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Delayed Renewal Process with Uncertain Random Inter-Arrival Times

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Abstract: An uncertain random variable is a tool used to research indeterminacy quantities involving randomness and uncertainty. The concepts of an 'uncertain random process' and an 'uncertain random renewal process' have been proposed in order to model the evolution of an uncertain random phenomena. This paper designs a new uncertain random process, called the uncertain random delayed renewal process. It is a special type of uncertain random renewal process, in which the first arrival interval is different from the subsequent arrival interval. We discuss the chance distribution of the uncertain random delayed renewal process. Furthermore, an uncertain random delay renewal theorem is derived, and the chance distribution limit of long-term expected renewal rate of the uncertain random delay renewal system is proved. Then its average uncertain random delay renewal rate is obtained, and it is proved that it is convergent in the chance distribution. Finally, we provide several examples to illustrate the consistency with the existing conclusions.

Keywords: delayed renewal process; uncertain random process; chance theory

Citation: Wang, X.; Shi, G.; Sheng, Y. Delayed Renewal Process with Uncertain Random Inter-Arrival Times. *Symmetry* **2021**, *13*, 1943. https://doi.org/10.3390/sym13101943

Academic Editor: Abraham A. Ungar

Received: 18 September 2021
Accepted: 12 October 2021
Published: 15 October 2021

1. Introduction

The delayed renewal process is a variation of the normal renewal process, which allows the first arrival time to be different from other processes. Traditionally, the arrival interval is regarded as a random variable, so the classical renewal process is also called a random renewal process. In practice, the probability theory can be applied—the estimated probability distribution is close to the cumulative frequency. However, sometimes we cannot get the real frequency. In this case, we have to invite some domain experts to give "possibility" in every event. Because this "possibility" is usually much larger than the range of probability distribution, it can not deal with probability theory [1]. To solve these problems, Liu [2] founded the uncertainty theory in 2007 and refined in 2009 [3]. Under the uncertainty theory framework, Liu [4] proposed a definition of uncertain process, and then an uncertain renewal process was proposed to simulate the sudden jump in uncertain systems. Then, Liu [5] further discusses this process and applies it to insurance models. In addition, Yao and Li [6] proposed an uncertain alternating renewal process, in which the closing time and opening time are regarded as uncertain variables. Zhang et al. [7] proposed an uncertain delayed renewal process.

In many practical problems, there are both uncertainty and randomness in complex systems. To describe such a system, Liu [8] founded the chance theory and proposed chance measure, defined uncertain random variables, gave their chance distribution, and defined the expected value and variance of uncertain random variables. Then, Liu [9] proposed the operation law of uncertain random variables. Following that, Yao and Gao [10], Gao and Sheng [11], and Sheng et al. [12] verified some laws of large numbers of uncertain random variable sequences based on different assumptions. Gao and Yao [13] researched an uncertain random process and an uncertain random renewal process. Yao and Zhou [14] further studied an uncertain random renewal reward process and applied it to the block replacement policy.

In this paper, we mainly study a delayed renewal process in a hybrid environment. In fact, all uncertain variables and uncertain random variables given in both uncertainty theory and chance theory are symmetrical. Therefore, this paper studies the uncertain random delayed renewal process under the framework of symmetry, and gives some properties of the uncertain random delayed renewal process, some renewal theorems, and delayed renewal rates. The contributions of this paper have three aspects. Firstly, the concept of the uncertain random delay renewal process is proposed, regarding the arrival interval as uncertain random variables, and allowing the chance distribution of the first arrival interval to be different from other times. Secondly, we prove a basic delay renewal theorem for the uncertain random delay renewal process, we discuss this uncertain random delay renewal process and average delay renewal rate. Thirdly, we study some properties of this process. Meanwhile, we prove that the average delayed renewal rate converges under the chance distribution. The rest of the paper is structured as follows. In Section 2, this paper introduces the preliminary knowledge of uncertain variables and uncertain random variables. In Section 3, we discuss the concept of the uncertain random delayed renewal process. In Section 4, we discuss the chance distribution of the uncertain random delayed renewal processes and some theorems about the average delayed renewal rate. Finally, in Section 5, conclusions are given.

