m+24\right] \\ &\left.+(m+1)^2(m+2)(m+3)(3m^2+5m+4)\right\},\end{aligned}$$

$$\begin{aligned}K_2 := &\frac{m}{3(m+1)(m+2)^3(m+3)(m+4)} \\ &\cdot\left\{\frac{1}{4}(m-1)(m-2)(3m+2) + m^3+4m^2+5m+2\right\},\end{aligned}$$

$$K_3 := \frac{1}{2(m+2)^2(m+3)(m+1)}\left[\frac{1}{4}m(m-1)+(m+1)^2\right],$$

$$\delta_1 := \frac{\sqrt{m+4}}{2(m+2)}$$

and

$$\delta_2 := \frac{\sqrt{m^2+2(m+1)}}{(m+2)\sqrt{2(m+1)}}.$$

Thus, by using the result of Shisha and Mond (see [26]; see also Remark 1), we get the following estimate:

$$\left|\left(M_m - D_m^{<\frac{1}{m}>}\right)(f;x)\right| \leq E_m^{(SM)}(f), \tag{10}$$

where

$$E_m^{(SM)}(f) = 2\left(\sqrt{\frac{m+1}{2(m+2)(m+3)}} + \sqrt{\frac{3m^2+3m+2}{4(m+1)(m+2)(m+3)}}\right)\|f'\|,\ f \in C^1[0,1].$$

Table 2 below gives the values of $E_m(f)$ and $E_m^{(SM)}(f)$ for certain specific values of m. We also note that, for this particular case, the estimate in Equation (9) is better than the estimate given by the Shisha–Mond result in Equation (10).

Table 2. Estimates for the difference of $M_m f$ and $D_m^{<\frac{1}{m}>} f$.

m	$E_m(f)$	$E_m^{(SM)}(f)$
10^2	1.5210054310	1.9330219770
10^3	0.4794548855	0.6237181803
10^4	0.1516781199	0.1976406539
10^5	0.0479680333	0.0625122598
10^6	0.0151689380	0.0197685170
10^7	0.0047968431	0.0062513668
10^8	0.0015168951	0.0019768561

Remark 6. *The earlier works [36,37] proposed certain general families of positive linear operators which reproduce only constant functions. Recently, as a continuation of these works, in [38] some positive linear operators reproducing linear functions were introduced and studied. Analogous further researches for this class of operators are possible.*

3. Conclusions

The studies of the differences of positive linear operators has become an interesting area of research in Approximation Theory. The present paper deals with the estimates of the differences of various positive linear operators, which are defined on bounded or unbounded intervals, in terms of the modulus of continuity. In several earlier papers, the results of the type which we have presented here were obtained for a class of positive linear operators constructed with the same fundamental functions. The novelty of this paper is that the fundamental functions of the positive linear operators can chosen to be different. Our present study makes use of the Baskakov type operators, the Szász–Mirakyan type operators, and the Durrmeyer type operators. In some illustrative numerical examples, we have shown that the estimates obtained in this study are better than the estimates given by the classical Shisha–Mond result. For a future work, we propose to obtain estimates for these operators involving some suitably weighted modulus of smoothness.

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