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Hamilton–Jacobi Equation for a Charged Test Particle in the Stäckel Space of Type (2.0)

Valeriy Obukhov [1,2,*]

[1] Department of Theoretical Physics, Tomsk State Pedagogical University, 60 Kievskaya St., 634041 Tomsk, Russia; obukhov@tspu.edu.ru; Tel.: +7-913-827-04-57

[2] Department of Theoretical Physics, Tomsk State University, 36 Lenin Ave., 634041 Tomsk, Russia

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Abstract: All electromagnetic potentials and space–time metrics of Stäckel spaces of type (2.0) in which the Hamilton–Jacobi equation for a charged test particle can be integrated by the method of complete separation of variables are found. Complete sets of motion integrals, as well as complete sets of killing vector and tensor fields, are constructed. The results can be used when studying solutions of field equations in the theory of gravity.

Keywords: theory of symmetry; separation of variables; Hamilton–Jacobi equation; integrals of motion

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1. Introduction

The construction of the theory of the separation of variables in the classical and quantum equations of test particles motion, as well as in the classical equations of mathematical physics, is one of the most important results in the theory of symmetry implementation. The main tool in the theory of the separation of variables is the Stäckel spaces. Thus, the Riemannian spaces, whose metrics in some (privileged) coordinate systems allow separation of variables in the free Hamilton–Jacobi equation were named after Paul Stäckel [1,2]. In the found metric, the equation admits a complete separation of variables in the orthogonal privileged coordinate system. The equation itself has a complete set of motion integrals quadratic in velocity. Among the articles devoted to the Stäckel spaces beside the works from Stäckel himself and from Eisenhart [3], we mention Levy-Civita [4], who formulated in a covariant form the condition for separation of variables in privileged coordinate systems, and Yarov-Yarovoy [5], who generalized the definition of Stäckel spaces in the case of non-orthogonal privileged coordinate systems. V.N. Shapovalov in a series of works [6–8] proved the main theorem of the theory of separation of variables—the theorem on the necessary and sufficient conditions for separation. In the case of the free Hamilton–Jacobi equation, such condition is met when there is a complete set of Killing fields in space. The set includes n geometric objects which are called Killing tensor fields not higher than the second rank, which mutually commute and meet some additional restrictions.

The results obtained by solving the appropriate Killing equations allowed finding all Stäckel metrics in the real coordinate systems. The work [9] provides proof for the generalization of the theorem in the case of complex privileged coordinate systems.

Even though there are known examples of spaces in which the quantum-mechanical equations of motion allow non-commutative integration (see, for example, [10]), these were Stäckel spaces that attracted the greatest interest of researchers due to their high level of symmetry and rich geometric content. Therefore, the types of Stäckel metrics for the flat space–time were used to construct the theory of separation of variables in the Klein–Gordon–Fock and Dirac equations. Using the complete sets of symmetry operators, all privileged coordinate systems, electromagnetic potentials, and many examples of the exact integration of equations by the method of complete separation of variables were found (see, for example: [11–15]).

A special place is taken by the Stäckel spaces in the theory of gravitation. The set of Stäckel spaces of type (2.0) include such important solutions of the Einstein equations as solutions found by Schwarzschild, Kerr, and by Taub–Newman–Unti–Tambourino (Taub–NUT solution). Moreover, the basic metric of cosmology, the Roberson–Walker metric also belongs to the set of Stäckel spaces, namely the spaces of type (1.0). A high level of symmetry of the metric allows provision of new approaches to solving cosmological problems, in particular, related to the consideration of dark energy or to the construction of modified gravity theories (see, for example, [16,17]). Also, it is possible to use methods of symmetry theory to justify model selection in the Extended Gravity Cosmology [18].

Hundreds of works have been devoted to the study of the Stäckel spaces geometry so as the problem of integrating the basic equations of mathematical physics and field equations of the theory of gravitation. A sufficiently detailed bibliography, can be supplemented by the numerous works of Russian authors, can be found in papers [19,20].

Recently, interest in the problem of separation of variables has grown significantly. So the work [21] separation of variables of type (2.0) is directed to obtain the exact solutions of the Maxwell equation in the Plebanski–Deminski space. A lot of work is devoted to the study of various effects in individual Stäckel spaces; see, for example, [22–30].

One of the conclusions of the necessary and sufficient conditions theorem is the partition of the set of Stäckel metrics into invariant subsets according to the type of the complete set of Killing fields. According to the definition of the metric of Stäckel spaces, the complete sets and the complete separation of variables are of type $(N.N_0)$, where N is the maximum number of independent Killing vector fields $Y_p{}^i$, included in the complete set; $N_0 = N - \det(Y_p{}^i Y_{q_i})$. In the case of the Lorentz signature $N_0 = 0, 1$. If $N_0 = 0$, the space is called non-null. Otherwise, null. There are seven disjoint sets of Stäckel spaces with the signature of the space–time.

(1) Non-null Stäckel spaces of type (3.0). The complete set includes three Killing vector fields. The coordinate hypersurface related to the non-ignored variable is non-null. (Ignored are the privileged variables (see the definition below) that occur linearly in the complete integral. Other variables are called non-ignored.)
(2) Null Stäckel spaces of type (3.1). The complete set includes three Killing vector fields. The coordinate hypersurface related to the non-ignored variable is null one.
(3) Non-null Stäckel spaces of type (2.0). The complete set includes two Killing vector fields. Coordinate hypersurfaces related to non-ignored variables are non-null.
(4) Null Stäckel spaces of type (2.1). The complete set includes two Killing vector fields. The coordinate hypersurface belonging to one of the non-ignored variables is null.
(5) Non-isotropic Stäckel spaces of type (1.0). The complete set includes one non-isotropic Killing vector field. Coordinate hypersurfaces related to non-ignored variables are non-null.
(6) Null Stäckel spaces of type (1.1). The set includes one isotropic Killing vector field. The coordinate hypersurface belonging to one of the non-ignored variables is null.
(7) Non-null Stäckel spaces of type (0.0). The complete set does not contain Killing vector fields. Coordinate hypersurfaces related to non-ignored variables are non-null.

The theorem about necessary and sufficient conditions for the separation of variables made it possible to develop a theory of complete separation of variables for single-particle classical and quantum-mechanical equations of motion in the theory of gravity in the presence of physical fields of various nature. A task occurred to systematically classify Stäckel spaces in the presence of these fields. By classification, we mean an enumeration of all non-equivalent relatively admissible (i.e., non-violating conditions for the complete separation of variables) coordinate transformations and gauge transformations of the potentials. Since the set of Stäckel spaces is divided into disjoint subsets consisting of spaces of type $(N.N_0)$, the classification is carried out separately for each type.