2. Preliminaries

2.1. Uncertain Variable

Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure [2] if it satisfies the following axioms: (1) Normality Axiom: $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ. (2) Duality axiom: $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ. (3) Subadditivity axiom: For every countable sequence of events $\Lambda_1,\Lambda_2,\cdots$, we have $\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$. Besides, let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{i=1}^{\infty}\Lambda_k\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$ where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$, respectively, so the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by Liu [15].

Definition 1 (Liu [2])**.** *Let ξ be an uncertain variable. Then its uncertainty distribution is*

$$\Phi(x)=\mathcal{M}\{\xi\le x\}$$

for any real number x.

Definition 2 (Liu [15])**.** *The uncertain variables $\xi_1,\xi_2,\cdots,\xi_m$ are independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i\in B_i)\right\}=\bigwedge_{k=1}^{m}\mathcal{M}\{\xi_i\in B_i\}$$

for any Borel sets $B_1,B_2,\cdots,B_m$ of real numbers.

Lemma 1 (Liu [5])**.** *Let $\Phi_1,\Phi_2,\cdots,\Phi_n$ be uncertainty distributions of independent uncertain variables $\xi_1,\xi_2,\cdots,\xi_n$, respectively. If f is a strictly increasing function, then $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ is an uncertain variable with uncertainty distribution*

$$\Phi(x)=\sup_{f(x_1,x_2,\cdots,x_n)=x}\min_{1\le i\le n}\Phi_i(x_i).$$

Definition 3 (Liu [2])**.** *The expected value of an uncertain variable ξ is*

$$E[\xi]=\int_0^{+\infty}\mathcal{M}\{\xi\ge x\}\mathrm{d}x-\int_{-\infty}^{0}\mathcal{M}\{\xi\le x\}\mathrm{d}x,$$

where at least one of the two integrals is finite.

Let Φ be uncertainty distribution of an uncertain variable ξ. Then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x.$$

Lemma 2 (Liu and Ha [16])**.** *Let $\Phi_1, \Phi_2, \cdots, \Phi_n$ be regular uncertainty distributions of independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha.$$

2.2. Uncertain Random Variable

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, let $(\Omega, \mathcal{A}, \Pr)$ be a probability space, then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is defined a chance space.

Definition 4 (Liu [8])**.** *Let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event on $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$. Then the chance measure of Θ is*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\}\mathrm{d}r.$$

The chance measure satisfies the following four properties [8,17]: (1) normality, i.e., $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$. (2) Duality, i.e., $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for and event Θ. (3) Monotonicity, i.e., $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$.. (4) Subadditivity, i.e., $\mathrm{Ch}\{\bigcup_{i=1}^{\infty} \Theta_i\} \leq \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}$ for a sequence of events $\Theta_1, \Theta_2, \cdots$.

Definition 5 (Liu [8])**.** *An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event for any Borel set B.*

Definition 6 (Liu [9])**.** *Let ξ be an uncertain random variable. Then its chance distribution is*

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\}$$

for any $x \in \Re$.

Lemma 3 (Liu [9])**.** *Let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables, and let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively. Then chance distribution of the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is*

$$\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m)\mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m),$$

where $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

Furthermore, the expected value operator of an uncertain random variable and a mean chance of an uncertain random event in [9] were given.

Definition 7 (Liu [16])**.** *Let ξ be an uncertain random variable. Then its expected value is*

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Let Φ denote the chance distribution of ξ, Liu [9] proved a formula to calculate the expected value of the uncertain random variable with chance distribution if $E[\xi]$ exists, then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x.$$

Lemma 4 (Liu [9])**.** *Let $\Psi_1, \Psi_2, \cdots, \Psi_m$ be probability distributions of independent random variables $\eta_1, \eta_2, \cdots, \eta_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables, then the expected value of the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is*

$$E[\xi] = \int_{\Re^m} E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m),$$

where $E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]$ is the expected value of $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

Definition 8 (Yao and Gao [10])**.** *Let $\Phi, \Phi_1, \Phi_2, \cdots$ be chance distributions of uncertain random variables $\xi, \xi_1, \xi_2, \cdots$, respectively. Then the sequence $\xi_1, \xi_2, \cdots$ is converged in distribution to ξ if*

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x)$$

for every number $x \in \Re$ at which Φ is continuous, which is denoted as $\xi_i \xrightarrow{d} \xi$.

