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Fuzzy Mixed Variational-like and Integral Inequalities for Strongly Preinvex Fuzzy Mappings

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Abstract: It is a familiar fact that convex and non-convex fuzzy mappings play a critical role in the study of fuzzy optimization. Due to the behavior of its definition, the idea of convexity plays a significant role in the subject of inequalities. The concepts of convexity and symmetry have a tight connection. We may use whatever we learn from one to the other, thanks to the significant correlation that has developed between both in recent years. Our aim is to consider a new class of fuzzy mappings (FMs) known as strongly preinvex fuzzy mappings (strongly preinvex-FMs) on the invex set. These FMs are more general than convex fuzzy mappings (convex-FMs) and preinvex fuzzy mappings (preinvex-FMs), and when generalized differentiable (briefly, G-differentiable), strongly preinvex-FMs are strongly invex fuzzy mappings (strongly invex-FMs). Some new relationships among various concepts of strongly preinvex-FMs are established and verified with the support of some useful examples. We have also shown that optimality conditions of G-differentiable strongly preinvex-FMs and the fuzzy functional, which is the sum of G-differentiable preinvex-FMs and non G-differentiable strongly preinvex-FMs, can be distinguished by strongly fuzzy variational-like inequalities and strongly fuzzy mixed variational-like inequalities, respectively. In the end, we have established and verified a strong relationship between the Hermite–Hadamard inequality and strongly preinvex-FM. Several exceptional cases are also discussed. These inequalities are a very interesting outcome of our main results and appear to be new ones. The results in this research can be seen as refinements and improvements to previously published findings.

Keywords: preinvex fuzzy mappings; strongly preinvex fuzzy mappings; strongly invex fuzzy mappings; strongly fuzzy monotonicity; strongly fuzzy mixed variational-like inequalities

1. Introduction

Recently, many generalizations and extensions have been studied for classical convexity. Polyak [1] introduced and studied the idea of strongly convex functions on the convex set, which have a significant impact on optimization theory and related fields. Karmardian [2] discussed how strongly convex functions can be used to solve nonlinear complementarity problems for the first time. Qu and Li [3] and Nikodem and Pales [4]

developed the convergence analysis for addressing equilibrium issues and variational inequalities, using strongly convex functions. For further study, we refer the reader to applications and properties of the strongly convex functions of [5–10], and the references therein. For differentiable functions, invex functions were introduced by Hanson [11], which played a significant role in mathematical programing. The concept of invex sets and preinvex functions were introduced and studied by Israel and Mond [12]. It is well known that differential preinvex function are invex functions. The converse also holds under Condition C [13]. Furthermore, Noor [14], studied the optimality conditions of differentiable preinex functions and proved that the minimum can be characterized by variational-like inequalities. Noor et al. [15,16] studied the properties of the strongly preinvex function and investigated its applications. For more applications and properties of strongly preinvex functions, see [17–19] and the references therein.

In [20], a large amount of research work on fuzzy sets and systems was devoted to the advancement of various fields, playing an important role in the analysis of broad class problems emerging in pure and applied sciences, such as operation research, computer science, decision sciences, control engineering, artificial intelligence, and management sciences. Convex analysis has made significant contributions to the improvement of several practical and pure science domains. In the same way, fuzzy convex analysis is a fundamental principle in fuzzy optimization and it is worthwhile to explore some basic principles of convex sets in fuzzy set theory. Many scholars have addressed fuzzy convex sets. Liu [21] investigated some properties of convex fuzzy sets and updated the definition of shadow of fuzzy sets with the support of useful examples. Lowen [22] gathered some well-known convex sets' results and proved the separation theorem for convex fuzzy sets. Ammar and Metz [23,24] investigated forms of convexity and established the generalized convexity of fuzzy sets. Furthermore, they used the principle of convexity to formulate a general fuzzy nonlinear programming problem.

A fuzzy number is a generalized version of an interval that can be discussed (in crisp set theory). Zadeh [20] defined fuzzy numbers, while Dubois and Prade [25] built on Zadeh's work by adding new fuzzy number conditions. Furthermore, Goetschel and Voxman [26] adjusted many conditions on fuzzy numbers to make them easier to handle. For example, in [25], one of the conditions for a fuzzy number is that it is a continuous function, whereas in [26], the fuzzy number is upper semi-continuous. The purpose is to establish a metric for a collection of fuzzy numbers, using the relaxation of requirements on fuzzy numbers, and then use this metric to examine some basic features of topological space. Nanda and Kar [27], Syau [28] and Furukawa [29] introduced the concept of convex-FMs from $\mathbb{R}^n$ to the set of fuzzy numbers. Furthermore, they also defined different type of convex-FMs, such as logarithmic convex-FMs and quasi-convex-FMs, as well studying Lipschitz continuity of fuzzy valued mappings. Yan and Xu [30] provided the notions of epigraphs and the convexity of FMs, as well as the characteristics of convex-FMs and quasi-convex-FMs, based on Goetschel and Voxman's concept of ordering [31]. The concept of fuzzy preinvex mapping on the invex set was introduced and studied by Noor [32]. He also demonstrated that variational inequalities may be used to specify the fuzzy optimality conditions of differentiable fuzzy preinex mappings. Syau [33], introduced notions of (ϕ_1, ϕ_2)−convexity, ϕ_1-B-vexity and ϕ_1-convexity-FMs through the so-called fuzzy max order among the fuzzy numbers, and proved that the ϕ_1-B-vexity and ϕ_1-convexity, B-vexity, convexity and preinvexity of FMs are the subclasses. Syau and Lee [34] examined various aspects of fuzzy optimization and discussed continuity and convexity through linear ordering and metrics defined on fuzzy integers. They also extended the Weirstrass theorem from real-valued functions to FMs. For recent applications, see [35–39] and the references therein.

On the other hand, integral inequalities have various applications in linear programing, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics, and the theory of relativity; see [40,41] and the references therein. The HH-inequality is a familiar, supreme and broadly useful inequality. This inequality has

fundamental significance [42,43], due to other classical inequalities, such as the Olsen, Gagliardo–Nirenberg, Hardy, Opial, Young, Linger, arithmetic–geometric, Ostrowski, Levinson, Minkowski, Beckenbach–Dresher, Ky Fan and Holer inequalities [44–49], which are closely linked to the classical HH-inequality. It can be stated as follows:

Let $\mathcal{H} : K \to \mathbb{R}$ be a convex function on a convex set K and , $\nu \in K$ with $u \leq \nu$. Then,

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u} \int_u^\nu \mathcal{H}(z)dz \leq \frac{\mathcal{H}(u)+\mathcal{H}(\nu)}{2}. \tag{1}$$

If $\mathcal{H}$ is a concave function, then inequality (1) is reversed.

There are several integrals that deal with FMs and have FMs as integrands. For FMs, Oseuna-Gomez et al. [50] and Costa et al. [51] constructed Jensen's integral inequality. Costa and Floures [52] used the same method to present Minkowski and Beckenbach's inequalities, where the integrands are fuzzy mappings. Costa et al. established a relationship between elements of fuzzy-interval space and interval space and introduced level-wise fuzzy order relation on fuzzy-interval space through Kulisch–Miranker order relation defined on an interval space. This was motivated by [48–53] and particularly [54], because Costa et al. established a relationship between elements of fuzzy-interval space and interval space and introduced a level-wise fuzzy order relation on fuzzy-interval space through the Kulisch–Miranker order relation defined on interval space. By using this relation on the fuzzy-interval space, we generalize integral inequality (1) by constructing fuzzy integral inequalities for strongly preinvex-FMs, where the integrands are strongly preinvex-FMs. Recently, Khan et al. [55] introduced the new class of convex-FMs, which is known as (h_1, h_2)-convex-FMs by means of the fuzzy order relation and presented the following new version of HH-type inequality for (h_1, h_2)-convex-FM involving fuzzy-interval Riemann integrals:

Theorem 1. *Let $\mathcal{H} : [u, \nu] \to \mathbb{F}_0$ be a (h_1, h_2) -convex-FM with $h_1, h_2 : [0,1] \to \mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$. If $\mathcal{H}$ is fuzzy Riemann integrable (in sort, FR -integrable), then the following holds:*

$$\begin{aligned} \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \mathcal{H}\left(\frac{u+\nu}{2}\right) &\preccurlyeq \frac{1}{\nu-u} \int_u^\nu \mathcal{H}(z)dz \\ &\preccurlyeq \left[\mathcal{H}(u) \widetilde{+} \mathcal{H}(\nu)\right] \int_0^1 h_1(\tau)h_2(1-\tau)d\tau. \end{aligned} \tag{2}$$

Theorem 1 reduces to the result for convex fuzzy-IVF:

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u} \int_u^\nu \mathcal{H}(z)dz \preccurlyeq \frac{H(u) \widetilde{+} H(\nu)}{2}. \tag{3}$$

For further information related to fuzzy integral inequalities, see [56–68].

Motivated by ongoing studies as well as the relevance of the concepts of invexity and preinvexity of FMs, in Section 2, we provide an overview of some fundamental concepts, preliminary notations, and findings that will be useful in further research. In the parts that follow, the key results are considered and discussed. Section 3 introduces the concepts of strongly preinvex-FMs and discusses some of their properties. Moreover, new relationships among various concepts of strongly preinvex-FMs are also investigated in Section 3. In Section 4, we introduce fuzzy variational-like and Hermite–Hadamard inequalities for strong preinvex-FMs.

2. Preliminaries

In this section, we first provide some definitions, preliminary notations and results, which will be helpful for further study.

A fuzzy set of $\mathbb{R}$ is a mapping $\Psi : \mathbb{R} \to [0, 1]$, for each fuzzy set and $\gamma \in (0, 1]$; then, γ-level sets of Ψ are denoted and defined as follows: $\Psi_\gamma = \{u \in \mathbb{R}|\ \Psi(u) \geq \gamma\}$. The support

of Ψ is denoted by supp(Ψ) and is defined as supp(Ψ) $= \{u \in \mathbb{R}|\ \Psi(u) \geq \gamma\}$. A fuzzy set is normal if there exist $u \in \mathbb{R}$ such that $\Psi(u) = 1$. A fuzzy set is convex and concave if $\Psi((1-\tau)u + \tau v) \geq min(\Psi(u), \Psi(v))$ and $\Psi((1-\tau)u + \tau v) \leq max(\Psi(u), \Psi(v))$ for $u, v \in \mathbb{R}$, $\tau \in [0,1]$, respectively. A fuzzy convex set is a generalization of the classical convex set.

A fuzzy set is said to be fuzzy number with the following properties.

(a) Ψ is normal. (b) Ψ is a convex fuzzy set. (c) Ψ is upper semi-continuous. (d) Ψ_0 is compact.