Apparently, the first example of a systematic classification was made by J. Iwata [31]. In the paper vacuum solutions for the Einstein equations for spaces of type (1.0) were presented. In works [32–35] a similar classification for the remaining types of vacuum Stäckel spaces was carried out. For electro-vacuum Stäckel spaces, the classification was made in papers [36–38].

However, to date, the classical classification problem for the case when the Hamilton–Jacobi equation for a charged test particle moving in an external electromagnetic field in the absence of additional restrictions for this field admits complete separation of variables, has not been solved. In this work, the classification is constructed for the type (2.0). All appropriate non-equivalent sets of space–time metrics and potentials of the external electromagnetic field are found.

2. Conditions for the Existence of a Complete Set of Motion Integrals

Let us consider spaces of type (2.0) in which the Hamilton–Jacobi equation for a charged test particle

$$g^{ij}P_iP_j = \tilde{\lambda}; \quad P_i = p_i + A_i = S_{,i} + A_i, \tag{1}$$

can be integrated by the complete separation of variables method. In the privileged coordinate set the complete integral has the additive form:

$$S = \sum s_i(u^i). \tag{2}$$

Let us recall that according to the definition the coordinate system, in which such form exists, is called privileged and denoted by variables: $u^i = (u^0, u^1, u^2, u^3)$. Throughout the text, the repeating upper and lower indices are summed up within the established limits for indices changes; $i, j = 0, 1, 2, 3$. We consider the field external when the electromagnetic potential contains at least one (free) function, which is not expressed through the metric tensor. If the complete set contains N Killing vector fields, the first N coordinates are ignored. We will denote their coordinate indices by the letters $p, q, = 0, 1, \cdots = N-1$. We will supply the non-ignored coordinates with the indices $\mu, \nu = N, \ldots, 3$. Functions that depend only on the variable u^2 will be denoted by the lowercase Greek letters, only from the variable u^3 by the e lowercase Latin letters. Lowercase Latin and Greek letters with a tilde icon indicate constants. Exceptions: $\delta^{ij}, \delta_{ij}, \delta^i_j$ are the Kronecker symbols, g^{ij}, g_{ij}—components of the metric tensor, $\varepsilon, \varepsilon_i = +1, -1; \lambda, \lambda_i = const$, $h^{ij}_\nu, h^i_\nu, h_\nu$ - functions of u^ν. In this notation, the metric tensor of the Stäckel space of type (2.0) can be written as:

$$g^{ij} = (\hat{\Phi}^{-1})^\nu_3 h^{ij}_\nu = \frac{\delta^i_p \delta^j_q (a^{pq} + \alpha^{pq}) + \varepsilon^\nu \delta^i_\nu \delta^j_\nu}{\Delta}, \tag{3}$$

$$(\hat{\Phi})^\mu_\nu = \begin{pmatrix} 1 & \phi \\ -1 & f \end{pmatrix}, \quad \Delta = \phi + f = \det \hat{\Phi}, \tag{4}$$

where $p, q, = 0, 1; \nu, \mu = 2, 3$. $\hat{\Phi}$—Stäckel matrix. Please note that the Stäckel space of type (2.0) admits two Killing vector fields: $Y^i_p = \delta^i_p$ and two Killing tensor vector fields (together with the metric tensor):

$$X^{ij}_2 = \frac{f\varepsilon^2(\delta^i_p\delta^j_q\alpha^{pq} + \delta^i_2\delta^j_2) - \phi\varepsilon^3(\delta^i_p\delta^j_q a^{pq} + \delta^i_3\delta^j_3)}{\Delta}. \quad X^{ij}_3 = g^{ij}. \tag{5}$$

The complete set of integrals of motion for the free Hamilton–Jacobi equation has the form:

$$\hat{X}_\nu = X^{ij}_\nu p_i p_j, \quad \hat{X}_q = Y^i_q p_i. \tag{6}$$

The integrals of motion of the Hamilton–Jacobi equation, quadratic in impulse, included in the full set in the presence of an external electromagnetic field have the form:

$$\hat{X}_2 = X_2^{ij} p_i p_j + 2X^i p_i + X, \quad \hat{X}_3 = g^{ij} P_i P_j. \tag{7}$$

The complete integral in the privileged coordinate system can be reduced to the form:

$$S = \sum_{q=0}^{1} \lambda_q u^p + \sum_{\nu=2}^{3} s_\nu. \tag{8}$$

The presence of a complete set of commuting motion integrals allows us to find (8) as a solution to the system of equations:

$$\hat{X}_i = \lambda_i. \tag{9}$$

Moreover, $\hat{X}_i$ have the form:

$$\hat{X}_q = \delta_q^i p^q = p_q, \quad \hat{X}_\nu = (\hat{\phi}^{-1})_\nu^\mu \hat{H}_\mu = (\hat{\phi}^{-1})_\nu^\mu (h_\mu^{ij} p_i p_i + 2h_\mu^i p_j + h_\mu), \tag{10}$$

where

$$h_2^{ij} = \delta_p^i \delta_q^j \alpha^{pq} + \varepsilon^2 \delta_2^i \delta_2^j, \quad h_2^i = \omega^i, \quad h_2 = \hat{\omega},$$
$$h_3^{ij} = \delta_p^i \delta_q^j a^{pq} + \varepsilon^3 \delta_3^i \delta_3^j, \quad h_3^i = h^i, \quad h_3 = \hat{h}.$$

Equating the coefficients before the impulses to the right and left in the (10) when $\nu = 3$ and vanishing with a gradient transformation of the function potential ω^2, h^3, we obtain:

$$A^p = \frac{\omega^p + h^p}{\Delta}, \quad A^\nu = 0, \quad A_i A^i = \frac{\hat{\omega} + \hat{h}}{\Delta}. \tag{11}$$

The expressions (11) determine the electromagnetic potential of A^i. A necessary and sufficient condition for Equation (1) to admit complete separation of variables is:

$$A^i A_i = \frac{(\omega^p + h^p)(\omega^q + h^q) g_{pq}}{\Delta} = (\hat{\omega} + \hat{h}). \tag{12}$$

By setting:

$$G^{pq} = g^{pq}\Delta, \quad G_{pq} = \frac{g_{pq}}{\Delta}, \quad G = \det(G^{pq}), \quad G^{pq} = \alpha^{pq} + a^{pq}, \tag{13}$$

we obtain the necessary and sufficient condition in the form of a functional equation:

$$(\alpha^{00} + a^{00})(\omega^1 + h^1)^2 + (\alpha^{11} + a^{11})(\omega^0 + h^0)^2 -$$
$$2(\alpha^{01} + a^{01})(\omega^1 + h^1)(\omega^0 + h^0) = \tag{14}$$
$$(det(\alpha^{pq}) + det(a^{pq}) + \alpha^{11}a^{00} + \alpha^{00}a^{11} - 2\alpha^{01}a^{01})(\hat{\omega} + \hat{h}).$$

