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New Fractional Dynamic Inequalities via Conformable Delta Derivative on Arbitrary Time Scales

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Abstract: Building on the work of Josip Pečarić in 2013 and 1982 and on the work of Srivastava in 2017. We prove some new α-conformable dynamic inequalities of Steffensen-type on time scales. In the case when $\alpha = 1$, we obtain some well-known time scale inequalities due to Steffensen inequalities. For some specific time scales, we further show some relevant inequalities as special cases: α-conformable integral inequalities and α-conformable discrete inequalities. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Steffensen's inequality; dynamic inequality; α-conformable calculus; time scale

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1. Introduction

Every nonempty arbitrary closed subset of the real numbers is called time-scale $\mathbb{T}$. We suppose that $\mathbb{T}$ has a standard topology on real numbers $\mathbb{R}$. More details about the definitions and concepts of time-scales calculus and α-conformable calculus can be found in [1–18]. We suppose that $\sigma : \mathbb{T} \to \mathbb{T}$, the forward jump operator, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad t \in \mathbb{T}, \tag{1}$$

and that $\rho : \mathbb{T} :\to \mathbb{T}$, the backward jump operator, by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \qquad t \in \mathbb{T}. \tag{2}$$

In (1) and (2), we set $\sup \mathbb{T} = \inf \emptyset$ (i.e., $\sigma(t) = t$ if t is the minimum of $\mathbb{T}$) and $\inf \mathbb{T} = \sup \emptyset$ (i.e., $\rho(t) = t$ if t is the maximum), where $\emptyset$ is the empty set.

Definition 1. *Let $\eta : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$, and $\alpha \in (0,1]$. For $t > 0$, we define $T^{\Delta}_{\alpha}(\eta)(t)$ as the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ neighbrhood $U_t \subset \mathbb{T}$ of t, $\delta > 0$, such that*

$$|[\eta(\sigma(t)) - \eta(s)]t^{1-\alpha} - T^{\Delta}_{\alpha}(\eta)(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U_t$. We call $T^{\Delta}_{\alpha}(\eta)(t)$ the conformable fractional derivative of η of order α at t, and we define a conformable fractional derivative on $\mathbb{T}$ at 0, as $T^{\Delta}_{\alpha}(\eta)(0) = \lim_{t \longrightarrow 0^+} T^{\Delta}_{\alpha}(\eta)(t)$.

For any time-scales $\mathbb{T}$, we have

$$(\eta)^{\Delta_\alpha}(t) = (\eta)^{\Delta}(t)t^{1-\alpha},$$
$$\int_a^b \eta(t)\Delta_\alpha t = \int_a^b \eta(t)t^{\alpha-1}\Delta t.$$

Recently, a massive range of dynamic inequalities on time scales has been investigated by using exclusive authors who have been inspired with the aid of a few applications (see [19–32]). Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities. Some authors created different results regarding fractional calculus on time scales to provide associated dynamic inequalities (see [33–36]).

In [37], Pečarić introduced the following result.

Theorem 1. *Let $\hat{\lambda}$, $\hat{\lambda_1}$, $\hat{\lambda_2} : [\ell_1, \ell_2] \to \mathbb{R}$ be integrable functions on $[\ell_1, \ell_2]$ such that $\hat{\lambda}/\hat{\lambda_2}$ is nonincreasing and $\hat{\lambda_2}$ is nonnegative. Furthermore, let $0 \leq \hat{\lambda_1}(\jmath) \leq 1$ $\forall \jmath \in [\ell_1, \ell_2]$. Then,*

$$\int_{\ell_2 - \hat{\Im}}^{\ell_2} \hat{\lambda}(\jmath) d\jmath \leq \int_{\ell_1}^{\ell_2} \hat{\lambda}(\jmath) \hat{\lambda_1}(\jmath) d\jmath$$

where $\hat{\Im}$ gives us the solution of

$$\int_{\ell_2 - \hat{\Im}}^{\ell_2} \hat{\lambda_2}(\jmath) d\jmath = \int_{\ell_1}^{\ell_2} \hat{\lambda_2}(\jmath) \hat{\lambda_1}(\jmath) d\jmath.$$

We obtain the reverse of (4), if $f\hat{(\jmath)}/h\hat{(\jmath)}$ is nondecreasing.