3. Uncertain Random Delayed Renewal Process

Gao and Yao [13] researched an uncertain random process to describe the evolution of the indeterminacy phenomena with time or space in 2015. Then, they further defined the uncertain random renewal process, and the chance distribution of the average renewal rate is given. On this basis, the definition of the uncertain random delay renewal process was proposed, and its average delay renewal rate was discussed.

Let $\eta_1, \eta_2, \cdots$ be random variables with probability distributions $\Phi_1(x), \Phi_2(x), \cdots$ respectively and $\tau_1, \tau_2, \cdots$ be uncertain variables with uncertainty distributions $\Upsilon_1(y), \Upsilon_2(y), \cdots$, respectively. Denote by $f(\cdot, \cdot)$ a measurable function of two variables. Define $S_0 = 0$ and

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n), \quad n = 1, 2, \cdots. \tag{1}$$

Definition 9 (Gao and Yao [13])**.** *Assume that $\eta_1, \eta_2, \cdots$ are independently and identically distributed random variables, and $\tau_1, \tau_2, \cdots$ are iid uncertain variables. If the function $f > 0$, then $N_t = \max_{n\geq 0}\{n | S_n \leq t\}$ is called an uncertain random renewal process.*

Following, we propose a concept of the uncertain random delay renewal process to describe a both uncertain and random system with delay.

Definition 10. *Let $\eta_1, \eta_2, \cdots$ be independent random variables, and $\tau_1, \tau_2, \cdots$ be independent uncertain variables. Assume that $\eta_2, \eta_3, \cdots$ are identically distributed with common probability distribution $\Phi(x)$, which is different from $\Phi_1(x)$, and $\tau_2, \tau_3, \cdots$ are identically distributed with common uncertainty distribution $\Upsilon(y)$, which is different from $\Upsilon_1(y)$. If the function f is positive and strictly monotone, then $D_t = \max_{n\geq 0}\{n | S_n \leq t\}$ is called an uncertain random renewal process with inter-arrival times $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$.*

It follows from Definition 10 that an uncertain random delayed renewal process is just like an uncertain random ordinary one, except that the first arrival time is different from the other inter-arrival times. It is clear that D_t is an uncertain random variable, and we call D_t the uncertain random delayed renewal variable.

Remark 1. *An uncertain random delayed renewal process D_t degenerates to an uncertain random renewal process N_t if τ_1 has the common uncertainty distribution as τ_2, τ_3, $\cdots$, and η_1 has the common probability distribution as η_2, η_3, $\cdots$.*

Remark 2. *If each of the uncertain sequence $\tau_1, \tau_2, \cdots$ degenerates into a crisp number, then the associated uncertain random delayed renewal process becomes a stochastic delayed renewal process since the uncertain random sequence $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$ degenerates into a random sequence.*

Remark 3. *If each of the random sequence $\eta_1, \eta_2, \cdots$ degenerates into a crisp number, then the associated uncertain random delayed renewal process becomes an uncertain delayed renewal process since the uncertain random sequence $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$ degenerates into an uncertain sequence.*