Please note that in the general case of an n-dimensional Stäckel space, it follows from the Shapovalovs theorem that Equation (1) admits complete separation of variables if and only if there exists a coordinate system in which:

$$g^{ij} = (\hat{\Phi}^{-1})_n^\nu h_\nu^{ij}, \quad A^i = (\hat{\Phi}^{-1})_n^\nu h_\nu^i,$$
$$A^i A_i = (\hat{\Phi}^{-1})_n^\nu h_\nu. \tag{15}$$

$\nu = 1, 2, \ldots, n$, h^{ij}_ν, h^i_ν, $h^{(0)}_\nu$, h_ν—functions only from u^ν, $\hat{\Phi}$—Stäckel matrix. If we introduce the additively arbitrary scalar field Ψ into Equation (1):

$$g^{ij}P_iP_j + \Psi = \tilde{m},$$

(see [39], where the classification problem for the natural Hamilton–Jacobi equation was considered), it is easy to ensure that the necessary and sufficient conditions are satisfied by a simple choice:

$$\Psi = (\hat{\Phi}^{-1})^\nu_n h^{(0)}_\nu - A^i A_i.$$

In our case, for the classification, it is necessary to solve a non-trivial functional Equation (15). It can be shown that the solution for the functional Equation (14) is equivalent to the solution of two algebraic equation systems, each containing 7 equations (6 of the second degree and one of the third degree). The first system includes functions only from the variable u^2, the second only from u^3. Both systems are overcrowded, and solutions are possible if there is an additional symmetry in Equation (14). As a similar symmetry, justified from a physical point of view, we require that the electromagnetic field be external. Since the variables u^2 and u^3 appear in (14) symmetrically, without loss of generality, we can assume that at least one free function is contained in ω^p. We also assume that all nonzero functions in (14) are fairly smooth, and there are points on the coordinate axes in the vicinity of which not one of them vanishes.

3. Solutions for the Case When Both Functions ω^p Are Free

To solve the classification problem, it is necessary to find and resolve the conditions for the existence of the free function. First, consider the case where both ω^p are free. We use the smoothness condition mentioned above and consider Equation (14) at a fixed point $\tilde{u}^3$ on the coordinate axis (u^3). Since $G \neq 0, \omega$ is expressed in terms of the function ω^p as follows:

$$\hat{\omega} = \gamma_{pq}\omega^p\omega^q + 2\gamma_p\omega^p + \gamma. \tag{16}$$

Here the functions $\gamma_{pq}, \gamma_p, \gamma$ are rational functions of α^{pq}. We substitute (16) into (14) and equate the coefficients of $\omega^p\omega^q$ and ω^p on the right and left sides in (14). Hence, we obtain the conditions for the existence of an external electromagnetic field. From the equality to zero of the coefficients before $\omega^p\omega^q$ it follows:

$$(\alpha^{pq} + a^{pq}) = \gamma_{pq}G, \quad G = \frac{1}{det(\gamma_{pq})}, \quad \to a^{pq} = 0 \to det(\alpha^{pq}) \neq 0. \tag{17}$$

The equality of coefficients before ω^p to zero gives:

$$\begin{cases} \alpha^{00}h^1 - \alpha^{01}h^0 = \gamma_1, \\ \alpha^{01}h^1 - \alpha^{11}h^0 = -\gamma_0. \end{cases} \tag{18}$$

As $det(\alpha_{pq}) \neq 0, h^p = \tilde{h}^p = 0 \to A^p = \frac{\omega^p}{\Delta}$, and the complete set integrals of motion has the form:

$$\hat{X}_q = p_q, \quad \hat{X}_2 = \frac{f\varepsilon^2(\alpha^{pq}p_pp_q + 2\omega^q p_q + \alpha_{pq}\omega^p\omega^q + p_2^2) - \phi\varepsilon^3 p_3^2}{\Delta}. \tag{19}$$

$$\hat{X}_3 = g^{ij}p_ip_j, \quad g^{ij} = \frac{\delta^i_p\delta^j_q\alpha^{pq} + \varepsilon^\nu\delta^i_\nu\delta^j_\nu}{\Delta}. \tag{20}$$

4. The Linear Dependence of ω^p on a Free Function

Now let ω^0 and ω^1 be connected to each other by a linear relation of the form:

$$\omega^p = \alpha^p \sigma + \sigma^p. \tag{21}$$

Here σ is a free function; σ^p, α^p are some rational functions of α^{pq}. Obviously, the function $\hat{\omega}$ in the relation to (14) depends on the free function as follows:

$$\hat{\omega} = \gamma\sigma^2 + 2\gamma_0\sigma + \omega_0. \tag{22}$$

$\gamma, \gamma_0, \omega_0$ are also rational functions of α^{pq}. We substitute the expressions (21) and (22) in (14). The equality to zero of the coefficients before $\sigma^b (b = 0, 1, 2)$ gives the system of equations:

$$\begin{gathered} a^{00}(\alpha^1)^2 + a^{11}(\alpha^0)^2 - 2a^{01}\alpha^0\alpha^1 + \\ \alpha^{11}(\alpha^0)^2 + \alpha^{00}(\alpha^1)^2 - 2\alpha^{01}\alpha^0\alpha^1 = G\gamma,. \end{gathered} \tag{23}$$

$$(\gamma = 0, 1)$$

$$\begin{gathered} (a^{00} + \alpha^{00})\alpha^1(h^1 + \sigma^1) + (a^{11} + \alpha^{11})\alpha^0(h^0 + \sigma^0) - \\ (a^{01} + \alpha^{01})(\alpha^0(h^1 + \sigma^1) + \alpha^1(h^0 + \sigma^0)) = G\gamma_0, \end{gathered} \tag{24}$$

$$\begin{gathered} (\alpha^{00} + a^{00})(\sigma^1 + h^1)^2 + (\alpha^{11} + a^{11})(\sigma^0 + h^0)^2 - \\ 2(\alpha^{01} + a^{01})(\sigma^1 + h^1)(\sigma^0 + h^0) = G(\omega_0 + h). \end{gathered} \tag{25}$$

The Equation (23) is a condition for the existence of a free function, which is superimposed on the metric tensor. Spaces of type (2.0) satisfying (23) already allow complete separation of variables in Equation (1). The remaining equations of the system serve to find the functions h^p. Please note that they have the obvious solution $h^p = 0$. To find other solutions of Equation (23), we classify the matrices $\hat{G} = (G^{pq})$, with respect to the group of admissible transformations of ignored variables: $u^p \to \tilde{c}^p_q u^q$. Let us list all equivalence classes of matrices $\hat{G}$. The classes will be denoted: $\hat{G}_\alpha$, where $\alpha = 1, \ldots, 5$.