Several inequalities such as Hardy's inequality [38,39], Hermite–Hadamard's inequality [40–42], Opial's inequality [43,44], and Steffensen's inequality [45] have been introduced. For example, in 2016, Anderson [46] gave an α-conformable version of Steffensen inequality as follows:

Theorem 2. *[46] [Fractional Steffensen's inequality] Suppose $\alpha \in (0, 1]$ and ℓ_1, $\ell_2 \in \mathbb{R}$ such that $0 \leq \ell_1 \leq \ell_2$. Suppose that $\prod : [\ell_1, \ell_2] \to [0, \infty)$ and $\Gamma : [\ell_1, \ell_2] \to [0, 1]$ are α-fractional integrable functions on $[\ell_1, \ell_2]$ with ϖ decreasing. We have*

$$\int_{\ell_2 - \mho}^{\ell_2} \varpi(\jmath) d_\alpha \jmath \leq \int_{\ell_1}^{\ell_2} \varpi(\jmath) \Gamma(\jmath) d_\alpha \jmath \leq \int_{\ell_1}^{\ell_1 + \mho} \varpi(\jmath) d_\alpha \jmath,$$

where $\mho = \frac{\alpha(\ell_2 - \ell_1)}{\ell_2^\alpha - \ell_1^\alpha} \int_{\ell_1}^{\ell_2} \Gamma(\jmath) d_\alpha \jmath \in [0, \ell_2 - \ell_1]$.

In 2017, Sarikaya et al. [45] gave a generalization for Theorem 2 as follows:

Theorem 3. *Suppose that $\alpha \in (0, 1]$ and ℓ_1, $\ell_2 \in \mathbb{R}$ such that $0 \leq \ell_1 \leq \ell_2$. Suppose that $\prod, \Gamma, \Xi : [\ell_1, \ell_2] \to [0, \infty)$ are α-fractional integrable functions on $[\ell_1, \ell_2]$ with ϖ decreasing and $0 \leq \Gamma \leq \Xi$. We have*

$$\int_{\ell_2 - \mho}^{\ell_2} \Xi(\jmath) \varpi(\jmath) d_\alpha \jmath \leq \int_{\ell_1}^{\ell_2} \varpi(\jmath) \Gamma(\jmath) d_\alpha \jmath \leq \int_{\ell_1}^{\ell_1 + \mho} \Xi(\jmath) \varpi(\jmath) d_\alpha \jmath,$$

where $\mho = \frac{(\ell_2 - \ell_1)}{\int_{\ell_1}^{\ell_2} \Xi(\jmath) d_\alpha \jmath} \int_{\ell_1}^{\ell_2} \Gamma(\jmath) d_\alpha \jmath \in [0, \ell_2 - \ell_1]$.

In this article, we explore new generalizations of the integral Steffensen inequality given in [37,47,48] via a conformable integral on a general time-scale measure space. We also retrieve some of the integral inequalities known in the literature as special cases of our tests.

2. Main Results

Next, we enroll the accompanying suppositions for the verifications of our primary outcomes:

($\Re_1$) $([\ell_1, \ell_2]_{\mathbb{T}}, \mathfrak{B}([\ell_1, \ell_2]_{\mathbb{T}}), \hat{\mu})$ is a time-scale measure space with a positive σ-finite measure on $\mathfrak{B}([\ell_1, \ell_2]_{\mathbb{T}})$.

($\Re_2$) $\beth, \Phi, F : [\ell_1, \ell_2]_{\mathbb{T}} \to \mathbb{R}$ is $\Delta_\alpha \jmath$-integrable functions on $[\ell_1, \ell_2]_{\mathbb{T}}$.

($\Re_3$) $\beth/F$ is nonincreasing, and F is nonnegative.

($\Re_4$) $0 \leq \Phi(\jmath) \leq 1$ for all $\jmath \in [\ell_1, \ell_2]_{\mathbb{T}}$.

($\Re_5$) $\hat{\Im}$ is a real number.

($\Re_6$) $\beth$ is nonincreasing.

($\Re_7$) $1 \leq \Phi(\jmath) \leq F(\jmath)$ for all $\jmath \in [\ell_1, \ell_2]_{\mathbb{T}}$.

($\Re_8$) $0 \leq \psi(\jmath) \leq \Phi(\jmath) \leq F(\jmath) - \psi(\jmath)$ for all $\jmath \in [\ell_1, \ell_2]_{\mathbb{T}}$.

($\Re_9$) $0 \leq M \leq \Phi(\jmath) \leq 1 - M$ for all $\jmath \in [\ell_1, \ell_2]_{\mathbb{T}}$.

($\Re_{10}$) $0 \leq \psi(\jmath) \leq \Phi(\jmath) \leq 1 - \psi(\jmath)$ for all $\jmath \in [\ell_1, \ell_2]_{\mathbb{T}}$.

$\hat{\Im}$ is the solution of the equations listed below:

($\Re_{11}$) $\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1, \ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath$.

($\Re_{12}$) $\int_{[\ell_2-\hat{\Im}, \ell_2]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1, \ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath$.

($\Re_{13}$) $\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1, \ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath = \int_{[\ell_2-\hat{\Im}, \ell_2]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath$.

($\Re_{14}$) $\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1, \ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath$.