Theorem 1. *Let D_t be a delayed renewal process with uncertain random inter-arrival times. $\tau_1, \tau_2, \cdots$ be independent uncertain variables. Assume that $\eta_2, \eta_3, \cdots$ are identically distributed with common probability distribution Φ, which is different from Φ_1, and $\tau_2, \tau_3, \cdots$ are identically distributed with common uncertainty distribution Υ, which is different from Υ_1, the function f is positive and strictly monotone. Then the chance distribution of D_t is*

$$\Psi_t(x) = 1 - \int_{\Re^\infty} \sup_{f(y_1,x_1)+\sum_{i=2}^{k+1} f(y_i,x_i)=t} \Big(\Upsilon_1(x_1) \bigwedge \Upsilon\Big(\frac{t-x_1}{\lfloor x \rfloor}\Big)\Big) \mathrm{d}\Phi_1(y_1)\mathrm{d}\Phi(y_2)\mathrm{d}\Phi(y_3)\cdots, \ x \in [0, +\infty),$$

where $\lfloor x \rfloor$ is the maximal integer less than or equal to x, we set $(t - x_1)/\lfloor x \rfloor = +\infty$ and $\Psi((t - x_1)/\lfloor x \rfloor) = 1$ when $\lfloor x \rfloor = 0$.

Proof. By Definition 4 and Definition 6, we have

$$\begin{aligned}
&\Psi_t(k) = \mathrm{Ch}\{D_t \le k\} = \mathrm{Ch}\{S_{k+1} \le t\} = 1 - \mathrm{Ch}\{S_{k+1} \le t\} \\
&= 1 - \mathrm{Ch}\Big\{f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_{k+1}, \tau_{k+1}) \le t\Big\} \\
&= 1 - \mathrm{Ch}\Big\{f(\eta_1, \tau_1) + \sum_{i=2}^{k+1} f(\eta_i, \tau_i) \le t\Big\} \\
&= 1 - \int_0^1 \Pr\Big\{\omega \in \Omega \mid \mathcal{M}\Big\{f(\eta_1(\omega), \tau_1) + \sum_{i=2}^{k+1} f(\eta_i(\omega), \tau_i) \le t\Big\} \ge r\Big\}\mathrm{d}r \\
&= 1 - \int_{\Re^{k+1}} \mathcal{M}\Big\{f(y_1, \tau_1) + \sum_{i=2}^{k+1} f(y_i, \tau_i) \le t\Big\}\mathrm{d}\Phi_1(y_1)\mathrm{d}\Phi(y_2)\cdots\mathrm{d}\Phi(y_{k+1})
\end{aligned}$$

for any integer $k \ge 2$. Using Lemma 1, we have

$$\begin{aligned}
&\mathcal{M}\Big\{f(y_1, \tau_1) + \sum_{i=2}^{k+1} f(y_i, \tau_i) \le t\Big\} \\
&= \sup_{f(y_1,x_1)+\sum_{i=2}^{k+1} f(y_i,x_i)=t} \Big(\Upsilon_1(x_1) \bigwedge \min_{i=2}^{k+1} \Upsilon(x_i)\Big) \\
&= \sup_{f(y_1,x_1)+\sum_{i=2}^{k+1} f(y_i,x_i)=t} \left(\Upsilon_1(x_1) \bigwedge \Upsilon\Big(\frac{t-x_1}{k}\Big)\right).
\end{aligned}$$

So we can obtain

$$\Psi_t(k) = 1 - \int_{\Re^{k+1}} \sup_{f(y_1,x_1)+\sum_{i=2}^{k+1} f(y_i,x_i)=t} \left(\Upsilon_1(x_1) \bigwedge \Upsilon\Big(\frac{t-x_1}{k}\Big)\right)\mathrm{d}\Phi_1(y_1)\mathrm{d}\Phi(y_2)\cdots\mathrm{d}\Phi(y_{k+1}).$$

We know that an uncertain random delay renewal process take integer values. So

$$\Psi_t(x) = \Psi_t(\lfloor x \rfloor)$$
$$=1 - \int_{\Re^{+\infty}} \sup_{f(y_1,x_1)+\sum_{i=2}^{\infty} f(y_i,x_i)=t} \left(\Upsilon_1(x_1) \bigwedge \Upsilon\Big(\frac{t-x_1}{\lfloor x \rfloor}\Big) \right) \mathrm{d}\Phi_1(y_1)\mathrm{d}\Phi(y_2)\mathrm{d}\Phi(y_3)\cdots,\ x \in [0,+\infty).$$

Thus the theorem is completed. $\square$

4. Elementary Uncertain Random Delayed Renewal Theorem

In the following, we prove an elementary uncertain random delayed renewal theorem. Note that this process D_t is the total renewal time before t. Therefore, D_t/t represents the average delayed renewal rate in the time interval $[0,t]$. Similar to the classical delayed renewal process, an important problem is to discuss the chance distribution of the average delayed renewal rate. In order to prove the main results, we first need two lemmaes.