$\mathbb{F}_0$ denotes the set of all fuzzy numbers. For a fuzzy number, it is convenient to distinguish the following γ-levels:

$$\Psi_\gamma = \{u \in \mathbb{R}|\ \Psi(u) \geq \gamma\},$$

From these definitions, we have the following:

$$\Psi_\gamma = [\Psi_*(\gamma),\ \Psi^*(\gamma)]$$

where

$$\Psi_*(\gamma) = inf\{u \in \mathbb{R}|\ \Psi(u) \geq \gamma\}, \Psi^*(\gamma) = sup\{u \in \mathbb{R}|\ \Psi(u) \geq \gamma\}.$$

Since each $r \in \mathbb{R}$ is also a fuzzy number, it is defined as follows:

$$\widetilde{r}(u) = \begin{cases} 1 \text{ if } u = r \\ 0 \text{ if } u \neq r \end{cases}.$$

It is also well known that for any $\Psi, \phi \in \mathbb{F}_0$ and $r \in \mathbb{R}$, the following holds:

$$\Psi \widetilde{+} \phi = \{(\Psi_*(\gamma) + \phi_*(\gamma), \Psi^*(\gamma) + \phi^*(\gamma), \gamma) : \gamma \in [0,\ 1]\}, \tag{4}$$

$$r\Psi = \{(r\Psi_*(\gamma), r\Psi^*(\gamma), \gamma) : \gamma \in [0,\ 1]\}. \tag{5}$$

Obviously, $\mathbb{F}_0$ is closed under addition and nonnegative scaler multiplication. Furthermore, for each scaler number $r \in \mathbb{R}$, the following holds:

$$\Psi \widetilde{+} r = \{(\Psi_*(\gamma) + r, \Psi^*(\gamma) + r, \gamma) : \gamma \in [0,\ 1]\}. \tag{6}$$

For any $\Psi, \phi \in \mathbb{F}_0$, we say that $\Psi \preccurlyeq \phi$ ("$\preccurlyeq$" relation between fuzzy numbers Ψ and ϕ if for all $\gamma \in (0,\ 1]$, $\Psi^*(\gamma) \leq \phi^*(\gamma)$ ("$\leq$" relation $\Psi^*(\gamma)$ and $\phi^*(\gamma)$) and $\Psi_*(\gamma) \leq \phi_*(\gamma)$. We say it is comparable if for any $\Psi, \phi \in \mathbb{F}_0$, we have $\Psi \preccurlyeq \phi$ or $\Psi \succcurlyeq \phi$; otherwise, they are non-comparable.

We can state that $\mathbb{F}_0$ is a partial ordered set under the relation $\preccurlyeq$ if we write $\Psi \preccurlyeq \phi$ instead of $\phi \succcurlyeq \Psi$. If $\Psi, \phi \in \mathbb{F}_0$, there exist $\omega \in \mathbb{F}_0$ such that $\Psi = \phi \widetilde{+} \omega$; then, we have the existence of the Hukuhara difference (in short, H-difference) of Ψ and ϕ, and we say that ω is the H-difference of Ψ and ϕ, denoted by $\Psi \widetilde{-} \phi$; see [37]. If this fuzzy operation exists, then we have the following:

$$(\omega)^*(\gamma) = (\Psi \widetilde{-} \phi)^*(\gamma) = \Psi^*(\gamma) - \phi^*(\gamma), (\omega)_*(\gamma) = (\Psi \widetilde{-} \phi)_*(\gamma) = \Psi_*(\gamma) - \phi_*(\gamma).$$

A mapping $\mathcal{H} : K \to \mathbb{F}_0$ is called fuzzy mapping (FM). For each $\gamma \in [0,\ 1]$, denote $[\mathcal{H}(u)]^\gamma = [\mathcal{H}_*(u, \gamma),\ \mathcal{H}^*(u, \gamma)]$ and in parameterized form, denote $\mathcal{H}(u) = \{(\mathcal{H}_*(u,\ \gamma), \mathcal{H}^*(u,\ \gamma), \gamma) : \gamma \in [0,\ 1]\}$.

Definition 1. *Let us say $I = (m,\ n)$ and $\sqcap \in (m,\ n)$ [35]. Then, FM $\mathcal{H} : (m,\ n) \to \mathbb{F}_0$ is said to be a generalized differentiable (briefly, G-differentiable) at $\sqcap$ if there exists an element $\mathcal{H}'(u) \in \mathbb{F}_0$*

such that for any $0 < \tau$, sufficiently small, there exist $\mathcal{H}(u+\tau)\widetilde{-}\mathcal{H}(u)$, $\mathcal{H}(u)\widetilde{-}\mathcal{H}(u-\tau)$, and the limits are (in the metric D) as follows:

$$\lim_{\tau\to 0^+}\frac{\mathcal{H}(u+\tau)\widetilde{-}\mathcal{H}(u)}{\tau} = \lim_{\tau\to 0^+}\frac{\mathcal{H}(u)\widetilde{-}\mathcal{H}(u-\tau)}{\tau} = \mathcal{H}'(u)$$

$$or \lim_{\tau\to 0^+}\frac{\mathcal{H}(u)\widetilde{-}\mathcal{H}(u+\tau)}{-\tau} = \lim_{\tau\to 0^+}\frac{\mathcal{H}(u-\tau)\widetilde{-}\mathcal{H}(u)}{-\tau} = \mathcal{H}'(u)$$

$$or \lim_{\tau\to 0^+}\frac{\mathcal{H}(u+\tau)\widetilde{-}\mathcal{H}(u)}{\tau} = \lim_{\tau\to 0^+}\frac{\mathcal{H}(u-\tau)\widetilde{-}\mathcal{H}(u)}{-\tau} = \mathcal{H}'(u)$$

$$or \lim_{\tau\to 0^+}\frac{\mathcal{H}(u)\widetilde{-}\mathcal{H}(u+\tau)}{-\tau} = \lim_{\tau\to 0^+}\frac{\mathcal{H}(u)\widetilde{-}\mathcal{H}(u-\tau)}{\tau} = \mathcal{H}'(u),$$

where the limits are taken in the metric space (E, D), for Ψ, $\phi \in \mathbb{F}_0$ as follows:

$$D(\Psi, \phi) = \sup_{0\le\gamma\le 1} H(\Psi_\gamma, \phi_\gamma),$$

and H denotes the well-known Hausdorff metric on the space of intervals.

Definition 2. *A FM $\mathcal{H} : K \to \mathbb{F}_0$ is said to be convex on the convex set K if the following holds [27]:*

$$\mathcal{H}((1-\tau)u + \tau v) \preccurlyeq (1-\tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(v), \forall u, v \in K, \tau \in [0, 1]. \tag{7}$$

Similarly, $\mathcal{H}$ is said to be concave-FM on K if inequality (7) is reversed.

Definition 3. *The set K_ξ in $\mathbb{R}$ is said to be invex set with respect to (w.r.t.) arbitrary bifunction $\xi(.,.)$, if the following holds [12]:*

$$u + \tau\xi(v, u) \in K_\xi, \forall u, v \in K_\xi, \tau \in [0, 1].$$

The invex set K_ξ is also known as a ξ-connected set. Note that each convex set with $v - u = \xi(v, u)$ is an invex set in the classical sense, but the reverse is not true. For instance, the following set $K_\xi = [-7, -2] \cup [2, 10]$ is an invex set w.r.t. non-trivial bi-function $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given as follows:

$$\xi(v, u) = v - u, v \ge 0, u \ge 0,$$
$$\xi(v, u) = v - u, 0 \ge v, 0 \ge u,$$
$$\xi(v, u) = -7 - u, v \ge 0 \ge u,$$
$$\xi(v, u) = 2 - u, u \ge 0 \ge v.$$

Definition 4. *A FM $\mathcal{H} : K_\xi \to \mathbb{F}_0$ is said to be preinvex on the invex set K_ξ w.r.t. bi-function ξ if the following holds [32]:*

$$\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq (1-\tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(v), \tag{8}$$

for all $u, v \in K_\xi$, $\tau \in [0, 1]$, where $\xi : K_\xi \times K_\xi \to \mathbb{R}$. $\mathcal{H}$ is said to be preconcave-FM on K_ξ if inequality (8) is reversed.

Lemma 1. *Let K_ξ be an invex set w.r.t. ξ and let $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be a FM, parameterized by the following [21]:*

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u, \gamma), \mathcal{H}^*(u, \gamma), \gamma) : \gamma \in [0, 1]\}, \forall u \in K_\xi$$

Then, $\mathcal{H}$ is preinvex on K_ξ if, and only if, for all $\gamma \in [0, 1]$,
$\mathcal{H}_*(u, \gamma)$ and $\mathcal{H}^*(u, \gamma)$ are preinvex w.r.t. ξ on K_ξ.
If $\xi(v, u) = v - u$, then Lemma 1 reduces to the following result:

"*Let K_ξ be a convex set and let $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be a FM parameterized by the following:*

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u, \gamma), \mathcal{H}^*(u, \gamma), \gamma) : \gamma \in [0, 1]\}, \; \forall \, u \in K_\xi$$

Then, $\mathcal{H}$ is convex on K_ξ if, and only if, for all $\gamma \in [0, 1]$, $\mathcal{H}_*(u, \gamma)$ and $\mathcal{H}^*(u, \gamma)$ are convex w.r.t. ξ on K_ξ.

Theorem 2. *If $\mathcal{H} : [c, d] \subset \mathbb{R} \to \mathcal{K}_C$ is an interval valued function on : $[c, d]$ such that $[\mathcal{H}_*, \mathcal{H}^*]$[54]. Then $\mathcal{H}$ is Riemann integrable over : $[c, d]$ if and only if, $\mathcal{H}_*$ and $\mathcal{H}^*$ both are Riemann integrable over: $[c, d]$ such that the following holds:*

$$(IR) \int_c^d \mathcal{H}(z)dz = \left[(R) \int_c^d \mathcal{H}_*(u)dz, \; (R) \int_c^d \mathcal{H}^*(u)dz \right] \tag{9}$$

From the above literature review, the following results can be concluded; see [31,32,53,54].

Definition 5. *Let $\mathcal{H} : [c, d] \subset \mathbb{R} \to \mathbb{F}_0$ be a FM [47]. The fuzzy Riemann integral of $\mathcal{H}$ over $[c, d]$, denoted by $(FR) \int_c^d \mathcal{H}(z)dz$, is defined by the following:*

$$\left[(FR) \int_c^d \mathcal{H}(z)dz \right]^\gamma = (IR) \int_c^d \mathcal{H}_\gamma(z)dz = \left\{ \int_c^d \mathcal{H}(z, \gamma)dz : \mathcal{H}(z, \gamma) \in \mathcal{R}_{[c, d]} \right\}, \tag{10}$$

for all $\gamma \in [0, 1]$, where $\mathcal{R}_{[c, d]}$ is the collection of end-point functions of IVFs. $\mathcal{H}$ is (FR)-integrable over $[c, d]$ if $(FR) \int_c^d \mathcal{H}(z)dz \in \mathbb{F}_0$. Note that, if both end-point functions are Lebesgue-integrable, then $\mathcal{H}$ is fuzzy Aumann-integrable.

Let K_ξ be a nonempty invex set in $\mathbb{R}$ for future investigation. Let $\xi : K_\xi \times K_\xi \to \mathbb{R}$ be an arbitrary bifunction and $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be an FM. We denote $\|.\|$ and $\langle ., . \rangle$ as the norm and inner product, respectively. Furthermore, throughout this article, FMs are discussed through the so-called "fuzzy-max" order among fuzzy numbers. As is well known, the fuzzy-max order is a partial order relation " $\preccurlyeq$ " on the set of fuzzy numbers.

3. Strongly Preinvex Fuzzy Mappings

In this section, we propose and study the class of strongly preinvex-FMs. We also establish the relationship between strongly preinvex-FMs, strongly monotone operators and strongly invex-FMs. Firstly, we define the following notion of strongly preinvex-FM.

Definition 6. *Let K_ξ be an invex set and ω be a positive number. Then, FM $\mathcal{H} : K_\xi \to \mathbb{F}_0$ is said to be strongly preinvex-FM on K_ξ w.r.t. bi-function $\xi(.,.)$ if the following holds:*

$$\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq (1 - \tau)\mathcal{H}(u) \widetilde{+} \tau\mathcal{H}(v) \widetilde{-} \omega\tau(1 - \tau)\|\xi(v, u)\|^2, \tag{11}$$

for all $u, v \in K_\xi$, $\tau \in [0, 1]$. $\mathcal{H}$ is said to be strongly preconcave-FM on K_ξ if inequality (11) is reversed. $\mathcal{H}$ is said to be strongly affine preinvex-FM on K_ξ if the following holds:

$$\mathcal{H}(u + \tau\xi(v, u)) = (1 - \tau)\mathcal{H}(u) \widetilde{+} \tau\mathcal{H}(v) \widetilde{-} \omega\tau(1 - \tau)\|\xi(v, u)\|^2, \tag{12}$$

for all $u, v \in K_\xi$, $\tau \in [0, 1]$.