($\Re_{15}$) $\int_{[\ell_2-\hat{\Im}, \ell_2]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1, \ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath$.

Now, we are ready to state and prove our main results.

Theorem 4. *Let $\Re_1$, $\Re_2$, $\Re_3$, $\Re_4$, and $\Re_{11}$ be satisfied. Then,*

$$\int_{[\ell_1, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \leq \int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} \beth(\jmath)\Delta_\alpha \jmath. \tag{3}$$

We obtain the reverse of (3) if $\beth/F$ is nondecreasing.

Proof.

$$\begin{aligned}
&\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} \beth(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)[1-\Phi(\jmath)]\frac{\beth(\jmath)}{F(\jmath)}\Delta_\alpha \jmath - \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&\geq \frac{\beth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)[1-\Phi(\jmath)]\Delta_\alpha \jmath - \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\beth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\left[\int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath\right] - \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\beth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\left[\int_{[\ell_1, \ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1, \ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath\right] - \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\beth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \int_{[\ell_1+\hat{\Im}, \ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\left(\frac{\beth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})} - \frac{\beth(\jmath)}{F(\jmath)}\right)\Delta_\alpha \jmath \geq 0.
\end{aligned}$$

The proof is complete. □

Corollary 1. *Putting $\mathbb{T} = \mathbb{R}$ in Theorem 4, we obtain*

$$\int_{[\ell_1, \ell_2]} \beth(\jmath)\Phi(\jmath)d_\alpha \jmath \leq \int_{[\ell_1, \ell_1+\hat{\Im}]} \beth(\jmath)d_\alpha \jmath.$$

Remark 1. *In the case of* $\alpha = 1$ *in Corollary 1, we recollect [37] [Theorem 1].*

Theorem 5. *Assumptions* $\Re_1$, $\Re_2$, $\Re_3$, $\Re_4$, *and* $\Re_{12}$ *imply*

$$\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Delta_\alpha \jmath \leq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath. \tag{4}$$

We obtain the reverse of (4) *if* $\gimel/F$ *is nondecreasing.*

Proof.

$$\begin{aligned}
&\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)[1-\Phi(\jmath)]\frac{\gimel(\jmath)}{F(\jmath)}\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&\leq \frac{\gimel(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)[1-\Phi(\jmath)]\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\gimel(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\left[\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath - \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath\right] - \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\gimel(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\left[\int_{[\ell_1,\ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath\right] - \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \frac{\gimel(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\
&= \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Phi(\jmath)\left(\frac{\gimel(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})} - \frac{\gimel(\jmath)}{F(\jmath)}\right)\Delta_\alpha \jmath \leq 0.
\end{aligned}$$

□

Corollary 2. *Putting* $\mathbb{T} = \mathbb{R}$ *in Theorem 5,*

$$\int_{[\ell_2-\hat{\Im},\ell_2]} \gimel(\jmath)d_\alpha \jmath \leq \int_{[\ell_1,\ell_2]} \gimel(\jmath)\Phi(\jmath)d_\alpha \jmath.$$

Remark 2. *In Corollary 2 and* $\alpha = 1$, *we recapture [37] [Theorem 2].*

We will need the following lemma to prove the subsequent results.

Lemma 1. *Let* $\Re_1$, $\Re_2$, *and* $\Re_5$ *hold such that*

$$\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath = \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath = \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} F(\jmath)\Delta_\alpha \jmath.$$

Then,

$$\begin{aligned}
\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \quad = \quad & \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Big(\gimel(\jmath)F(\jmath) - [\gimel(\jmath)-\gimel(\ell_1+\hat{\Im})][F(\jmath)-\Phi(\jmath)]\Big)\Delta_\alpha \jmath \\
& + \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} [\gimel(\jmath)-\gimel(\ell_1+\hat{\Im})]\Phi(\jmath)\Delta_\alpha \jmath,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \quad = \quad & \int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} [\gimel(\jmath)-\gimel(\ell_2-\hat{\Im})]\Phi(\jmath)\Delta_\alpha \jmath \\
& + \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \Big(\gimel(\jmath)F(\jmath) - [\gimel(\jmath)-\gimel(\ell_2-\hat{\Im})][F(\jmath)-\Phi(\jmath)]\Big)\Delta_\alpha \jmath.
\end{aligned} \tag{6}$$

Proof. The suppositions of the Lemma imply that

$$\ell_1 \leq \ell_1+\hat{\Im} \leq \ell_2 \qquad \text{and} \qquad \ell_1 \leq \ell_2-\hat{\Im} \leq \ell_2.$$