$$\left\{ \begin{array}{l} \hat{G}_1 = \begin{pmatrix} a_0 + \alpha^{00} & a + \alpha^{01} \\ a + \alpha^{01} & a_1 + \alpha^{11} \end{pmatrix}, \\ \\ \hat{G}_2 = \begin{pmatrix} a_0 + \alpha^{00} & a + \alpha^{01} \\ a + \alpha^{01} & -a_0 + \alpha^{11} \end{pmatrix}, \\ \\ \hat{G}_3 = \begin{pmatrix} a_0 + \alpha^{00} & \alpha^{01} \\ \alpha^{01} & a_1 + \alpha^{11} \end{pmatrix}, \\ \\ \hat{G}_4 = \begin{pmatrix} a_0 + \alpha^{00} & a + \alpha^{01} \\ a + \alpha^{01} & \alpha^{11} \end{pmatrix}, \\ \\ \hat{G}_5 = \begin{pmatrix} a + \alpha^{00} & \alpha^{01} \\ \alpha^{01} & \varepsilon a + \alpha^{11} \end{pmatrix}, \quad \varepsilon = 0, -1. \end{array} \right. \tag{26}$$

Here $a^{00} = a_0$, $a^{01} = a$, $a^{11} = a_1$. Functions a_0, a, a_1 are linearly independent.

4.1. Finding the Metric Tensor

As already noted, Equation (23) does not contain functions defining the electromagnetic potential and, being a necessary and sufficient condition for the existence of a free function, allows us to find the metric tensor. Let us present (23) in the following form:

$$(a_0{\alpha^1}^2 + a_1{\alpha^0}^2 - 2a\alpha^0\alpha^1) + (\alpha^{11}{\alpha^0}^2 + \alpha^{00}{\alpha^1}^2 - 2\alpha^{01}\alpha^0\alpha^1) = \\ (det(\alpha^{pq}) + det(a^{pq}) + \alpha^{11}a_0 + \alpha^{00}a_1 - 2\alpha^{01}a)\tilde{\gamma}, \tag{27}$$

and consider it separately for options: $\tilde{\gamma} = 1$ and $\tilde{\gamma} = 0$.

I $\tilde{\gamma} = 1$.

We show that in this case all the solutions of Equation (23) for all classes $\hat{G}_\alpha$ from the expressions (26) except $\hat{G}_5$ can be represented as:

$$\hat{G} = \begin{pmatrix} (a^0)^2 + \varepsilon(\alpha^0)^2 & a^0a^1 + \varepsilon\alpha^0\alpha^1 \\ a^0a^1 + \varepsilon\alpha^0\alpha^1 & (a^1)^2 + \varepsilon(\alpha^1)^2 \end{pmatrix}. \tag{28}$$

By marking: $\beta^{pq} = \alpha^p\alpha^q - \alpha^{pq}$, we bring the equation (23) to the form:

$$\det(a^{pq}) + det(\beta^{pq}) + a_1\beta^{00} + a_0\beta^{11} - 2a\beta^{01} = 0. \tag{29}$$

To prove the statement, let us consider (29) for all $\hat{G}_\alpha$ from the expressions (26) with numbers $\alpha = 1, \dots, 4$.

(a) $\hat{G} = \hat{G}_1$.

Let us fix the variable u^2 to the point $u^2 = \tilde{u}^2$ in the functional Equation (24). As result, we get the expression:

$$det(a^{pq}) = \tilde{\gamma}_{pq}a^{pq} + \tilde{a},$$

which after the shift of the functions $\alpha^{pq} \to \alpha^{pq} + \gamma_{pq}$, and vanishing $\tilde{\gamma}_{pq}$ can be present in the form: $det(a^{pq}) = \tilde{a}$. Since all a^{pq} are linearly independent, from (29) it follows:

$$\beta^{pq} = 0 \to \alpha^{pq} = \alpha^p\alpha^q, \quad \tilde{a} = det(a^{pq}) = 0.$$

From $deta^(pq)0$ we get: $a^{pq} = a^pa^q$. Therefore, the matrix $\hat{G}$ can be represented as (28). The statement is proved.

(b) $\hat{G} = \hat{G}_2$.

The functional equation follows from (29):

$$\det(a^{pq}) + det(\beta^{pq}) + a_0(\beta^{11} - \beta^{00}) - 2a\beta^{01} = 0. \tag{30}$$

Since in a_0 and a are linearly independent, from (29) it follows:

$$a^2 + a_0^2 = \tilde{a}^2, \quad \beta^{pp} = \tilde{a}, \quad \beta^{01} = 0.$$

The elements of the matrix $\hat{G}$ can be represented as:

$$a^{00} = \tilde{a} + a_0, a^{11} = \tilde{a} - a_0, a^{01} = a, \alpha^{pq} = \alpha^p\alpha^q.$$

Thus, $det(a^{pq}) = det(\alpha^{pq}) = 0$, The equality to zero of the coefficients before $(\sigma)^b (b = 0, 1, 2)$ gives the system of equations: and the matrix $\hat{G}_2$ are reduced to the form (27).

(c) $\hat{G} = \hat{G}_3$.

The functional equation follows from (29):

$$a_0 a_1 + \det \beta^{pq} + a_0 \beta^{11} + a\beta^{00} = 0.$$

Since $aa_0 \neq 0$, we get: $\beta^{pq} = \tilde{\beta}^{pq}$, $(a_0 + \tilde{\beta}^{00})(a_1 + \tilde{\beta}^{11}) = \tilde{a}^2$, Therefore, the matrix $\hat{G}$ can be represented as:

$$\hat{G} = \begin{pmatrix} a + \varepsilon \alpha^{0^2} & \tilde{a} + \alpha^0 \alpha^1 \\ \tilde{a} + \alpha^0 \alpha^1 & (\frac{\tilde{a}^2}{a} + \varepsilon \alpha^{0^2}) \end{pmatrix}. \tag{31}$$

and thus, result in the form (27).

(d) $\hat{G} = \hat{G}_4$.

The functional equation follows from (29):

$$a_0 \beta^{11} - 2a\beta^{01} + det\beta^{pq} = \varepsilon a^2.$$

Since $a, a_0 \neq const$ we get: $a_0 = \varepsilon a^2, \alpha^{00} = (\alpha^0)^2, \alpha^{01} = \alpha^0 \alpha^1, \alpha^{11} = 1 + (\alpha^1)^2$, and the matrix $\hat{G}$ can be represented as:

$$\hat{G} = \begin{pmatrix} \varepsilon a^2 + (\alpha^0)^2 & a + \alpha^0 \alpha^1 \\ a + \alpha^0 \alpha^1 & \varepsilon + (\alpha^1)^2 \end{pmatrix}. \tag{32}$$

Thus, it has the form (27).