Remark 1. *Strongly preinvex-FMs, such as preinvex-FMs, have the following highly desirable features:*

(1) $Y\mathcal{H}$ *is also strongly preinvex for $Y \geq 0$, if $\mathcal{H}$ is strongly preinvex-FM.*

(2) $max(\mathcal{H}(u), \omega(u))$ *is also strongly preinvex-FM if $\mathcal{H}$ and ω both are strongly preinvex-FMs.*

Now, we discuss some special cases of strongly preinvex-FMs:

If $\xi(v,u) = v - u$, then strongly preinvex-FM becomes strongly convex-FM, that is

$$\mathcal{H}((1-\tau)u + \tau v) \preccurlyeq (1-\tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(v)\widetilde{-}\omega\tau(1-\tau)\|v-u\|^2, \,\forall\, u,\, v \in K_\xi,\, \tau \in [0,\, 1].$$

If $\omega = 0$, then inequality (11) reduces to inequality (8).
If $\omega = 0$ and $\xi(v,u) = v - u$, then inequality (11) reduces to inequality (7).
The following result characterizes the definition of strongly preinvex-FMs and establishes the relationship between strongly preinvex-FMs and end-point functions. With the help of this theorem, we can easily handle the upcoming results.

Theorem 3. *Let* $\mathcal{H} : K_\xi \to \mathbb{F}_0$ *be a FM parametrized by the following:*

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u,\,\gamma), \mathcal{H}^*(u,\,\gamma), \gamma) : \gamma \in [0,\,1]\}, \,\forall\, u \in K_\xi. \tag{13}$$

Then, $\mathcal{H}$ *is strongly preinvex on* K *w.r.t.* ξ, *with modulus* ω *if and only if, for all* $\gamma \in [0,\,1]$,
$\mathcal{H}_*(u,\,\gamma)$ *and* $\mathcal{H}^*(u,\,\gamma)$ *are strongly preinvex w.r.t.* ξ *and modulus* ω.

Proof. Assume that for each $\gamma \in [0,\,1]$, $\mathcal{H}_*(u,\,\gamma)$ and $\mathcal{H}^*(u,\,\gamma)$ are strongly preinvex w.r.t. ξ and modulus ω on K_ξ. Then, from (11), for all $u,v \in K_\xi,\, \tau \in [0,\,1]$, we have the following:

$$\mathcal{H}_*(u + \tau\xi(v,u),\,\gamma) \leq (1-\tau)\mathcal{H}_*(u,\,\gamma) + \tau\mathcal{H}_*(v,\,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2$$

and

$$\mathcal{H}^*(u + \tau\xi(v,u),\,\gamma) \leq (1-\tau)\mathcal{H}^*(u,\,\gamma) + \tau\mathcal{H}^*(v,\,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2.$$

Then, by (13), (4), (5) and (6), we obtain the following:
$$\mathcal{H}(u + \tau\xi(v,u)) = \{(\mathcal{H}_*(u + \tau\xi(v,u),\,\gamma), \mathcal{H}^*(u + \tau\xi(v,u),\,\gamma), \gamma) : \gamma \in [0,\,1]\},$$
$$\preccurlyeq \{((1-\tau)\mathcal{H}_*(u,\,\gamma), (1-\tau)\mathcal{H}^*(u,\,\gamma), \gamma) : \gamma \in [0,\,1]\}\widetilde{+}\{(\tau\mathcal{H}_*(v,\,\gamma), \tau\mathcal{H}^*(v,\,\gamma), \gamma) : \gamma \in [0,\,1]\}$$
$$\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2,$$
$$= (1-\tau)\mathcal{H}(u)\widetilde{+}\mathcal{H}(v)\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2.$$

Hence, $\mathcal{H}$ is strongly preinvex-FM on K_ξ with modulus ω. $\square$

Conversely, let $\mathcal{H}$ be a strongly preinvex-FM on K_ξ with modulus ω. Then, for all $u,v \in K_\xi$ and $\tau \in [0,\,1]$, we have $\mathcal{H}(u + \tau\xi(v,u)) \preccurlyeq (1-\tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(v)\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2$. From (13), we have the following:

$$\mathcal{H}(u + \tau\xi(v,u)) = \{(\mathcal{H}_*(u + \tau\xi(v,u),\,\gamma), \mathcal{H}^*(u + \tau\xi(v,u),\,\gamma), \gamma) : \gamma \in [0,\,1]\}.$$

Again, from (13), (4), (5) and (6), we obtain the following:

$$(1-\tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(u)\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2$$
$$= \{((1-\tau)\mathcal{H}_*(u,\,\gamma), (1-\tau)\mathcal{H}^*(u,\,\gamma), \gamma) : \gamma \in [0,\,1]\}$$
$$\widetilde{+}\{(\tau\mathcal{H}_*(v,\,\gamma), \tau\mathcal{H}^*(v,\,\gamma), \gamma) : \gamma \in [0,\,1]\}\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2,$$

for all $u,v \in K_\xi$ and $\tau \in [0,\,1]$. Then, by strongly preinvexity of $\mathcal{H}$, we have for all $u,v \in K_\xi$ and $\tau \in [0,\,1]$ such that the following holds:

$$\mathcal{H}_*(u + \tau\xi(v,u),\,\gamma) \leq (1-\tau)\mathcal{H}_*(u,\,\gamma) + \tau\mathcal{H}_*(v,\,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$

and

$$\mathcal{H}^*(u + \tau\xi(v,u),\ \gamma) \leq (1-\tau)\mathcal{H}^*(u,\ \gamma) + \tau\mathcal{H}^*(v,\ \gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$

for each $\gamma \in [0,\ 1]$. Hence, the result follows.

Example 1. *We consider the FM* $\mathcal{H} : [0,\ 1] \rightarrow \mathbb{F}_0$ *defined by the following:*

$$\mathcal{H}(u)(\sigma) = \begin{cases} \frac{\sigma}{2u^2} & \sigma \in [0,\ 2u^2] \\ \frac{4u^2 - \sigma}{2u^2} & \sigma \in (2u^2,\ 4u^2] \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

Then, for each $\gamma \in [0,\ 1]$, we have $\mathcal{H}_\gamma(u) = [2\gamma u^2, (4 - 2\gamma)u^2]$. Since $\mathcal{H}_*(u,\gamma)$, $\mathcal{H}^*(u,\gamma)$ are strongly preinvex functions for each $\gamma \in [0,1]$, $\mathcal{H}(u)$ is strongly preinvex-FM w.r.t. the following:

$$\xi(v,u) = v - u,$$

with $0 < \omega = \gamma \leq 1$. It can be easily seen that for each $\omega \in (0, 1]$, there exists a strongly preinvex-FM, and $\mathcal{H}(u)$ is neither a convex FM nor a preinvex-FM w.r.t. bifunction $\xi(v,u) = v - u$ with $0 < \omega \leq 1$.

Now, we show that the difference between a strongly preinvex-FM and a strongly affine preinvex-FM is, again, a preinvex-FM for a strongly preinvex-FM.

Theorem 4. *Let FM* $f : K_\xi \rightarrow \mathbb{F}_0$ *be a strongly affine preinvex w.r.t.* ξ *and* $0 \leq \omega$. *Then,* $\mathcal{H}$ *is strongly preinvex-FM w.r.t. the same bi-function* ξ *if, and only if,* $\varpi = \mathcal{H} - f$ *is a preinvex-FM.*

Proof. The "If" part is obvious. To prove the "only if", assume that $f : K_\xi \rightarrow \mathbb{F}_0$ is a strongly fuzzy affine preinvex w.r.t. the non-negative bi-function ξ and $0 \leq \omega$. Then, the following holds:

$$f(u + \tau\xi(v,u)) = (1-\tau)f(u)\widetilde{+}\tau f(v)\widetilde{-}\omega\tau(1-\tau)\|\xi(v,u)\|^2 \tag{15}$$

Therefore, for each $\gamma \in [0,\ 1]$, we have the following:

$$f_*(u + \tau\xi(v,u),\gamma) = (1-\tau)f_*(u,\gamma) + \tau f_*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$
$$f^*(u + \tau\xi(v,u),\gamma) = (1-\tau)f^*(u,\gamma) + \tau f^*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2.$$

Since $\mathcal{H}$ is strongly preinvex-FM w.r.t. the same bi-function ξ, then, for each $\gamma \in [0,\ 1]$, we have the following:

$$\mathcal{H}_*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)\mathcal{H}_*(u,\gamma) + \tau\mathcal{H}_*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)\mathcal{H}^*(u,\gamma) + \tau\mathcal{H}^*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2. \tag{16}$$

From (15) and (16), we have the following:

$$\mathcal{H}_*(u + \tau\xi(v,u),\gamma) - f_*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)\mathcal{H}_*(u,\gamma) + \tau\mathcal{H}_*(v,\gamma)$$
$$-(1-\tau)f_*(u,\gamma) - \tau f_*(v,\gamma),$$
$$\mathcal{H}^*(u + \tau\xi(v,u),\gamma) - f^*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)\mathcal{H}^*(u,\gamma) + \tau\mathcal{H}^*(v,\gamma)$$
$$-(1-\tau)f^*(u,\gamma) - \tau f^*(v,\gamma),$$

$$\mathcal{H}_*(u + \tau\xi(v,u),\gamma) - f_*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)(\mathcal{H}_*(u,\gamma) - f_*(u,\gamma))$$
$$+\tau(\mathcal{H}_*(v,\gamma) - f_*(v,\gamma)),$$
$$\mathcal{H}^*(u + \tau\xi(v,u),\gamma) - f^*(u + \tau\xi(v,u),\gamma) \leq (1-\tau)(\mathcal{H}^*(u,\gamma) - f^*(u,\gamma))$$
$$+\tau(\mathcal{H}^*(v,\gamma) - f^*(v,\gamma)),$$

from which it follows that

$$\varpi_*(u + \tau\xi(v, u), \gamma) = \mathcal{H}_*(u + \tau\xi(v, u), \gamma) - f_*(u + \tau\xi(v, u), \gamma),$$
$$\varpi^*(u + \tau\xi(v, u), \gamma) = \mathcal{H}^*(u + \tau\xi(v, u), \gamma) - f^*(u + \tau\xi(v, u), \gamma),$$
$$\varpi_*(u + \tau\xi(v, u), \gamma) \leq (1 - \tau)\varpi_*(u, \gamma) + \tau\varpi_*(v, \gamma),$$
$$\varpi_*(u + \tau\xi(v, u), \gamma) \leq (1 - \tau)\varpi^*(u, \gamma) + \tau\varpi^*(v, \gamma),$$

that is

$$\varpi(u + \tau\xi(v, u)) \preccurlyeq (1 - \tau)\varpi(u)\widetilde{+}\tau\varpi(u),$$

showing that $\varpi = \mathcal{H} - f$ is preinvex-FM. $\square$

We know that under certain condition invex-FMs, we obtain a solution of the fuzzy optimization problem because with the help of these FMs, we can obtain the relationship between the fuzzy variational inequalities and optimization problems.