Now, we prove (5), and we see that

$$
\begin{aligned}
&\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Big(\gimel(\jmath)\digamma(\jmath) - [\gimel(\jmath) - \gimel(\ell_1+\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)\Delta_\alpha\jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&= \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Big(\gimel(\jmath)\digamma(\jmath) - \gimel(\jmath)\Phi(\jmath) - [\gimel(\jmath) - \gimel(\ell_1+\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)\Delta_\alpha\jmath \\
&\quad + \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&= \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \gimel(\ell_1+\hat{\Im})\,[\digamma(\jmath) - \Phi(\jmath)]\,\Delta_\alpha\jmath - \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&= \gimel(\ell_1+\hat{\Im})\left(\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \digamma(\jmath)\Delta_\alpha\jmath - \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath\right) - \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath. \qquad (7)
\end{aligned}
$$

Since

$$\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \digamma(\jmath)\Delta_\alpha\jmath = \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath,$$

we have

$$
\begin{aligned}
&\gimel(\ell_1+\hat{\Im})\left(\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \digamma(\jmath)\Delta_\alpha\jmath - \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath\right) - \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&= \gimel(\ell_1+\hat{\Im})\left(\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath - \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath\right) - \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&= \gimel(\ell_1+\hat{\Im}) \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha\jmath - \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath\Bigg) \\
&= \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} [\gimel(\ell_1+\hat{\Im}) - \gimel(\jmath)]\,\Phi(\jmath)\Delta_\alpha\jmath. \qquad (8)
\end{aligned}
$$

A combination of (7) and (8) led to the required integral identity (5) asserted by the Lemma. The integral identity (10) can be proved similarly. The proof is completed. □

Corollary 3. *Putting $\mathbb{T} = \mathbb{R}$ in Lemma 1, we obtain*

$$
\begin{aligned}
\int_{[\ell_1,\ell_2]} \gimel(\jmath)\Phi(\jmath)d_\alpha\jmath &= \int_{[\ell_1,\ell_1+\hat{\Im}]} \Big(\gimel(\jmath)\digamma(\jmath) - [\gimel(\jmath) - \gimel(\ell_1+\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)d_\alpha\jmath \\
&\quad + \int_{[\ell_1+\hat{\Im},\ell_2]} [\gimel(\jmath) - \gimel(\ell_1+\hat{\Im})]\,\Phi(\jmath)d_\alpha\jmath, \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{[\ell_1,\ell_2]} \gimel(\jmath)\Phi(\jmath)d_\alpha\jmath &= \int_{[\ell_1,\ell_2-\hat{\Im}]} [\gimel(\jmath) - \gimel(\ell_2-\hat{\Im})]\,\Phi(\jmath)d_\alpha\jmath \qquad (10) \\
&\quad + \int_{[\ell_2-\hat{\Im},\ell_2]} \Big(\gimel(\jmath)\digamma(\jmath) - [\gimel(\jmath) - \gimel(\ell_2-\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)d_\alpha\jmath.
\end{aligned}
$$

such that

$$\int_{[\ell_1,\ell_1+\hat{\Im}]} \digamma(\jmath)d_\alpha\jmath = \int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha\jmath = \int_{[\ell_2-\hat{\Im},\ell_2]} \digamma(\jmath)d_\alpha\jmath.$$

Theorem 6. *Suppose that $\Re_1$, $\Re_2$, $\Re_6$, $\Re_7$, and $\Re_{13}$ give*

$$
\begin{aligned}
\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \gimel(\jmath)\digamma(\jmath)\Delta_\alpha\jmath &\le \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \Big(\gimel(\jmath)\digamma(\jmath) - [\gimel(\jmath) - \gimel(\ell_2-\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)\Delta_\alpha\jmath \\
&\le \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \gimel(\jmath)\Phi(\jmath)\Delta_\alpha\jmath \\
&\le \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Big(\gimel(\jmath)\digamma(\jmath) - [\gimel(\jmath) - \gimel(\ell_1+\hat{\Im})]\,[\digamma(\jmath) - \Phi(\jmath)]\Big)\Delta_\alpha\jmath \\
&\le \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \gimel(\jmath)\digamma(\jmath)\Delta_\alpha\jmath.
\end{aligned}
$$

Proof. In the perspective of the considerations that the function $\daleth$ is nonincreasing on $[\ell_1, \ell_2]$ and $0 \leq \Phi(\jmath) \leq F(\jmath)$ for all $\jmath \in [\ell_1, \ell_2]$, we infer that

$$\int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}} \left[\daleth(\jmath) - \daleth(\ell_2 - \hat{\Im})\right]\Phi(\jmath)\Delta_\alpha \jmath \geq 0, \tag{11}$$

and

$$\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \left[\daleth(\ell_2 - \hat{\Im}) - \daleth(\jmath)\right]\left[F(\jmath) - \Phi(\jmath)\right]\Delta_\alpha \jmath \geq 0. \tag{12}$$