(e) Now consider the case when $\hat{G} = \hat{G}_5$. The function $a \neq const, \rightarrow \varepsilon = 0$, and the functional equation follows from (29):

$$a(\alpha^{11} - \alpha^{1^2}) = (\alpha^{00} - \alpha^{0^2})(\alpha^{11} - \alpha^{1^2}) - (\alpha^{01} - \alpha^0 \alpha^1)^2.$$

From here: $\alpha^{11} = \alpha^{1^2}, \alpha^{01} = \alpha^0 \alpha^1$, and the matrix $\hat{G}$ takes the form:

$$\hat{G} = \begin{pmatrix} a + \alpha + \alpha^{0^2} & \alpha^0 \alpha^1 \\ \alpha^0 \alpha^1 & \alpha^{1^2} \end{pmatrix}. \tag{33}$$

II $\gamma = 0$.

In case when the matrices $\hat{G}_\rho$ belong to the first three variants ($\rho = 1, \ldots, 3$), Equation (23) has no nonzero solutions and ω^p does not contain any free function. Consider the remaining options.

(a) $\hat{G} = \hat{G}_4$.

From Equation (23) it follows:

$$a_0 \alpha^{1^2} - 2a\alpha^0 \alpha^1 + \alpha^{11} \alpha^{0^2} + \alpha^{00} \alpha^{1^2} - 2\alpha^{01} \alpha^0 \alpha^1 = 0. \tag{34}$$

From here: $\alpha^1 = \alpha^{11} = 0$, and we get the matrix $\hat{G}$ in the form:

$$\hat{G} = \begin{pmatrix} a_0 + \alpha_0 & a + \alpha \\ a + \alpha & 0 \end{pmatrix}. \tag{35}$$

(b) $\hat{G} = \hat{G}_5$.

From Equation (23) it follows:

$$\alpha^{1^2} + \varepsilon\alpha^{0^2} = 0, \quad \alpha^{11}\alpha^{0^2} + \alpha^{00}\alpha^{1^2} - 2\alpha^{01}\alpha^0\alpha^1 = 0. \tag{36}$$

First, let $\varepsilon = -1 \rightarrow \alpha^0 = \alpha^1, \alpha^{11} = -\alpha^{00} + 2\alpha^{01}$, and the matrix $\hat{G}$ takes the form:

$$\hat{G} = \begin{pmatrix} (a_0 + \alpha_0) & \alpha \\ \alpha & -(a_0 + \alpha_0) + \alpha \end{pmatrix}. \tag{37}$$

By replacing the variables: $\hat{u^0} \rightarrow \frac{1}{\sqrt{2}}(u^0 + u^1), \hat{u^1} \rightarrow \frac{1}{\sqrt{2}}(u^0 - u^1)$, we bring the solution to the form:

$$\hat{G} = \begin{pmatrix} \alpha & (a_0 + \alpha_0) \\ (a_0 + \alpha_0) & 0 \end{pmatrix}. \tag{38}$$

Now let $\varepsilon = 0$. It is easy to show that in this case $\alpha^1 = 0$ and $\hat{G}$ has the form:

$$\hat{G} = \begin{pmatrix} a_0 + \alpha_0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad \alpha^1 = 0. \tag{39}$$

Thus, both solutions are special cases of the solution (35).

4.2. Building an Electromagnetic Potential

To complete the classification, it is necessary to establish the dependence of the electromagnetic potential on the variable u^3 for this, it is necessary to consider the remaining Equations (24) and (25).

(a) Matrix $\hat{G}_1$.

We substitute the matrix $\hat{G}_1$ into the equation (24). After some transformation we get the equation:

$$a^0 h^1 - a^1 h^0 = a^0(\gamma\alpha^1 - \sigma^1) - a^1(\gamma\alpha^0 - \sigma^0).$$

This implies:

$$a^0(h^1 - \tilde{c}^1) = a^0(h^0 - \tilde{c}^0), \quad \sigma^p = \gamma\alpha^p.$$

By the admissible gradient transformations of the potential, the values $\tilde{c}^p$ and γ can be set to zero. The solution has the form:

$$A^p = \frac{a^p h + \alpha^p \omega}{\Delta}, \quad A^\nu = 0,$$

$$(g^{ij}) = \begin{pmatrix} \frac{(a^0)^2 + \varepsilon(\alpha^0)^2}{\Delta} & \frac{a^0 a^1 + \varepsilon\alpha^0\alpha^1}{\Delta} & 0 & 0 \\ \frac{a^0 a^1 + \varepsilon\alpha^0\alpha^1}{\Delta} & \frac{(a^1)^2 + \varepsilon(\alpha^1)^2}{\Delta} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon_2}{\Delta} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon_3}{\Delta} \end{pmatrix}. \tag{40}$$

This result was first obtained by Carter [38].

(b) Matrix $\hat{G}_3$.

We substitute the matrix $\hat{G}_3$ into Equation (24). After some transformations in Equation (24) we get: $h^1 + \omega^1 = \gamma(\alpha^{01} + a^{01})$. Hence, $\gamma = \tilde{\gamma}$. By the admissible gradient transformation of the potential $\tilde{\gamma}$ can be set to zero. The final solution is:

$$A^0 = \frac{h+\omega}{\Delta} \quad A^1 = A^\nu = 0, \quad g^{ij} = \begin{pmatrix} \frac{a_0+\alpha_0}{\Delta} & \frac{a+\alpha}{\Delta} & 0 & 0 \\ \frac{a+\alpha}{\Delta} & 0 & 0 & 0 \\ 0 & 0 & \frac{\varepsilon_2}{\Delta} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon_3}{\Delta} \end{pmatrix}. \tag{41}$$

(c) Matrix $\hat{G}_5$.