Lemma 5 (Sheng et al. [12])**.** *Let $\eta_1, \eta_2, \cdots$ and $\tau_1, \tau_2, \cdots$ be independent random variables and independent uncertain variables, respectively. Assume that the function f is strictly monotone with the first argument. If for any $y \in \Re$, $\sum_{n=1}^{\infty} \frac{Var[f(\eta_n,y)]}{n^2} < \infty$ and meanwhile $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x,\tau_i)\mathrm{d}\Phi_i(x)$ exists in probability distribution. Then $\{S_n/n\}$ converges in chance distribution to*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x,\tau_i)\mathrm{d}\Phi_i(x) \quad as \quad n \to +\infty.$$

Lemma 6 (Kolmogorov's Large Number Law [18])**.** *Assume that η_1 has a different probability distribution from $\eta_2, \eta_3, \cdots$ which are identically distributed, and $E|\eta_1|$, $E|\eta_2|$ are finite. If $\sum_{n=1}^{\infty} \frac{Var[\eta_n]}{n^2} < \infty$, then the sequence $\frac{1}{n}\sum_{i=1}^{n} \eta_i$ converges almost sure to $E[\eta_2]$, which is indicated by*

$$\frac{1}{n}\sum_{i=1}^{n} \eta_i \to E\eta_2 \quad (a.s.).$$

Theorem 2. *Let $\eta_1, \eta_2, \cdots$ and $\tau_1, \tau_2, \cdots$ be independent random variables and independent uncertain variables, respectively. Assume that $\eta_2, \eta_3, \cdots$ are identically distributed with common probability distribution $\Phi(x)$, which is different from $\Phi_1(x)$, and $\tau_2, \tau_3, \cdots$ are identically distributed with common uncertainty distribution $\Upsilon(y)$, which is different from $\Upsilon_1(y)$. Let the function g be strictly monotone. If, for any $y \in \Re$, $E|g(\eta_1,y)|$, $E|g(\eta_2,y)|$ are finite, and $\sum_{n=1}^{\infty} \frac{Var[f(\eta_n,y)]}{n^2} < \infty$, then we have*

$$\frac{S_n}{n} \xrightarrow{d} \int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)$$

in the sense of convergence in chance distribution as $t \to +\infty$.

Proof. For any given $y \in \Re$, $g(\eta_1,y), g(\eta_2,y), \cdots$ are obviously independent random variables. It follows from Lemma 6 that, for any $y \in \Re$,

$$\frac{1}{n}\sum_{i=1}^{n} g(\eta_i,y) \to \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x) \quad (a.s.).$$

In addition, for each $x \in \Re$, we have

$$\mathcal{M}\{g(x,\tau_1) \le g(x,y)\} = \Upsilon_1(y)$$

and

$$\mathcal{M}\{g(x,\tau_i) \le g(x,y)\} = \Upsilon(y),\ i = 2,3,\cdots,$$

as a result of which, we have

$$\mathcal{M}\left\{\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x) \le \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x)\right\} = \Upsilon(y).$$

Further, it follows from Lemma 5 that

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x)\right\}\\
=&\mathrm{Ch}\left\{\lim_{n\to\infty} \frac{S_n}{n} \le \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x)\right\}\\
=&\mathrm{Ch}\left\{\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x) \le \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x)\right\}\\
=&\mathcal{M}\left\{\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x) \le \int_{-\infty}^{+\infty} g(x,y)\mathrm{d}\Phi(x)\right\}.
\end{aligned}$$

That is, the sequence $\{S_n/n\}$ converges in distribution to $\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)$. □