Using (5), (11), and (12), we find that

$$\begin{aligned} \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath &\leq \int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \Big(\daleth(\jmath)F(\jmath) - \left[\daleth(\jmath) - \daleth(\ell_2 - \hat{\Im})\right]\left[F(\jmath) - \Phi(\jmath)\right]\Big)\Delta_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath. \end{aligned} \tag{13}$$

$$\begin{aligned} \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath &\leq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \Big(\daleth(\jmath)F(\jmath) - \left[\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right]\left[F(\jmath) - \Phi(\jmath)\right]\Big)\Delta_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath, \end{aligned} \tag{14}$$

The confirmation is finished by joining the integral inequalities (13) and (14). □

Corollary 4. *Putting $\mathbb{T} = \mathbb{R}$ in Theorem 6, we have*

$$\begin{aligned} \int_{[\ell_2-\hat{\Im},\ell_2]} \daleth(\jmath)F(\jmath)d_\alpha \jmath &\leq \int_{[\ell_2-\hat{\Im},\ell_2]} \Big(\daleth(\jmath)F(\jmath) - \left[\daleth(\jmath) - \daleth(\ell_2 - \hat{\Im})\right]\left[F(\jmath) - \Phi(\jmath)\right]\Big)d_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_2]} \daleth(\jmath)\Phi(\jmath)d_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_1+\hat{\Im}]} \Big(\daleth(\jmath)F(\jmath) - \left[\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right]\left[F(\jmath) - \Phi(\jmath)\right]\Big)d_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_1+\hat{\Im}]} \daleth(\jmath)F(\jmath)d_\alpha \jmath. \end{aligned}$$

Remark 3. *We can reclaim [48] [Theorem 1] in Corollary 4 by taking $\alpha = 1$.*

Theorem 7. *Assume that $\Re_1$, $\Re_2$, $\Re_6$, $\Re_8$, and $\Re_{13}$ are fulfilled. Then,*

$$\begin{aligned} &\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Big|\left[\daleth(\jmath) - \daleth(\ell_2 - \hat{\Im})\right]\psi(\jmath)\Big|\Delta_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \\ &\leq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Big|\left[\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right]\psi(\jmath)\Big|\Delta_\alpha \jmath, \end{aligned} \tag{15}$$

Proof. Clearly, function $\daleth$ is nonincreasing on $[\ell_1, \ell_2]$ and $0 \leq \psi(\jmath) \leq \Phi(\jmath) \leq F(\jmath) - \psi(\jmath)$ for all $\jmath \in [\ell_1, \ell_2]$, and we obtain

$$\begin{aligned} &\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \left[\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right]\left[F(\jmath) - \Phi(\jmath)\right]\Delta_\alpha \jmath + \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \left[\daleth(\ell_1 + \hat{\Im}) - \daleth(\jmath)\right]\Phi(\jmath)\Delta_\alpha \jmath \\ &= \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \left|\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right|\left[F(\jmath) - \Phi(\jmath)\right]\Delta_\alpha \jmath + \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \left|\daleth(\ell_1 + \hat{\Im}) - \daleth(\jmath)\right|\Phi(\jmath)\Delta_\alpha \jmath \\ &\geq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \left|\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right|\psi(\jmath)\Delta_\alpha \jmath + \int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}} \left|\daleth(\ell_1 + \hat{\Im}) - \daleth(\jmath)\right|\psi(\jmath)\Delta_\alpha \jmath \\ &\geq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Big|\left[\daleth(\jmath) - \daleth(\ell_1 + \hat{\Im})\right]\psi(\jmath)\Big|\Delta_\alpha \jmath. \end{aligned}$$

Additionally,

$$\int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}}\left[\daleth(\jmath)-\daleth(\ell_1+\hat{\Im})\right]\left[F(\jmath)-\Phi(\jmath)\right]\Delta_\alpha\jmath+\int_{[\ell_1+\hat{\Im},\ell_2]_{\mathbb{T}}}\left[\daleth(\ell_1+\hat{\Im})-\daleth(\jmath)\right]\Phi(\jmath)\Delta_\alpha\jmath$$
$$\geq\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\left|\left[\daleth(\jmath)-\daleth(\ell_1+\hat{\Im})\right]\psi(\jmath)\right|\Delta_\alpha\jmath. \tag{16}$$

Similarly, we find that

$$\int_{[\ell_1,\ell_2-\hat{\Im}]_{\mathbb{T}}}\left[\daleth(\jmath)-\daleth(\ell_2-\hat{\Im})\right]\Phi(\jmath)\Delta_\alpha\jmath+\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}}\left[\daleth(\ell_2-\hat{\Im})-\daleth(\jmath)\right]\left[F(\jmath)-\Phi(\jmath)\right]\Delta_\alpha\jmath$$
$$\geq\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\left|\left[\daleth(\jmath)-\daleth(\ell_2-\hat{\Im})\right]\psi(\jmath)\right|\Delta_\alpha\jmath. \tag{17}$$