Substitute it in Equation (24). Denote: $H_p = h^p + \sigma^p$. After the reduction, we get:

$$H_1 = \sigma^1 \rightarrow h^1 = 0.$$

The Equation (25) can be reduced to the form:

$$(H_0 - \alpha^0\sigma)^2 = (\rho + p)A, \quad (A = a + \alpha). \tag{42}$$

Equation (42) has a unique solution: $H_0 = \alpha^0\sigma + \hat{e}A$. By the admissible gradient transformation of the potential we vanish $\tilde{e}$. The solution has the form

$$A^0 = \frac{a^0\omega}{\Delta}, \quad A^1 = A^\nu = 0.$$

5. Quadratic Dependence between Free Functions

To complete the classification, we must consider the variant of a quadratic dependence between the functions ω^p. Using the same technique as before, from Equation (14) we find the relation connecting the functions ω^p. Without limiting the generality, we assume that the free function is ω^0. Then:

$$\omega^1 = \gamma_0 + \gamma_1\omega^0 + \Sigma \quad (\Sigma^2 = \phi_2{\omega^0}^2 + \phi_1\omega^0 + \phi_0). \tag{43}$$

In the relations (43), all functions except the free function ω^0 are expressed in terms of α^{pq}. If Σ^2 is a full square, we get the already considered version of the linear relationship between ω^p. Therefore:

$$(\phi_1)^2 - \phi_0\phi_2 \neq 0. \tag{44}$$

Obviously, under this condition

$$\omega^0, \quad {\omega^0}^2, \quad \omega^0\Sigma, \quad \Sigma \tag{45}$$

are linearly independent functions (with coefficients depending on α^{pq}). These functions are included in the function ω^0 as follows (see (14)):

$$\hat{\omega} = \rho + \tau_2{\omega^0}^2 + 2\tau_1\omega^0 + \tau_0 + 2\xi_1\omega^0\Sigma + 2\xi_0\Sigma. \tag{46}$$

Substituting (46) into the system (14) and equating the coefficients to the independent functions (46) to zero, we obtain the following systems of equations after some transformations:

$$\begin{cases} G^{00}\gamma_1 - G^{01} = \xi_1 G, \\ G^{00}(\phi_2 - {\gamma_1}^2) + G^{11} = (\tau_2 - 2\xi_1\gamma_1)G; \end{cases} \tag{47}$$

$$\begin{cases} G^{00}h^1 - G^{01}h^0 = \xi_0 G - G^{00}\gamma_0, \\ G^{01}h^1 - G^{11}h^0 = (-\tau_1 + \gamma_1\xi_0)G - G^{01}\gamma_0 + G^{00}\phi_1; \end{cases} \tag{48}$$

$$G^{00}((h^1 + \gamma_0)^2 + \phi_0) + G^{11}{h^0}^2 - 2G^{01}h^0(h^1 + \gamma_0) = (\rho + p)G. \tag{49}$$

The Equation (47) are solved in the same way as in the case of linear dependence of the functions ω^p. Using the classification (26), the matrix $\hat{G}$ is found. A distinctive feature is that the completion of this stage does not mean, as before, automatically obtaining a particular case of the desired potential, since the system (48) and Equation (49) non-trivially include the functions defining ω^1. Now, to find the potential, it is necessary to satisfy all the remaining equations. To do this, the found matrices $\hat{G}_a$ are substituted in them and a solution is sought. The second distinguishing feature is that the system (48) immediately implies absence in the potential of a free function that depends only on u^3 since from $\det \hat{G} \neq 0 \rightarrow$ the functions h^p are expressed in terms of a^{pq}. The third feature is that the obtained solutions must satisfy the condition (44).

Omitting the obvious, but rather cumbersome calculations, we present the final solutions for the (47)–(49):

$$\gamma_0 = \gamma_1 = \phi_1 = 0, \quad \phi_0 = -\varepsilon, \quad \phi_2 = \tilde{a}. \tag{50}$$

$$(\omega^1)^2 = \sqrt{\tilde{a} - \varepsilon(\omega^0)^2}. \tag{51}$$

$$\hat{G} = \begin{pmatrix} a & 0 \\ 0 & \varepsilon a \end{pmatrix}, \quad \varepsilon = +1, -1. \tag{52}$$

The metric tensor and electromagnetic potential have the form:

$$\begin{cases} (g^{ij}) = \begin{pmatrix} \frac{a}{\Delta} & 0 & 0 & 0 \\ 0 & \frac{\varepsilon a}{\Delta} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon_2}{\Delta} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon_3}{\Delta} \end{pmatrix}; \\ \mathbf{a})\varepsilon = 1, \quad A^0 = \frac{\tilde{e} sin\sigma}{\Delta}, \quad A^1 = \frac{\tilde{e} cos\sigma}{\Delta}, \quad A^\nu = 0; \\ \mathbf{b})\varepsilon = -1, \quad A^0 = \frac{\tilde{e} sh\sigma}{\Delta}, \quad A^1 = \frac{\tilde{e} ch\sigma}{\Delta}, \quad A^\nu = 0. \end{cases} \tag{53}$$

6. Discussion

The main result of the classification is to obtain all metrics and potentials of the external electromagnetic field that have sufficient symmetry to perform an exact integration of the Hamilton–Jacobi equation for a charged test particle. Currently, many exact solutions of Einstein's equations are known [40]. However, only a small part of them have such property, and they are the main object of research. The additional symmetry of Stäckel spaces, which allows us to separate variables in the Hamilton–Jacobi equation for a charged test particle, can be applied for integrating the vacuum equations of the gravitational field both in General relativity (for example, when studying the problem of axion fields [41]) and in alternative theories. For convenience of use metrics: $(ds)^2 = g_{ij} du^i du^j$, covariant components of electromagnetic potentials: A_i, and separated systems whose solutions determine the complete integrals of Equation (1)

$$\hat{H}_2 = \lambda_3 \phi + \lambda_2, \quad \hat{H}_3 = \lambda_3 f - \lambda_2. \tag{54}$$

are given below.

The separated systems have the form:

$$\hat{H}_2 = \lambda_3 \phi + \lambda_2, \quad \hat{H}_3 = \lambda_3 f - \lambda_2. \tag{55}$$

In conclusion, we would like to give a complete summary of the results. For each solution of Equation (14) we give the metric $(ds)^2 = g_{ij}du^i du^j$, the covariant components of the electromagnetic potential A_p and the function $\hat{H}_\nu$, which define the integrals of motion quadratic in momenta in accordance with:

$$\hat{X}_2 = \frac{fH_2 - \phi H_3}{\Delta} \quad , \tag{56}$$

where

$$\hat{H}_2 = \alpha^{pq} p_p p_q + \varepsilon_2 p_2^2 + 2\omega^q p_q + \hat{\omega}, \quad \hat{H}_3 = h^{pq} p_p p_q + \varepsilon_3 p_3^2 + 2h^q p_q + \hat{h}.$$

Recall that $\quad A_2 = A_3 = 0, \quad \Delta = \phi + f.$

I

$$\begin{cases} (ds)^2 = (\dfrac{(\alpha^{1^2} + \varepsilon a^{1^2})du^{0^2} + (\alpha^{0^2} + \varepsilon a^{0^2})du^{1^2} - 2(\alpha^0\alpha^1 + \varepsilon a^0 a^1)du^0 du^1}{(\alpha^1 a^0 - a^1\alpha^0)^2} + \\ \varepsilon_2 du^{2^2} + \varepsilon_3 du^{3^2})\Delta. \\ A_0 = \dfrac{\varepsilon h\alpha^1 - \omega a^1}{(\alpha^1 a^0 - a^1\alpha^0)}, \\ A_1 = \dfrac{-\varepsilon h\alpha^0 + \omega\alpha^0}{(\alpha^1 a^0 - a^1\alpha^0)}. \\ \hat{H}_2 = \varepsilon_2 {p_2}^2 + \alpha^p\alpha^q p_p p_q + 2\alpha^q p_q\omega + \varepsilon\omega^2 \\ \hat{H}_3 = \varepsilon_3 {p_3}^2 + \varepsilon a^p a^q p_p p_q + 2a^q p_q h + h^2. \end{cases} \tag{57}$$