Theorem 3 (Uncertain Random Elementary Delayed Renewal Theorem)**.** *Assume D_t is an uncertain random delayed renewal process with inter-arrival times $g(\eta_1,\tau_1), g(\eta_2,\tau_2),\cdots$. If for any $y \in \Re$, $E|g(\eta_1,y)|$, $E|g(\eta_2,y)|$ are finite, and $\sum_{n=1}^{\infty} \frac{Var[g(\eta_n,y)]}{n^2} < \infty$, then we have*

$$\frac{D_t}{t} \xrightarrow{d} \left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1}$$

in the sense of convergence in chance distribution as $t \to +\infty$.

Proof. Since y is a continuous point of

$$\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1},$$

so we can obtain that $1/y$ is a continuous point of

$$\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x).$$

By Definition 10 that

$$\mathrm{Ch}\left\{\frac{D_t}{t} \le y\right\} = \mathrm{Ch}\{D_t \le ty\} = \mathrm{Ch}\{D_t \le \lfloor ty \rfloor\} = \mathrm{Ch}\left\{S_{\lfloor ty \rfloor+1} > t\right\} = \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{t}{\lfloor ty \rfloor+1}\right\}$$

where $\lfloor ty \rfloor$ represents the maximal integer less than or equal to ty. Note that, $\lfloor ty \rfloor \le ty < \lfloor ty \rfloor + 1$ and for each $y > 0$, $\lfloor ty \rfloor \to +\infty$ as $t \to +\infty$. Therefore, we have

$$\frac{\lfloor ty \rfloor}{\lfloor ty \rfloor+1}\frac{1}{y} \le \frac{t}{\lfloor ty \rfloor+1} < \frac{1}{y}$$

and

$$\mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{1}{y}\right\} \le \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{t}{\lfloor ty \rfloor+1}\right\} \le \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{\lfloor ty \rfloor}{\lfloor ty \rfloor+1}\frac{1}{y}\right\}.$$

Further, by Theorem 2 that

$$\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} \xrightarrow{d} \int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x), \quad t \to +\infty.$$

Since

$$\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor} = \frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} \cdot \frac{\lfloor ty \rfloor+1}{\lfloor ty \rfloor}$$

and

$$\frac{\lfloor ty \rfloor+1}{\lfloor ty \rfloor} \to 1 \quad as \quad t \to +\infty,$$

it is obtained that

$$\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor} \xrightarrow{d} \int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x), \quad \text{as } t \to +\infty.$$

Thus we can obtain

$$\begin{aligned}
&\lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{1}{y}\right\} = 1 - \lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} \le \frac{1}{y}\right\} \\
=&1 - \mathcal{M}\left\{\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x) \le \frac{1}{y}\right\} = 1 - \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} \ge y\right\} \\
=&\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} < y\right\} = \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} \le y\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{\lfloor ty \rfloor}{\lfloor ty \rfloor+1}\frac{1}{y}\right\} = \lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor} > \frac{1}{y}\right\} \\
=&1 - \lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor} \le \frac{1}{y}\right\} = 1 - \mathcal{M}\left\{\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x) \le \frac{1}{y}\right\} \\
=&1 - \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} \ge y\right\} = \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} < y\right\} \\
=&\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} \le y\right\}.
\end{aligned}$$

For any continuous point y of $\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1}$, we have

$$\lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{D_t}{t} \le y\right\} = \lim_{t\to+\infty} \mathrm{Ch}\left\{\frac{S_{\lfloor ty \rfloor+1}}{\lfloor ty \rfloor+1} > \frac{t}{\lfloor ty \rfloor+1}\right\} = \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1} \le y\right\}.$$

So, we can obtain that the average delayed renewal rate is

$$\frac{D_t}{t} \xrightarrow{d} \left(\int_{-\infty}^{+\infty} g(x,\tau_2)\mathrm{d}\Phi(x)\right)^{-1}$$

in the sense of convergence in chance distribution as $t \to +\infty$. □

Remark 4. *Assume that $\eta_1, \eta_2, \cdots$ are positive and independent random variables and η_1 has a different probability distribution from $\eta_2, \eta_3, \cdots$, which are identically distributed. Let D_t be a delayed renewal process with inter-arrival times $\eta_1, \eta_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E[\eta_2]}, \quad as \ t \to +\infty.$$