By combining (5), (10), (16), and (17), we arrive at the inequality (15) asserted by Theorem 7. □

Corollary 5. *Putting $\mathbb{T}=\mathbb{R}$ in Theorem 7, we have*

$$\begin{aligned}&\int_{[\ell_2-\hat{\Im},\ell_2]}\daleth(\jmath)F(\jmath)d_\alpha\jmath+\int_{[\ell_1,\ell_2]}\left|\left[\daleth(\jmath)-\daleth(\ell_2-\hat{\Im})\right]\psi(\jmath)\right|d_\alpha\jmath\\&\leq\int_{[\ell_1,\ell_2]}\daleth(\jmath)\Phi(\jmath)d_\alpha\jmath\\&\leq\int_{[\ell_1,\ell_1+\hat{\Im}]}\daleth(\jmath)F(\jmath)d_\alpha\jmath-\int_{[\ell_1,\ell_2]}\left|\left[\daleth(\jmath)-\daleth(\ell_1+\hat{\Im})\right]\psi(\jmath)\right|d_\alpha\jmath.\end{aligned}$$

Remark 4. *If we take $\alpha=1$, in Corollary 5, we recapture [48] [Theorem 2].*

Theorem 8. *Let $\Re_1$, $\Re_2$, $\Re_6$, and $\Re_9$ be satisfied, and*

$$0\leq\hat{\Im}_1\leq\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath\leq\hat{\Im}_2\leq\ell_2-\ell_1.$$

Then,

$$\begin{aligned}&\int_{[\ell_2-\hat{\Im}_1,\ell_2]_{\mathbb{T}}}\daleth(\jmath)\Delta_\alpha\jmath+\daleth(\ell_2)\left(\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath-\hat{\Im}_1\right)\\&+M\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\left|\daleth(\jmath)-f\left(\ell_2-\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath\right)\right|\Delta_\alpha\jmath\\&\leq\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\daleth(\jmath)\Phi(\jmath)\Delta_\alpha\jmath\\&\leq\int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}}\daleth(\jmath)\Delta_\alpha\jmath-\daleth(\ell_2)\left(\hat{\Im}_2-\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath\right)\\&-M\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\left|\daleth(\jmath)-f\left(\ell_1+\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath\right)\right|\Delta_\alpha\jmath.\end{aligned} \tag{18}$$

Proof. By using straightforward calculations, we have

$$\begin{aligned}&\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\daleth(\jmath)\Phi(\jmath)\Delta_\alpha\jmath-\int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}}\daleth(\jmath)\Delta_\alpha\jmath+\daleth(\ell_2)\left(\hat{\Im}_2-\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath\right)\\&=\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\daleth(\jmath)\Phi(\jmath)\Delta_\alpha\jmath-\int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}}\daleth(\jmath)\Delta_\alpha\jmath+\int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}}\daleth(\ell_2)\Delta_\alpha\jmath-\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\daleth(\ell_2)\Phi(\jmath)\Delta_\alpha\jmath\\&=\int_{[\ell_1,\ell_2]_{\mathbb{T}}}[\daleth(\jmath)-\daleth(\ell_2)]\Phi(\jmath)\Delta_\alpha\jmath-\int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}}[\daleth(\jmath)-\daleth(\ell_2)]\Delta_\alpha\jmath\\&\leq\int_{[\ell_1,\ell_2]_{\mathbb{T}}}[\daleth(\jmath)-\daleth(\ell_2)]\Phi(\jmath)\Delta_\alpha\jmath-\int_{[\ell_1,\ell_1+\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha\jmath]}[\daleth(\jmath)-\daleth(\ell_2)]\Delta_\alpha\jmath,\end{aligned} \tag{19}$$

where we used the theorem's hypotheses

$$\ell_1 \le \ell_1 + \hat{\Im}_1 \le \ell_1 + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath \le \ell_1 + \hat{\Im}_2 \le \ell_2$$

and

$$\beth(\jmath) - \beth(\ell_2) \ge 0 \quad \textit{for all} \quad \jmath \in [\ell_1, \ell_2].$$