Definition 7. *The G-differentiable FM $\mathcal{H} : K_\xi \to \mathbb{F}_0$ on K_ξ is said to be strongly invex-FM w.r.t. bi-function ξ if there exist a constant $0 \leq \omega$ such that the following holds:*

$$\mathcal{H}(v)\widetilde{-}\mathcal{H}(u) \succcurlyeq F'(u),\ \xi(v, u)\widetilde{+}\omega\|\xi(v, u)\|^2,\ for\ all\ u, v \in K_\xi. \tag{17}$$

Example 2. *We consider the FMs $\mathcal{H} : (0, 1) \to \mathbb{F}_0$ defined by, $\mathcal{H}_\gamma(u) = [2\gamma u^2, (4 - 2\gamma)u^2]$, as in Example 1; then, $\mathcal{H}(u)$ is strongly invex-FM w.r.t. bifunction $\xi(v, u) = v - u$, with $0 < \omega = \gamma \leq 1$, where $u \leq v$. We have $\mathcal{H}_*(u,\ \gamma) = \gamma u^2$ and $\mathcal{H}^*(u,\ \gamma) = (2 - \gamma)u^2$. Now, we compute the following:*

$$\mathcal{H}_*(v,\ \gamma) - \mathcal{H}_*(u,\ \gamma) = \gamma v^2 - \gamma u^2,$$

while

$$\langle \mathcal{H}_*{}'(u, \gamma),\ \xi(v, u)\rangle + \omega\|\xi(v, u)\|^2 = 2\gamma(v - u) + \omega\|v - u\|^2.$$

and $\gamma v^2 - \gamma u^2 \geq 2\gamma(v - u) + \omega v - u^2$, with $0 < \omega \leq 1$, where $u \leq v$.
Similarly, it can be easily shown that

$$\mathcal{H}^*(v,\ \gamma) - \mathcal{H}^*(u,\ \gamma) \geq \langle \mathcal{H}^*{}'(u, \gamma),\ \xi(v, u)\rangle + \omega\|\xi(v, u)\|^2$$

Hence, $\mathcal{H}(u)$ is strongly invex-FM w.r.t. bifunction $\xi(v, u) = v - u$, with $0 < \omega \leq 1$. It can be easily seen that $\mathcal{H}(u)$ is not invex-FM w.r.t. bifunction $\xi(v, u) = v - u$.

Definition 8. *The G-differentiable FM $\mathcal{H} : K_\xi \to \mathbb{F}_0$ on K_ξ is said to be strongly pseudo invex-FM w.r.t. bi-function ξ if there exists a constant $0 \leq \omega$ such that the following holds:*

$$\langle \mathcal{H}'(u),\ \xi(v, u)\rangle\widetilde{+}\omega\|\xi(v, u)\|^2 \succcurlyeq \widetilde{0}\ \Rightarrow\ \mathcal{H}(v)\widetilde{-}\mathcal{H}(u) \succcurlyeq \widetilde{0},\ for\ all\ u, v \in K_\xi. \tag{18}$$

If $\omega = 0$, then from Definition 7 and Definition 8, we obtain the classical definitions of invex-FM and pseudo invex-FM, respectively. If $\xi(v, u) = v - u$, then Definition 7 and Definition 8 reduce to known ones.

Example 3. *We consider the FMs $\mathcal{H} : (0, \infty) \to \mathbb{F}_0$ defined by, $\mathcal{H}_\gamma(u) = [\gamma u, (3 - 2\gamma)u]$, then $\mathcal{H}(u)$ is strongly pseudo invex-FM w.r.t. bifunction $\xi(v, u) = v - u$, with $0 \leq \omega = \gamma$, where $u \leq v$. We have $\mathcal{H}_*(u,\ \gamma) = \gamma u$ and $\mathcal{H}^*(u,\ \gamma) = (3 - 2\gamma)u$. Now we compute the following:*

$$\langle \mathcal{H}_*{}'(u, \gamma),\ \xi(v, u)\rangle + \omega\|\xi(v, u)\|^2 = \gamma(v - u) + \omega\|v - u\|^2 \geq 0,$$

for all $u, v \in K_\xi$ and $\gamma \in [0,\ 1]$ with $u \leq v, 0 \leq \omega$; which implies the following:

$$\mathcal{H}_*(v,\ \gamma) = \gamma v \geq \gamma u = \mathcal{H}_*(u,\ \gamma),$$
$$\mathcal{H}_*(v,\ \gamma) \geq \mathcal{H}_*(u,\ \gamma),$$

Similarly, it can be easily shown that the following holds:

$$\langle \mathcal{H}_*'(u, \gamma), \ \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2 = (3 - 2\gamma)(v - u) + \omega \|v - u\|^2 \geq 0,$$

for all $u, v \in K_\xi$ and $\gamma \in [0, 1]$ with $u \leq v, 0 \leq \omega$. This means that the following holds:

$$\mathcal{H}^*(v, \ \gamma) = (3 - 2\gamma)v \geq \gamma u = \mathcal{H}^*(u, \ \gamma),$$

from which, it follows that

$$\mathcal{H}^*(v, \ \gamma) \geq \mathcal{H}^*(u, \ \gamma)$$

Hence, the FM $\mathcal{H}_\gamma(u) = [\gamma u, (3 - 2\gamma)u]$ is strongly pseudo invex-FM w.r.t. $\xi(v, u) = v - u$, with $0 \leq \omega$, where $u \leq v$. It can be easily seen that $\mathcal{H}(u)$ is not a pseudo invex-FM w.r.t. ξ.

Theorem 5. *Let $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be a G-differentiable and strongly preinvex-FM then $\mathcal{H}$ is a strongly invex-FM.*

Proof. Let $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be G-differentiable strongly preinvex-FM. Since $\mathcal{H}$ is strongly preinvex, then for each $u, v \in K_\xi$ and $\tau \in [0, 1]$, we have the following:

$$\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq (1 - \tau)\mathcal{H}(u) \widetilde{+} \tau\mathcal{H}(v) \widetilde{-} \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$
$$\preccurlyeq \mathcal{H}(u) \widetilde{+} \tau(\mathcal{H}(v) \widetilde{-} \mathcal{H}(u)) \widetilde{-} \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$

Therefore, for every $\gamma \in [0, 1]$, we have the following:

$$\mathcal{H}_*(u + \tau\xi(v, u), \gamma) \leq \mathcal{H}_*(u, \gamma) + \tau(\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma)) - \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(v, u), \gamma) \leq \mathcal{H}_*(u, \gamma) + \tau(\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma)) - \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$

which implies that the following:

$$\tau(\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma)) \geq \mathcal{H}_*(u + \tau\xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) + \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$
$$\tau(\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma)) \geq \mathcal{H}^*(u + \tau\xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma) + \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$
$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \geq \frac{\mathcal{H}_*(u + \tau\xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma)}{\tau} + \omega(1 - \tau)\|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \geq \frac{\mathcal{H}^*(u + \tau\xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma)}{\tau} + \omega(1 - \tau)\|\xi(v, u)\|^2.$$

Taking the limit in the above inequality as $\tau \to 0$, we have the following:

$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \geq \langle \mathcal{H}_*'(u, \gamma), \ \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \geq \langle \mathcal{H}^{*'}(u, \gamma), \ \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$

that is,

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \ \xi(v, u) \rangle \widetilde{+} \omega \|\xi(v, u)\|^2.$$

As a special case of Theorem 5, when $\omega = 0$, we have the following. $\square$

Corollary 1. *Let $\mathcal{H} : K_\xi \to \mathbb{F}_0$ be a G-differentiable preinvex-FM on K_ξ [32]. Then, $\mathcal{H}$ is an invex-FM.*

It is well known that the differentiable preinvex functions are invex functions, but the converse is not true. However, Mohan and Neogy [13] showed that the preinvex functions and invex functions are equivalent under Condition C. Similarly, the converse of Theorem 5 is not valid; the natural question is how to obtain a strongly preinvex-FM from strongly invex-FM. To prove the converse, we need the following assumption regarding the bi-function ξ, which plays an important role in G-differentiation of the main results.

Condition C.

$$\xi(v, u + \tau\xi(v, u)) = (1 - \tau)\xi(v, u),$$
$$\xi(u, u + \tau\xi(v, u)) = -\tau\xi(v, u).$$

Clearly for $\tau = 0$, we have $\xi(v, u) = 0$ if, and only if, $v = u$ for all u, $v \in K_\xi$. Additionally, note that from Condition C, we have the following:

$$\xi(u + \tau_2\xi(v, u), u + \tau_1\xi(v, u)) = (\tau_2 - \tau_1)\xi(v, u)$$

For the application of Condition C, see [13–17].

The following Theorem 6 gives the result of the converse of Theorem 5.

Theorem 6. *Let* $\mathcal{H} : K_\xi \to \mathbb{F}_0$ *be a G-differentiable FM on* K_ξ *. Let Condition C holds and* $\mathcal{H}(u)$ *satisfies the following condition:*

$$\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq \mathcal{H}(v), \tag{19}$$

and then, the following are equivalent:
(a) $\mathcal{H}$ *is strongly preinvex-FM.*

$$(b) \; \mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \mathcal{H}'(u), \; \xi(v, u) \widetilde{+} \omega \|\xi(v, u)\|^2, \; \text{for all } u, \; v \in K_\xi, \tag{20}$$

$$\langle (c) \, \mathcal{H}'(u), \; \xi(v, u) \rangle \Big\langle \widetilde{+} \mathcal{H}'(v), \; \xi(u, v) \Big\rangle \preccurlyeq \widetilde{-} \omega \Big\{ \|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big\} \tag{21}$$

for all u, $v \in K_\xi$.

Proof. (a) implies (b) $\square$

The demonstration is analogous to the demonstration of Theorem 5.

(b) implies (c). Let (b) hold. Then, for every $\gamma \in [0, 1]$ *, we have the following:*

$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \geq \langle \mathcal{H}_*'(u, \gamma), \; \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \geq \langle \mathcal{H}^{*'}(u, \gamma), \; \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2, \tag{22}$$

Then, by replacing v by u and u by v in (22), we obtain the following:

$$\mathcal{H}_*(u, \gamma) - \mathcal{H}_*(v, \gamma) \geq \langle \mathcal{H}_*'(v, \gamma), \; \xi(u, v) \rangle + \omega \|\xi(u, v)\|^2,$$
$$\mathcal{H}^*(u, \gamma) - \mathcal{H}^*(v, \gamma) \geq \langle \mathcal{H}^{*'}(v, \gamma), \; \xi(u, v) \rangle + \omega \|\xi(u, v)\|^2. \tag{23}$$

Adding (22) and (23), we have the following:

$$\langle \mathcal{H}_*'(u, \gamma), \; \xi(v, u) \rangle + \langle \mathcal{H}_*'(v, \gamma), \; \xi(u, v) \rangle \leq -\omega \Big(\|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big),$$
$$\langle \mathcal{H}^{*'}(u, \gamma), \; \xi(v, u) \rangle + \langle \mathcal{H}^{*'}(v, \gamma), \; \xi(u, v) \rangle \leq -\omega \Big(\|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big),$$

That is, the following:

$$\langle \mathcal{H}'(u), \; \xi(v, u) \rangle \widetilde{+} \langle \mathcal{H}'(v), \; \xi(u, v) \rangle \preccurlyeq \widetilde{-} \omega \Big\{ \|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big\}$$

(c) implies (b). Assume that (21) holds. Then, for every $\gamma \in [0, 1]$, we have the following:

$$\langle \mathcal{H}_*'(v, \gamma), \; \xi(u, v) \rangle \leq -\langle \mathcal{H}_*'(u, \gamma), \; \xi(v, u) \rangle - \omega \Big(\|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big),$$
$$\langle \mathcal{H}^{*'}(v, \; \gamma), \; \xi(u, v) \rangle \leq -\langle \mathcal{H}^{*'}(u, \; \gamma), \; \xi(v, u) \rangle - \omega \Big(\|\xi(v, u)\|^2 + \|\xi(u, v)\|^2 \Big). \tag{24}$$