Remark 5. *Assume that $\tau_1, \tau_2, \cdots$ are positive and independent uncertain variables and τ_1 has a different uncertainty distribution from $\tau_2, \tau_3, \cdots$, which are identically distributed. Let D_t be a delayed renewal process with inter-arrival times $\tau_1, \tau_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{\tau_2}, \quad \textit{as} \;\; t \to +\infty.$$

Remark 6. *When an uncertain random delayed renewal process D_t degenerates to an uncertain random renewal process, then the average delayed renewal rate degenerates to the average renewal rate, i.e.,*

$$\frac{D_t}{t} \xrightarrow{d} \left(\int_{-\infty}^{+\infty} f(x, \tau_1) \mathrm{d}\Phi(x) \right)^{-1}, \quad \textit{as} \;\; t \to +\infty$$

which is consistent with the result of Gao and Yao [13].

Example 1. *Let $\eta_1, \eta_2, \cdots$ be positive and independent random variables and $\tau_1, \tau_2, \cdots$ be positive and independent uncertain variables, respectively. Let D_t be an uncertain random delayed renewal process with uncertain random inter-arrival times $\eta_1 + \tau_1, \eta_2 + \tau_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E[\eta_2] + \tau_2}, \quad \textit{as} \;\; t \to +\infty.$$

In fact, by Theorem 3, we have

$$\begin{aligned}
\frac{D_t}{t} &\xrightarrow{d} \left(\int_{-\infty}^{+\infty} (x + \tau_2) \mathrm{d}\Phi(x) \right)^{-1} \\
&= \left(\int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x) + \int_{-\infty}^{+\infty} \tau_2 \mathrm{d}\Phi(x) \right)^{-1} \\
&= \left(E[\eta_2] + \tau_2 \int_{-\infty}^{+\infty} \mathrm{d}\Phi(x) \right)^{-1} \\
&= (E[\eta_2] + \tau_2)^{-1} = \frac{1}{E[\eta_2] + \tau_2}, \quad \text{as } t \to +\infty.
\end{aligned}$$

Further, by Remark 6, if random variables $\eta_1, \eta_2, \cdots$ are also identically distributed and uncertain variables $\tau_1, \tau_2, \cdots$ are also identically distributed, then we have

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E[\eta_1]\tau_1}, \quad \text{as} \;\; t \to +\infty.$$

Example 2. *Let $\eta_1, \eta_2, \cdots$ be positive and independent random variables and $\tau_1, \tau_2, \cdots$ be positive and independent uncertain variables, respectively. Let D_t be an uncertain random delayed renewal process with uncertain random inter-arrival times $\eta_1\tau_1, \eta_2\tau_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E[\eta_2]\tau_2}, \quad \textit{as} \;\; t \to +\infty.$$

In fact, by Theorem 3, we have

$$\begin{aligned}
\frac{D_t}{t} &\xrightarrow{d} \left(\int_{-\infty}^{+\infty} (x\tau_2) \mathrm{d}\Phi(x) \right)^{-1} = \left(\tau_2 \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x) \right)^{-1} \\
&= (E[\eta_2]\tau_2)^{-1} = \frac{1}{E[\eta_2]\tau_2}, \quad \text{as } t \to +\infty.
\end{aligned}$$

By Remark 6, further, if random variables $\eta_1, \eta_2, \cdots$ are also identically distributed and uncertain variables $\tau_1, \tau_2, \cdots$ are also identically distributed, then we have

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E[\eta_1] + \tau_1}, \quad \text{as } t \to +\infty.$$