II

$$\begin{cases} (ds)^2 = (\dfrac{2(\alpha + a)du^0 du^1 - (\alpha_0 + a_0)du^{1^2}}{(\alpha + a)} + \\ \varepsilon_2 du^{2^2} + \varepsilon_3 du^{3^2})\Delta. \\ A_0 = 0, \qquad A_1 = \dfrac{h + \omega}{(\alpha + a)}. \\ \hat{H}_2 = \varepsilon_2 {p_2}^2 + \alpha_0 p_0^2 + 2\alpha p_0 p_1 + 2\omega p_0, \\ \hat{H}_3 = \varepsilon_3 {p_3}^2 + a_0 p_0^2 + 2a p_0 p_1 + 2h p_0 \end{cases} \tag{58}$$

III

$$\begin{cases} (ds)^2 = (\dfrac{(\alpha + a)du^{1^2} + \varepsilon(\alpha^0 du^1 - \alpha^1 du^0)^2}{(\alpha + a)\alpha^{1^2}} + \\ \varepsilon_2 du^{2^2} + \varepsilon_3 du^{3^2})\Delta. \\ A_0 = 0, \qquad A_1 = \dfrac{\omega}{\alpha^1}. \\ \hat{H}_2 = \varepsilon_2 {p_2}^2 + \alpha p_0^2 + (\alpha^q p_q + \omega)^2 du^0)^2 \quad \hat{H}_3 = \varepsilon_3 {p_3}^2 + a p_0^2 \end{cases} \tag{59}$$

IV

$$\begin{cases} (ds)^2 = (\dfrac{\alpha^{11}du^{0^2} + \alpha^{00}du^{1^2} - 2\alpha^{01}du^0du^1}{(\alpha^{00}\alpha^{11} - \alpha^{01^2})} + \\ \varepsilon_2 du^{2^2} + \varepsilon_3 du^{3^2})\Delta. \\ A_0 = \dfrac{\omega^0\alpha^{11} - \omega^1\alpha^{01}}{(\alpha^{00}\alpha^{11} - \alpha^{01^2})}. \\ A_1 = \dfrac{\omega^1\alpha^{00} - \omega^0\alpha^{01}}{(\alpha^{00}\alpha^{11} - \alpha^{01^2})}. \\ \tilde{H}_2 = \varepsilon_2 {p_2}^2 + \alpha^{pq}p_pp_q + 2\omega^p p_p + \\ \frac{\alpha^{00}(\omega^1)^2 + \alpha^{11}(\omega^0)^2 - 2\alpha^{01}\omega^0\omega^1}{(\alpha^{00}\alpha^{11} - \alpha^{01^2})}. \quad \hat{H}_3 = \varepsilon_3 {p_3}^2 \end{cases} \tag{60}$$

V

$$\begin{cases} (ds)^2 = (\dfrac{du^{0^2} + \varepsilon du^{1^2}}{a} + \varepsilon_2 du^{2^2} + \varepsilon_3 du^{3^2})\Delta. \\ \mathbf{a}) \quad \varepsilon = -1, \quad A_0 = \dfrac{sh\omega}{a}, \quad A_1 = \dfrac{ch\omega}{a}. \\ \tilde{H}_2 = \varepsilon_2 {p_2}^2 + 2(p_0 sh\omega - p_1 ch\omega), \quad \tilde{H}_3 = \varepsilon {p_3}^2 + a({p_0}^2 - {p_1}^2) + \dfrac{1}{a^2}, \\ \mathbf{b}) \quad \varepsilon = 1, \quad A_0 = \dfrac{sin\omega}{a}, \quad A_1 = \dfrac{cos\omega}{a}. \\ \tilde{H}_2 = \varepsilon_2 {p_2}^2 + 2(p_0 sin\omega + p_1 cos\omega), \quad \tilde{H}_3 = \varepsilon {p_3}^2 + a({p_0}^2 + {p_1}^2) + \frac{1}{a^2}, \end{cases} \tag{61}$$

7. Conclusions

Thus, all space–time metrics and electromagnetic potentials that allow complete separation of variables of type (2.0) in the Hamilton–Jacobi Equation (1) for a charged test particle moving in an external electromagnetic field are found. The complete sets of mutually commuting vector and Killing tensor fields and the complete sets of motion integrals are defined. Please note that the same problem has been solved for the Stäckel spaces of type (1.0) ([42]) and for the Stäckel spaces of type (2.1) ([43]).

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References

1. Stäckel, P. Uber die intagration der Hamiltonschen differentialechung mittels separation der variablen. *Math. Ann.* **1897**, *49*, 145–147. [CrossRef]
2. Stäckel, P. Ueber Die Bewegung Eines Punktes In Einer N-Fachen Mannigfaltigkeit. *Math. Ann.* **1893**, *42*, 537–563. [CrossRef]
3. Eisenhart, L.P. Separable systems of stackel. *Ann. Math.* **1934**, *35*, 284–305. [CrossRef]
4. Levi-Civita, T. Sulla Integraziome Della Equazione Di Hamilton-Jacobi Per Separazione Di Variabili. *Math. Ann.* **1904**, *59*, 383–397. [CrossRef]
5. Jarov-Jrovoy, M.S. Integration of Hamilton-Jacobi equation by complete separation of variables method. *J. Appl. Math. Mech.* **1963**, *27*, 173–219.