The function $\beth(\jmath) - \beth(\ell_2)$ is nonincreasing and integrable on $[\ell_1, \ell_2]$, and by applying Theorem 7 with $F(\jmath) = 1$, $\psi(\jmath) = M$ and $\beth(\jmath)$ replaced by $\beth(\jmath) - \beth(\ell_2)$,

$$\begin{aligned}&\int_{[\ell_1,\ell_2]_{\mathbb{T}}} [\beth(\jmath) - \beth(\ell_2)]\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_1+\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha \jmath]} [\beth(\jmath) - \beth(\ell_2)]\Delta_\alpha \jmath \\ &\le -M\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\beth(\jmath) - f\left(\ell_1 + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath\right)\right|\Delta_\alpha \jmath.\end{aligned} \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned}&\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_1+\hat{\Im}_2]_{\mathbb{T}}} \beth(\jmath)\Delta_\alpha \jmath + \beth(\ell_2)(\hat{\Im}_2 - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath) \\ &\le -M\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\beth(\jmath) - f\left(\ell_1 + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath\right)\right|\Delta_\alpha \jmath,\end{aligned} \tag{21}$$

which is the right-hand side inequality in (18).
Similarly, one can show that

$$\begin{aligned}&\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \beth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath - \int_{[\ell_2-\hat{\Im}_1,\ell_2]_{\mathbb{T}}} \beth(\jmath)\Delta_\alpha \jmath + \beth(\ell_2)\left(\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath - \hat{\Im}_2\right) \\ &\ge \int_{[\ell_1,\ell_2]_{\mathbb{T}}} [\beth(\jmath) - \beth(\ell_2)]\Phi(\jmath)\Delta_\alpha \jmath + \int_{[\ell_2-\int_{[\ell_1,\ell_2]_{\mathbb{T}}}\Phi(\jmath)\Delta_\alpha \jmath,\ell_2]} [\beth(\ell_2) - \beth(\jmath)]\Delta_\alpha \jmath \\ &\ge M\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\beth(\jmath) - f\left(\ell_2 - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \Phi(\jmath)\Delta_\alpha \jmath\right)\right|\Delta_\alpha \jmath,\end{aligned} \tag{22}$$

which is the left-hand side inequality in (18). □

Corollary 6. *Putting $\mathbb{T} = \mathbb{R}$ in Theorem 8, we obtain*

$$\begin{aligned}&\int_{[\ell_2-\hat{\Im}_1,\ell_2]} \beth(\jmath)d_\alpha \jmath + \beth(\ell_2)\left(\int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha \jmath - \hat{\Im}_1\right) \\ &+M\int_{[\ell_1,\ell_2]} \left|\beth(\jmath) - f\left(\ell_2 - \int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha \jmath\right)\right|d_\alpha \jmath \\ &\le \int_{[\ell_1,\ell_2]} \beth(\jmath)\Phi(\jmath)d_\alpha \jmath \\ &\le \int_{[\ell_1,\ell_1+\hat{\Im}_2]} \beth(\jmath)d_\alpha \jmath - \beth(\ell_2)\left(\hat{\Im}_2 - \int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha \jmath\right) \\ &-M\int_{[\ell_1,\ell_2]} \left|\beth(\jmath) - f\left(\ell_1 + \int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha \jmath\right)\right|d_\alpha \jmath,\end{aligned}$$

such that

$$0 \le \hat{\Im}_1 \le \int_{[\ell_1,\ell_2]} \Phi(\jmath)d_\alpha \jmath \le \hat{\Im}_2 \le \ell_2 - \ell_1.$$

Remark 5. *[48] [Theorem 3] can be obtained if we put $\alpha = 1$ in Corollary 6.*

Theorem 9. *If $\Re_1$, $\Re_2$, $\Re_6$, $\Re_7$, and $\Re_{14}$ hold, then*

$$\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \leq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\left(\daleth(\jmath) - \daleth(\ell_1+\hat{\Im})\right)\psi(\jmath)\right|\Delta_\alpha \jmath. \quad (23)$$

Proof. Follow a similar to the proof of the right-hand side inequality in Theorem 7. □

Corollary 7. $\mathbb{T}=\mathbb{R}$ *in Theorem, and we get*

$$\int_{[\ell_1,\ell_2]} \daleth(\jmath)\Phi(\jmath)d_\alpha \jmath \leq \int_{[\ell_1,\ell_1+\hat{\Im}]} \daleth(\jmath)F(\jmath)d_\alpha \jmath - \int_{[\ell_1,\ell_2]} \left|\left(\daleth(\jmath) - \daleth(\ell_1+\hat{\Im})\right)\psi(\jmath)\right|d_\alpha \jmath.$$

Remark 6. *If we take $\alpha = 1$, in Corollary 7, we recapture [47] [Theorem 2.12].*

Corollary 8. *Hypotheses $\Re_1$, $\Re_2$, $\Re_3$, $\Re_{10}$, and $\Re_{11}$ yield*

$$\int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath \leq \int_{[\ell_1,\ell_1+\hat{\Im}]_{\mathbb{T}}} \daleth(\jmath)\Delta_\alpha \jmath - \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\left(\frac{\daleth(\jmath)}{F(\jmath)} - \frac{\daleth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\right)F(\jmath)\psi(\jmath)\right|\Delta_\alpha \jmath. \quad (24)$$