Since, $v_\tau = u + \tau\xi(v, u) \in K_{\widetilde{\xi}}$ for all $u, v \in K_{\widetilde{\xi}}$ and $\tau \in [0, 1]$. Taking $v = v_\tau$ in (24), we obtain the following:

$$\langle \mathcal{H}_*'(u + \tau\xi(v, u), \gamma), \xi(u, u + \tau\xi(v, u)) \rangle \le -\langle \mathcal{H}_*'(u, \gamma), \xi(u + \tau\xi(v, u), u) \rangle$$
$$-\omega\Big(\|\xi(u + \tau\xi(v, u), u)\|^2 + \|\xi(u, u + \tau\xi(v, u))\|^2 \Big),$$
$$\langle \mathcal{H}^{*'}(u + \tau\xi(v, u), \gamma), \xi(u, u + \tau\xi(v, u)) \rangle \le -\langle \mathcal{H}^{*'}(u, \gamma), \xi(u + \tau\xi(v, u), u) \rangle$$
$$-\omega\Big(\|\xi(u + \tau\xi(v, u), u)\|^2 + \|\xi(u, u + \tau\xi(v, u))\|^2 \Big),$$

by using Condition C, we have the following:

$$\langle \mathcal{H}_*'(u + \tau\xi(v, u), \gamma), \tau\xi(v, u) \rangle \ge \langle \mathcal{H}_*'(u, \gamma), \tau\xi(v, u) \rangle + 2\omega\tau^2 \|\xi(v, u)\|^2,$$
$$\langle \mathcal{H}^{*'}(u + \tau\xi(v, u), \gamma), \tau\xi(v, u) \rangle \ge \langle \mathcal{H}^{*'}(u, \gamma), \tau\xi(v, u) \rangle + 2\omega\tau^2 \|\xi(v, u)\|^2,$$

$$\langle \mathcal{H}_*'(u + \tau\xi(v, u), \gamma), \xi(v, u) \rangle \ge \langle \mathcal{H}_*'(u, \gamma), \xi(v, u) \rangle + 2\omega\tau \|\xi(v, u)\|^2,$$
$$\langle \mathcal{H}^{*'}(u + \tau\xi(v, u), \gamma), \xi(v, u) \rangle \ge \langle \mathcal{H}^{*'}(u, \gamma), \xi(v, u) \rangle + 2\omega\tau \|\xi(v, u)\|^2, \tag{25}$$

Let the following hold:

$$H_*(\tau) = \mathcal{H}_*(u + \tau\xi(v, u), \gamma),$$
$$H^*(\tau) = \mathcal{H}^*(u + \tau\xi(v, u), \gamma).$$

Taking the derivative w.r.t. τ, we obtain the following:

$$H_*'(\tau) = \mathcal{H}_*'(u + \tau\xi(v, u), \gamma).\xi(v, u) = \langle \mathcal{H}_*'(u + \tau\xi(v, u), \gamma), \xi(v, u) \rangle,$$
$$H^{*'}(\tau) = \mathcal{H}^{*'}(u + \tau\xi(v, u), \gamma).\xi(v, u) = \langle \mathcal{H}^{*'}(u + \tau\xi(v, u), \gamma), \xi(v, u) \rangle,$$

from which, using (25), we have the following:

$$H_*'(\tau) \ge \langle \mathcal{H}_*'(u, \gamma), \xi(v, u) \rangle + 2\omega\tau \|\xi(v, u)\|^2,$$
$$H^{*'}(\tau) \ge \langle \mathcal{H}^{*'}(u, \gamma), \xi(v, u) \rangle + 2\omega\tau \|\xi(v, u)\|^2. \tag{26}$$

By integrating (26) between 0 to 1, w.r.t. τ, we obtain the following:

$$H_*(1) - H_*(0) \ge \langle \mathcal{H}_*'(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$H^*(1) - H^*(0) \ge \langle \mathcal{H}^{*'}(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2.$$

$$\mathcal{H}_*(u + \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) \ge \langle \mathcal{H}_*'(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(u + \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma) \ge \langle \mathcal{H}^{*'}(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2.$$

Using (19), we have the following:

$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \ge \langle \mathcal{H}_*'(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \ge \langle \mathcal{H}^{*'}(u, \gamma), \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$

that is, the following:

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \tau\xi(v, u) \rangle \widetilde{+} \omega \|\xi(v, u)\|^2, \text{ for all } u, \ v \in K_{\widetilde{\xi}}.$$

(*b*) implies (*a*). Assume that (20) holds. Since $K_{\widetilde{\xi}}$, $v_\tau = u + \tau\xi(v, u) \in K_{\widetilde{\xi}}$ for all $u, v \in K_{\widetilde{\xi}}$ and $\tau \in [0, 1]$. Taking $v = v_\tau$ in (20), we obtain the following:

$$\mathcal{H}(u + \tau\xi(v, u)) \widetilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \xi(u + \tau\xi(v, u), u) \rangle \widetilde{+} \omega \|\xi(u + \tau\xi(v, u), u)\|^2.$$

Therefore, for every $\gamma \in [0, 1]$, we have the following:
$$\mathcal{H}_*(u + \tau\xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) \ge \langle \mathcal{H}_*'(u, \gamma), \xi(u + \tau\xi(v, u), u) \rangle + \omega \|\xi(u + \tau\xi(v, u), u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(v, u), \ \gamma) - \mathcal{H}^*(u, \gamma) \ge \langle \mathcal{H}^{*'}(u, \gamma), \ \xi(u + \tau\xi(v, u), u) \rangle + \omega \|\xi(u + \tau\xi(v, u), u)\|^2.$$

Using Condition C, we have the following:

$$\mathcal{H}_*(u + \tau\xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) \geq (1 - \tau)\langle\mathcal{H}_*'(u, \gamma), \xi(v, u)\rangle + \omega(1 - \tau)^2\|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma) \geq (1 - \tau)\langle\mathcal{H}^{*\prime}(u, \gamma), \xi(v, u)\rangle + \omega(1 - \tau)^2\|\xi(v, u)\|^2.$$

(27)

In a similar way, we have the following:

$$\mathcal{H}_*(u, \gamma) - \mathcal{H}_*(u + \tau\xi(v, u), \gamma) \geq -\tau\langle\mathcal{H}_*'(u, \gamma), \xi(v, u)\rangle + \omega\tau^2\|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(u, \gamma) - \mathcal{H}^*(u + \tau\xi(v, u), \gamma) \geq -\tau\langle\mathcal{H}^{*\prime}(u, \gamma), \xi(v, u)\rangle + \omega\tau^2\|\xi(v, u)\|^2.$$

(28)

Multiplying (27) by τ and (28) by $(1 - \tau)$, and adding the resultant, we have the following:

$$\mathcal{H}_*(u + \tau\xi(v, u), \gamma) \leq (1 - \tau)\mathcal{H}_*(u, \gamma) + \tau\mathcal{H}_*(v, \gamma) - \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(v, u), \gamma) \leq (1 - \tau)\mathcal{H}^*(u, \gamma) + \tau\mathcal{H}^*(v, \gamma) - \omega\tau(1 - \tau)\|\xi(v, u)\|^2,$$

That is, the following holds:

$$\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq (1 - \tau)\mathcal{H}(u)\widetilde{+}\tau\mathcal{H}(v)\widetilde{-}\omega\tau(1 - \tau)\|\xi(v, u)\|^2.$$

Hence, $\mathcal{H}$ is strongly preinvex-FM w.r.t.

Theorems 5 and 6, enable us to define the followings new definitions.

Definition 9. *A G-differentiable FM $\mathcal{H} : K_\xi \to \mathbb{F}_0$ is said to be as follows:*

(i) *Strongly monotone w.r.t. bi-function ξ if, and only if, there exists a constant $0 \leq \omega$ such that the following is true:*

$$\langle\mathcal{H}'(u), \xi(v, u)\rangle\widetilde{+}\langle\mathcal{H}'(v), \xi(u, v)\rangle \preccurlyeq \widetilde{-}\omega\Big\{\|\xi(v, u)\|^2 + \|\xi(u, v)\|^2\Big\}, \text{for all } u, v \in K_\xi$$

(ii) *Strongly pseudo monotone w.r.t. bi-function ξ if, and only if, there exists a constant $0 \leq \omega$ such that the following is true:*

$$\langle\mathcal{H}'(u), \xi(v, u)\rangle\widetilde{+}\omega\|\xi(v, u)\|^2 \succcurlyeq \widetilde{0} \Rightarrow \widetilde{-}\langle\mathcal{H}'(v), \xi(u, v)\rangle \succcurlyeq \widetilde{0}, \text{ for all } u, v \in K_\xi.$$

If $\xi(v, u) = -\xi(u, v)$, then Definition 9. reduces to new one.

Example 4. *We consider the FMs $\mathcal{H} : (0, \infty) \to \mathbb{F}_0$ defined by the following:*

$$\mathcal{H}(u)(\sigma) = \begin{cases} \frac{\sigma}{2u^2} & \sigma \in [0, 2u^2] \\ \frac{5u^2 - \sigma}{3u^2} & \sigma \in (2u^2, 5u^2] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\gamma \in [0, 1]$, we have $\mathcal{H}_\gamma(u) = [2\gamma u^2, (5 - 3\gamma)u^2]$, where $\mathcal{H}(u)$ is strongly fuzzy pseudomonotone w.r.t. bifunction $\xi(v, u) = u - v$, with $1 \leq \omega$, where $v \leq u$. We have $\mathcal{H}_*(u, \gamma) = 2\gamma u^2$ and $\mathcal{H}^*(u, \gamma) = (5 - 3\gamma)u^2$. Now, we compute the following:

$$\langle\mathcal{H}_*'(u, \gamma), \xi(v, u)\rangle + \omega\|\xi(v, u)\|^2 = 4\gamma u(u - v) + \omega u - v^2 \geq 0,$$

for all $u, v \in K_\xi$ and $\gamma \in [0, 1]$ with $v \leq u, 1 \leq \omega$; which implies he following:

$$-\langle\mathcal{H}_*'(v, \gamma), \xi(u, v) = -4\gamma u(v - u)\rangle = 4\gamma v(u - v) \geq 0, \forall u, v \in K_\xi,$$
$$-\langle\mathcal{H}^{*\prime}(v, \gamma), \xi(u, v)\rangle \geq 0.$$

Similarly, it can be easily shown that the following holds:

$$\langle\mathcal{H}^{*\prime}(u, \gamma), \xi(v, u)\rangle + \omega\|\xi(v, u)\|^2 = 2(5 - 3\gamma)u(u - v) + \omega\|u - v\|^2 \geq 0,$$

for all $u, v \in K_{\xi}$ and $\gamma \in [0, 1]$ with $v \leq u, 1 \leq \omega$. This means the following is true:

$$- \langle \mathcal{H}^{*\prime}(v, \gamma), \ \xi(u,v) \rangle = -2(5 - 3\gamma)u(v - u) = 2(5 - 3\gamma)v(u - v) \geq 0, \ \forall \ u, \ v \in K_{\xi},$$

From which, it follows that

$$- \langle \mathcal{H}^{*\prime}(v, \gamma), \ \xi(u,v) \rangle \geq 0.$$

Hence, the G-differentiable FM $\mathcal{H}_{\gamma}(u) = [\gamma u, (5 - 4\gamma)u]$ is strongly fuzzy pseudomonotone w.r.t. $\xi(v,u) = u - v$, with $1 \leq \omega$, where $v \leq u$. It can be easily noted that $\mathcal{H}^{\prime}(u)$ is neither fuzzy pseudomonotone nor fuzzy quasimonotone w.r.t. ξ.
If $\omega = 0$, then from Theorem 6, we obtain following result.

Corollary 2. *[36] Let $\mathcal{H} : K_{\xi} \to \mathbb{F}_0$ be a G-differentiable FM on K_{ξ}. Let Condition C holds and $\mathcal{H}(u)$ satisfies the following condition:*

$$\mathcal{H}(u + \tau \xi(v, u)) \preccurlyeq \mathcal{H}(v),$$

and then, the following are equivalent:

(a)　$\mathcal{H}$ *is invex-FM.*
(b)　$\mathcal{H}^{\prime}$ *is monotone.*

Theorem 7. *Let $\mathcal{H} : K_{\xi} \to \mathbb{F}_0$ be FM on K_{ξ} w.r.t. ξ and Condition C hold. Let $\mathcal{H}(u)$ is G-differentiable on K_{ξ} with the following conditions*:
(a)　$\mathcal{H}(u + \tau \xi(v, u)) \preccurlyeq \mathcal{H}(v)$.
(b)　$\mathcal{H}^{\prime}(u)$ *is a strongly fuzzy pseudomonotone.*

Then, $\mathcal{H}$ is a strongly pseudo invex-FM.