6. Shapovalov, V.N. Symmetry of motion equations of free particle in riemannian space. *Sov. Phys. J.* **1975**, *18*, 1650–1654. [CrossRef]
7. Shapovalov, V.N. Symmetry and separation of variables in the Hamilton-Jacobi equation. *Sov. Phys. J.* **1978**, *21*, 1124–1132. [CrossRef]
8. Shapovalov, V.N. Stackel's spaces. *Sib. Math. J.* **1979**, *20*, 1117–1130. [CrossRef]
9. Bagrov, V.G.; Obukhov, V.V. Complete separation of variables in the free Hamilton-Jacobi equation. *Theor. Math. Phys.* **1993**, *97*, 1275–1289. [CrossRef]
10. Shapovalov, A.V.; Shirokov, I.V. Noncommutative Integration Method For Linear Partial Differential Equations. Functional Algebras And Dimensional Reduction. *Theor. Math. Phys.* **1996**, *106*, 1–10. [CrossRef]
11. Shapovalov, V.N.; Eckle, G.G. Separation of Variables in the Dirac Equation. *Sov. Phys. J.* **1973**, *16*, 818–823. [CrossRef]
12. Shapovalov, V.N. Symmetry and separation of variables in a linear second-order differential equation. I, II. *Sov. Phys. J.* **1978**, *21*, 645–650, 693–695.
13. Bagrov, V.G.; Meshkov, A.G.; Shapovalov, V.N.; Shapovalov, A.V. Separation of variables in the Klein-Gordon equations I. *Sov. Phys. J.* **1973**, *16*, 1533–1538. [CrossRef]
14. Bagrov, V.G.; Meshkov, A.G.; Shapovalov, V.N.; Shapovalov, A.V. Separation of variables in the Klein-Gordon equations II. *Sov. Phys. J.* **1973**, *16*, 1659–1665. [CrossRef]
15. Bagrov, V.G.; Meshkov, A.G.; Shapovalov, V.N.; Shapovalov, A.V. Separation of variables in the Klein-Gordon equations III. *Sov. Phys. J.* **1974**, *17*, 812–815. [CrossRef]
16. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. *Phys. Rep.* **2017**, *692*. [CrossRef]
17. Bamba, K.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. Dark energy cosmology: The equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **2012**, *342*, 155. [CrossRef]
18. Capozziello, S.; De Laurentis, M.; Odintsov, D. Hamiltonian dynamics and Noether symmetries in extended gravity cosmology. *Eur. Phys. J. C* **2012**, *72*, 2068. [CrossRef]
19. Benenti, S. Separability In Riemannian Manifolds, Symmetry, Integrability and geometry: Methods and applications. *SIGMA* **2016**, *12*, 13. [CrossRef]
20. Miller, W. *Symmetry and Separation of Variables*; Cambridge University Press: Cambridge, UK, 1984; 318p.
21. Frolov, V.P.; Krtouš, P.; Kubizňák, D. Separation of variables in maxwell equations in Plebanski-Demianski spacetime. *Phys. Rev. D* **2018**, *97*, 101701. [CrossRef]
22. Chong, Z.W.; Gibbons, G,W.; Pope, C.N. Separability and Killing tensors in Kerr-Taub-Nut-De Sitter metrics in higher dimensions. *Phys. Lett. B* **2005**, *609*, 124–132. [CrossRef]
23. Vasudevan, M.; Stevens, K.A.; Page, D.N. Separability Of The Hamilton-Jacobi And Klein-Gordon Equations In Kerr-De Sitter Metrics. *Class. Quant. Grav.* **2005**, *22*, 339–352. [CrossRef]
24. Miller, W., Jr.; Post, S.; Winternitz, P. Classical Aad Qqantum superintegrability with applications. *J. Phys. A Math. Theor.* **2013**, *46*, 423001. [CrossRef]
25. Osetrin, E.K.; Osetrin, K.E.; Filippov, A.E. Stationary homogeneous models of Stackel spaces of type (3.1). *Russ. Phys. J.* **2020**, *63*, 51–56.
26. Maharaj, S.D.; Goswami, R.; Chervon, S.V.; Nikolaev, A.V. Exact solutions for scalar field cosmology in f(R) gravity. *Mod. Phys. Lett. A* **2017**, *32*, 1750164. [CrossRef]
27. Fomin, I.V.; Chervon, S.V. Exact and approximate solutions in the Friedmann cosmology. *Russ. Phys. J.* **2017**, *60*, 427–440. [CrossRef]
28. Osetrin, E.K.; Osetrin, K.E.; Filippov, A.E. Stationary homogeneous models of Stackel spaces of type (2.1). *Russ. Phys. J.* **2020**, *63*, 57–65.
29. Osetrin, K.E.; Filippov, A.E.; Osetrin, E.R. The spacetime models with dust matter that admit separation of variables in Hamilton-Jacobi equations of a test particle. *Mod. Phys. Lett. A* **2016**, *31*, 1650027. [CrossRef]
30. Breev, A.I.; Shapovalov, A.V. The Dirac equation in an external electromagnetic field: Symmetry algebra and exact integration. *J. Phys. Conf. Ser.* **2016**, *21*, 012015. [CrossRef]
31. Ivata, G. Empty spaces of Stackel. *Nat. Sci. Rep. Ochonomisu Univ.* **1969**, *9*, 79–93.
32. Obukhov, V.V. Classes of exact solutions of Einstein equations. *Sov. Phys. J.* **1977**, *2*, 73–77. [CrossRef]
33. Obukhov, V.V. Some classes of exact solutions of Einstein equations. *Sov. Phys. J.* **1977**, *5*, 148–150. [CrossRef]
34. Obukhov, V.V. Classes of exact solutions of Einstein equations II. *Sov. Phys. J.* **1978**, *5*, 56–59. [CrossRef]

35. Boyer, C.P.; Kalnins, E.G.; Miller, W. Separation of variables in Einstein spaces. I. Two ignorable and one null coordinate. *J. Phys. Math. Gen.* **1981**, *14*, 1675–1684. [CrossRef]
36. Bagrov, V.G.; Obukhov, V.V.; Shapovalov, A.V. Special Stackel electrovac spacetimes. *Pramana J. Phys.* **1986**, *26*, 93–108. [CrossRef]
37. Bagrov, V.G.; Obukhov, V.V. Classes of exact solutions of the Einstein-Maxwell equations. *Ann. Phys.* **1983**, *40*, 181–188. [CrossRef]
38. Carter, B. New family of Einstein spaces. *Phys. Lett A* **1968**, *25*, 399–400. [CrossRef]
39. Benenti, S.; Chanu, C.; Rastelli, G. Variables Separation for Natural Hamiltonians with Scalar and Vector Potentials on Riemannian Manifolds. *J. Math. Phys.* **2001**, *42*, 2065–2091. [CrossRef]
40. Stephani, H.; Kramer, D.; Mac Callum, M.; Hoenselaers, C.; Herlt, E. *Exact Solutions of Einstein's Field Equations*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2003; 732p.
41. Balakin, A.B. The Extended Einstein-Maxwell-Aether-Axion Model: exact solutions for axionically controlled pp-Wave aether modes. *Mod. Phys. Lett. A* **2018**, *33*, 1850050. [CrossRef]
42. Obukhov, V.V. Integration of Hamilton-Jacobi and Maxwell equations in diagonal metrics of Stackel spaces. *Sov. Phys. J.* **2020**, *7*, 33–35.
43. Obukhov, V.V. Hamilton-Jacobi equation for a charged test particle in the Stackel space of type (2.1). **2020**, in press.