Proof. Insert $\Phi(\jmath) \mapsto F(\jmath)\Phi(\jmath)$, $\daleth(\jmath) \mapsto \daleth(\jmath)/F(\jmath)$ and $\psi(\jmath) \mapsto F(\jmath)\psi(\jmath)$ in Theorem 9. □

Corollary 9. $\mathbb{T}=\mathbb{R}$ *in Corollary 8, we have*

$$\int_{[\ell_1,\ell_2]} \daleth(\jmath)\Phi(\jmath)d_\alpha \jmath \leq \int_{[\ell_1,\ell_1+\hat{\Im}]} \daleth(\jmath)d_\alpha \jmath - \int_{[\ell_1,\ell_2]} \left|\left(\frac{\daleth(\jmath)}{F(\jmath)} - \frac{\daleth(\ell_1+\hat{\Im})}{F(\ell_1+\hat{\Im})}\right)F(\jmath)\psi(\jmath)\right|d_\alpha \jmath.$$

Remark 7. *[47] [Corollary 2.3] can be recovered with the help of $\alpha = 1$, in Corollary 9.*

Theorem 10. *If $\Re_1$, $\Re_2$, $\Re_6$, $\Re_7$, and $\Re_{15}$ hold, then*

$$\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \daleth(\jmath)F(\jmath)\Delta_\alpha \jmath + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\left(\daleth(\jmath) - \daleth(\ell_2-\hat{\Im})\right)\psi(\jmath)\right|\Delta_\alpha \jmath \leq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath. \quad (25)$$

Proof. Carry out the same proof of the left-hand side inequality in Theorem 7. □

Corollary 10. $\mathbb{T}=\mathbb{R}$ *in Theorem 10, and we have*

$$\int_{[\ell_2-\hat{\Im},\ell_2]} \daleth(\jmath)F(\jmath)d_\alpha \jmath + \int_{[\ell_1,\ell_2]} \left|\left(\daleth(\jmath) - \daleth(\ell_2-\hat{\Im})\right)\psi(\jmath)\right|d_\alpha \jmath \leq \int_{[\ell_1,\ell_2]} \daleth(\jmath)\Phi(\jmath)d_\alpha \jmath.$$

Remark 8. *If we take $\alpha = 1$, in Corollary 10, we recapture [47] [Theorem 2.13].*

Corollary 11. *Let $\Re_1$, $\Re_2$, $\Re_3$, $\Re_9$, and $\Re_{12}$ be fulfilled. Then,*

$$\int_{[\ell_2-\hat{\Im},\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Delta_\alpha \jmath + \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \left|\left(\frac{\daleth(\jmath)}{F(\jmath)} - \frac{\daleth(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\right)F(\jmath)\psi(\jmath)\right|\Delta_\alpha \jmath \leq \int_{[\ell_1,\ell_2]_{\mathbb{T}}} \daleth(\jmath)\Phi(\jmath)\Delta_\alpha \jmath. \quad (26)$$

Proof. The proof can be completed by taking $\Phi(\jmath) \mapsto F(\jmath)\Phi(\jmath)$, $\daleth(\jmath) \mapsto \daleth(\jmath)/F(\jmath)$, and $\psi(\jmath) \mapsto F(\jmath)\psi(\jmath)$ in Theorem 10. □

Corollary 12. $\mathbb{T}=\mathbb{R}$ *in Corollary 11, and we have*

$$\int_{[\ell_2-\hat{\Im},\ell_2]} \daleth(\jmath)d_\alpha \jmath + \int_{[\ell_1,\ell_2]} \left|\left(\frac{\daleth(\jmath)}{F(\jmath)} - \frac{\daleth(\ell_2-\hat{\Im})}{F(\ell_2-\hat{\Im})}\right)F(\jmath)\psi(\jmath)\right|d_\alpha \jmath \leq \int_{[\ell_1,\ell_2]} \daleth(\jmath)\Phi(\jmath)d_\alpha \jmath.$$

Remark 9. *By letting $\alpha = 1$, in Corollary 12, we recapture [47] [Corollary 2.4].*

3. Conclusions

In this important work, we discussed some new dynamic inequalities of Steffensen-type using delta integral on time scales. By employing the conformable fractional α-integral on time scales, several α-conformable Steffensen-type inequalities on time scales are proved. Our proposed results show the potential for producing some original continuous, discrete, and quantum inequalities. We further presented some relevant inequalities as special cases: discrete inequalities and integral inequalities. These results may be used to obtain more generalized results of several obtained inequalities before. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

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