Proof. Let $\mathcal{H}^{\prime}$ be strongly pseudomonotone. Then, for all $u, \ v \in K_{\xi}$, we have the following:

$$\langle \mathcal{H}^{\prime}(u), \ \xi(v,u) \rangle \widetilde{+} \omega \|\xi(v,u)\|^2 \succcurlyeq \widetilde{0}.$$

Therefore, for every $\gamma \in [0, 1]$, we have the following:

$$\langle \mathcal{H}_*{}^{\prime}(u, \gamma), \xi(v,u) \rangle + \omega \|\xi(v,u)\|^2 \geq 0,$$
$$\langle \mathcal{H}^{*\prime}(u, \gamma), \xi(v,u) \rangle + \omega \|\xi(v,u)\|^2 \geq 0,$$

which implies that the following is true:

$$
\begin{aligned}
-\langle \mathcal{H}_*{}^{\prime}(v, \gamma), \ \xi(u,v) \rangle \geq 0, \\
-\langle \mathcal{H}^{*\prime}(v, \gamma), \ \xi(u,v) \rangle \geq 0.
\end{aligned}
\tag{29}
$$

Since $v_{\tau} = u + \tau \xi(v, u) \in K_{\xi}$ for all $u, v \in K_{\xi}$ and $\tau \in [0, 1]$. Taking $v = v_{\tau}$ in (29), we obtain the following:

$$
\begin{aligned}
-\langle \mathcal{H}_*{}^{\prime}(u + \tau \xi(v, u), \gamma), \ \xi(u, u + \tau \xi(v, u)) \rangle \geq 0, \\
-\langle \mathcal{H}^{*\prime}(u + \tau \xi(v, u), \gamma), \ \xi(u, u + \tau \xi(v, u)) \rangle \geq 0.
\end{aligned}
$$

By using Condition C, we have the following:

$$
\begin{aligned}
\langle \mathcal{H}_*{}^{\prime}(u + \tau \xi(v, u), \gamma), \ \xi(v, u) \rangle \geq 0, \\
\langle \mathcal{H}^{*\prime}(u + \tau \xi(v, u), \gamma), \ \xi(v, u) \rangle \geq 0.
\end{aligned}
\tag{30}
$$

Assume the following:

$$H_*(\tau) = \mathcal{H}_*(u + \tau\xi(v, u), \gamma),$$
$$H^*(\tau) = \mathcal{H}^*(u + \tau\xi(v, u), \gamma),$$

taking G-derivative w.r.t. τ, then using (30), we have the following:

$$\begin{aligned} H_*{}'(\tau) &= \langle \mathcal{H}_*{}'(u + \tau\xi(v, u), \gamma), \; \xi(v, u) \rangle \geq 0, \\ H^{*\prime}(\tau) &= \langle \mathcal{H}^{*\prime}(u + \tau\xi(v, u), \gamma), \; \xi(v, u) \rangle \geq 0, \end{aligned} \tag{31}$$

Integrating (31) between 0 to 1 w.r.t. τ, we obtain the following:

$$\begin{aligned} H_*(1) - H_*(0) &\geq 0, \\ H^*(1) - H^*(0) &\geq 0, \end{aligned}$$

which implies the following:

$$\begin{aligned} \mathcal{H}_*(u + \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) &\geq 0, \\ \mathcal{H}^*(u + \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma) &\geq 0. \end{aligned}$$

From condition (i), we have the following:

$$\begin{aligned} \mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) &\geq 0, \\ \mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) &\geq 0, \end{aligned}$$

that is,

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \widetilde{0}, \; \forall \, u, v \in K_\xi.$$

Hence, $\mathcal{H}$ is a strongly pseudo invex-FM.

If $\omega = 0$, then Theorem 7 reduces to the following result. $\square$

Corollary 3. *Let* $\mathcal{H} : K_\xi \to \mathbb{F}_0$ *be a FM on* K_ξ *w.r.t.* ξ *and Condition C hold [36]. Let* $\mathcal{H}(u)$ *be G-differentiable on* K_ξ *with the following conditions:*

(a) $\mathcal{H}(u + \tau\xi(v, u)) \preccurlyeq \mathcal{H}(v)$.
(b) $\mathcal{H}'(.)$ *is fuzzy pseudomonotone.*

Then, $\mathcal{H}$ is a pseudo invex-FM.

The fuzzy optimality requirement for G-differentiable strongly preinvex-FMs, which is the fundamental impetus for our findings, is now discussed.

4. Fuzzy Mixed Variational-like and Integral Inequalities

The variational inequality problem has a close relationship with the optimization problem, which is a well-known fact in mathematical programming. Similarly, the fuzzy variational inequality problem and the fuzzy optimization problem have a strong link.

Consider the following unconstrained fuzzy optimization problem:

$$\min_{u \in K_\xi} \mathcal{H}(u),$$

where K_ξ is a subset of $\mathbb{R}$, $\mathcal{H} : K_\xi \to \mathbb{F}_0$ and is a FM.

A feasible point is defined, as $u \in K_\xi$ is called an optimal solution, a global optimal solution, or simply a solution to the fuzzy optimization problem if $u \in K_\xi$ and no $v \in K_\xi$, $\mathcal{H}(u) \preccurlyeq \mathcal{H}(v)$.

The fuzzy optimality criterion for G-differentiable preinvex-FMs is discussed in the following theorems, and this is the fundamental rationale for the results.

Theorem 8. *Let $\mathcal{H}$ be a G-differentiable strongly preinvex-FM modulus $0 \leq \omega$. If $u \in K_\xi$ is the minimum of the FM $\mathcal{H}$, then the following holds:*

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \omega \|\xi(v, u)\|^2, \text{ for all } u, v \in K_\xi. \tag{32}$$

Proof: Let $u \in K$ be a minimum of $\mathcal{H}$. Then

$\mathcal{H}(u) \preccurlyeq \mathcal{H}(v)$, for all $v \in K_\xi$.

Therefore, for every $\gamma \in [0, 1]$, we have the following:

$$\begin{aligned} \mathcal{H}_*(u, \gamma) &\leq \mathcal{H}_*(v, \gamma), \\ \mathcal{H}^*(u, \gamma) &\leq \mathcal{H}^*(v, \gamma). \end{aligned} \tag{33}$$

For all $u, v \in K_\xi$, $\tau \in [0, 1]$, we have the following:

$$v_\tau = u + \tau \xi(v, u) \in K_\xi$$

Taking $v = v_\tau$ in (33), and dividing by "τ", we obtain the following:

$$\begin{aligned} 0 &\leq \frac{\mathcal{H}_*(u + \tau \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma)}{\tau}, \\ 0 &\leq \frac{\mathcal{H}^*(u + \tau \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma)}{\tau}. \end{aligned}$$

Taking limit in the above inequality as $\tau \to 0$, we obtain the following:

$$\begin{aligned} 0 &\leq \langle \mathcal{H}_*{}'(u, \gamma), \, \xi(v, u) \rangle, \\ 0 &\leq \langle \mathcal{H}^*{}'(u, \gamma), \, \xi(v, u) \rangle. \end{aligned} \tag{34}$$

Since $\mathcal{H} : K_\xi \to \mathbb{F}_0$ is a G-differentiable strongly preinvex-FM, we have the following:

$$\begin{aligned} \mathcal{H}_*(u + \tau \xi(v, u), \gamma) &\leq (1 - \tau) \mathcal{H}_*(u, \gamma) + \tau \mathcal{H}_*(v, \gamma) - \omega \tau (1 - \tau) \|\xi(v, u)\|^2, \\ \mathcal{H}^*(u + \tau \xi(v, u), \gamma) &\leq (1 - \tau) \mathcal{H}^*(u, \gamma) + \tau \mathcal{H}^*(v, \gamma) - \omega \tau (1 - \tau) \|\xi(v, u)\|^2, \\ \mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) &\geq \frac{\mathcal{H}_*(u + \tau \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma)}{\tau} + \omega (1 - \tau) \|\xi(v, u)\|^2, \\ \mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) &\geq \frac{\mathcal{H}^*(u + \tau \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma)}{\tau} + \omega (1 - \tau) \|\xi(v, u)\|^2, \end{aligned}$$

Again, taking the limit in the above inequality as $\tau \to 0$, we obtain the following:

$$\begin{aligned} \mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) &\geq \langle \mathcal{H}_*{}'(u, \gamma), \, \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2, \\ \mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) &\geq \langle \mathcal{H}^*{}'(u, \gamma), \, \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2, \end{aligned}$$

from which, using (34), we have the following:

$$\begin{aligned} \mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) &\geq \omega \|\xi(v, u)\|^2 \geq 0, \\ \mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) &\geq \omega \|\xi(v, u)\|^2 \geq 0, \end{aligned}$$

that is,

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \widetilde{0}.$$

Hence, the result follows. □

Theorem 9. *Let $\mathcal{H}$ be a G-differentiable strongly preinvex-FM modulus $0 \leq \omega$, and*

$$\langle \mathcal{H}'(u), \, \xi(v, u) \rangle \widetilde{+} \omega \|\xi(v, u)\|^2 \succcurlyeq \widetilde{0}, \text{ for all } u, v \in K_\xi, \tag{35}$$

then $u \in K_\xi$ is the minimum of the FM $\mathcal{H}$.

Proof. Let $\mathcal{H} : K_{\tilde{\xi}} \to \mathbb{F}_0$ be a G-differentiable strongly preinvex-FM and $u \in K_{\tilde{\xi}}$ satisfies (35)**.** Then, by Theorem 5, we have the following:

$$\mathcal{H}(v) \widetilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \, \xi(v, u) \rangle \widetilde{+} \omega \|\xi(v, u)\|^2,$$

Therefore, for every $\gamma \in [0, \, 1]$, we have the following:

$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \geq \langle \mathcal{H}_*'(u, \gamma), \, \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \geq \langle \mathcal{H}^{*'}(u, \gamma), \, \xi(v, u) \rangle + \omega \|\xi(v, u)\|^2,$$

from which, using (35), we have the following:

$$\mathcal{H}_*(v, \gamma) - \mathcal{H}_*(u, \gamma) \geq 0,$$
$$\mathcal{H}^*(v, \gamma) - \mathcal{H}^*(u, \gamma) \geq 0,$$

that is,

$$\mathcal{H}(u) \preccurlyeq \mathcal{H}(v).$$

$\square$

If $\omega = 0$, then Theorem 9 reduces to the following result:

Corollary 4. *Let $\mathcal{H}$ be a G-differentiable preinvex-FM w.r.t. ξ [32]. Then, $u \in K_{\tilde{\xi}}$ is the minimum of $\mathcal{H}$ if, and only if, $u \in K_{\tilde{\xi}}$ satisfies the following:*

$$\langle \mathcal{H}'(u), \, \xi(v, u) \rangle \succcurlyeq \widetilde{0}, \text{for all } u, v \in K_{\tilde{\xi}}.$$

Remark 2. *The inequality of the type (35) is called a strongly variational-like inequality. It is very important to note that the optimality condition of preinvex-FMs cannot be obtained with the help of (35). So, this idea inspires us to introduce a more general form of a fuzzy variational-like inequality of which (35) is a special case. To be more unambiguous, for given FM Ψ, bi function $\xi(., .)$ and a $0 \leq \omega$, consider the problem of finding $u \in K_{\tilde{\xi}}$, such that the following holds:*

$$\langle \Psi(u), \, \xi(v, u) \rangle \widetilde{+} \omega \|\xi(v, u)\|^2 \succcurlyeq \widetilde{0}, \, \forall \, v \in K_{\tilde{\xi}}. \tag{36}$$

This inequality is called a strongly fuzzy variational-like inequality.
We look at the functional $I(v)$, which is defined as follows:

$$I(v) = \mathcal{H}(v) \widetilde{+} \mathcal{J}(v), \, \forall \, v \in \mathbb{R}, \tag{37}$$

where $\mathcal{H}$ is a G-differentiable preinvex-FM and $\mathcal{J}$ is a strongly preinvex-FM, which is non-G-differentiable.