Example 3. *Let $\eta_1, \eta_2, \cdots$ be positive and independent random variables and $\tau_1, \tau_2, \cdots$ be positive and independent uncertain variables, respectively. Let D_t be an uncertain random delayed renewal process with uncertain random inter-arrival times $\eta_1/\tau_1, \eta_2/\tau_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{\tau_2}{E[\eta_2]}, \quad \textit{as } t \to +\infty.$$

In fact, by Theorem 3, we have

$$\begin{aligned} \frac{D_t}{t} &\xrightarrow{d} \left(\int_{-\infty}^{+\infty} \frac{x}{\tau_2} \mathrm{d}\Phi(x) \right)^{-1} = \left(\frac{1}{\tau_2} \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x) \right)^{-1} \\ &= \left(\frac{1}{\tau_2} E[\eta_2] \right)^{-1} = \frac{\tau_2}{E[\eta_2]}, \quad \text{as } t \to +\infty. \end{aligned}$$

Further, by Remark 6, if random variables $\eta_1, \eta_2, \cdots$ are also identically distributed and uncertain variables $\tau_1, \tau_2, \cdots$ are also identically distributed, then we have

$$\frac{D_t}{t} \xrightarrow{d} \frac{\tau_1}{E[\eta_1]}, \quad \text{as } t \to +\infty.$$

Example 4. *Let $\eta_1, \eta_2, \cdots$ be positive and independent random variables and $\tau_1, \tau_2, \cdots$ be positive and independent uncertain variables, respectively. Let D_t be an uncertain random delayed renewal process with uncertain random inter-arrival times $\tau_1/\eta_1, \tau_2/\eta_2, \cdots$. Then we have*

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E\left[\frac{1}{\eta_2}\right]\tau_2}, \quad \textit{as } t \to +\infty.$$

In fact, by Theorem 3, we have

$$\begin{aligned} \frac{D_t}{t} &\xrightarrow{d} \left(\int_{-\infty}^{+\infty} \frac{\tau_2}{x} \mathrm{d}\Phi(x) \right)^{-1} = \left(\tau_2 \int_{-\infty}^{+\infty} \frac{1}{x} \mathrm{d}\Phi(x) \right)^{-1} \\ &= \left(E\left[\frac{1}{\eta_2}\right] \tau_2 \right)^{-1} = \frac{1}{E\left[\frac{1}{\eta_2}\right]\tau_2}, \quad \textit{as } t \to +\infty. \end{aligned}$$

Further, by Remark 6, if random variables $\eta_1, \eta_2, \cdots$ are also identically distributed and uncertain variables $\tau_1, \tau_2, \cdots$ are also identically distributed, then we have

$$\frac{D_t}{t} \xrightarrow{d} \frac{1}{E\left[\frac{1}{\eta_1}\right]\tau_1}, \quad \textit{as } t \to +\infty.$$

5. Conclusions

In this paper, to describe an uncertain random process with a delayed—by employing uncertain random variables to describe the inter-arrival times—the uncertain random delayed renewal process was proposed and the chance distribution of the delay renewal process was obtained. Furthermore, we studied the average renewal rate of the special process and a useful theorem named the uncertain random elementary delay renewal theorem was established. We found that the average delayed renewal rate is convergent in chance distribution. Finally, we provided some examples to illustrate the uncertain random delayed renewal theorem.

Author Contributions: Conceptualization, X.W. and G.S.; methodology, X.W.; software, G.S.; validation, X.W., G.S. and Y.S.; formal analysis, X.W.; investigation, X.W.; resources, Y.S.; data curation, X.W.; writing—original draft preparation, X.W.; writing—review and editing, X.W., G.S. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grants Nos. 12061072, 62162059).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors especially thank the editors and anonymous referees for their kindly review and helpful comments. In addition, the authors would like to acknowledge the gracious support of this work by the National Natural Science Foundation of China—Joint Key Program of Xinjiang (Grants No. U1703262) and Cooperative and collaborative education project of the Ministry of Education (Grants No. 201902146050).

Conflicts of Interest: We declare that we have no relevant or material financial interests that relate to the research described in this paper. The manuscript has neither been published before, nor has it been submitted for consideration of publication in another journal.

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