The following theorem shows that the functional $I(v)$ minimum can be distinguished by a class of variational-like inequalities.

Theorem 10. *Let $\mathcal{H} : K_{\tilde{\xi}} \to \mathbb{F}_0$ be a G-differentiable preinvex-FM and $\mathcal{J} : K_{\tilde{\xi}} \to \mathbb{F}_0$ be a non-G-differentiable strongly preinvex-FM. Then, the functional $I(v)$ has minimum $u \in K_{\tilde{\xi}}$, if and only if $u \in K_{\tilde{\xi}}$ satisfies the following:*

$$\langle \mathcal{H}'(u), \, \xi(v, u) \rangle \widetilde{+} \mathcal{J}(v) \widetilde{-} \mathcal{J}(u) \widetilde{+} \omega \|\xi(v, u)\|^2 \succcurlyeq \widetilde{0}, \, \forall \, v \in K_{\tilde{\xi}}. \tag{38}$$

Proof: Let $u \in K_{\tilde{\xi}}$ be the smallest value of I.

Therefore, for every $\gamma \in [0, 1]$, we have the following:

$$
\begin{aligned}
I_*(u, \gamma) &\leq I_*(v, \gamma), \\
I^*(u, \gamma) &\leq I^*(v, \gamma).
\end{aligned}
\tag{39}
$$

Since $v_\tau = u + \tau \xi(v, u)$, for all $u, v \in K_\xi$ and $\tau \in [0, 1]$. Replacing v by v_τ in (39), we obtain the following:

$$
\begin{aligned}
I_*(u, \gamma) &\leq I_*(u + \tau \xi(v, u), \gamma), \\
I^*(u, \gamma) &\leq I^*(u + \tau \xi(v, u), \gamma).
\end{aligned}
$$

which implies that, using (37), the following holds:

$$
\begin{aligned}
\mathcal{H}_*(u, \gamma) + \mathcal{J}_*(u, \gamma) &\leq \mathcal{H}_*(u + \tau \xi(v, u), \gamma) + \mathcal{J}_*(u + \tau \xi(v, u), \gamma), \\
\mathcal{H}^*(u, \gamma) + \mathcal{J}^*(u, \gamma) &\leq \mathcal{H}^*(u + \tau \xi(v, u), \gamma) + \mathcal{J}^*(u + \tau \xi(v, u), \gamma).
\end{aligned}
$$

Since $\mathcal{J}$ is strongly preinvex-FM, then the following holds:

$$
\begin{aligned}
\mathcal{H}_*(u, \gamma) + \mathcal{J}_*(u, \gamma) &\leq \mathcal{H}_*(u + \tau \xi(v, u), \gamma) + (1 - \tau)\mathcal{J}_*(u, \gamma) + \tau \mathcal{J}_*(v, \gamma) \\
&\quad + \omega \tau (1 - \tau)\|\xi(v, u)\|^2, \\
\mathcal{H}^*(u, \gamma) + \mathcal{J}^*(u, \gamma) &\leq \mathcal{H}^*(u + \tau \xi(v, u), \gamma) + (1 - \tau)\mathcal{J}^*(u, \gamma) + \tau \mathcal{J}^*(v, \gamma) \\
&\quad + \omega \tau (1 - \tau)\|\xi(v, u)\|^2,
\end{aligned}
$$

that is

$$
\begin{aligned}
0 &\leq \mathcal{H}_*(u + \tau \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma) + \tau(\mathcal{J}_*(v, \gamma) - \mathcal{J}_*(u, \gamma)) + \omega \tau (1 - \tau)\|\xi(v, u)\|^2, \\
0 &\leq \mathcal{H}^*(u + \tau \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma) + \tau(\mathcal{J}^*(v, \gamma) - \mathcal{J}^*(u, \gamma)) + \omega \tau (1 - \tau)\|\xi(v, u)\|^2,
\end{aligned}
$$

Now dividing by "τ" and taking $\lim\limits_{\tau \to 0}$, we have the following:

$$
\begin{aligned}
0 &\leq \lim_{\tau \to 0}\left\{ \frac{\mathcal{H}_*(u + \tau \xi(v, u), \gamma) - \mathcal{H}_*(u, \gamma)}{\tau} + \mathcal{J}_*(v, \gamma) - \mathcal{J}_*(u, \gamma) + \omega(1 - \tau)\|\xi(v, u)\|^2 \right\}, \\
0 &\leq \lim_{\tau \to 0}\left\{ \frac{\mathcal{H}^*(u + \tau \xi(v, u), \gamma) - \mathcal{H}^*(u, \gamma)}{\tau} + \mathcal{J}^*(v, \gamma) - \mathcal{J}^*(u, \gamma) + \omega(1 - \tau)\|\xi(v, u)\|^2 \right\},
\end{aligned}
$$

then

$$
\begin{aligned}
0 &\leq \langle \mathcal{H}_*{}'(u, \gamma),\ \xi(v, u) \rangle + \mathcal{J}_*(v, \gamma) - \mathcal{J}_*(u, \gamma) + \omega\|\xi(v, u)\|^2, \\
0 &\leq \langle \mathcal{H}^*{}'(u, \gamma),\ \xi(v, u) \rangle + \mathcal{J}^*(v, \gamma) - \mathcal{J}^*(u, \gamma) + \omega\|\xi(v, u)\|^2,
\end{aligned}
$$

that is,

$$
\widetilde{0} \preccurlyeq \langle \mathcal{H}'(u),\ \xi(v, u) \rangle \widetilde{+} \mathcal{J}(v) \widetilde{-} \mathcal{J}(u) \widetilde{+} \omega\|\xi(v, u)\|^2.
$$

Conversely, let (38) be satisfied to prove that $u \in K_\xi$ is a minimum of I. Assume that for all $v \in K_\xi$, we have $I(u) \widetilde{-} I(v) = \mathcal{H}(u) \widetilde{+} \mathcal{J}(u) \widetilde{-} \mathcal{H}(v) \widetilde{-} \mathcal{J}(v), = \mathcal{H}(u) \widetilde{-} \mathcal{H}(v) \widetilde{+} \mathcal{J}(u) \widetilde{-} \mathcal{J}(v)$, Therefore, for every $\gamma \in [0, 1]$, we have the following:

$$
\begin{aligned}
I_*(u, \gamma) - I_*(v, \gamma) &= \mathcal{H}_*(u, \gamma) - \mathcal{H}_*(v, \gamma) + \mathcal{J}_*(u, \gamma) - \mathcal{J}_*(v, \gamma), \\
I^*(u, \gamma) - I^*(v, \gamma) &= \mathcal{H}^*(u, \gamma) - \mathcal{H}^*(v, \gamma) + \mathcal{J}^*(u, \gamma) - \mathcal{J}^*(v, \gamma).
\end{aligned}
$$

By Corollary 1, we have the following:

$$
\begin{aligned}
I_*(u, \gamma) - I_*(v, \gamma) &\leq -[\langle \mathcal{H}_*{}'(u, \gamma),\ \xi(v, u) \rangle + \mathcal{J}_*(v, \gamma) - \mathcal{J}_*(u, \gamma)], \\
I^*(u, \gamma) - I^*(v, \gamma) &\leq -[\langle \mathcal{H}^*{}'(u, \gamma),\ \xi(v, u) \rangle + \mathcal{J}^*(v, \gamma) - \mathcal{J}^*(u, \gamma)],
\end{aligned}
$$

from which, using (38), we have the following:

$$
\begin{aligned}
I_*(u, \gamma) - I_*(v, \gamma) &\leq -\omega\|\xi(v, u)\|^2 \leq 0, \\
I^*(u, , \gamma) - I^*(v, \gamma) &\leq -\omega\|\xi(v, u)\|^2 \leq 0,
\end{aligned}
$$

that is, $I(u) \widetilde{-} I(v) \preccurlyeq \widetilde{0}$, hence, $I(u) \preccurlyeq I(v)$. $\square$

Note that (38) are called strongly fuzzy mixed variational-like inequalities. This result shows that the minimum of fuzzy functional $I(v)$ can be characterized by a strongly fuzzy mixed variational-like inequality. It is very important to observe that the optimality conditions of preinvex-FMs and strongly preinvex-FMs cannot be obtained with the help of (38). This idea encourages us to introduce a more general type of fuzzy variational-like inequality of which (38) is a particular case. In order to be more precise, for given FMs Ψ, ϖ, bi function $\xi(.,.)$ and a $0 \le \omega$, consider the problem of finding $u \in K_\xi$, such that the following holds:

$$\langle \Psi(u), \xi(v,u) \rangle \widetilde{+} \varpi(v) \widetilde{-} \varpi(u) \widetilde{+} \omega \|\xi(v,u)\|^2 \succcurlyeq \widetilde{0}, \forall \, v \in K_\xi. \tag{40}$$

This inequality is called a strongly fuzzy mixed variational-like inequality.

Now, we look at a few specific types of strongly fuzzy mixed variational-like inequalities:

If $\xi(v,u) = v - u$, then (40) is called a strongly fuzzy mixed variational inequality such as the following:

$$\langle \Psi(u), v - u \rangle \widetilde{+} \varpi(v) \widetilde{-} \varpi(u) \widetilde{+} \omega \|v - u\|^2 \succcurlyeq \widetilde{0}, \, \forall \, v \in K_\xi.$$

If $\omega = 0$, then (40) is called fuzzy mixed variational-like inequality such as the following:

$$\langle \Psi(u), \xi(v,u) \rangle \widetilde{+} \varpi(v) \widetilde{-} \varpi(u) \succcurlyeq \widetilde{0}, \, \forall \, v \in K_\xi.$$

If $\xi(v,u) = v - u$ and $\omega = 0$, then (40) is called a fuzzy mixed variational inequality such as the following:

$$\langle \Psi(u), \, v - u \rangle \widetilde{+} \varpi(v) \widetilde{-} \varpi(u) \succcurlyeq \widetilde{0}, \, \forall \, v \in K_\xi.$$

Similarly, we can obtain a fuzzy variational inequality and fuzzy variational-like inequality in [32] as special cases of (40). In a similar way, some special cases of strongly fuzzy variational-like inequality (36) can also be discussed.

Remark 3. *The inequalities (36) and (40) show that the variational-like inequalities arise naturally in connection with the minimization of the G-differentiable preinvex-FMs, subject to certain constraints.*

The Theorem 11 provides the Hermite–Hadamard inequality for strongly preinvex-FM. This inequality provides a lower and an upper estimation for the average of strongly preinvex-FM defined on a compact interval.

Theorem 11. *Let $\mathcal{H} : [u, \, u + \xi(v, \, u)] \to \mathbb{F}_0$ be a strongly preinvex-FM with $\mathcal{H}(z) \succcurlyeq \widetilde{0}$. If $\mathcal{H}$ is fuzzy integrable and $\xi(., \, .)$ satisfies Condition C, then the following holds:*

$$\mathcal{H}\left(\frac{2u + \xi(v, \, u)}{2}\right) \widetilde{+} \frac{\omega}{12} \|\xi(v,u)\|^2 \preccurlyeq \frac{1}{\xi(v, \, u)} \, (FR) \int_u^{u+\xi(v, \, u)} \mathcal{H}(z) dz \preccurlyeq \frac{\mathcal{H}(u) \widetilde{+} \mathcal{H}(v)}{2} \widetilde{-} \frac{\omega}{6} \|\xi(v,u)\|^2. \tag{41}$$

If $\mathcal{H}$ is preconcave FM then, inequality (41) reduces to the following inequality:

$$\mathcal{H}\left(\frac{2u + \xi(v, \, u)}{2}\right) \widetilde{+} \frac{\omega}{12} \|\xi(v,u)\|^2 \succcurlyeq \frac{1}{\xi(v, \, u)} \, (FR) \int_u^{u+\xi(v, \, u)} \mathcal{H}(z) dz \succcurlyeq \frac{\mathcal{H}(u) \widetilde{+} \mathcal{H}(v)}{2} \widetilde{-} \frac{\omega}{6} \|\xi(v,u)\|^2.$$

Proof. Let $\mathcal{H} : [u, \, u + \xi(v, \, u)] \to \mathbb{F}_0$ be a strongly preinvex-FM. Then, by hypothesis, we have the following:

$$2\mathcal{H}\left(\frac{2u+\xi(v, \, u)}{2}\right)$$
$$\preccurlyeq \mathcal{H}(u + (1 - \tau)\xi(v, \, u))$$
$$\widetilde{+} \mathcal{H}(u + \tau\xi(v, \, u)) \widetilde{-} \frac{\omega}{2}(1 - 2\tau)^2 \|\xi(v,u)\|^2.$$

Therefore, for every $\gamma \in (0, 1]$, we have the following:

$$2\mathcal{H}_*\left(\frac{2u+\xi(v, u)\|}{2}, \gamma\right) \leq \mathcal{H}_*(u + (1-\tau)\xi(v, u), \gamma) + \mathcal{H}_*(u + \tau\xi(v, u), \gamma)$$
$$-\frac{\omega}{2}(1-2\tau)^2\|\xi(v,u)\|^2$$
$$2\mathcal{H}^*\left(\frac{2u+\xi(v, u)\|}{2}, \gamma\right) \leq \mathcal{H}^*(u + (1-\tau)\xi(v, u), \gamma) + \mathcal{H}^*(u + \tau\xi(v, u), \gamma)$$
$$-\frac{\omega}{2}(1-2\tau)^2\|\xi(v,u)\|^2.$$

Then

$$2\int_0^1 \mathcal{H}_*\left(\frac{2u+\xi(v, u)}{2}, \gamma\right)d\tau \leq \int_0^1 \mathcal{H}_*(u + (1-\tau)\xi(v, u), \gamma)d\tau + \int_0^1 \mathcal{H}_*(u + \tau\xi(v, u), \gamma)d\tau$$
$$-\frac{\omega}{6}\|\xi(v,u)\|^2,$$
$$2\int_0^1 \mathcal{H}^*\left(\frac{2u+\xi(v, u)}{2}, \gamma\right)d\tau \leq \int_0^1 \mathcal{H}^*(u + (1-\tau)\xi(v, u), \gamma)d\tau + \int_0^1 \mathcal{H}^*(u + \tau\xi(v, u), \gamma)d\tau$$
$$-\frac{\omega}{2}\|\xi(v,u)\|^2.$$

It follows that

$$\mathcal{H}_*\left(\frac{2u+\xi(v, u)\|}{2}, \gamma\right) + \frac{\omega}{12}\|\xi(v,u)\|^2 \leq \frac{1}{\xi(v, u)}\int_u^{u+\xi(v, u)}\mathcal{H}_*(z, \gamma)dz,$$
$$\mathcal{H}^*\left(\frac{2u+\xi(v, u)\|}{2}, \gamma\right) + \frac{\omega}{12}\|\xi(v,u)\|^2 \leq \frac{1}{\xi(v, u)}\int_u^{u+\xi(v, u)}\mathcal{H}^*(z, \gamma)dz.$$

That is

$$\left[\mathcal{H}_*\left(\frac{2u+\xi(v, u)}{2}, \gamma\right), \mathcal{H}^*\left(\frac{2u+\xi(v, u)}{2}, \gamma\right)\right] + \frac{\omega}{12}\|\xi(v,u)\|^2$$
$$\leq_I \frac{1}{\xi(v, u)}\left[\int_u^{u+\xi(v, u)}\mathcal{H}_*(z, \gamma)dz, \int_u^{u+\xi(v, u)}\mathcal{H}^*(z, \gamma)dz\right].$$

Thus,

$$\mathcal{H}\left(\frac{2u+\xi(v, u)}{2}\right) + \frac{\omega}{12}\|\xi(v,u)\|^2 \preccurlyeq \frac{1}{\xi(v, u)} (FR)\int_u^{u+\xi(v, u)}\mathcal{H}(z)dz. \tag{42}$$

In a similar way as above, we have the following:

$$\frac{1}{\xi(v, u)} (FR)\int_u^{u+\xi(v, u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \widetilde{\mp} \mathcal{H}(v)}{2} - \frac{\omega}{6}\|\xi(v,u)\|^2. \tag{43}$$

Combining (42) and (43), we have the following:

$$\mathcal{H}\left(\frac{2u+\xi(v, u)}{2}\right) \widetilde{+} \frac{\omega}{12}\|\xi(v,u)\|^2 \preccurlyeq \frac{1}{\xi(v, u)} (FR)\int_u^{u+\xi(v, u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \widetilde{\mp} \mathcal{H}(v)}{2} \widetilde{-} \frac{\omega}{6}\|\xi(v,u)\|^2.$$

This completes the proof. □

Remark 4. *If $\omega = 0$, then Theorem 11 reduces to the result for preinvex convex-FM as follows:*

$$\mathcal{H}\left(\frac{2u+\xi(v, u)}{2}\right) \preccurlyeq \frac{1}{\xi(v, u)} (FR)\int_u^{u+\xi(v, u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \widetilde{\mp} \mathcal{H}(v)}{2}.$$

If $\xi(v, u) = v - u$, then Theorem 11 reduces to the result for strongly convex-FM as follows:

$$\mathcal{H}\left(\frac{u+v}{2}\right) \widetilde{+} \frac{\omega}{12}v - u^2 \preccurlyeq \frac{1}{v - u} (FR)\int_u^v \mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \widetilde{\mp} \mathcal{H}(v)}{2} \widetilde{-} \frac{\omega}{6}\|v - u\|^2.$$

If $\xi(v, u) = v - u$ and $\omega = 0$, then Theorem 11 reduces to the result for convex-FM in [55] as follows:

$$\mathcal{H}\left(\frac{u+v}{2}\right) \preccurlyeq \frac{1}{v-u} \ (FR) \int_u^v \mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \, \widetilde{+} \, \mathcal{H}(v)}{2}. \tag{44}$$

If $\mathcal{H}_*(u, \gamma) = \mathcal{H}^*(v, \gamma)$ with $\omega = 0$ and $\gamma = 1$, then Theorem 11 reduces to the result for preinvex function as follows (see [36]):

$$\mathcal{H}\left(\frac{2u+\xi(v, u)}{2}\right) \leq \frac{1}{\xi(v, u)} \ (R) \int_u^{u+\xi(v, u)} \mathcal{H}(z)dz \leq \frac{\mathcal{H}(u) + \mathcal{H}(v)}{2}. \tag{45}$$

If $\mathcal{H}_*(u, \gamma) = \mathcal{H}^*(v, \gamma)$ with $\xi(v, u) = v - u$, $\omega = 0$ and $\gamma = 1$, then Theorem 11 reduces to the result for convex function as follows (see [42,43]):

$$\mathcal{H}\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \ (R) \int_u^v \mathcal{H}(z)dz \leq \frac{\mathcal{H}(u) + \mathcal{H}(v)}{2}. \tag{46}$$

Example 5. *We consider the fuzzy-IVF $\mathcal{H} : [u, \ u + \xi(v, \ u)] = [0, \ \xi(2, \ 0)] \to \mathbb{F}_0$ defined by the following:*

$$\mathcal{H}(z)(\sigma) = \begin{cases} \frac{\sigma}{2z^2}, & \sigma \in [0, \ 2z^2], \\ \frac{4z^2-\sigma}{2z^2}, & \sigma \in (2z^2, \ 4z^2], \\ 0, & otherwise, \end{cases}$$

Then, for each $\gamma \in [0, \ 1]$, we have $\mathcal{H}_\gamma(z) = [2\gamma z^2, (4-2\gamma)z^2]$. Since for each $\gamma \in [0,1]$, $\mathcal{H}_*(z, \gamma) = 2\gamma z^2$, $\mathcal{H}^*(z, \ \gamma) = (4-2\gamma)z^2$ are preinvex functions w.r.t. $\xi(v, \ u) = v - u$ and $\omega = \frac{2}{3}\gamma$. Hence $\mathcal{H}(z)$ is preinvex fuzzy-IVF w.r.t. $\xi(v, \ u) = v - u$. We now compute the following:

$$\mathcal{H}_*\left(\frac{2u+\xi(v, \ u)}{2}, \ \gamma\right) + \frac{\omega}{12}\|\xi(v,u)\|^2 = \mathcal{H}_*(1, \ \gamma) = \frac{8\gamma}{3},$$

$$\frac{1}{\xi(v, \ u)} \int_u^{u+\xi(v, \ u)} \mathcal{H}_*(z, \ \gamma)dz = \frac{1}{2} \int_0^2 2\gamma z^2 dz = \frac{8\gamma}{3},$$

$$\frac{\mathcal{H}_*(u,\gamma)+\mathcal{H}_*(v, \ \gamma)}{2} - \frac{\omega}{6}\|\xi(v,u)\|^2 = \frac{32\gamma}{9},$$

for all $\gamma \in [0, \ 1]$. That means the following holds:

$$\frac{8\gamma}{3} \leq \frac{8\gamma}{3} \leq \frac{32\gamma}{9}.$$

Similarly, it can be easily shown that the following holds:

$$\mathcal{H}^*\left(\frac{2u+\xi(v, \ u)}{2}, \ \gamma\right) \leq \frac{1}{\xi(v, \ u)} \int_u^{u+\xi(v, \ u)} \mathcal{H}^*(z, \ \gamma)dz \leq \frac{\mathcal{H}^*(u, \ \gamma) + \mathcal{H}^*(v, \ \gamma)}{2}.$$

for all $\gamma \in [0, \ 1]$, such that we have the following:

$$\mathcal{H}^*\left(\frac{2u+\xi(v, \ u)}{2}, \ \gamma\right) + \frac{\omega}{12}\xi(v,u)^2 = \mathcal{H}_*(1, \ \gamma) = \frac{36-16\gamma}{9},$$

$$\frac{1}{\xi(v, \ u)} \int_u^{u+\xi(v, \ u)} \mathcal{H}^*(z, \ \gamma)dz = \frac{1}{2} \int_0^2 (4-2\gamma)z^2 dz = \frac{8(2-\gamma)}{3},$$

$$\frac{\mathcal{H}^*(u, \ \gamma)+\mathcal{H}^*(v, \ \gamma)}{2} - \frac{\omega}{6}\|\xi(v,u)\|^2 = \frac{72-22\gamma}{9}.$$

From which, it follows that

$$\frac{36-16\gamma}{9} \leq \frac{8(2-\gamma)}{3} \leq \frac{72-22\gamma}{9},$$

that is

$$\left[\frac{8\gamma}{3}, \frac{36-16\gamma}{9}\right] \ \leq_I \ \left[\frac{8\gamma}{3}, \frac{8(2-\gamma)}{3}\right] \ \leq_I \ \left[\frac{32\gamma}{9}, \frac{72-22\gamma}{9}\right], \ for \ all \ \gamma \in [0, 1],$$

and hence, the Theorem 11 is verified.

5. Conclusions

In this study, we introduced and studied a new class of preinvex-FMs called strongly preinvex-FMs. Using Condition C, we obtained the equivalence relation between strongly preinvex- and strongly invex-FMs. To characterize the optimality condition of the sum of preinvex-FMs and strongly preinvex-FMs, we introduced the strong fuzzy mixed variational-like inequality. Moreover, we established a strong relationship between strongly preinvex-FM and the Hermite–Hadamard inequality. There is much room for further study to explore this concept in fuzzy convex and non-convex theory, such as the existence of a unique solution of strong fuzzy mixed variational-like inequalities and some iterative algorithms, which can also obtained under some mild conditions. From last two sections, we can conclude that these classes of FMs will play an important and significant role in fuzzy optimization and their related areas.

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